Group: MI4

Machine Intelligence 1 — **Exercise Sheet 2**
WS 2020/21, Obermayer/Kashef — due: 16.11.2020 at 23:55

Connectionist Neurons and Multi Layer Perceptrons

Please remember to upload exactly *one* ZIP file per group and name the file according to the respective group name: `yourgroupname.zip`
The ZIP file should contain a single Jupyter notebook source file as well as a single PDF file that is generated from the Jupyter notebook. Please do **not** include any folder structure, exercise PDF or data files.

Exercise H2.1: Connectionist Neuron (homework, 6 points)

The dataset[1] `applesOranges.csv` contains 200 measurements (`x.1` and `x.2`) from two types of objects as indicated by the column `y`. In this exercise, you will use a connectionist neuron with a "binary" transfer function $f(h)$ to classify the objects, i.e., obtain the predicted class y for a data point $\underline{\mathbf{x}} \in \mathbb{R}^2$ by

$$y(\underline{\mathbf{x}}) := f(\underline{\mathbf{w}}^\top \underline{\mathbf{x}} - \theta)$$

with

$$f(h) := \begin{cases} 1 & \text{for } h \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

where $h := \underline{\mathbf{w}}^\top \underline{\mathbf{x}} - \theta$ is the total input to the neuron.

```
In [ ]: import numpy as np
        import math
        import matplotlib.pyplot as plt
        %matplotlib inline
```

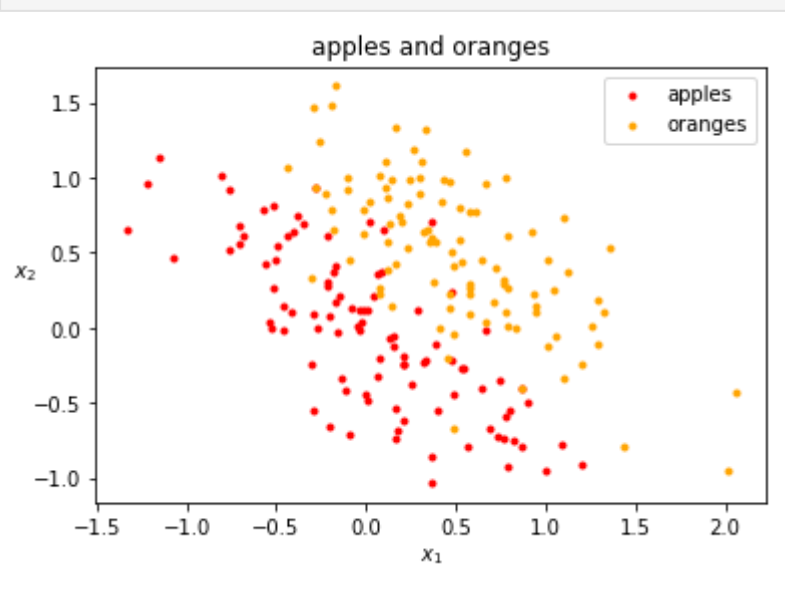
```
In [ ]: from google.colab import drive
        drive.mount('/content/drive/')
```

```
Drive already mounted at /content/drive/; to attempt to forcibly remount, call drive.mount("/content/driv
e/", force_remount=True).
```

```
In [ ]: PATH_TO_DATA = '/content/drive/My Drive/BCCN/Courses/Machine Intelligence/Machine Intelligence I/Exercise Sh
```

(a) Plot the data in a scatter plot (x2 vs. x1). Mark the points with different colors to indicate the type of each object.

```
In [ ]: raw_data = np.genfromtxt(fname=PATH_TO_DATA, delimiter=',', dtype=np.float, skip_header=1)
        x = raw_data[:,:2]
        y = raw_data[:,-1]
        plt.scatter(x[y==0,0], x[y==0,1], marker='.', c='r', label='apples')
        plt.scatter(x[y==1,0], x[y==1,1], marker='.', c='orange', label='oranges')
        lgd = plt.legend()
        plt.xlabel('$x_1$')
        plt.ylabel('$x_2$', rotation='horizontal')
        plt.title("apples and oranges")
        plt.show()
```



