Assignment I - CompStat2023

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Set up

```
library(ggplot2)
library(patchwork)
library(tidyverse)
```

```
## -- Attaching packages --------------------------------------- tidyverse 1.3.2 --
## v tibble  3.1.7     v dplyr   1.0.9
## v tidyr   1.2.0     v stringr 1.4.0
## v readr   2.1.2     v forcats 0.5.2
## v purrr   0.3.4
## -- Conflicts ------------------------------------------ tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
```

```
library(mixtools)
```

```
## mixtools package, version 2.0.0, Released 2022-12-04
## This package is based upon work supported by the National Science Foundation under Grant No. SES-0518
```

Simulation problem: `RISIKO!`

Exeercise 1

Point a

Compute $P[\max(X_1, X_2) > Y_1]$.

We know that

$$Z = max(X_1, X_2) \in (1, 2, 3, 4, 5, 6)$$

and that

$$P(Z = z) = \frac{(2(z-1)+1)}{36}$$

. In other word we want to find $P(Z > y) = 1 - P(Z \leq y)$. $P(Z \leq y) =$
$P(Z = 1 \cap y = 1) +$
$P(Z = 1 \cap y = 2) + P(Z = 2 \cap y = 2) +$
$P(Z = 1 \cap y = 3) + P(Z = 2 \cap y = 3) + P(Z = 3 \cap y = 3) +$
$P(Z = 1 \cap y = 4) + P(Z = 2 \cap y = 4) + P(Z = 3 \cap y = 4) + P(Z = 4 \cap y = 4) +$
$P(Z = 1 \cap y = 5) + P(Z = 2 \cap y = 5) + P(Z = 3 \cap y = 5) + P(Z = 4 \cap y = 5) + P(Z = 5 \cap y = 5) +$
$P(Z = 1 \cap y = 6) + P(Z = 2 \cap y = 6) + P(Z = 3 \cap y = 6) + P(Z = 4 \cap y = 6) + P(Z = 5 \cap y = 6) + P(Z = 6 \cap y = 6)$

So we have $P(Z \leq y) = 6\frac{1}{36}\frac{1}{6} + 5\frac{3}{36}\frac{1}{6} + 4\frac{5}{36}\frac{1}{6} + 3\frac{7}{36}\frac{1}{6} + 2\frac{9}{36}\frac{1}{6} + \frac{11}{36}\frac{1}{6} \approx 0,421$ and

$$P(Z > y) = 1 - 0,421 = 0,579$$

Point b

Here there are some code to simulate a generic `Risiko!` game for different values of competing units. Starting with a function, *combat_round*, that take as arguments: *att_units*, the number of attacking units, *def_units*, the number of defending units and *sim*, the number of scenarios to simulate.

```
set.seed(123)
combat_round <- function(att_units,def_units,sim=10000) {
  Results = rep(NA,sim)
  AS<-att_units
  DS<-def_units
  for(i in 1:sim){
    while(def_units>0 & att_units>0){
      Dnum <- sort(sample(1:6, min(def_units,3),replace = TRUE),decreasing = T)
      Anum <- sort(sample(1:6, min(att_units,3),replace = TRUE),decreasing = T)
      for (j in 1:min(length(Dnum),length(Anum))){
        if(Anum[j]>Dnum[j]){
          def_units<-def_units-1
        }
        else{
          att_units<-att_units-1
        }
      }
    }
    Results[i]<- ifelse(att_units>0,1,0)
    att_units<-AS
    def_units<-DS
  }
  return(mean(Results))
}
```

