Name:—————————— **Entry No:**————————————

MTL 100: Calculus

Minor II Exam

Indian Institute of Technology Delhi

Time: 1 Hour Max. Marks: 25

Write your name and entry number at the place specified above. Attempt all questions. All notations are standard. All parts of a question must be answered at one place. Exhibit clearly all the steps. Use of any electronic gadget including calculator is NOT allowed. Attach the question paper with the answer book. No query will be entertained.

1. (a) Is the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{a_n}}{n}$ convergent, where $\sum_{n=1}^{\infty} a_n$ is a convergent series with non negative terms?

   (b) What is the radius of convergence of the series $\sum_{n=1}^{\infty} n!(\frac{x}{n})^n$?

2. (a) Determine for what values $p \in \mathbb{R}$, the integral I converges, where $I = \int_1^{\infty} x^p e^{-x} dx$.

   (b) Evaluate the limit, if it exists

$$\lim_{n\to\infty} \int_0^1 \frac{ny^{n-1}}{1+y} dy.$$

3. Evaluate the integral I, if it exists where $I = \int_1^{\infty} \frac{x^{1/4}+x^{3/4}}{x(1+x)} dx$.

4. Prove for real numbers a, b s.t. a>0, $|b| < a$,

$$\int_0^{\pi} \frac{dx}{a+bcosx} = \frac{\pi}{\sqrt{a^2-b^2}}.$$

   Hence or otherwise evaluate the integral

$$\int_0^{\pi} \frac{cosx\ dx}{(5+3cosx)^2}.$$

5. Find the perimeter, and the area enclosed by the curve $r = 1 - cos\theta$.

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Model solutions

1(a) The series is absolutely convergent, hence convergent.

For, observe that

$$\left|(-1)^{n-1}\frac{\sqrt{a_n}}{n}\right| = \sqrt{a_n \times \frac{1}{n^2}} \leq \frac{1}{2}a_n + \frac{1}{2}\cdot\frac{1}{n^2} \quad \forall n \geq 1$$

[1 Mark].

and that the series $\frac{1}{2}\sum_{n=1}^{\infty} a_n$, and $\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}$ are convergent.

[$1\frac{1}{2}$ Mark].

(b) $\frac{|a_{n+1}|}{|a_n|} = \frac{|x|}{\left(1+\frac{1}{n}\right)^n} \to \frac{|x|}{e}$, as $n \to \infty$, where $a_n = n!\left(\frac{x}{n}\right)^n$.

[2 Marks]

The given series is convergent, if $|x| < e$, divergent if $|x| > e$. Hence, radius of convergence $R = e$ [$\frac{1}{2}$ Mark]

2(a). The integral converges for all $p \in \mathbb{R}$.

For $I(p) = \int_1^{\infty} x^p e^{-x}\,dx < \int_1^{\infty} x^n e^{-x}\,dx \quad \forall n \geq p, n \in \mathbb{N}$.

Since $x^{n+2}e^{-x} = \frac{x^{n+2}}{e^x} \to 0$, as $n \to \infty$, we get

$x^{n+2}e^{-x}$ to be bounded, say by H.

Hence $x^{n+2}e^{-x} \leq H$ or $x^n e^{-x} \leq \frac{H}{x^2} \quad \forall x \in (1, \infty)$

$\Rightarrow I(p) \leq H\int_1^{\infty}\frac{dx}{x^2}$ (which is always convergent).

[$2\frac{1}{2}$ Mark]

(b) Integrating by parts.

$$\int_0^1 \frac{ny^{n-1}}{1+y}\,dy = \left[\frac{n}{1+y} \times \frac{y^n}{n}\right]_{y=0}^{1} - n\int_0^1 -\frac{1}{(1+y)^2} \times \frac{y^n}{n}\,dy$$

[1 Mark]

$$= \frac{1}{2} + \int_0^1 \frac{y^n}{(1+y)^2}\,dy$$

Observe that

$$0 \leq \int_0^1 \frac{y^n}{(1+y)^2}\,dy \leq \int_0^1 y^n\,dy = \frac{1}{n+1}$$

$\Rightarrow \lim_{n\to\infty}\int_0^1 \frac{y^n}{(1+y)^2}\,dy = 0$, Hence $\int_0^1 \frac{ny^{n-1}}{1+y}\,dy \to \frac{1}{2}$, as $n \to \infty$.