(b) Set the bias θ = 0. Create a set of 19 weight vectors w = (w1, w2) pointing from the origin to the upper semi-circle with radius 1 (i.e. if γ denotes the angle between the weight vector and the x-axis, for each γ = 0, 10, . . . , 180 (equally spaced) such that ||w||2 = 1, w1 ∈ [−1, 1], w2 ∈ [0, 1]). For each of these weight vectors w,

(i) determine % correct classifications ρ of the corresponding neuron and

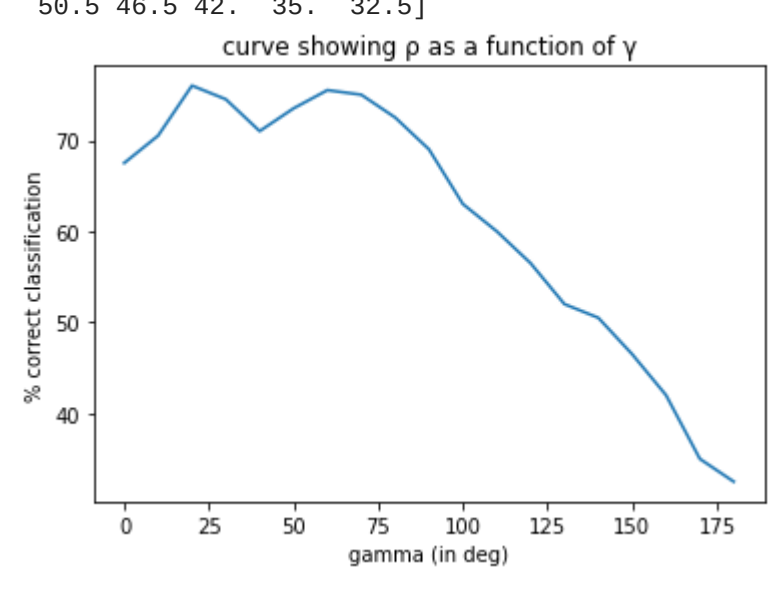
(ii) plot a curve showing ρ as a function of γ.

```
In [ ]: theta = 0

        gamma = np.arange(0, (190*math.pi)/180, (10*math.pi)/180)
        w = np.zeros((19,2))
        w1 = np.round(np.cos(gamma), 4)
        w2 = np.round(np.sin(gamma), 4)
        w[:,0] = w1
        w[:,1] = w2

        print(w)
```

```
[[ 1.      0.    ]
 [ 0.9848  0.1736]
 [ 0.9397  0.342 ]
 [ 0.866   0.5   ]
 [ 0.766   0.6428]
 [ 0.6428  0.766 ]
 [ 0.5     0.866 ]
 [ 0.342   0.9397]
 [ 0.1736  0.9848]
 [ 0.      1.    ]
 [-0.1736  0.9848]
 [-0.342   0.9397]
 [-0.5     0.866 ]
 [-0.6428  0.766 ]
 [-0.766   0.6428]
 [-0.866   0.5   ]
 [-0.9397  0.342 ]
 [-0.9848  0.1736]
 [-1.      0.    ]]
```

```
In [ ]: h = np.dot(w,x.T) - theta

        #activation function
        f = np.where(h > 0, 1, 0)

        percentage_correct_classification = (np.sum(f==y, axis=1)/(200))*100
        print(percentage_correct_classification)

        plt.plot(gamma*180/math.pi, percentage_correct_classification.T)
        plt.xlabel('gamma (in deg)')
        plt.ylabel('% correct classification')
        plt.title("curve showing ρ as a function of γ")
        plt.show()
```

```
[67.5 70.5 76.  74.5 71.  73.5 75.5 75.  72.5 69.  63.  60.  56.5 52.
 50.5 46.5 42.  35.  32.5]
```



