

Information-theoretically Secure Bit Commitment over Noisy Channels

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The Game of Chess

A problem!



A boy named Bob and a girl named Alice are playing chess at a table. A crescent moon and three stars are visible in the night sky above them. Alice is thinking about her next move, indicated by a thought bubble with a question mark.

Bob

Alice

Alice's turn, but its bed time



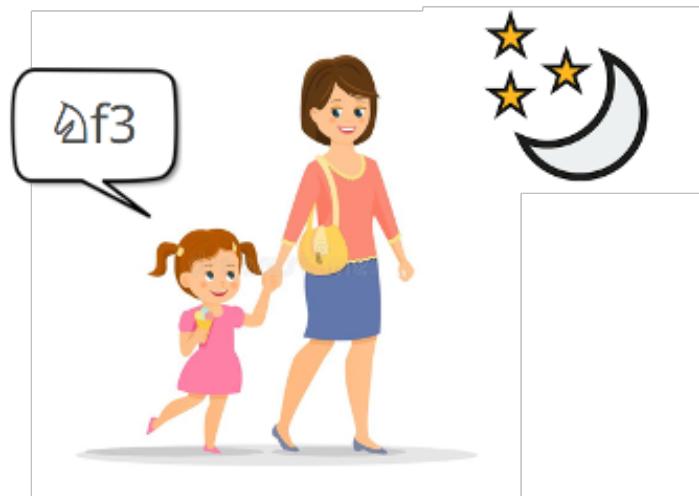
Alice is shown lying in bed, looking thoughtful. A thought bubble above her head contains a 3D-style illustration of a chessboard with pieces.

Alice can think about her next move for the whole night

The Game of Chess

Potential Solution!

That night:



Alice “commits” move to Mom.

Guarantee: the move is **concealed** from Bob

The next morning:



The move is “revealed” to Bob.

Guarantee: Alice is **bound** to her initial choice

“Question” : What if no such **Trusted third Party** is available?

Answer:

Commitment Protocol!

What is Commitment?

- **Security Protocol**
- **Two users:** sender/committer (*Alice*) and a receiver/verifier (*Bob*)
- **Two Phases:** “*Commit Phase*” followed by a “*Reveal phase*”
- **Security Guarantees:** Soundness
Concealment
Bindingness
- **Applications:** Secure Multiparty Computation
Zero-Knowledge Proofs
Coin Tossing

Computational Security (CS)

VS

Information-theoretic Security (ITS)

- [Blum '83] – Introduced Commitment
- [Brassard et. al]— (comp. binding and IT concealing)
- [Ostrovksy et. al] – (comp. concealing and IT binding)

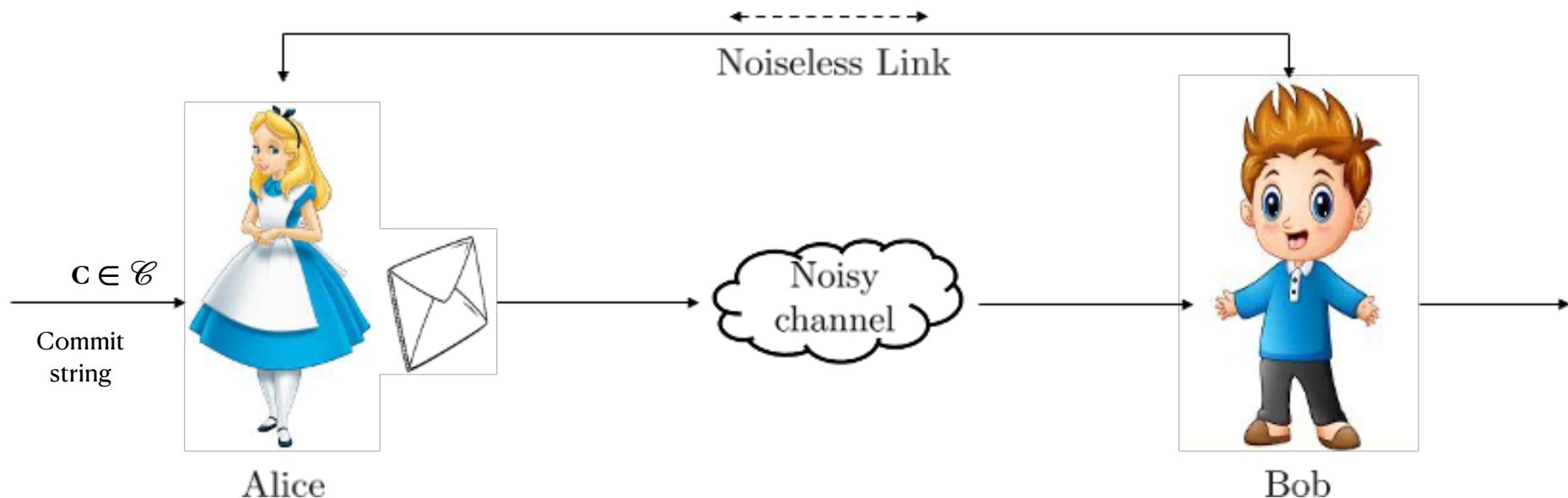
And many others....

What if both the users are “computationally unbounded” ?

- Noisy Communication Channel comes as a relief ! — [Creapau & Kilian '88]