Point c

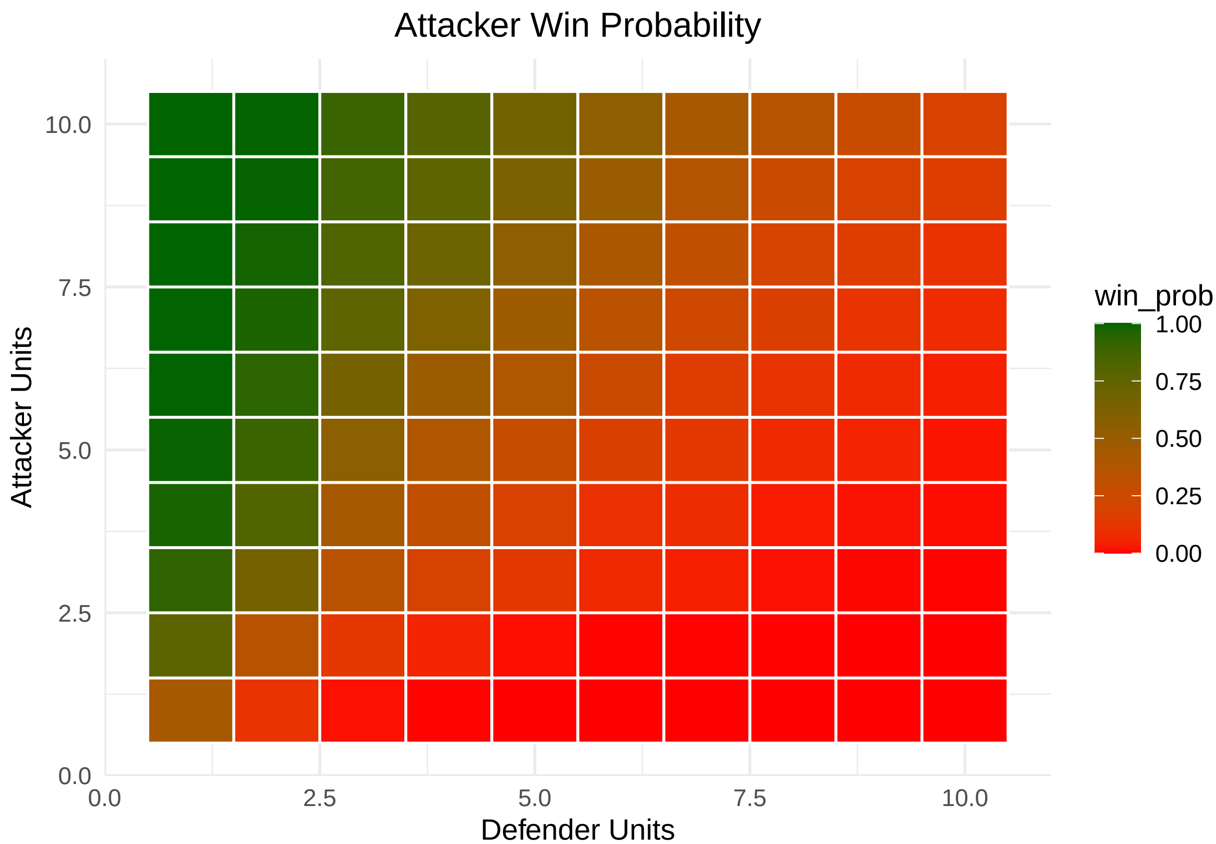
Now we replicate the results a sufficient amount of time equal to 10 to estimate the probability that the attacker has to win the battle for different values of *att_units* and *def_units*.

```
prob_att_win <- function(){
  res<-sapply(1:10, function(x) {
  sapply(1:10, function(y) {
    combat_round(y,x,1000)
    })
  })
  return(res)
}

att_prob<- data.frame(
  attacker_unit<- rep(1:10,10),
  defender_unit<- rep(1:10,each= 10),
  win_prob<- as.vector(prob_att_win())
)
```

Point d

Finally, we display the results in a plot.

```
ggplot(att_prob,aes(x=defender_unit,y=attacker_unit,fill=win_prob))+
  geom_tile()+
  scale_fill_gradient(low="red",high="darkgreen")+
  labs(title = "Attacker Win Probability",x="Defender Units",y="Attacker Units")+
  geom_tile(color = "white",lwd = 0.5,linetype = 1)+
  theme_minimal()+
  theme(plot.title = element_text(hjust = 0.5))
```



The probability we obtain when $att\_units = 2$ and $def\_units = 1$ in the *combat_round* function is different from the one computed analytically in point a.

```
print(combat_round(def_units=1,att_units=2,sim=10000))
```

```
## [1] 0.7561
```

This discrepancy arises due to a difference in the calculations. In point a, we specifically calculated the probability of the attacker winning the battle in the first round. However, in point d, we calculate the probability of the attacker winning the battle overall, without necessarily considering it happening in the first round alone.

Monte Carlo simulations I

Point a

An old and naive algorithm for the generation of Normally distributed random numbers is the following: 1. Generate independent variables as $U_1, ..., U_{12} \sim U[-\frac{1}{2}, \frac{1}{2}]$ 2. Set $Z = \sum_{i=1}^{12} U_i$
The rational here is that 12 realizations are usually enough to exploit the CLT.

We want to show analytically that $E(X) = 0$ and $V(Z) = 1$.

We know that

$$E(X) = \int_a^b \frac{x}{b-a} dx = \frac{b^2 - a^2}{2(b-a)}$$

So $E(Z) = E(\sum_{i=1}^{12} U_i)$
$= \sum_{i=1}^{12} E(U_i)$ independent
$= 12E(U_i)$ identically distributed
$12\frac{\frac{1}{2}^2 - (-\frac{1}{2})^2}{2(\frac{1}{2} - (-\frac{1}{2}))} = 0$

And

$$V(Z) = E[(X - E(X))^2] = \int_a^b (x - \frac{a+b}{2})^2 \frac{dx}{b-a} = \frac{(b-a)^2}{12}$$

So $V(Z) = V(\sum_{i=1}^{12} U_i)$
$= \sum_{i=1}^{12} V(U_i)$ variable independent
$= 12V(U_i)$ U_i identically distributed
$= 12\frac{(\frac{1}{2} - (-\frac{1}{2}))^2}{12}$
$= 12\frac{(\frac{1}{2} + \frac{1}{2})^2}{12} = 1$

Point b

```r
set.seed(13)


normal_functn_gen = function(sim){
  n<-12
  Uz<-runif(sim*n,min = -0.5,max = 0.5)
  Uz<-matrix(Uz,nrow=N,ncol=n)
  Z<-apply(Uz,1,sum)
  return(Z)
}

N <- 10000

gen_norm<-normal_functn_gen(sim=10000)
mean(gen_norm)
```