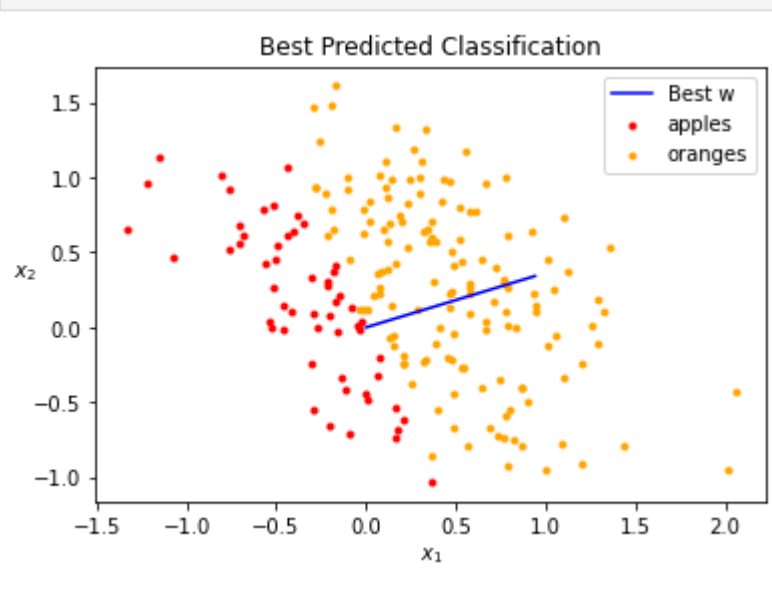
(c) Out of the 19 weight vectors from above, pick the w that yields the best performance. Now, vary the bias θ ∈ [−3, 3] and pick the value of θ that gives the best performance.

```
In [ ]: index_p_max = np.argmax(percentage_correct_classification)
        w_best = w[index_p_max,:]

        p_max = -1
        for theta_ in range(-3, 4, 1):
          h_temp = np.dot(w_best,x.T) - theta_
          f_temp = np.where(h_temp > 0, 1, 0)
          p_temp = np.sum(f_temp == y)
          if p_temp > p_max:
            p_max = p_temp
            theta_best = theta_
            f_best = f_temp

        print(theta_best)
```

```
0
```

(d) Plot the data points and color them according to the predicted classification when using the w and θ that led to the highest performance. Plot the weight vector w in the same plot. How do you interpret your results?

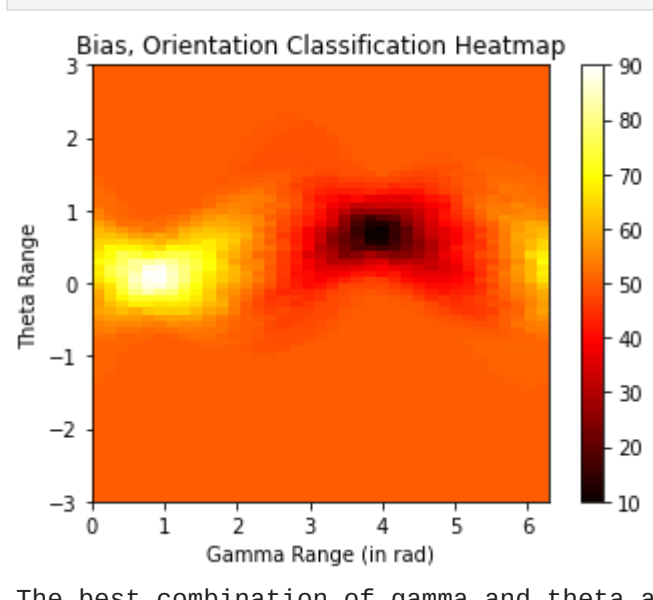
```
In [ ]: plt.scatter(x[f_best==0,0], x[f_best==0,1], marker='.', c='r', label='apples')
        plt.scatter(x[f_best==1,0], x[f_best==1,1], marker='.', c='orange', label='oranges')
        plt.xlabel('$x_1$')
        plt.ylabel('$x_2$', rotation='horizontal')
        plt.title("Best Predicted Classification")
        plt.plot([0, w_best[0]] , [0, w_best[1]] , c='blue', label='Best w')
        lgd = plt.legend()
        plt.show()
```



The weight vector defines the optimal decision boundary (at least in the grid space defined) that maximises the classification accuracy of apples/oranges.