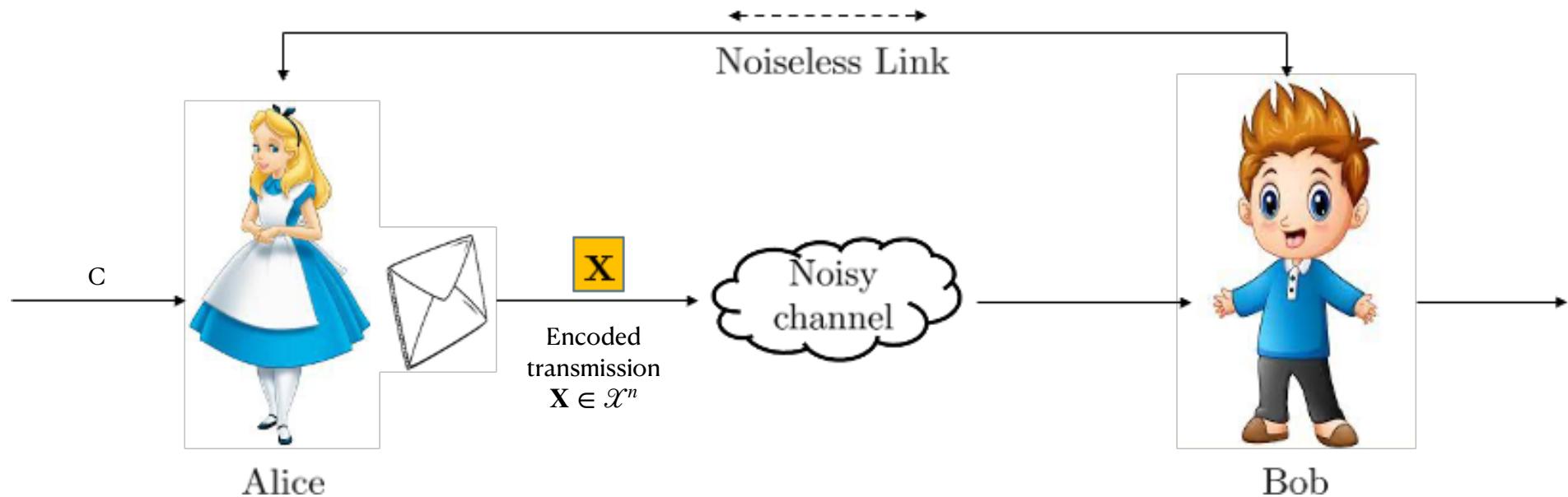
Commitment over Noisy Channels

Model



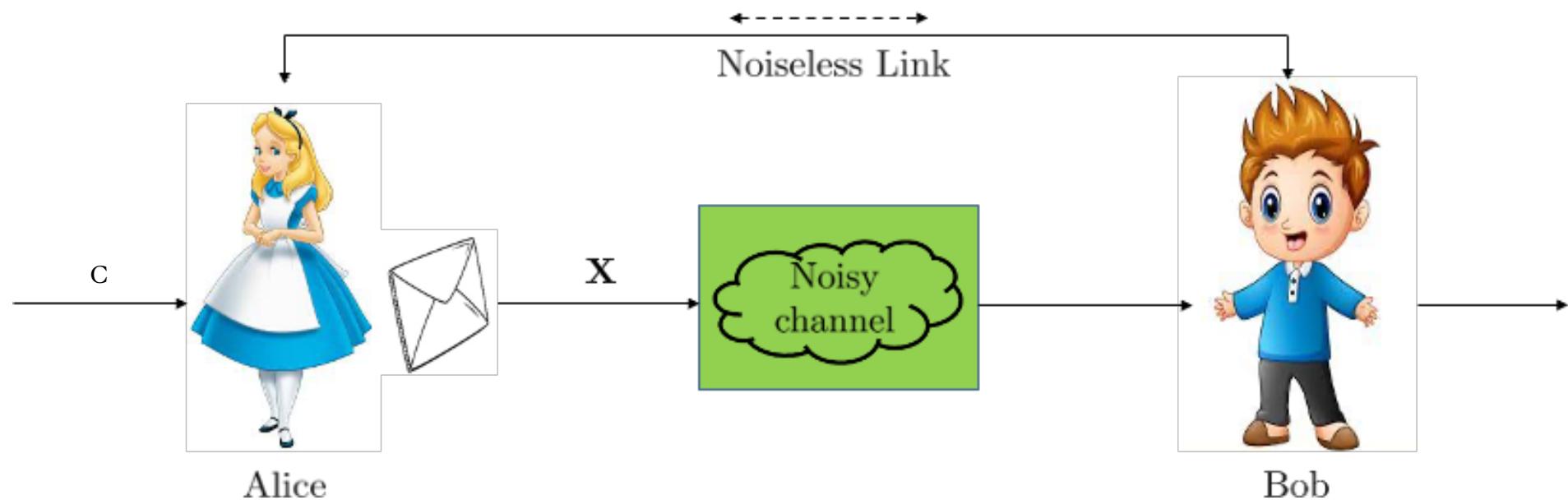
Commitment over Noisy Channels

Commit Phase



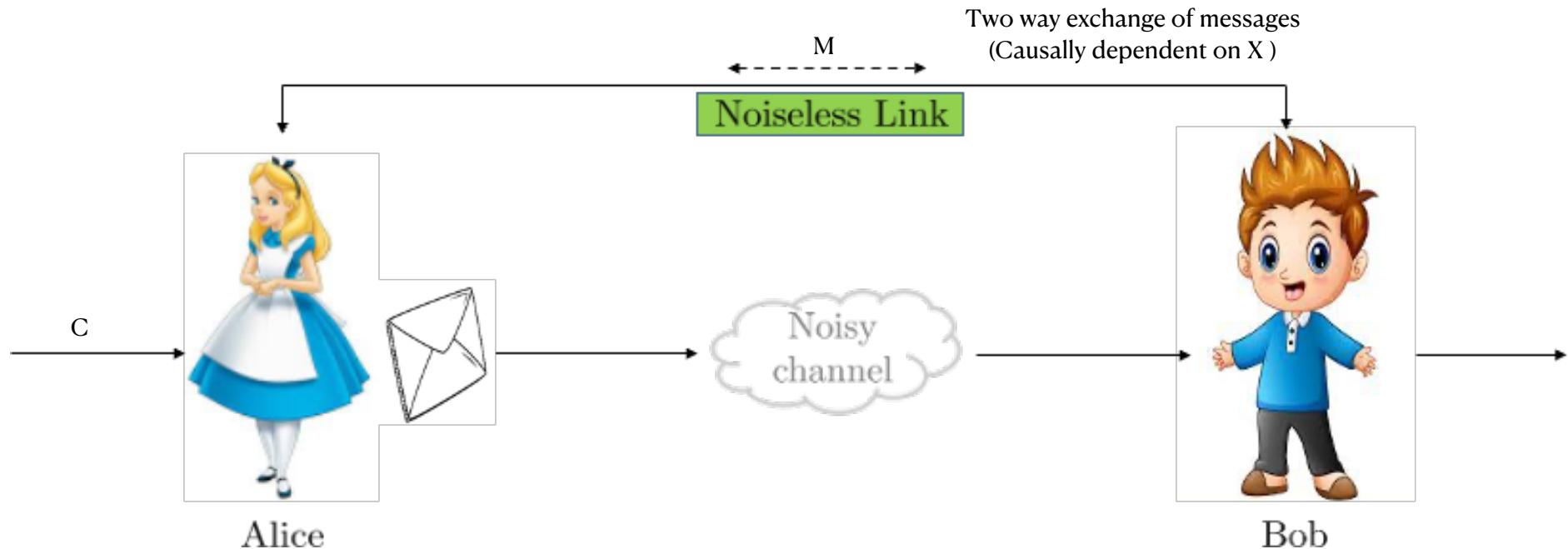
Commitment over Noisy Channels

Commit Phase



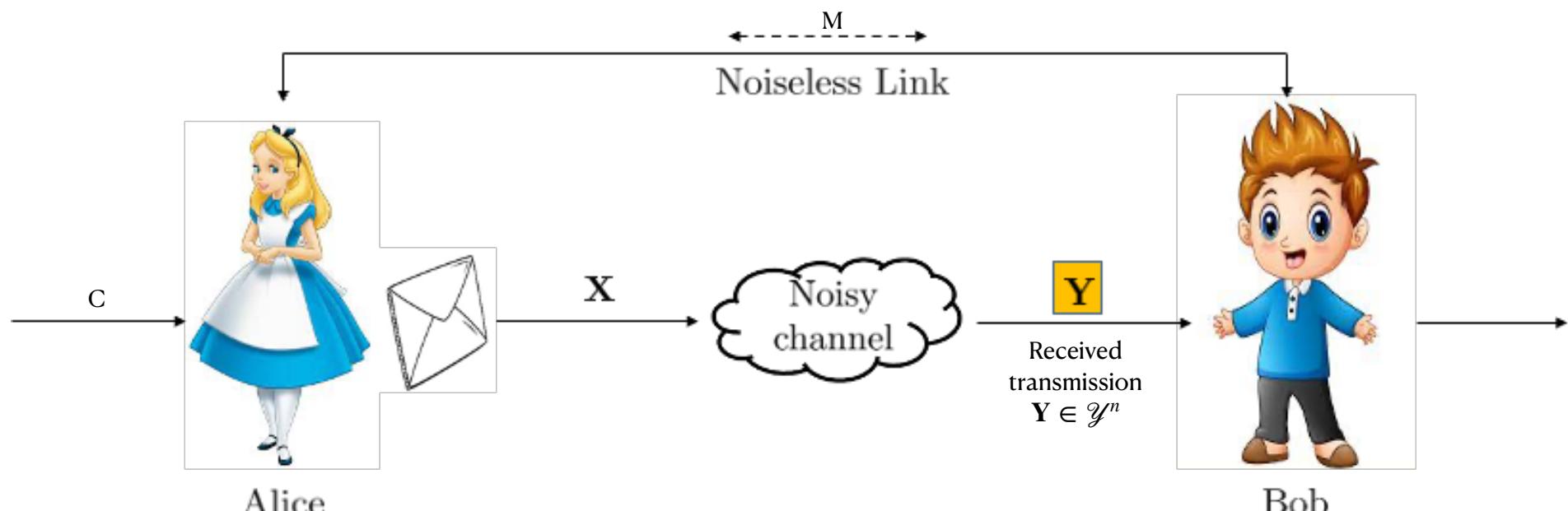
Commitment over Noisy Channels

Commit Phase



Commitment over Noisy Channels

Commit Phase

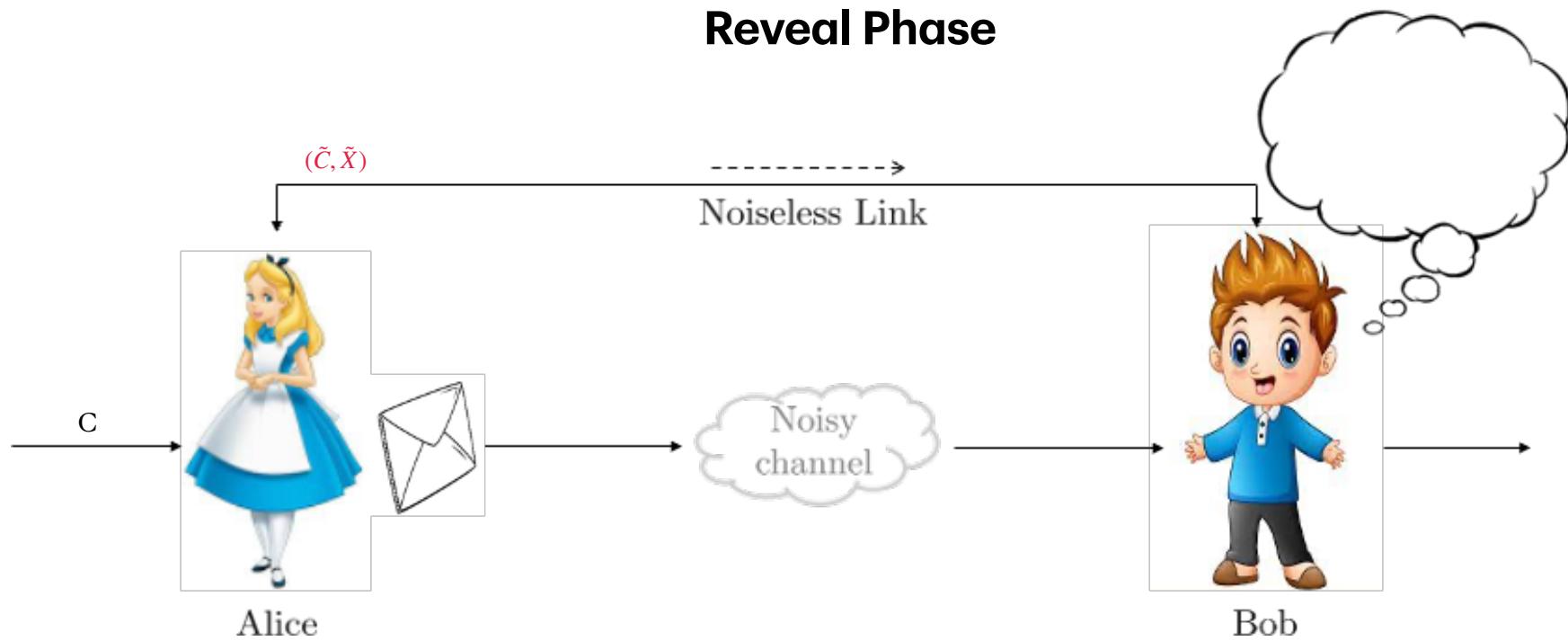