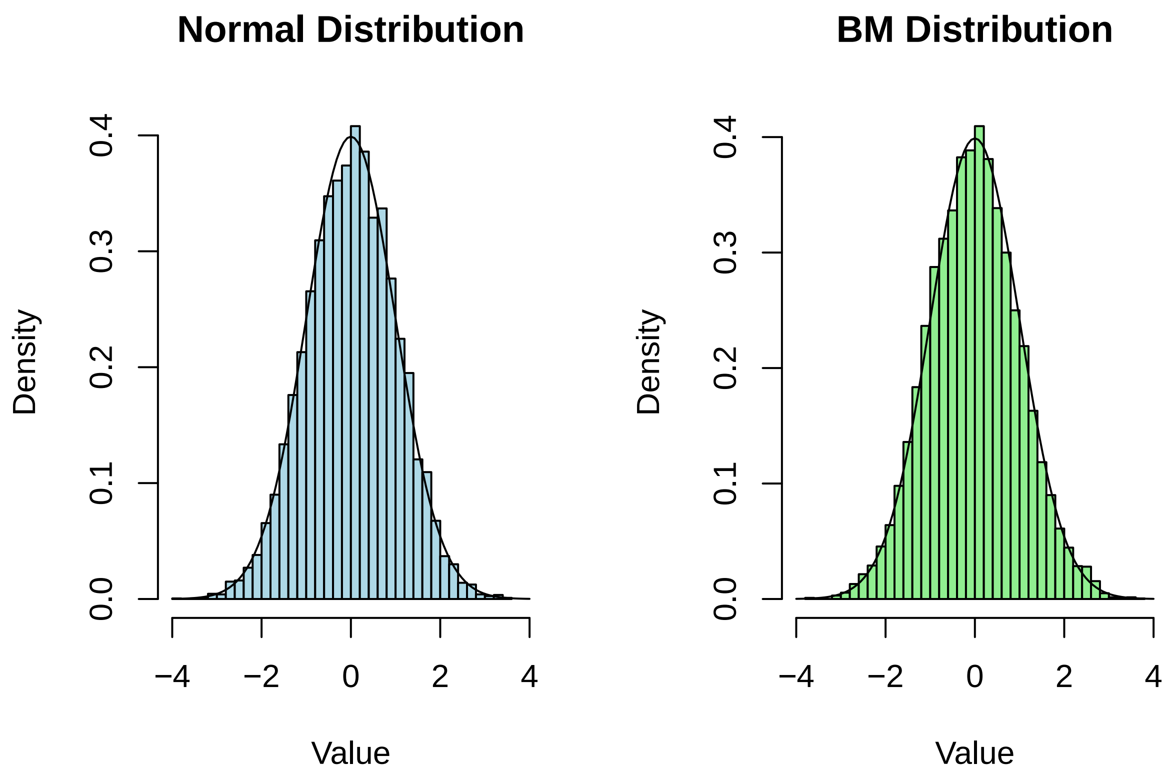
```
## [1] 0.01682814
```

```r
var(gen_norm)
```

```
## [1] 0.9994385
```

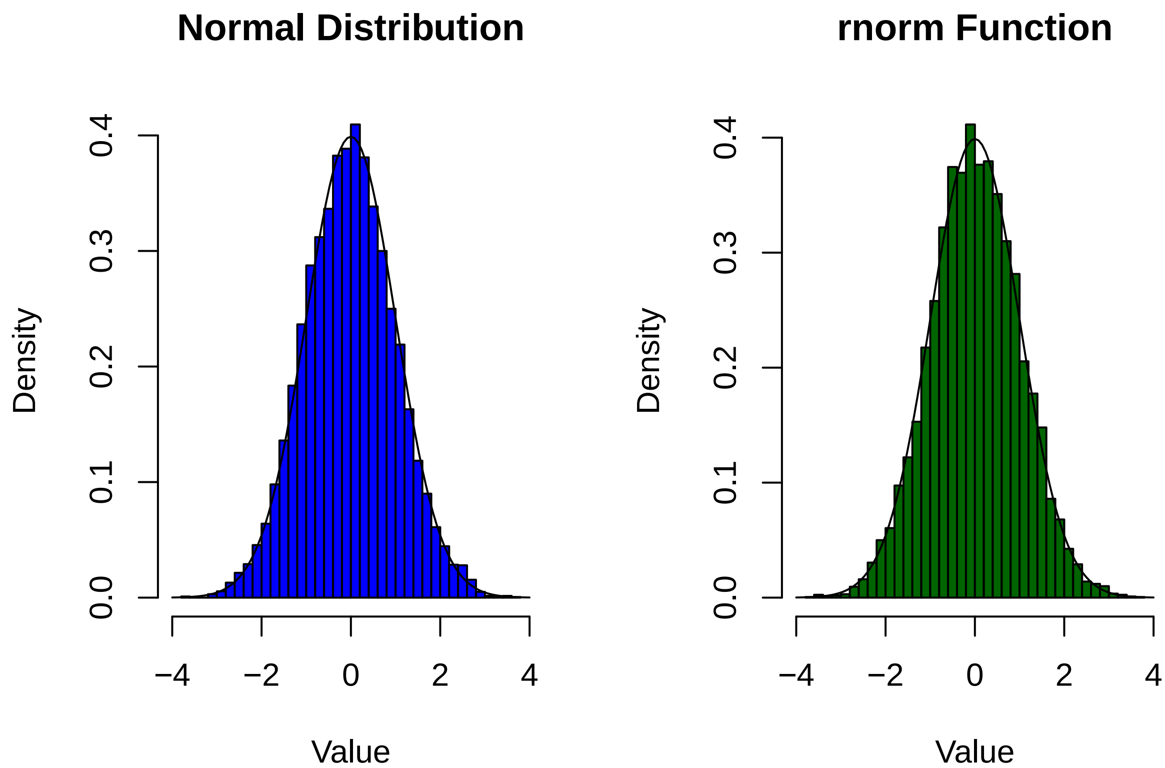
```r
U1 <- runif(N)
U2 <- runif(N)
X1 <- sqrt( -2*log(U1) )*cos(2*pi*U2)
```

```
par(mfrow=c(1,2))
hist(gen_norm, main="Normal Distribution", xlab="Value",col="lightblue" ,breaks=30,freq = F,xlim=c(-4,4
curve(dnorm(x,0,1),add=T)
hist(X1, main="BM Distribution", xlab="Value",col="lightgreen" ,breaks=30,freq = F,xlim=c(-4,4))
curve(dnorm(x,0,1),add=T)
```