(e) Find the best combination of w and θ by exploring all combinations of γ and θ (within a reasonable range and precision). Compute and plot the performance of all combinations in a heatmap.

```
In [86]: theta_array = np.arange(-3, 4, 0.1)
         gamma_array = np.arange(0, (370*math.pi)/180, (10*math.pi)/180)

         percentage_correct_classification_best = -1

         match_mat = np.empty([len(theta_array), len(gamma_array)])


         for i in range(len(theta_array)):

             for j in range(len(gamma_array)):

                 w1 = np.round(np.cos(gamma_array[j]), 4)
                 w2 = np.round(np.sin(gamma_array[j]), 4)
                 weight = np.zeros((1,2))
                 weight[:,0] = w1
                 weight[:,1] = w2

                 h = np.dot(weight,x.T) - theta_array[i]
                 f = np.where(h > 0, 1, 0)

                 percentage_correct_classification = (np.sum(f==y, axis=1)/(200))*100

                 match_mat[i][j] = percentage_correct_classification


                 if percentage_correct_classification>percentage_correct_classification_best:
                     percentage_correct_classification_best = percentage_correct_classification
                     gamma_best = gamma_array[j]
                     theta_best = theta_array[i]

         plt.imshow(match_mat, cmap='hot', interpolation='nearest', extent = [0, 6.28, -3, 3])
         plt.colorbar()
         plt.xlabel('Gamma Range (in rad)')
         plt.ylabel('Theta Range')
         plt.title("Bias, Orientation Classification Heatmap")
         plt.show()
         print("The best combination of gamma and theta are %s and %s respectively." % (gamma_best, theta_best))
         print(np.round(np.cos(gamma_best), 4))
         print("The best weight is w1: %s, w2:%s with a correct classification of %s percent." % (np.round(np.cos(gam
```




```
The best combination of gamma and theta are 0.6981317007977318 and 0.300000000000000293 respectively.
0.766
The best weight is w1: 0.766, w2:0.6428 with a correct classification of 90.0 percent.
```

Exercise H2.2: Multilayer Perceptrons (MLP) (homework, 4 points)

For an MLP with input $x \in \mathbb{R}$ and 1 hidden layer and 1 output node. The input-output function can be computed as

$$y(x) = \sum_{i=1}^{N_{\text{hid}}} w_{1i}^{21} f(w_{i1}^{10} x - b_i))$$

with output weights w_{1i}^{21} and parameters w_{i1}^{10} and b_i for the i-th hidden unit. In this case, the output node has no bias.

(a) Create 50 independent MLPs with $N_{\text{hid}} = 10$ hidden units by sampling for each MLP a set of random parameters $\{w_{1i}^{21}, w_{i1}^{10}, b_i\}, i = 1, ..., 10$.