'Alice's view' $V_A = (C, \mathbf{X}, K_A, M)$

'Bob's view' $V_B = (\mathbf{Y}, K_B, M)$

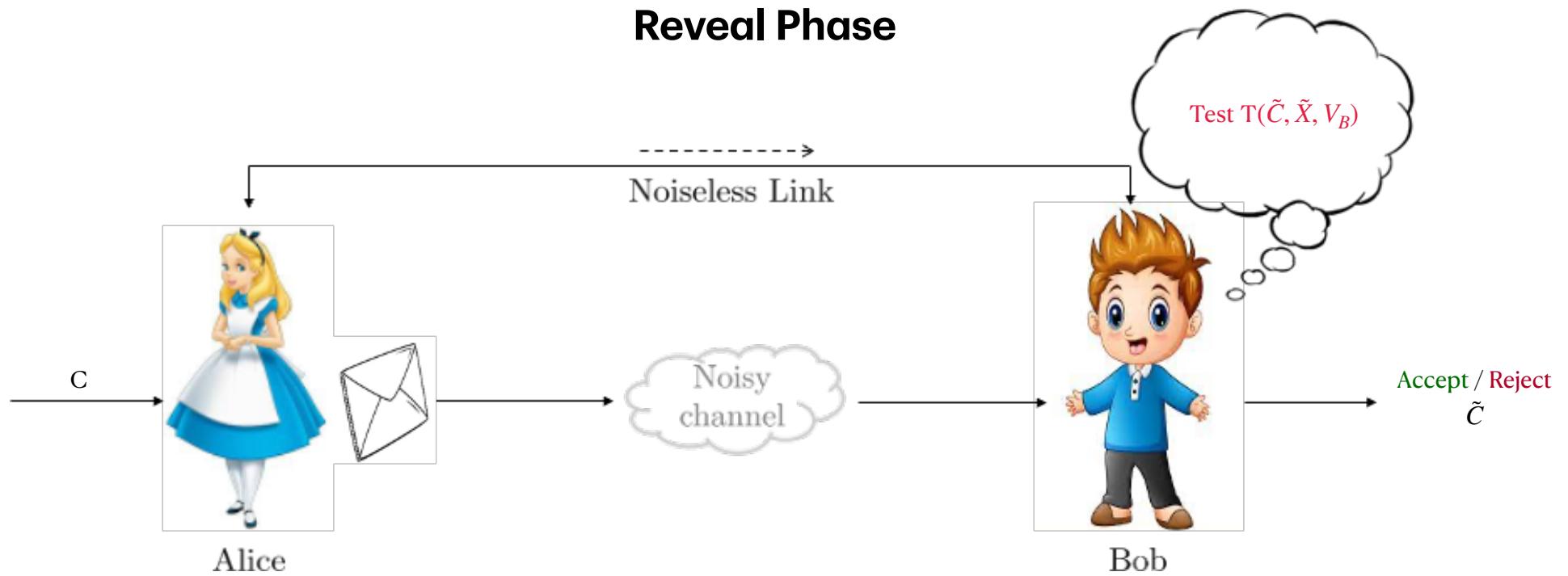
Commitment over Noisy Channels

Reveal Phase



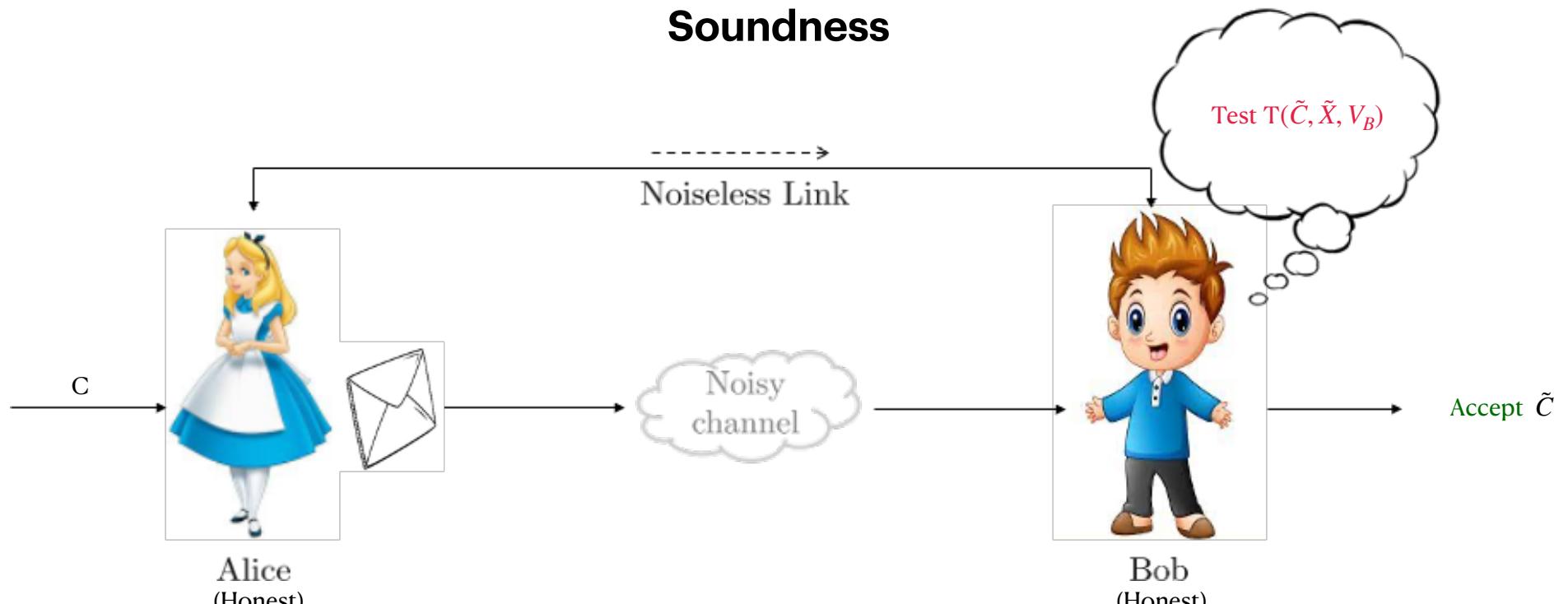
Commitment over Noisy Channels

Reveal Phase



Security Guarantees

Soundness

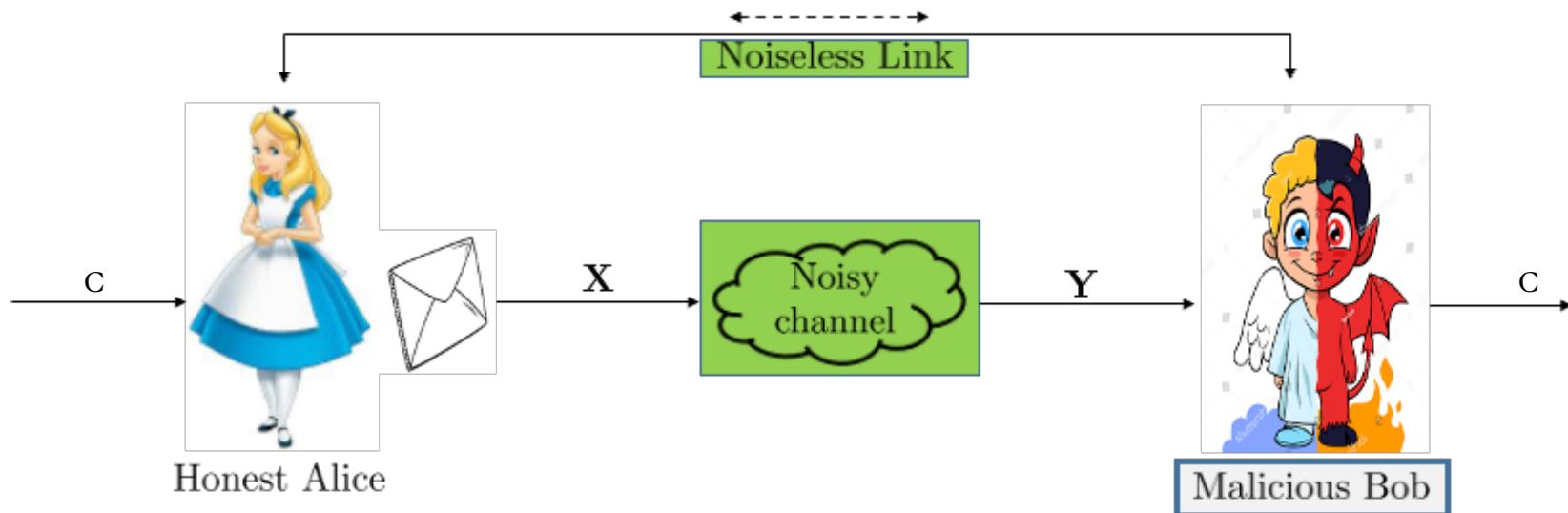


$$\mathbb{P}(T(C, X, V_B) = \text{REJECT}) \leq \epsilon(n)$$

Security Guarantees

Concealment

At the end of Commit Phase:

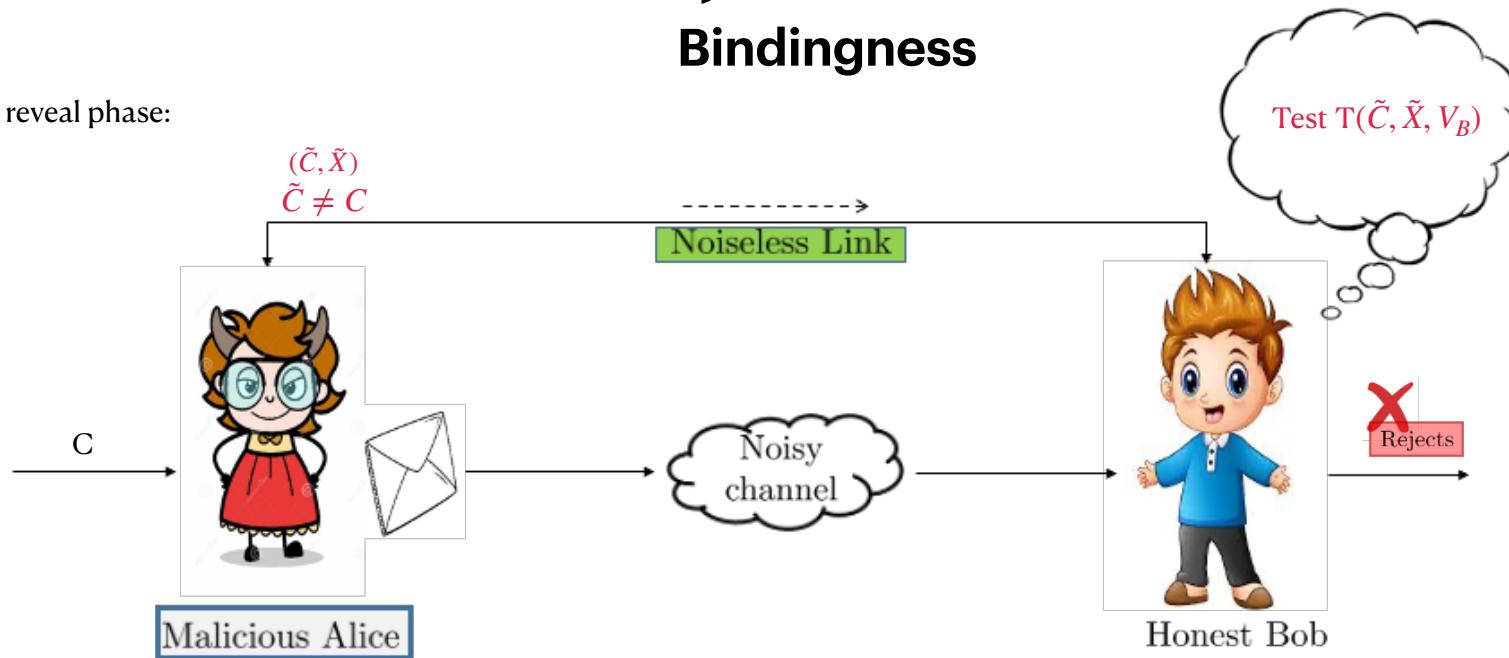


$$I(C; V_B) \leq \epsilon(n)$$

Security Guarantees

Bindingness

In the reveal phase:

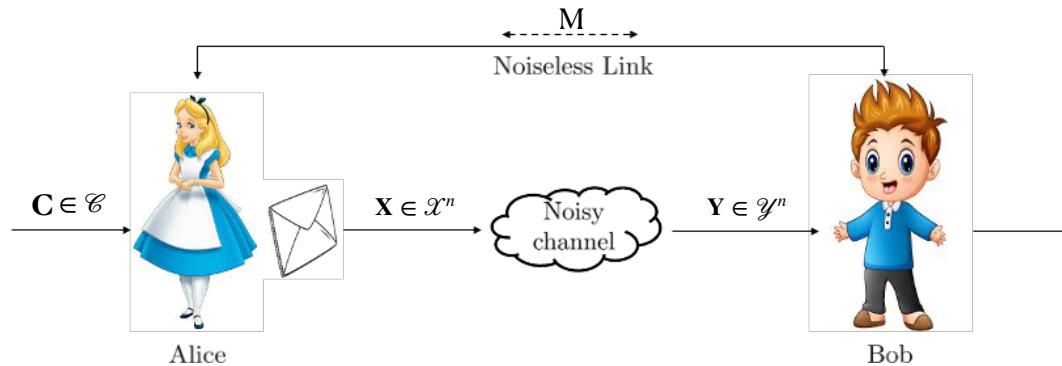


Bob's Test rejects dishonest Alice's cheating string

$$\begin{aligned} &\mathbb{P} \left(T(\hat{C}, \hat{X}, V_B) = \text{"Accept"} \ \& \ T(\tilde{C}, \tilde{X}, V_B) = \text{"Accept"} \right) \leq \epsilon(n) \\ &\forall (\hat{C}, \hat{X}), (\tilde{C}, \tilde{X}) : \hat{C} \neq \tilde{C} \end{aligned}$$

Commitment over Noisy Channels

Commitment Rate & Capacity



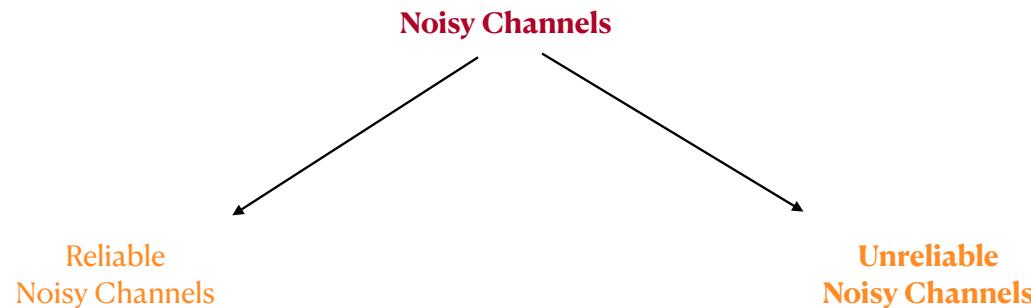
$$\text{Commitment Rate (R)} := \frac{\log |\mathcal{C}|}{n}$$

Rate $R > 0$ is “achievable” if $\forall \epsilon > 0, \forall n$ sufficiently large. \exists an $(n; R)$ –commitment protocol \mathcal{P} : \mathcal{P} is ϵ –sound, ϵ – binding and ϵ –concealing.

$$\mathbb{C} := \sup\{R : R \text{ is achievable}\}$$

Commitment over Noisy Channels

Commitment Capacity



- Perfectly characterised by a fixed transition function
- Examples: DMC($W_{Y|X}$), BSC(p), BEC(p), AWGN($0, \sigma^2$), etc..
- Poorly characterised Channels
- Users are unaware of the precise channel behaviour.
- Examples: Compound DMC ($\{W_{Y|X}\}_{s \in \mathcal{S}}$), AVC($w_{Y|X,s}$), UNC[γ, δ], **Elastic Channel (EC)**-[γ, δ], **Reverse Elastic Channels (REC)**-[γ, δ].

Commitment over Noisy Channels

Known Results on Capacity over Channels

Discrete Memoryless Channels (DMC) $\{W_{Y|X}\}$

$$C_{DMC} = \max_{P_X} H(X | Y)$$

[Winter et. al '04 (IMA ICCC)]

$$C_{BSC} = H(p)$$

Cost-Constrained DMC $\{\rho_X, \Gamma, W_{Y|X}\}$:

$$C(\Gamma) = \max_{P_X: \mathbb{E}[\rho_X(X)] \leq \Gamma} H(X | Y)$$

$$C(\Gamma) = \min_{\gamma \geq 0} \max_{Q_Y} \log \left[\sum_{x \in \mathcal{X}} 2^{-D(W_{Y|X}(\cdot|x)||Q_Y(\cdot)) + \gamma(\Gamma - \rho_X(x))} \right]$$

[MYBM '21 (ISIT)]

Compound- Discrete Memoryless Channels (C-DMC) $\{W_{Y|X}\}_{s \in \mathcal{S}}$

$$C_{C-DMC} = \max_{P_X} \min_{s \in \mathcal{S}} H(X | Y)$$

$$C_{BSC[p,q]} = H(p)$$

[YMBM'21 (NCC)]

State-aware compound channels (honest-but-curious users):