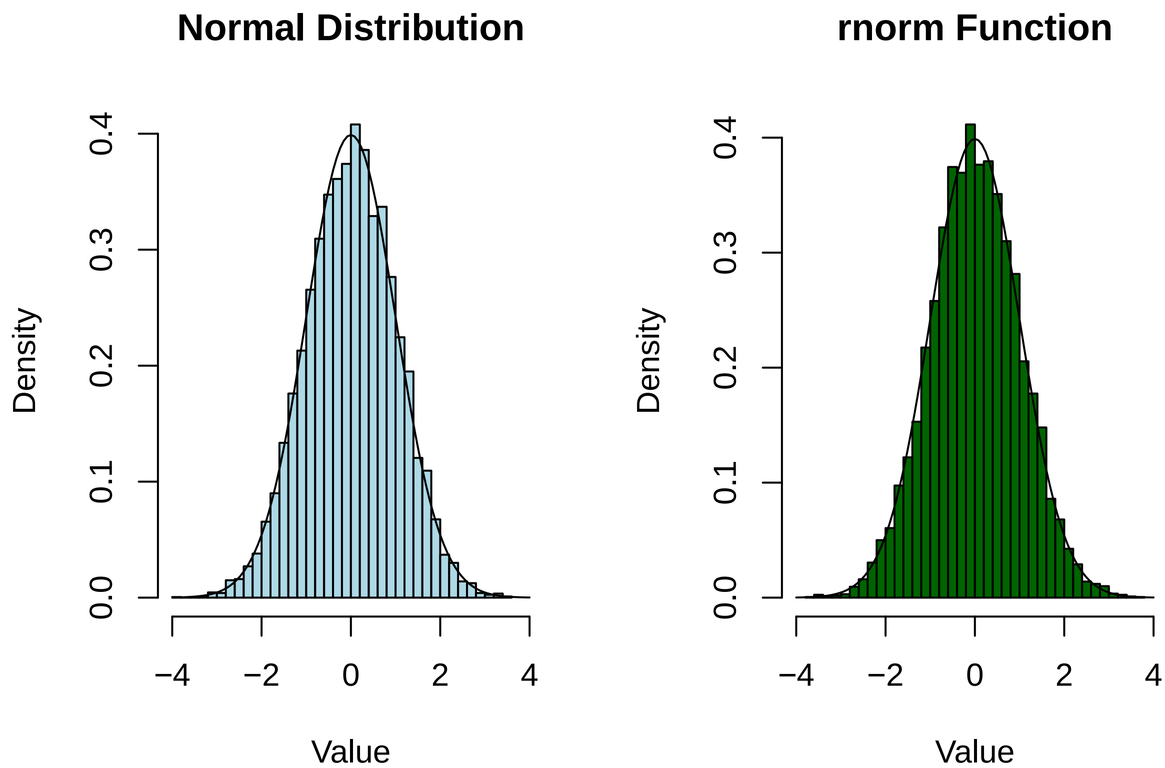


Point c

```
normgen <- rnorm(10000)
par(mfrow=c(1,2))
hist(X1, main="Normal Distribution", xlab="Value",col="blue" ,breaks=30,freq = F,xlim=c(-4,4))
curve(dnorm(x,0,1),add=T)
hist(normgen,xlab="Value",col="darkgreen" ,breaks=30,freq = F,xlim=c(-4,4), main = "rnorm Function")
curve(dnorm(x,0,1),add=T)
```




```
par(mfrow=c(1,2))
hist(gen_norm, main="Normal Distribution", xlab="Value",col="lightblue" ,breaks=30,freq = F,xlim=c(-4,4)
curve(dnorm(x,0,1),add=T)
hist(normgen,xlab="Value",col="darkgreen" ,breaks=30,freq = F,xlim=c(-4,4), main = "rnorm Function")
curve(dnorm(x,0,1),add=T)
```