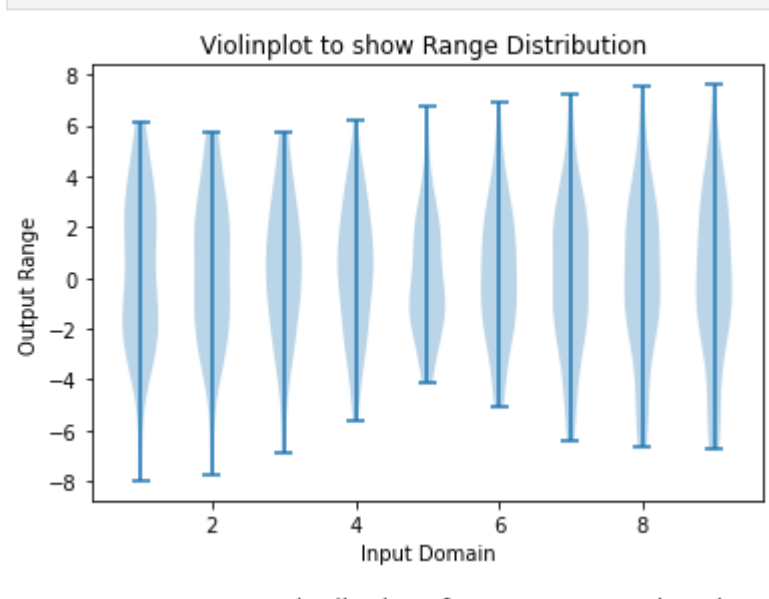
- Use $f(\cdot) := \tanh(\cdot)$ as the transfer function.
- Use normally distributed $w_{1i}^{21} \sim \mathcal{N}(0, 1)$
- Use normally distributed $w_{i1}^{10} \sim \mathcal{N}(0, 2)$ and uniformly distributed $b_i \sim \mathcal{U}(-2, 2)$.

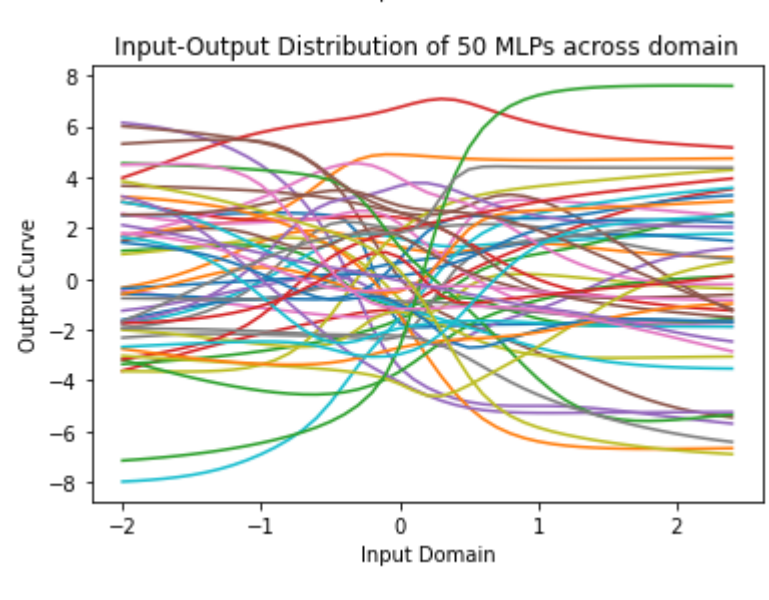
```
In [ ]: def random_weight_initializer(m1, m2 , s1, s2, num, N_hid=10):

            #num : number of independent MLPs
            # m1 : mean 1
            # m2 : mean 2
            # s1 : s.d 1
            # s2 : s.d 2

            w10  = np.random.normal(m1 , s1 , (N_hid, num))
            w21  = np.random.normal(m2 , s2 , (N_hid, num))
            b    = np.random.uniform(-2, 2 , (N_hid, num))

            return w10, w21, b

        def MLP(x, w10, w21, b):

            y = np.diag(np.matmul(w21.T , np.tanh(w10*x - b))  )
            return y


        #test
        w10 ,  w21 , b = random_weight_initializer(0,0,2,1, 50)
        MLP_ = [MLP(x, w10, w21, b) for x in np.random.randint(-2,2,1)]
        MLP_
```

```
Out[ ]: [array([ 0.85052722,  2.12286348, -2.05343418,  0.44298145, -4.30034271,
                -0.01106092,  1.68153694, -1.2280478 , -0.32625692,  1.68217435,
                -3.83349852,  0.78835377, -0.59292338,  2.03789786, -5.06746315,
                -3.97072335, -2.9379153 , -2.03443286,  0.08076328, -0.01056753,
                -1.79420396,  2.29550856, -1.69563985, -0.06470114,  4.62183284,
                -0.81793529,  1.26330974, -0.01733686, -0.59931302,  0.1720737 ,
                -3.10596353, -1.0619503 ,  1.80300426, -4.51539172,  0.15563199,
                 1.78963338,  3.97862954,  0.81315732, -3.4434027 ,  2.52241046,
                -4.87489973,  2.29547619, -1.18013282,  0.28474837, -3.29063026,
                -0.49797493,  3.39983255, -2.95291341, -0.99519493, -5.05552752])]
```

