Assignment I - CompStat2023

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Set up

Exeercise 1

Point a

Compute $P[\max(X_1, X_2) > Y_1]$.

We know that $Z = max(X_1, X_2) \in (1, 2, 3, 4, 5, 6)$ and that $P(Z = z) = (2(z-1)+1)/36$. In other word we want to find $P(Z > y) = 1 - P(Z \leq y)$. $P(Z \leq y) =$
$P(Z = 1 \cap y = 1) +$
$P(Z = 1 \cap y = 2) + P(Z = 2 \cap y = 2) +$
$P(Z = 1 \cap y = 3) + P(Z = 2 \cap y = 3) + P(Z = 3 \cap y = 3) +$
$P(Z = 1 \cap y = 4) + P(Z = 2 \cap y = 4) + P(Z = 3 \cap y = 4) + P(Z = 4 \cap y = 4) +$
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$P(Z = 1 \cap y = 6) + P(Z = 2 \cap y = 6) + P(Z = 3 \cap y = 6) + P(Z = 4 \cap y = 6) + P(Z = 5 \cap y = 6) + P(Z = 6 \cap y = 6)$

So we have $P(Z \leq y) = 6\frac{1}{36}\frac{1}{6} + 5\frac{3}{36}\frac{1}{6} + 4\frac{5}{36}\frac{1}{6} + 3\frac{7}{36}\frac{1}{6} + 2\frac{9}{36}\frac{1}{6} + \frac{11}{36}\frac{1}{6} \approx 0,421$ and

$$P(Z > y) = 1 - 0,421 = 0,579$$

Point b-c-d

Here there are some code to simulate a generic Risiko! game for different values of competing units.

Exercise 3

Point a

The Pareto distribution is defined by a density $f(x; \gamma) = \gamma x^{-(\gamma+1)}$ over $(1; +\infty)$, with $\gamma > 0$.

It can be generated as the $-\frac{1}{\gamma}$ power of a uniform r.v.

Cumulative distribution function of Pareto distribution $\int_1^x \gamma z^{-(\gamma+1)} dz = \gamma \int_1^x z^{-1-\gamma} dz = -[x^{-\gamma} - 1^{-\gamma}] = -x^{-\gamma} + 1 = 1 - (\frac{1}{x})^{\gamma}$

We will use the following theorem: if $X \sim F(x)$ then $U = F(x) \sim U(0, 1)$ $F(X) = 1 - (\frac{1}{x})^{\gamma} = U$ $(1 - U)^{-\frac{1}{\gamma}} = (x^{\gamma})^{-\frac{1}{\gamma}}$ $x = (1 - U)^{-\frac{1}{\gamma}} = U^{-\frac{1}{\gamma}}$

Pont b

Reference: Wikipedia The Pareto distribution is related to the exponential distribution as follows. If X is Pareto-distributed with minimum x_m and index α, then $Y = log(\frac{X}{x_m})$ is exponentially distributed with rate parameter α.

$Y = log(\frac{X}{x_m}) \sim Exp(\gamma)$ $Y = log(X) \sim Exp(\gamma)$

##Point c

Implement two samplers, one for X and one for Y . Plot the histogram and the density and comment on the results exploring different values of Y.

Exercise 4

Point a

We write the probability of a standard Normal r.v. X as an integral thanks to its probability density function.

$P(X > 20) = \int_{20}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

Reference: Wikipedia

The crude Monte Carlo estimation of this quantity is deemed to fail because the region of integration is so far out in the tails of the standard normal distribution.

Point b

Rewrite the integral employing the change of variable $Y = \frac{1}{X}$.

$dy = -\frac{1}{x^2} dx$
$dx = -x^2 dy = -\frac{1}{y^2} dy$
$y = Y_{20} = 0,05$
$y = Y_{\infty} = 0$

So $\int_{20}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$
$= \int_{0,05}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2y^2}} - \frac{1}{y^2} dy$
$= \int_{0}^{0,05} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2y^2}} \frac{1}{y^2} dy$