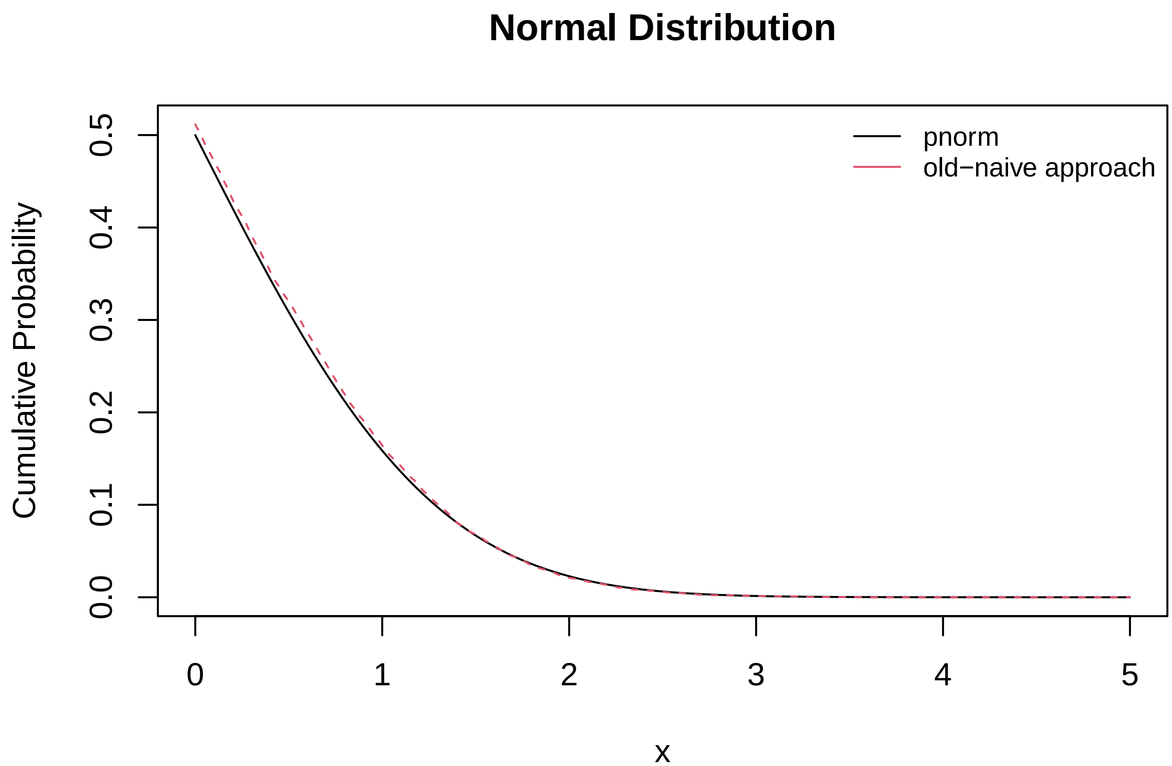


Point d

```
upper_tail_prob<- function(gen_norm,theta){
  prob=sum(gen_norm>theta)/length(gen_norm)
  return(prob)
}
x <- seq(0, 5, by = 0.01)
R<- sapply(seq(0, 5, by = 0.01), function(x) upper_tail_prob(gen_norm,x))
# Calculate cumulative probabilities using pnorm
prob <- 1-pnorm(x,0,1)
```

Point e

```
matplot(x, cbind(prob,R),
        type = "l",
        xlab = "x",
        ylab = "Cumulative Probability",
        main = "Normal Distribution")
legend("topright", legend = c("pnorm", "old-naive approach"),
       col = 1:3, lty = 1, bty = "n", cex = 0.8)
```



Monte Carlo simulations II

Point a

The Pareto distribution is defined by a density $f(x;\gamma) = \gamma x^{-(\gamma+1)}$ over $(1;+\infty)$, with $\gamma > 0$.

It can be generated as the $-\frac{1}{\gamma}$ power of a uniform r.v.

Cumulative distribution function of Pareto distribution

$$\int_1^x \gamma z^{-(\gamma+1)} dz = \gamma \int_1^x z^{-1-\gamma} dz = -[x^{-\gamma} - 1^{-\gamma}] = -x^{-\gamma} + 1 = 1 - (\frac{1}{x})^\gamma$$

We will use the following theorem: if $X \sim F(x)$ then $U = F(x) \sim U(0,1)$
$F(X) = 1 - (\frac{1}{x})^\gamma = U$
$(1-U)^{-\frac{1}{\gamma}} = (x^\gamma)^{-\frac{1}{\gamma}}$
$x = (1-U)^{-\frac{1}{\gamma}} = U^{-\frac{1}{\gamma}}$