(b) Plot the input-output functions (i.e. the response y(x)) of these 50 MLPs for x ∈ [−2, 2].

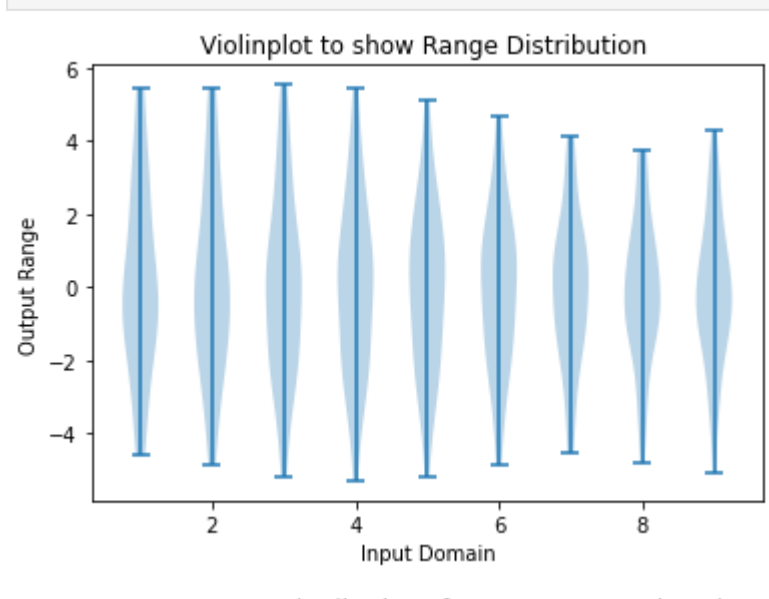
```
In [ ]: w10 ,  w21 , b = random_weight_initializer(0,0,2,1, 50)

        def generate_response(x, w10, w21, b):
          response_array = np.empty([len(x),50])
          for i in range(len(x)) :
              response_array[i] = MLP(x[i], w10, w21, b)
          return response_array

        def violin_plot(input_array):
          plt.violinplot(input_array.T)
          plt.xlabel('Input Domain')
          plt.ylabel('Output Range')
          plt.title('Violinplot to show Range Distribution ')
          plt.show()

        def input_output_distribution(x, input_array):
          for i in range(50):
              plt.plot(x, input_array[:,i])
          plt.xlabel('Input Domain')
          plt.ylabel('Output Curve')
          plt.title('Input-Output Distribution of 50 MLPs across domain')
          plt.show()

        x = np.arange(-2,2.5,0.5)
        violin_plot(generate_response(x, w10, w21, b))

        x = np.arange(-2,2.5,0.1)
        input_output_distribution(x, generate_response(x, w10, w21, b))
```

