Regularized AFT models

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This has been adapted from the Survival package vignette.

Model definition

A standard AFT model is defined as follows:

$$\log(T_i) = x_i^T \beta + \sigma\epsilon_i \tag{1}$$

Where x_i are the covariates, Y_i is the observed time (output). $\epsilon_i \sim f$, where f is the probability density.
Here on, we assume that σ is fixed, and is ignored. We let y_i be the transformed data vector obtained by taking log of T_i. Hence, we have:

$$e_i = \frac{y_i - x_i^T \beta}{\sigma} \sim f \tag{2}$$

For interval regression with censored data, we are given time intervals $\{\underline{t}_i, \overline{t}_i\}$ and covariates x_i for $i = 1 : n$, where $\underline{t}_i$ may be $-inf$ (left censoring) and $\overline{t}_i$ may be inf (right censoring).

Likelihood

For calculating likelihood, in the observations with no censoring, the pdf is used, and in censored observations, the cdf is used. Hence, the likelihood is given as:

$$l = \left(\prod_{exact} f(e_i)/\sigma\right)\left(\prod_{right} 1 - F(e_i)\right)\left(\prod_{left} F(e_i)\right)\left(\prod_{interval} F(e_i^u) - F(e_i^l)\right) \tag{3}$$

where, "exact", "left", "right", and "interval" refer to uncensored, left censored, right censored and interval censored observations respectively, and F is the cdf of the distribution. e_i^u, and e_i^l are upper and lower endpoints for interval censored data.
Hence the log likelihood is given as:

$$LL = \sum_{exact} g_1(e_i) - \log(\sigma) + \sum_{right} g_2(e_i) + \sum_{left} g_3(e_i) + \sum_{interval} g_4(e_i^l, e_i^u) \tag{4}$$

where $g_1 = \log(f)$, $g_2 = \log(1 - F)$, $g_3 = \log(F)$, $g_4(e_i^l, e_i^u) = \log(F(e_i^u) - F(e_i^l))$.

Score and Hessian

Derivatives of the LL with respect to the regression parameters are:

$$\frac{\partial LL}{\partial \beta_j} = \sum_{i=1}^{n} \frac{\partial g}{\partial \eta_i} \frac{\partial \eta_i}{\partial \beta_j} = \sum_{i=1}^{n} x_{ij} \frac{\partial g}{\partial \eta_i} \tag{5}$$

$$\frac{\partial^2 LL}{\partial \beta_j \beta_k} = \sum_{i=1}^{n} x_{ij} x_{ik} \frac{\partial^2 g}{\partial \eta_i^2} \tag{6}$$

where $\eta_i = x_i^T \beta$ is the vector of linear predictors.

Define $\mu_i = \frac{\partial g}{\partial \eta_i}$, where g is one of g_1 to g_4 depending on type of censoring in the i^{th} observation, and $\mu = [\mu_1, ...\mu_n]^T$. Then, partial derivative of log-likelihood is given as:

$$\frac{\partial l(\beta)}{\partial \beta_j} = \sum_{i=1}^{n} x_{ij} \mu_i \tag{7}$$

Hence, the score (gradient of log likelihood) is given as:

$$S = \nabla_\beta LL(\beta) = X^T \mu = \sum_{i=1}^{n} \mu_i \overline{x}_i \tag{8}$$

The hessian can be written as:

$$H = \sum_{i=1}^{n} \overline{x}_i \overline{x}_i^T \frac{\partial^2 g}{\partial \eta_i^2} = \sum_{i=1}^{n} \overline{x}_i \overline{x}_i^T w_i \tag{9}$$

Define $W = diag(w_1, ...w_n)$.