Pont b

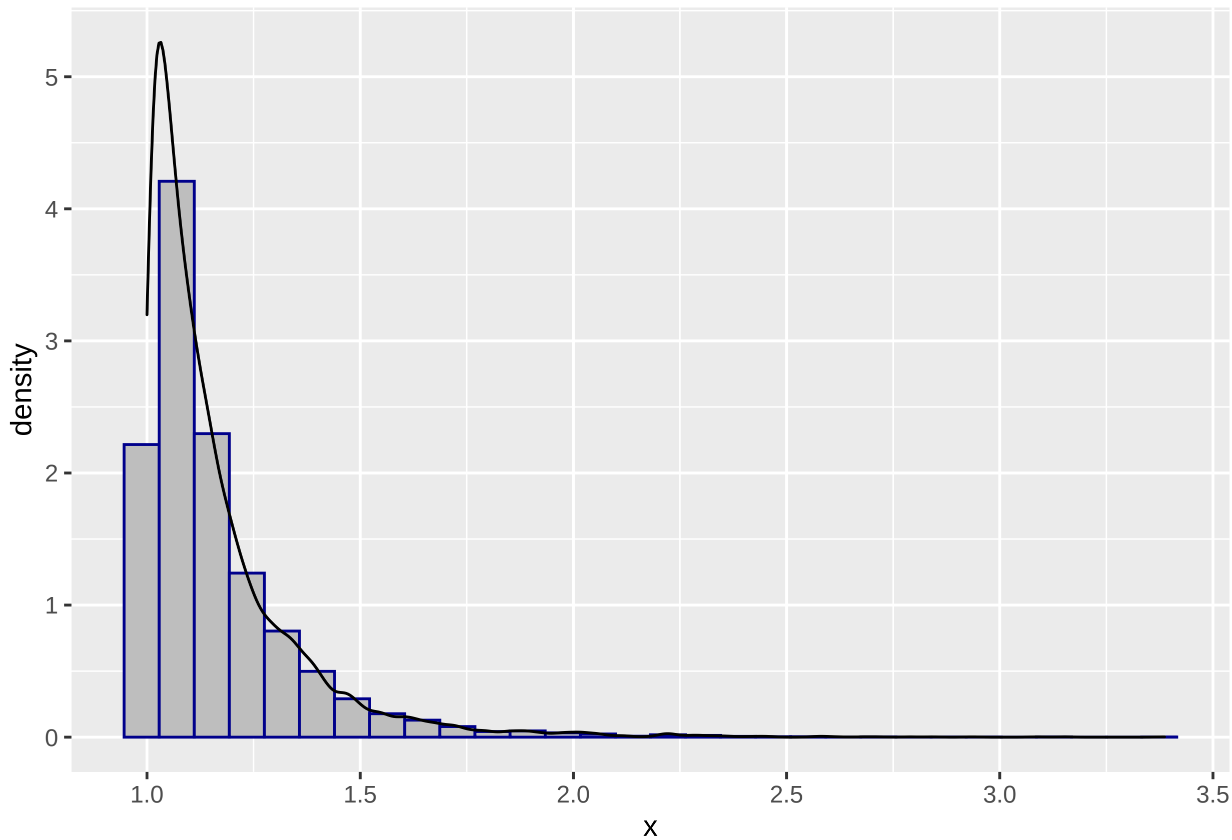
The Pareto distribution is related to the exponential distribution as follows. If X is Pareto-distributed with minimum x_m and index α, then $Y = log(\frac{X}{x_m})$ is exponentially distributed with rate parameter α.