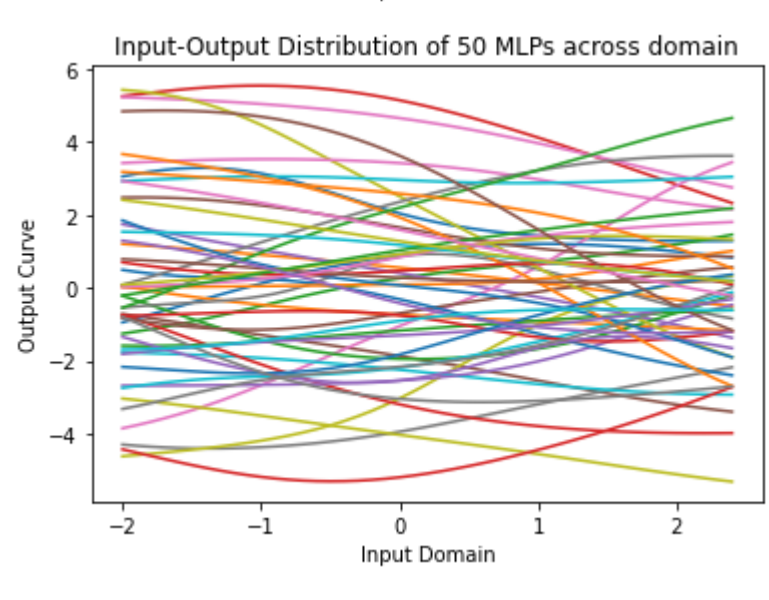




(c) Repeat this procedure using a different intialization scheme for the weights of the hidden neurons:
$w_{i1}^{10} \sim \mathcal{N}(0, 0.5)$. What difference can you observe?

```
In [ ]: w10_, w21_, b_ = random_weight_initializer(0,0,0.5,1, 50)

        response_array = np.empty([len(x),50])
        for i in range(len(x)) :
            response_array[i] = MLP(x[i], w10_,  w21_, b_ )

        x = np.arange(-2,2.5,0.5)
        violin_plot(generate_response(x, w10_, w21_, b_))

        x = np.arange(-2,2.5,0.1)
        input_output_distribution(x, generate_response(x, w10_, w21_, b_))
```





With a lower standard deviation on our weight distribution, we get more weights centered around 0. As a result, in the limit of small input values x, the linear combination of across hidden nodes with W_10 is also small. And in the small limit, hyperbolic tangent behaves linearly and therefore, we see that the distribution of MLPs is more uniformly distributed. In other words, the MLPs do a poorer job at being universal function approximators across the input domain, something we expect of MLPs with linear activation functions. (Higher the standard deviation on the weight initialisation, denser do we cover the possible functions on a specified domain.)

(d) Compute the mean squared error (MSE) between each of these 2 × 50 (50 from each of the above two initialization procedures) input-output functions and the function g(x) = −x. For each of the two initialization procedures, which MLP approximates g best? Plot y(x) for these two MLPs.

```
In [ ]: x=np.linspace(-2,2.5,100)

        g=-x

        MLP_ = np.empty([len(x),50])
        MLP_new= np.empty([len(x),50])
        mse_1=np.empty([50,1])
        mse_2=np.empty([50,1])
        for i in range(len(x)) :
            MLP_[i,:]= MLP(x[i], w10, w21, b)
            MLP_new[i,:]=MLP(x[i], w10_, w21_, b_)

        for i in range(50):
            difference_array_1= np.subtract(g, MLP_[:,i])
            squared_array_1 = np.square(difference_array_1)
            mse_1[i] = squared_array_1.mean()
            difference_array_2 = np.subtract(g, MLP_new[:,i])
            squared_array_2 = np.square(difference_array_2)
            mse_2[i] = squared_array_2.mean()

        print("The best function approximator of g(x) with initialisation in part b is MLP(",np.argmin(mse_1),") and
        print("The best function approximator of g(x) with initialisation in part c is MLP(",np.argmin(mse_2),")and

        plt.plot(x,MLP_[:,np.argmin(mse_1)],c='g', label='Initialisation from part b')
        plt.plot(x,MLP_new[:,np.argmin(mse_2)],c='b', label='Initialisation from part c')
        plt.plot(x,g,c='r', label='g(x)')
        plt.xlabel("x")
        plt.ylabel("y(x)")
        plt.title("Best function approximators")
        lgd = plt.legend()
        plt.show()
```

```
The best function approximator of g(x) with initialisation in part b is MLP( 44 ) and the mean square error
is: [0.0076547]
The best function approximator of g(x) with initialisation in part c is MLP( 30 )and the mean square error i
s: [0.18400767]
```