Receiver is state-aware:

$$C_{C-DMC} = \max_{P_X} \min_{s \in \mathcal{S}} H(X | Y)$$

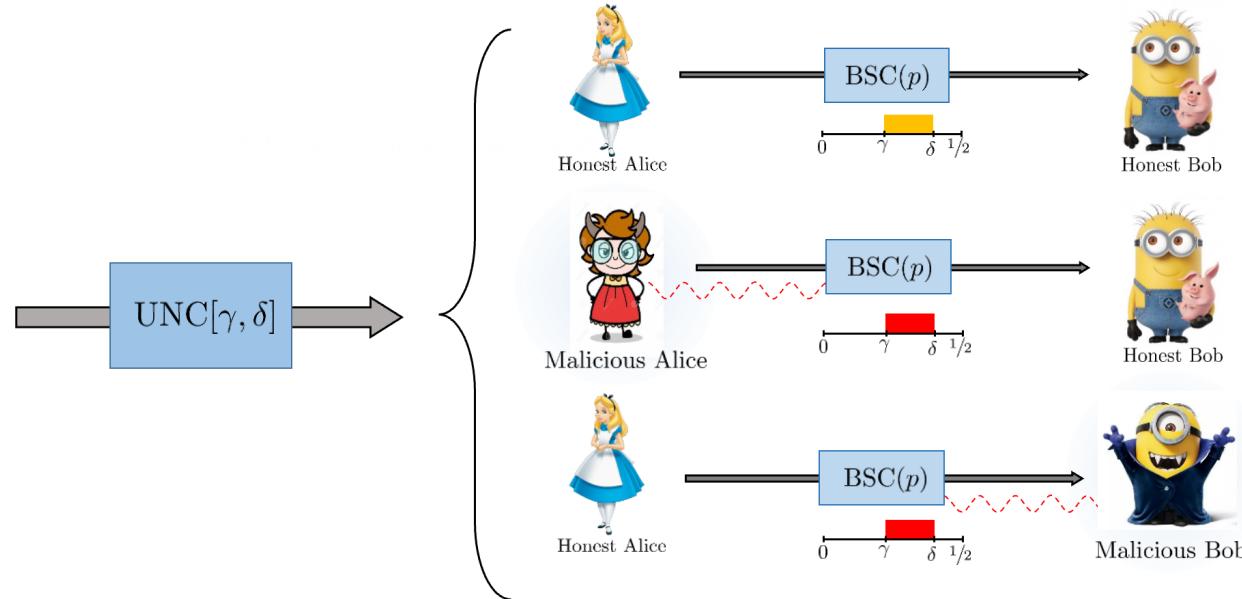
Sender is state-aware:

$$C_{C-DMC} = \min_{P_X} \max_{s \in \mathcal{S}} H(X | Y)$$

[YMBM'22 (COMSNETS)]

Unreliable Noisy Channels

Unfair Noisy Channel (UNC)- $[\gamma, \delta]$
[Damgård et. al '99 (EUROCRYPT)]



If $\delta \geq \gamma * \gamma = 2\gamma(1 - \gamma)$. Then, $C_{UNC} = 0$.

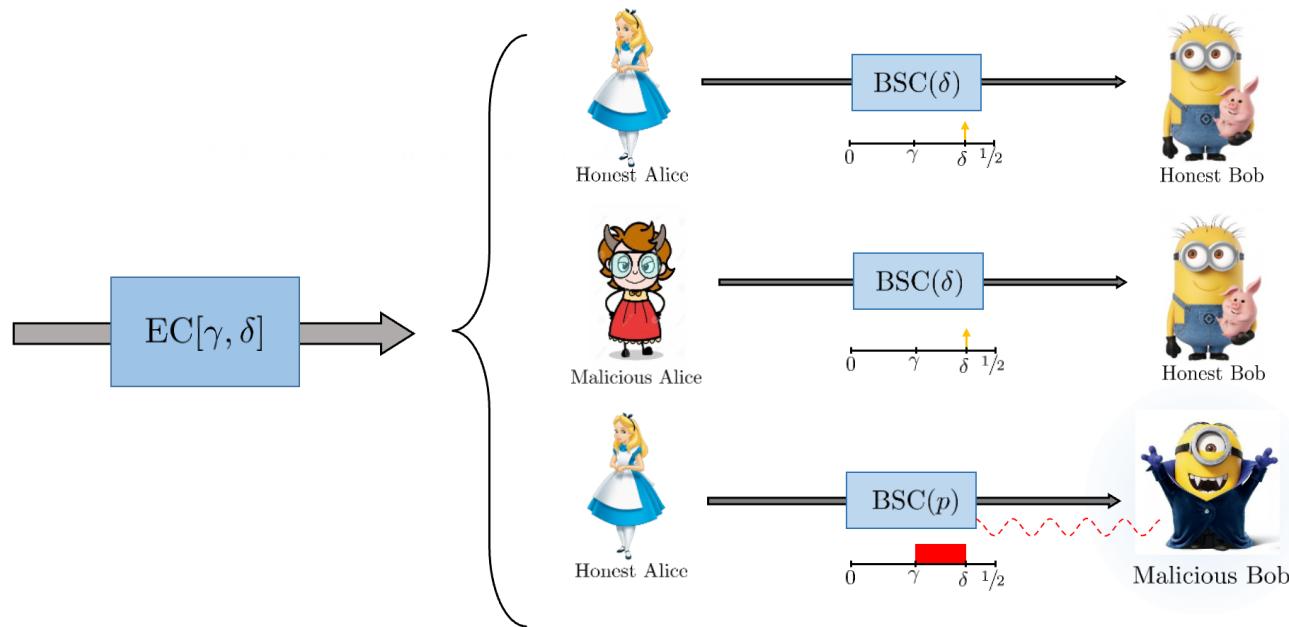
[Damgård et. al '99]

If $\delta < 2\gamma(1 - \gamma)$. Then, $C_{UNC} = H(\gamma) - H\left(\frac{\delta - \gamma}{1 - 2\gamma}\right)$

[Crepeau et. al '20 (Trans. IT)]

Unreliable Noisy Channels

Elastic Channel (EC) - $[\gamma, \delta]$
[Khurana et. al '16 (EUROCRYPT)]



$$C_{EC} \geq 0$$

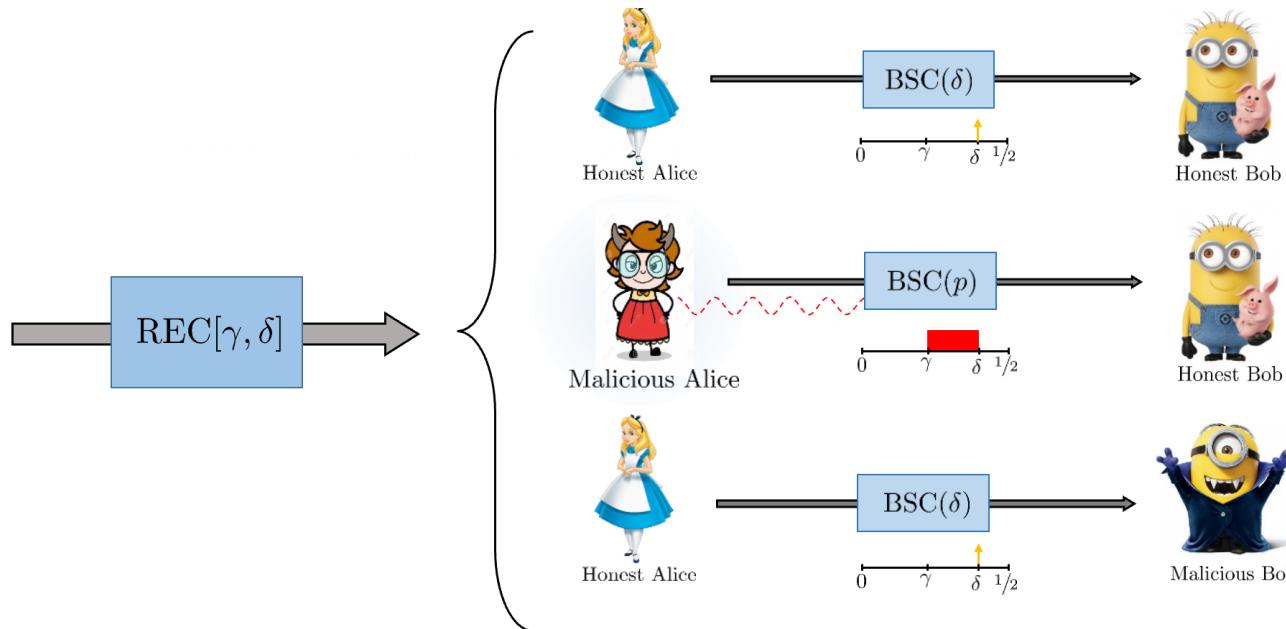
[Khurana et. al '16]

$$C_{EC} = H(\gamma)$$

[Crepeau et. al '20]

Unreliable Noisy Channels

Reverse Elastic Channel (EC) - $[\gamma, \delta]$



$$C_{REC} = H(\delta) - H\left(\frac{\delta - \gamma}{1 - 2\gamma}\right)$$

Conjectured by [Crepeau et. al '20]

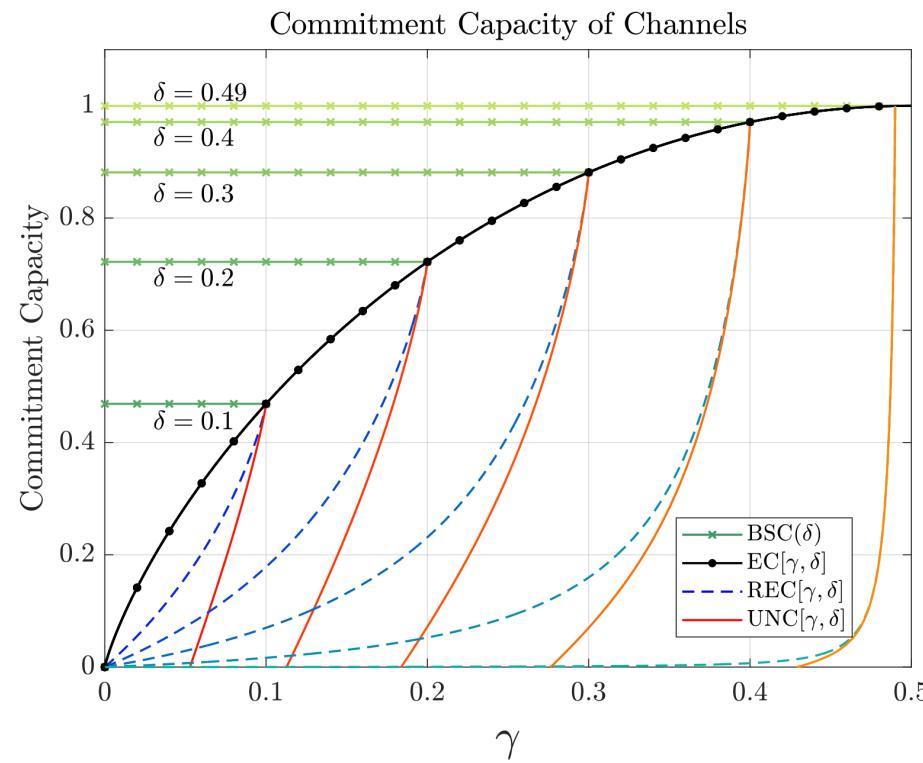
Commitment Capacity of REC- $[\gamma, \delta]$