$Y = log(\frac{X}{x_m}) \sim Exp(\gamma)$
$Y = log(X) \sim Exp(\gamma)$

Point c

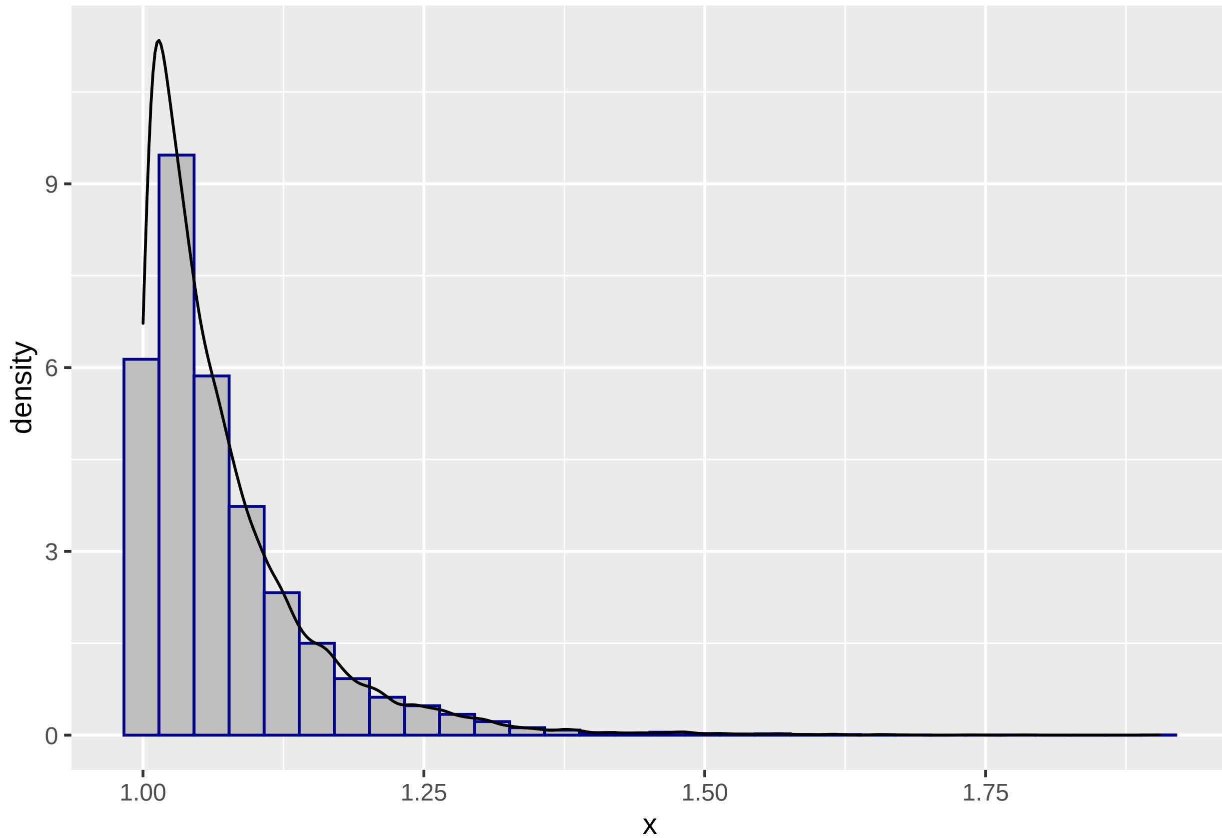
```r
#sampler for Pareto
xsampler<- function(n, gamma){
  u<- runif(n)
  x<-1/((1-u)^(1/gamma))
  return(x)
}

#sampler for Exponential
ysampler<- function(x){
  y<- log(x)
  return(y)
}
i=0
gamma = c(7,15,25)

x1=xsampler(10000,10)
y1=ysampler(x1)
x2<-xsampler(10000,10)
y2<- ysampler(x2)


ggplot(data.frame(x=xsampler(10000,gamma[1])), aes(x)) +
  geom_histogram(aes(y=..density..), fill="gray",col="darkblue")+
  geom_density()
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```



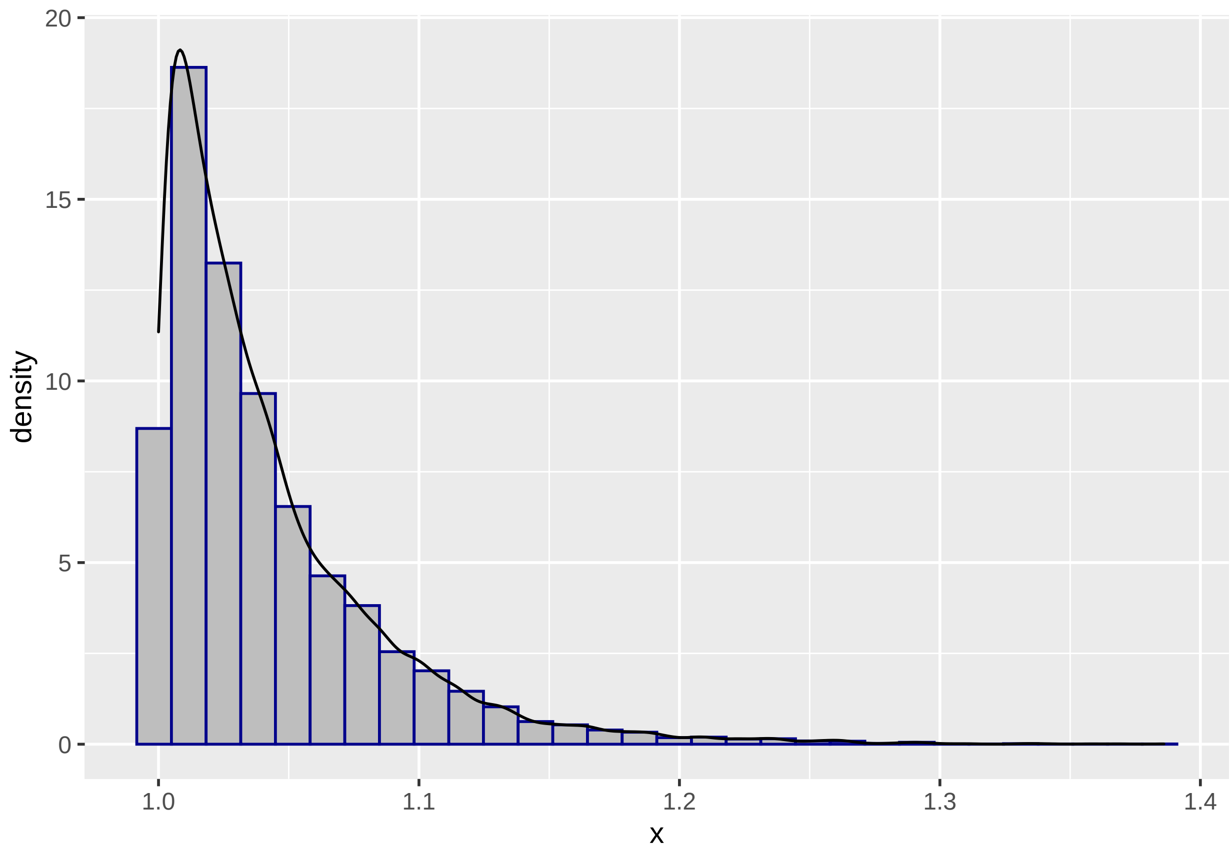