$$H = X^T W X \tag{10}$$

IRLS

We use Newton's algorithm to find MLE for the AFT model, using negative loglikelihood (NLL). The Newton update is as follows:

$$\begin{aligned} \beta &= \widetilde{\beta} - H^{-1} \widetilde{S} \\ &= \widetilde{\beta} - (X^T \widetilde{W} X)^{-1} X^T \widetilde{\mu} \\ &= (X^T \widetilde{W} X)^{-1} ((X^T \widetilde{W} X) \widetilde{\beta} - X^T \widetilde{\mu}) \\ &= (X^T \widetilde{W} X)^{-1} X^T (\widetilde{W} X \widetilde{\beta} - \widetilde{\mu}) \\ &= (X^T \widetilde{W} X)^{-1} X^T (\widetilde{W} X \widetilde{\beta} - \widetilde{\mu}) \end{aligned} \tag{11}$$

where, define the working response $\widetilde{z} = X\widetilde{\beta} - \widetilde{W}^{-1}\widetilde{\mu}$. Here, the tilde denotes that the respective values are evaluated using the parameters from the previous step.

Hence, at each step we are solving a weighted least squares problem, which is a minimizer of:

$$\sum_{i=1}^{n} \widetilde{w}_i(\widetilde{z}_i - \overline{x}_i^T\beta)^2 \tag{12}$$

This algorithm is the iteratively reweighted least squares (IRLS), since at each iteration we solve a weighted least squares problem.

Elastic net penalty and coordinate descent

We define the elastic net (L1 + L2) penalty as follows:

$$\lambda P_\alpha(\beta) = \lambda(\alpha\|\beta\|_1 + 1/2(1-\alpha)\|\beta\|_2^2) \tag{13}$$

Adding the elastic net (L1 + L2) penalty, we get the following penalized weighted least squares objective:

$$M = \sum_{i=1}^{n} \widetilde{w}_i(\widetilde{z}_i - \overline{x}_i^T\beta)^2 + \lambda P_\alpha(\beta) \tag{14}$$

The subderivative of the optimization objective is given as:

$$\frac{\partial M}{\partial \beta_k} = \sum_{i=1}^{n} \widetilde{w}_i x_{ik}(\widetilde{z}_i - \overline{x}_i^T\beta) + \lambda\alpha \ \mathrm{sgn}(\beta_k) + \lambda(1-\alpha)\beta_k \tag{15}$$

where, $\mathrm{sgn}(\beta_k)$ is 1 if $\beta_k > 1$, -1 if $\beta_k < 0$ and 0 if $\beta_k = 0$.

Using the subderivative, three cases of solutions for β_k may be obtained. The solution is given by:

$$\hat{\beta}_k = \frac{S\left(\sum_{i=1}^{n} \widetilde{w}_i x_{ik}\left[\widetilde{z}_i - \sum_{j\neq k} x_{ij}\beta_j\right], \lambda\alpha\right)}{\sum_{i=1}^{p} \widetilde{w}_i x_{ik}^2 + \lambda(1-\alpha)} \tag{16}$$

where, S is the soft thresholding operator, and w_i and z_i are given in 9 and 12 respectively.

The coordinate descent algorithm works by cycling through each β_j in turn, keeping the others constant, and using the above estimate to calculate the optimal value $\hat{\beta}_j$. This is repeated until convergence.

Pathwise solution

This section is borrowed from section 2.3 of [3]. The iregnet function will return solutions for an entire path of vaules of λ, for a fixed α. We begin with λ sufficiently large to set the solution $\beta = 0$, and decrease λ until we arrive near the unregularized solution. The solutions for each value of λ are used as the initial estimates of β for the next λ value. This is known as warm starting, and makes the algorithm efficient and stable. To choose initial value of λ, we use Equation 16, and notice that for $\frac{1}{n}\sum_{i=1}^{n} w_i(0)x_{ij}z(0)_i < \alpha\lambda$ for all j, then $\beta = 0$ minimizes the objective 14. Thus,

$$\lambda_{max} = max_j \frac{1}{n\alpha}\sum_{i=1}^{n} w_i(0)x_{ij}z(0)_i \tag{17}$$

We will set $\lambda_{min} = \epsilon\lambda_{max}$, and compute solutions over a grid of m values, where $\lambda_j = \lambda_{max}(\lambda_{min}/\lambda_{max})^{j/m}$ for $j = 0, .., m$.