Key results from [BJMY '22 (JSAC)]

- $C_{REC} \geq 0. \forall 0 < \gamma < \delta < 1/2$
- $C_{REC} = H(\delta) - H\left(\frac{\delta - \gamma}{1 - 2\gamma}\right)$
- $C_{REC} \leq C_{EC}$ i.e., Asymmetry in Commitment Capacity in channels with one-sided elasticity.
- For honest-but-curious users, we have $C_{REC} = C_{EC}$

Comparison of Commitment Capacities

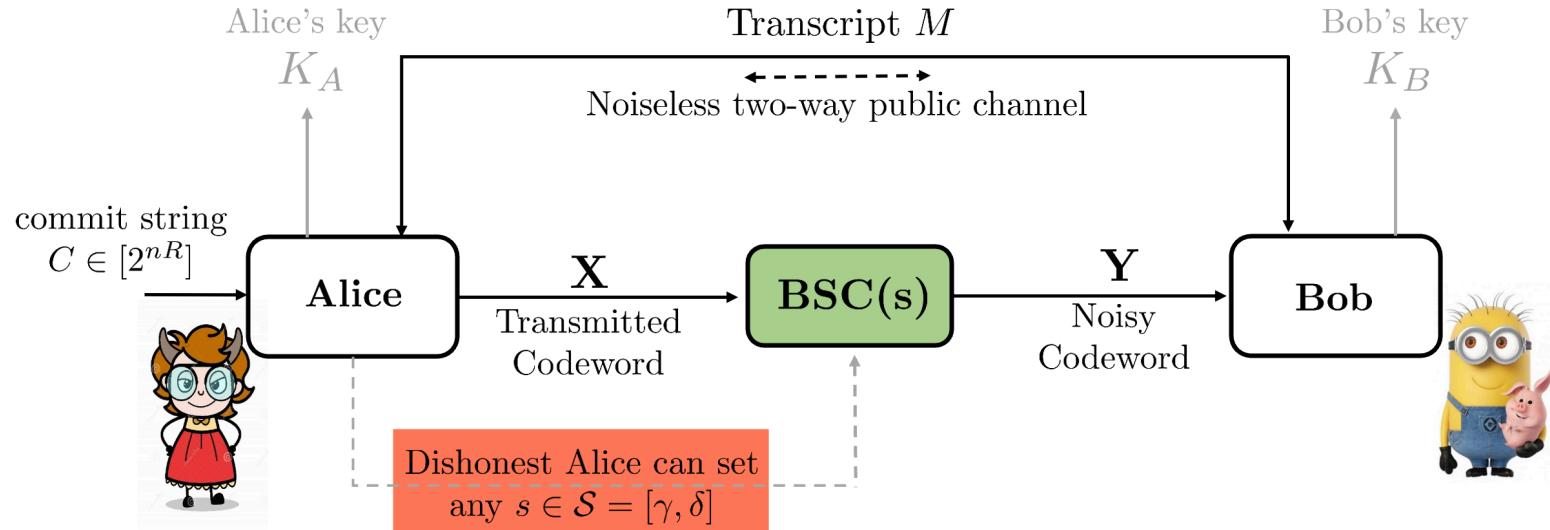
Key results



$$C_{BSC(\delta)} \geq C_{EC[\gamma, \delta]} \geq C_{REC[\gamma, \delta]} \geq C_{UNC[\gamma, \delta]}$$

Commitment over REC- $[\gamma, \delta]$

Revisiting Problem Setup



Reverse Elastic channel EC- $[\gamma, \delta]$
 $(0 < \gamma < \delta < \frac{1}{2})$