```
ggplot(data.frame(x=xsampler(10000,gamma[2])), aes(x)) +
  geom_histogram(aes(y=..density..), fill="gray",col="darkblue")+
  geom_density()
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```




```
ggplot(data.frame(x=xsampler(10000,gamma[3])), aes(x)) +
  geom_histogram(aes(y=..density..), fill="gray",col="darkblue")+
  geom_density()
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```



MC integration

Point a

We write the probability of a standard Normal r.v. X as an integral thanks to its probability density function.

$$P(X > 20) = \int_{20}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Reference: Wikipedia

The crude Monte Carlo estimation of this quantity is deemed to fail because the region of integration is so far out in the tails of the standard normal distribution.

Point b

Rewrite the integral employing the change of variable $Y = \frac{1}{X}$.

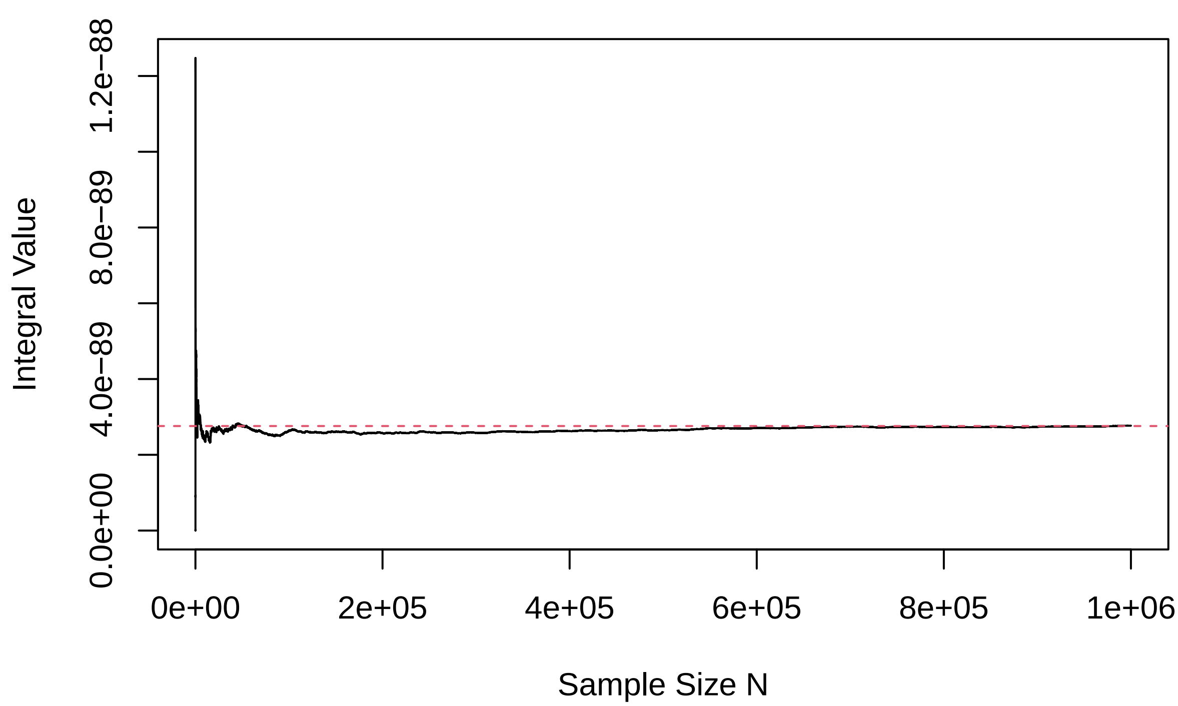
$dy = -\frac{1}{x^2} dx$
$dx = -x^2 dy = -\frac{1}{y^2} dy$
$y = Y_{20} = 0,05$
$y = Y_{\infty} = 0$

So $\int_{20}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$
$= \int_{0,05}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2y^2}} - \frac{1}{y^2} dy$
$= \int_{0}^{0,05} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2y^2}} \frac{1}{y^2} dy$

```r
theme_set(theme_bw())

set.seed(155523)
n <- 1000000
a <- 0
b <-  0.05
u <-runif(n, a, b)

intfunc <- function(y){
  f = (1/sqrt(2*pi)) * (1/(y^2)) * (exp(-1/(2*(y^2))))
}



H1 = (b-a)* intfunc(u)
psi_hat_MC <- mean(H1)
psi_hat_MC
```

```
## [1] 2.767355e-89
```

```r
set.seed(1234)
i_1n <- 1:n
plot( i_1n, cummean(H1), type='l', lwd=1,
      xlab="Sample Size N", ylab="Integral Value",col=1)
abline(h=integrate(intfunc, a, b)$value, lwd=1, col=2, lty=2)
```



Point c

```r
H2 <- (b-a)*intfunc(0.05-u)
MC1 <- mean(H1)
MC2 <- mean(H2)

MCA <- (MC1+MC2)/2


MCE_MC1 <- sd(H1)/sqrt(n)
MCE_MC1
```

```
## [1] 3.908867e-91
```

```r
MCE_MC2 <- sd(H2)/sqrt(n)
MCE_MC2
```

```
## [1] 3.943859e-91
```

```r
MCE_MCA <- sqrt(1+cor(H1,H2))*sd(H1)/sqrt(2*n)
MCE_MCA
```

```
## [1] 2.757021e-91
```

```r
effgain<- MCE_MCA/MCE_MC1
effgain
```

```
## [1] 0.7053249
```

Point d

```r
integrate(intfunc, a, b)
```

```
## 2.759158e-89 with absolute error < 5.4e-89
```

```r
pnorm(20,0,1,lower.tail = F)
```

```
## [1] 2.753624e-89
```