[$1\frac{1}{2}$ Mark]

3(a). $I = \int_1^\infty \frac{x^{-\frac{1}{4}} + x^{-\frac{3}{4}}}{1+x}\,dx = B\left(\frac{1}{4}, \frac{3}{4}\right) = \Gamma\tfrac{1}{4}\,\Gamma\left(1-\tfrac{1}{4}\right) = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \pi\sqrt{2}$

For, $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad (m,n>0)$, Put $x = \frac{1}{1+y}$

$$B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy \text{ [1 Mark]}$$

observe that

$$\int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt \quad \text{Put } y = \frac{1}{t} \text{ [1 Mark]}$$

$$\Rightarrow B(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\,dy + \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}}\,dy \quad \text{[2 Marks]}$$

For $m = \frac{1}{4}, n = \frac{3}{4}$, we get the result. [1 Mark]

Aliter:

$$I = \int_1^\infty \frac{x^{\frac{1}{4}} + x^{\frac{3}{4}}}{x(1+x)}\,dx \quad \text{Put } x = \tan^2\theta \text{ [1 Mark]}$$

$$= 2\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin\theta + \cos\theta}{\sqrt{\sin 2\theta}}\,d\theta, \text{ put } \phi = \frac{\pi}{2} - \theta \text{ [1 Mark]}$$

$$I = \sqrt{2}\int_0^{\frac{\pi}{2}} \frac{\sin\theta + \cos\theta}{\sqrt{\sin 2\theta}}\,d\theta = \frac{1}{2}\times 2\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta\,\cos^{-\frac{1}{2}}\theta\,d\theta + \frac{1}{2}\times 2\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}}\theta\,\cos^{\frac{1}{2}}\theta\,d\theta \text{ [2 Marks]}$$

$$\Rightarrow I = \frac{1}{2}B\left(\frac{3}{4}, \frac{1}{4}\right) + \frac{1}{2}B\left(\frac{1}{4}, \frac{3}{4}\right) = B\left(\frac{1}{4}, \frac{3}{4}\right) = \Gamma\tfrac{1}{4}\,\Gamma\left(1-\tfrac{1}{4}\right) = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)}$$

[1 Mark]

or $I = \pi\sqrt{2}$.

4. $I=\int_0^\pi \frac{dx}{a+b\cos x} = \int_0^\pi \frac{dx}{a(\cos^2 x/2+\sin^2 x/2)+b(\cos^2 x/2-\sin^2 x/2)}\,dx$

$= \int_0^\pi \frac{\sec^2 x/2}{(a+b)+(a-b)\tan^2 x/2}\,dx = \int_0^\infty \frac{2\,dt}{(a+b)+(a-b)t^2}$, Put $t=\tan x/2$ [1 Mark]

$\Rightarrow I(b) = \frac{2}{\sqrt{a^2-b^2}}\left[\tan^{-1} t\sqrt{\frac{a-b}{a+b}}\right]_{t=0}^{\infty} = \frac{\pi}{a^2-b^2}$ [$1\frac{1}{2}$ Mark]

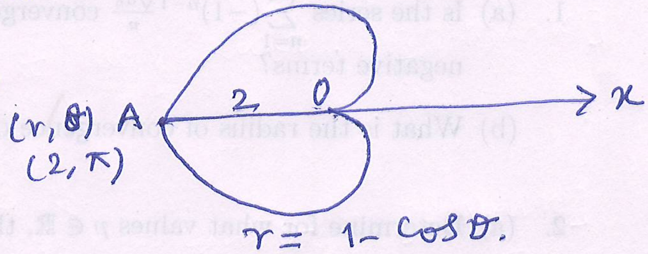
Diff. w.r.t. b, the integral, and the RHS, treating a as a constant.

$\frac{d\,I(b)}{db} = \int_0^\pi \left[\frac{\partial}{\partial b}\left(\frac{1}{a+b\cos x}\right)\right]dx = \frac{d}{db}\left(\frac{\pi}{\sqrt{a^2-b^2}}\right)$

we get $\frac{d\,I(b)}{db} = \int_0^\pi \frac{\cos x}{(a+b\cos x)^2}\,dx = -\frac{\pi b}{(a^2-b^2)^{3/2}}$ [$1\frac{1}{2}$ Mark]

$a=5,\ b=3$, gives $\int_0^\pi \frac{\cos x}{(a+b\cos x)^2}\,dx = -\frac{3\pi}{16^{3/2}} = -\frac{3\pi}{4}$ [1 Mark]

5. The curve is symmetric about the initial line.



Hence, the perimeter in polar form is given by

$S = 2\int ds = 2\int_{\theta=0}^{\pi} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$, $r = 1-\cos\theta$ [1 Mark]

[$1\frac{1}{2}$ Mark]

$= 4\int_0^\pi \sin(\theta/2)\,d\theta = 8$

And, the area bounded by the curve in the polar form:

$\mathcal{A} = 2\int_{\theta=0}^{\pi} \frac{1}{2} r^2\,d\theta = 4\int_{\theta=0}^{\pi} \sin^4(\theta/2)\,d\theta = 4\times 2\int_0^{\pi/2} \sin^4\phi\,d\phi$ [1 Mark]

Put $\frac{\theta}{2}=\phi$

[$1\frac{1}{2}$ Mark]

$\Rightarrow \mathcal{A} = 4\,B\left(\frac{5}{2},\frac{1}{2}\right) = 4\frac{\Gamma(5/2)\,\Gamma(1/2)}{\Gamma 3} = \frac{3\pi}{2}$