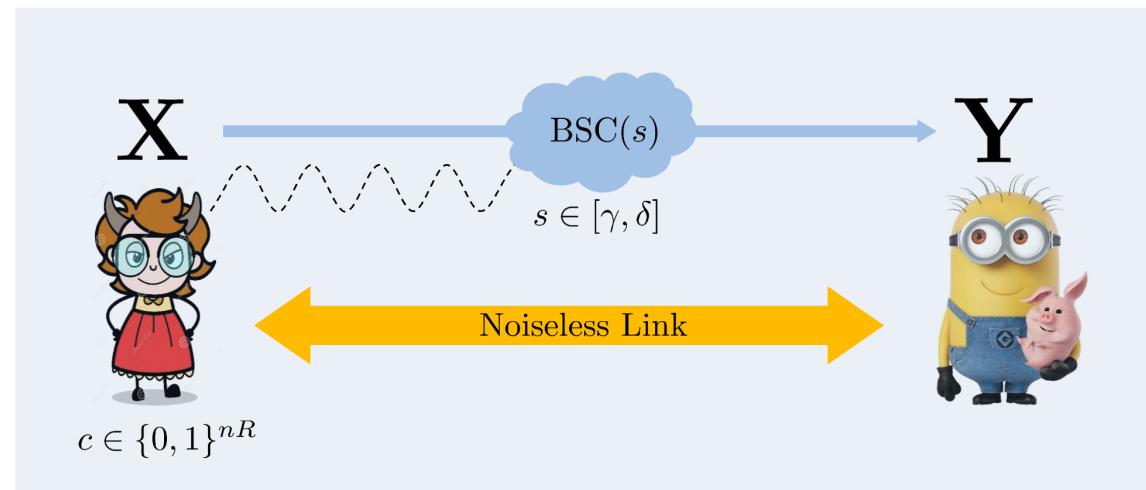
Commitment over REC- $[\gamma, \delta]$

Converse

Alice's Cheating Strategy

Alice sets the channel to be a BSC(s), $s \in [\gamma, \delta]$

This allows her some room to cheat



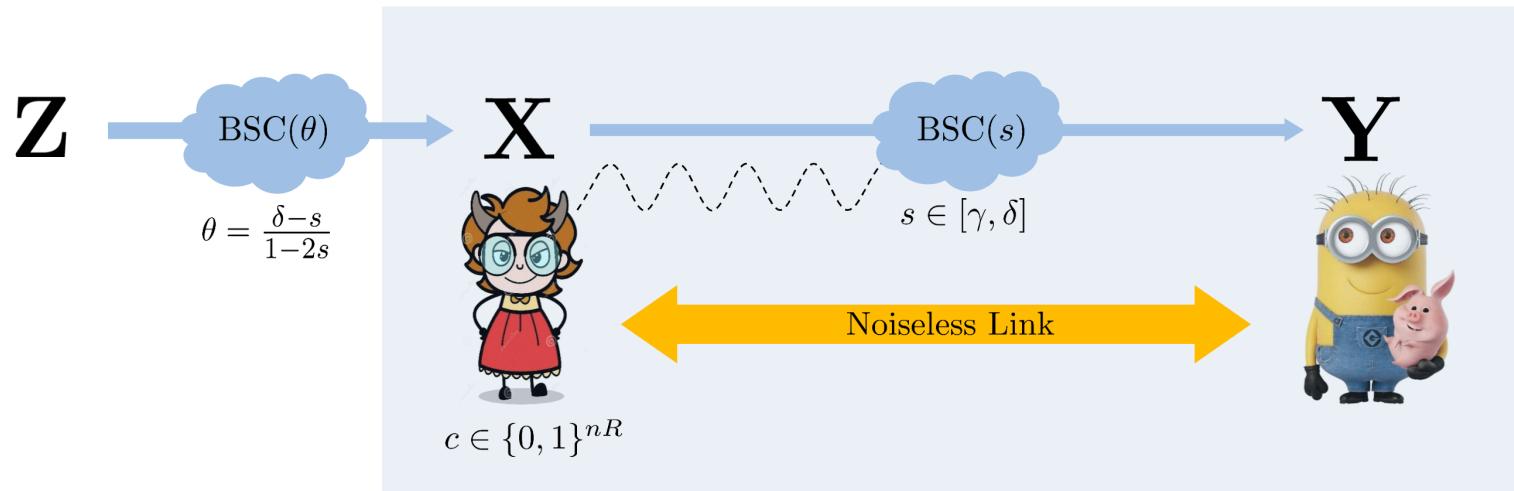
Commitment over REC- $[\gamma, \delta]$

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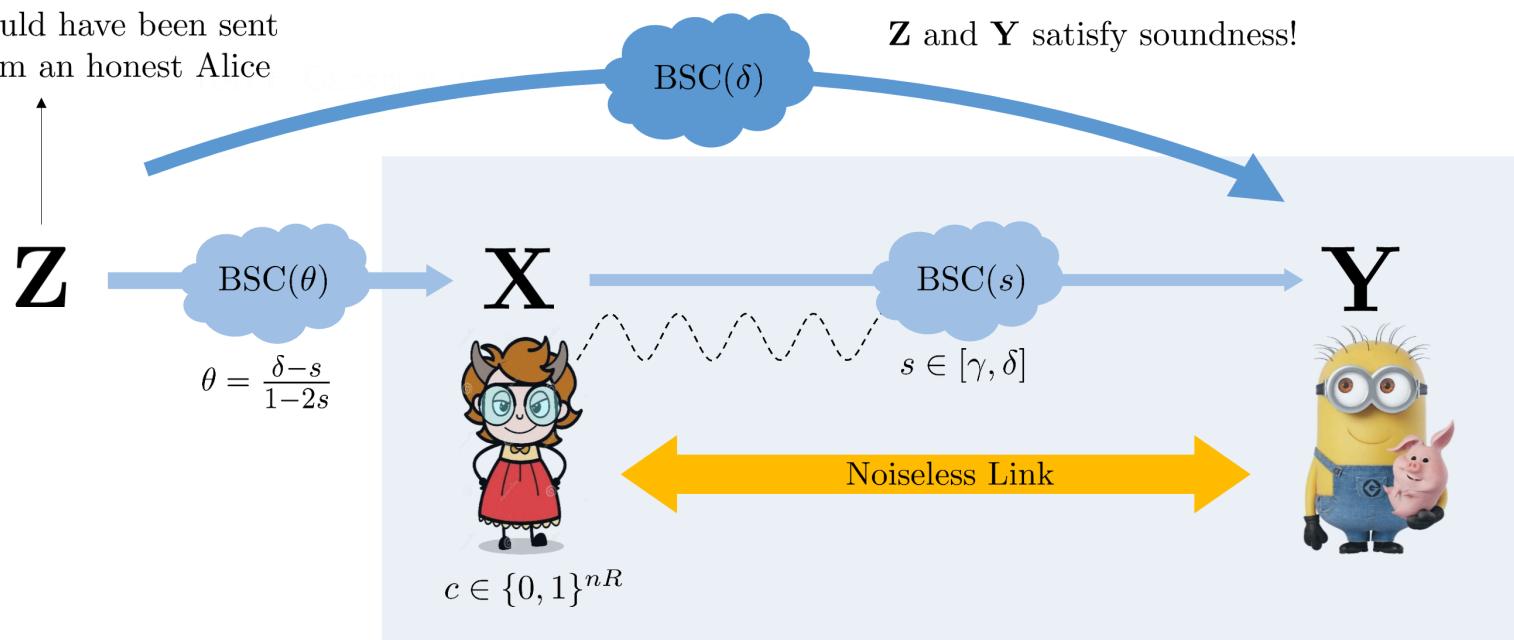


Commitment over REC- $[\gamma, \delta]$

Converse

Alice's Cheating Strategy

Could have been sent from an honest Alice



Commitment over REC- $[\gamma, \delta]$

Converse

A rate R scheme: $\epsilon_n - sound$, $\epsilon_n - concealing$ and $\epsilon_n - binding$ $\left(\epsilon_n \xrightarrow{n \rightarrow \infty} 0 \right)$

$$nR = H(C)$$

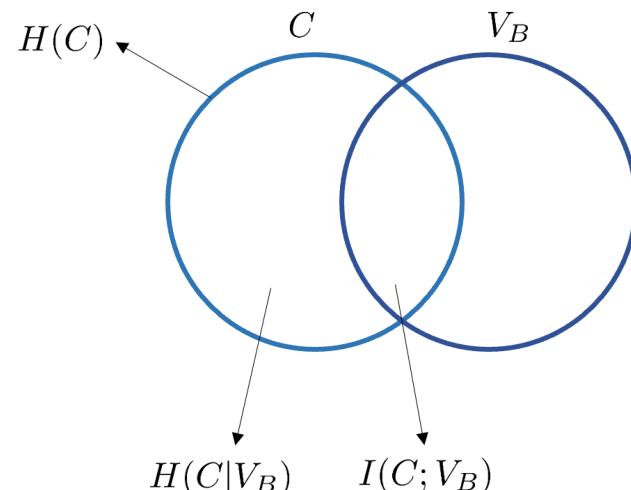
Because $C \in \{0, 1\}^{nR}$

Now, we analyse this expression **assuming Alice executes the cheating strategy described previously**

Commitment over REC- $[\gamma, \delta]$

Converse

$$\begin{aligned} nR &= H(C) \\ &= H(C|V_B) + I(C; V_B) \quad \text{Inner Loop Condition} \\ &\leq H(C|\mathbf{Y}, M, K_B) + \epsilon_n \\ &= H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\ &\quad + H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon_n \\ &\leq H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon'' + \epsilon_n \\ &= H(C|\mathbf{Y}, M, K_B) - H(C|M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\ &\quad + H(C|M, K_B) + \epsilon'' + \epsilon_n \\ &= I(C; \mathbf{YZ}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon \\ &= I(C; \tilde{\mathbf{Z}}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon \end{aligned}$$



Commitment over REC- $[\gamma, \delta]$

Converse

$$\begin{aligned} nR &= H(C) \\ &= H(C|V_B) + I(C; V_B) \quad \text{inner-outer Conjecture} \\ &\leq H(C|\mathbf{Y}, M, K_B) + \epsilon_n && I(C; V_B) \leq \epsilon_n \text{ by concealment} \\ &= H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\ &\quad + H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon_n \\ &\leq H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon'' + \epsilon_n \\ &= H(C|\mathbf{Y}, M, K_B) - H(C|M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\ &\quad + H(C|M, K_B) + \epsilon'' + \epsilon_n \\ &= I(C; \mathbf{YZ}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon \\ &= I(C; \tilde{\mathbf{Z}}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon \end{aligned}$$

Commitment over REC- $[\gamma, \delta]$

Converse

$$\begin{aligned} nR &= H(C) \\ &= H(C|V_B) + I(C; V_B) \quad \text{Since } C \text{ is a function of } V_B \\ &\leq H(C|\mathbf{Y}, M, K_B) + \epsilon_n \\ &= H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\ &\quad + H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon_n \\ &\leq H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon'' + \epsilon_n \\ &= H(C|\mathbf{Y}, M, K_B) - H(C|M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\ &\quad + H(C|M, K_B) + \epsilon'' + \epsilon_n \\ &= I(C; \mathbf{YZ}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon \\ &= I(C; \tilde{\mathbf{Z}}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon \end{aligned}$$

Adding and subtracting

Commitment over REC- $[\gamma, \delta]$

Converse

$$\begin{aligned} nR &= H(C) \\ &= H(C|V_B) + I(C; V_B) \quad \text{Under Cost Constraints} \\ &\leq H(C|\mathbf{Y}, M, K_B) + \epsilon_n \\ &= H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\ &\quad + H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon_n \quad \xrightarrow{\quad} H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \leq \epsilon'' \\ &\leq H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon'' + \epsilon_n \\ &= H(C|\mathbf{Y}, M, K_B) - H(C|M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\ &\quad + H(C|M, K_B) + \epsilon'' + \epsilon_n \\ &= I(C; \mathbf{YZ}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon \\ &= I(C; \tilde{\mathbf{Z}}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon \end{aligned}$$

Commitment over REC- $[\gamma, \delta]$

Converse

$$\begin{aligned} nR &= H(C) \\ &= H(C|V_B) + I(C; V_B) \quad \text{Rate-Cost Constraints} \\ &\leq H(C|\mathbf{Y}, M, K_B) + \epsilon_n \\ &= H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\ &\quad + H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon_n \\ &\leq H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon'' + \epsilon_n \\ &= H(C|\mathbf{Y}, M, K_B) - H(C|M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\ &\quad + H(C|M, K_B) + \epsilon'' + \epsilon_n \quad \boxed{\text{Adding and subtracting } H(C|M, K_B)} \\ &= I(C; \mathbf{YZ}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon \\ &= I(C; \tilde{\mathbf{Z}}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon \end{aligned}$$

Commitment over REC- $[\gamma, \delta]$

Converse

$$\begin{aligned}
nR &= H(C) \\
&= H(C|V_B) + I(C; V_B) \quad \text{Under Cost Constraint} \\
&\leq H(C|\mathbf{Y}, M, K_B) + \epsilon_n \\
&= H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\
&\quad + H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon_n \\
&\leq H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon'' + \epsilon_n \\
&= H(C|\mathbf{Y}, M, K_B) - H(C|M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\
&\quad + H(C|M, K_B) + \epsilon'' + \epsilon_n \\
&= I(C; \mathbf{YZ}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon \quad \boxed{\phantom{I(C; \mathbf{YZ}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon}} \\
&= I(C; \tilde{\mathbf{Z}}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon
\end{aligned}$$

Grouping 3^{rd} with 4^{th} term and
 1^{st} with 2^{nd} term

Commitment over REC- $[\gamma, \delta]$

Converse

$$\begin{aligned} nR &= H(C) \\ &= H(C|V_B) + I(C; V_B) \quad \text{Inner Coding Constraints} \\ &\leq H(C|\mathbf{Y}, M, K_B) + \epsilon_n \\ &= H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\ &\quad + H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon_n \\ &\leq H(C|\mathbf{Y}, M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) + \epsilon'' + \epsilon_n \\ &= H(C|\mathbf{Y}, M, K_B) - H(C|M, K_B) - H(C|\mathbf{Y}, \mathbf{Z}, M, K_B) \\ &\quad + H(C|M, K_B) + \epsilon'' + \epsilon_n \\ &= I(C; \mathbf{YZ}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon \\ &= I(C; \tilde{\mathbf{Z}}|M, K_B) - I(C; \mathbf{Y}|M, K_B) + \epsilon'' + \epsilon \end{aligned}$$