Algorithm

The algorithm to be followed for fitting the distribution is:

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Calculate $\lambda_{max}$ using equation 17 ;
foreach $\lambda_j$ in $j = 0, ..., m$ do
    Initialise $\widetilde{\beta}$ using solution $\hat{\beta}$ from previous $\lambda$;
    repeat
        Compute $\widetilde{w}_i$ and $\widetilde{z}_i$ ;
        Find $\hat{\beta}$ by solving the penalized weighted least square problem
          defined in equation 14 using coordinate descent ;
        Set $\widetilde{\beta} = \hat{\beta}$ ;
    until convergence of $\hat{\beta}$;
end
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Algorithm 1: Overall optimization algorithm

Scale parameter

So far, I have ignored the σ parameter from the calculations and equations. This is only reasonable if we treat σ as fixed. However, in other cases, σ needs to estimated along with the parameters β, by using the derivatives as listed below.

Derivatives

Iterations are done with respect to $\log(\sigma)$ to prevent numerical underflow.

$$
\begin{aligned}
\frac{\partial g_1}{\partial \eta} &= -\frac{1}{\sigma}\left[\frac{f'(z)}{f(z)}\right] \\
\frac{\partial g_4}{\partial \eta} &= -\frac{1}{\sigma}\left[\frac{f(z^u) - f(z^l)}{F(z^u) - F(z^l)}\right] \\
\frac{\partial^2 g_1}{\partial \eta^2} &= -\frac{1}{\sigma^2}\left[\frac{f''(z)}{f(z)}\right] - (\partial g_1/\partial \eta) \\
\frac{\partial^2 g_4}{\partial \eta^2} &= -\frac{1}{\sigma^2}\left[\frac{f'(z^u) - f'(z^l)}{F(z^u) - F(z^l)}\right] - (\partial g_4/\partial \eta)^2 \\
\frac{\partial g_1}{\partial \log \sigma} &= -\left[\frac{z f'(z)}{f(z)}\right] \\
\frac{\partial g_4}{\partial \log \sigma} &= -\left[\frac{z^u f(z^u) - z^l f(z^l)}{F(z^u) - F(z^l)}\right] \\
\frac{\partial^2 g_1}{\partial (\log \sigma)^2} &= \left[\frac{z^2 f''(z) + z f'(z)}{f(z)}\right] - (\partial g_1/\partial \log \sigma)^2 \\
\frac{\partial^2 g_4}{\partial (\log \sigma)^2} &= \left[\frac{(z^u)^2 f'(z^u) - (z^l)^2 f'(z^l)}{F(z^u) - F(z^l)}\right] - (\partial g_1/\partial \log \sigma)(1 + \partial g_1/\partial \log \sigma) \\
\frac{\partial^2 g_1}{\partial \eta \partial \log \sigma} &= \left[\frac{z f''(z)}{\sigma f(z)}\right] - (\partial g_1/\partial \eta)(1 + \partial g_1/\partial \log \sigma) \\
\frac{\partial^2 g_4}{\partial \eta \partial \log \sigma} &= \left[\frac{z^u f'(z^u) - z^l f'(z^l)}{\sigma[F(z^u) - F(z^l)]}\right] - (\partial g_4/\partial \eta)(1 + \partial g_4/\partial \log \sigma)
\end{aligned}
\tag{18}
$$

Derivatives for g_2 can be obtained by setting z_u to inf in the equations for g_4, and similarly for g_3.
The distribution specific values of $f(z)$, etc. are omitted.

Bibliography

[1] Survival - Terry M Therneau

[2] Machine Learning: A Probabilistic Perspective - Kevin Murphy

[3] Regularization Paths for Cox's Proportional Hazards Model via Coordinate Descent - Simon, Friedman, Hastie, Tibshirani

[4] AFT - TD Hocking