Denoting the pair of random variables (\mathbf{Y}, \mathbf{Z}) as $\tilde{\mathbf{Z}}$

Commitment over REC- $[\gamma, \delta]$

Converse

$$\begin{aligned} nR &= H(C) \\ &= I(C; \tilde{\mathbf{Z}} | M, K_B) - I(C; \mathbf{Y} | M, K_B) + \epsilon'' + \epsilon \\ \implies R &\leq I(\mathbf{X}; \tilde{\mathbf{Z}}) - I(\mathbf{X}; \mathbf{Y}) + \frac{\epsilon''}{n} + \frac{\epsilon}{n} \end{aligned}$$

Using the result from seminal work [Csizar and Korner '78]
Followed by few non-trivial information-theoretic
reductions

Commitment over REC- $[\gamma, \delta]$

Converse

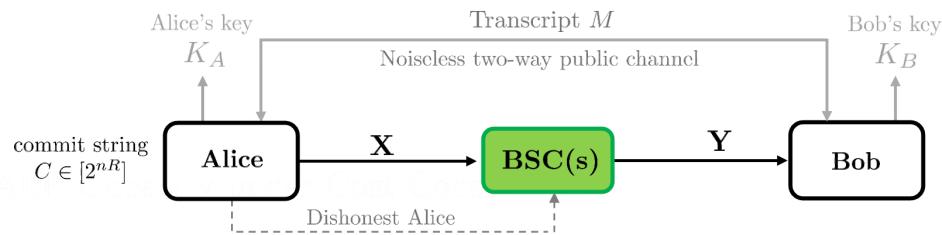
$$\begin{aligned} nR &= H(C) \\ &= I(C; \tilde{\mathbf{Z}} | M, K_B) - I(C; \mathbf{Y} | M, K_B) + \epsilon'' + \epsilon \\ \implies R &\leq I(\mathbf{X}; \tilde{\mathbf{Z}}) - I(\mathbf{X}; \mathbf{Y}) + \frac{\epsilon''}{n} + \frac{\epsilon}{n} \\ \implies R &\leq \min_{s \in [\gamma, \delta]} [I(\mathbf{X}; \tilde{\mathbf{Z}}) - I(\mathbf{X}; \mathbf{Y})] \\ &\leq \max_{P_X} \min_{s \in [\gamma, \delta]} [I(\mathbf{X}; \tilde{\mathbf{Z}}) - I(\mathbf{X}; \mathbf{Y})] \\ &\leq H(\delta) - H(\theta) \end{aligned}$$

Because the inequality holds for *all* cheating behaviours of Alice, it must also hold for the minimum

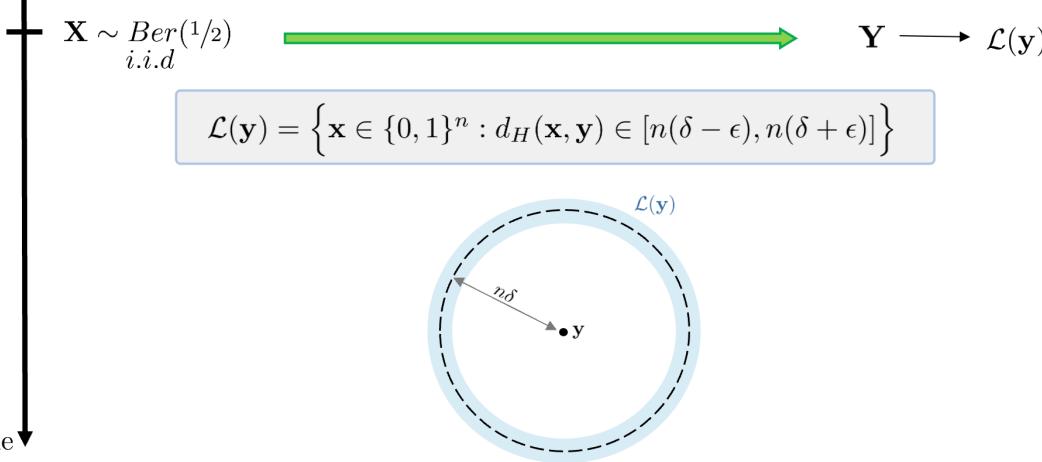
Commitment over REC- $[\gamma, \delta]$

Achievability: Protocol: Commit phase

The Protocol



Time

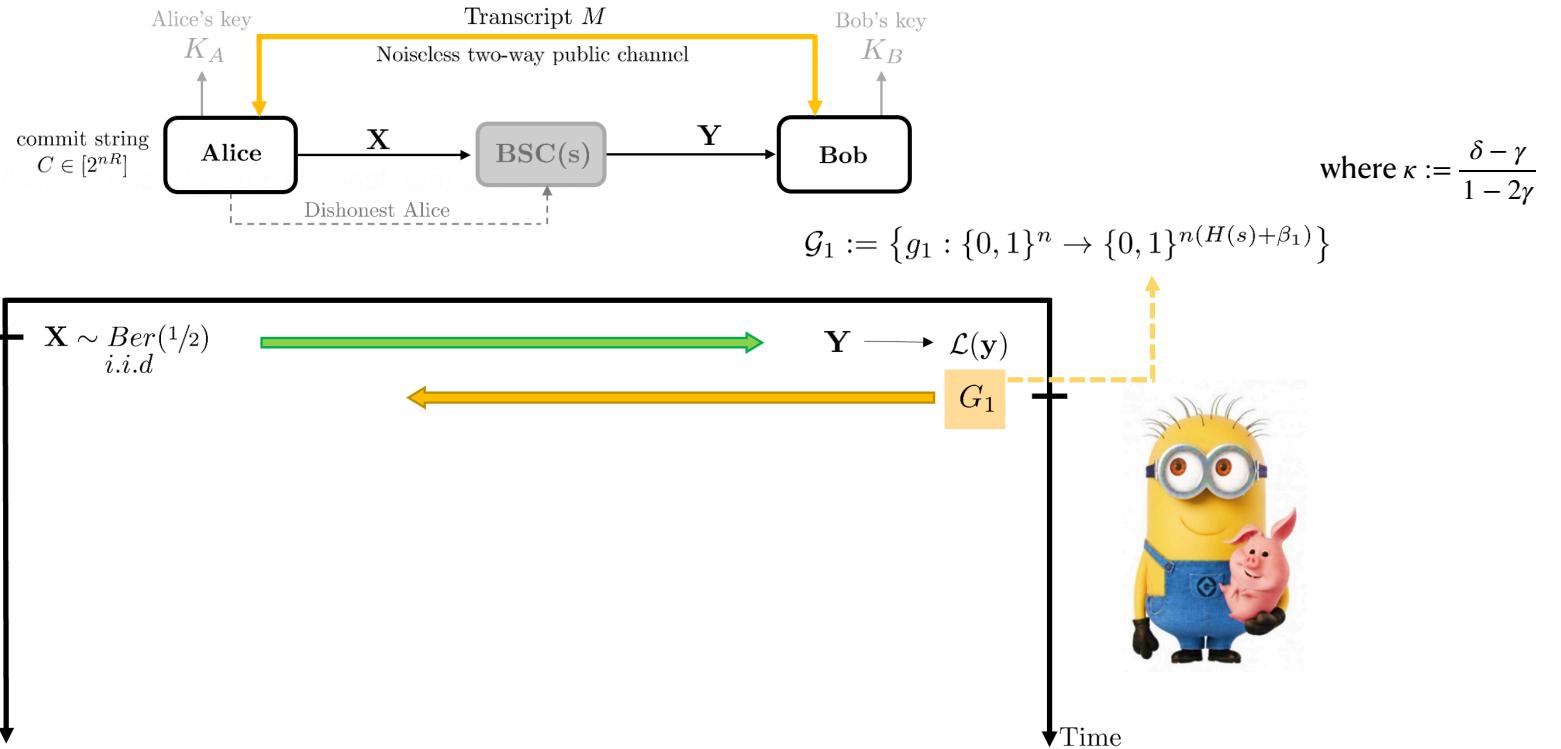


Time

Commitment over REC- $[\gamma, \delta]$

Achievability: Protocol: Commit phase

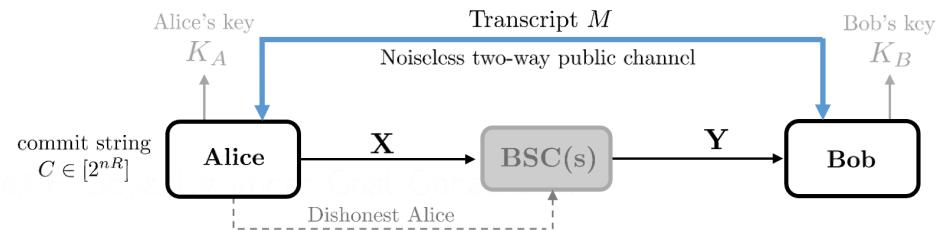
The Protocol



Commitment over REC- $[\gamma, \delta]$

Achievability: Protocol: Commit phase

The Protocol



Time



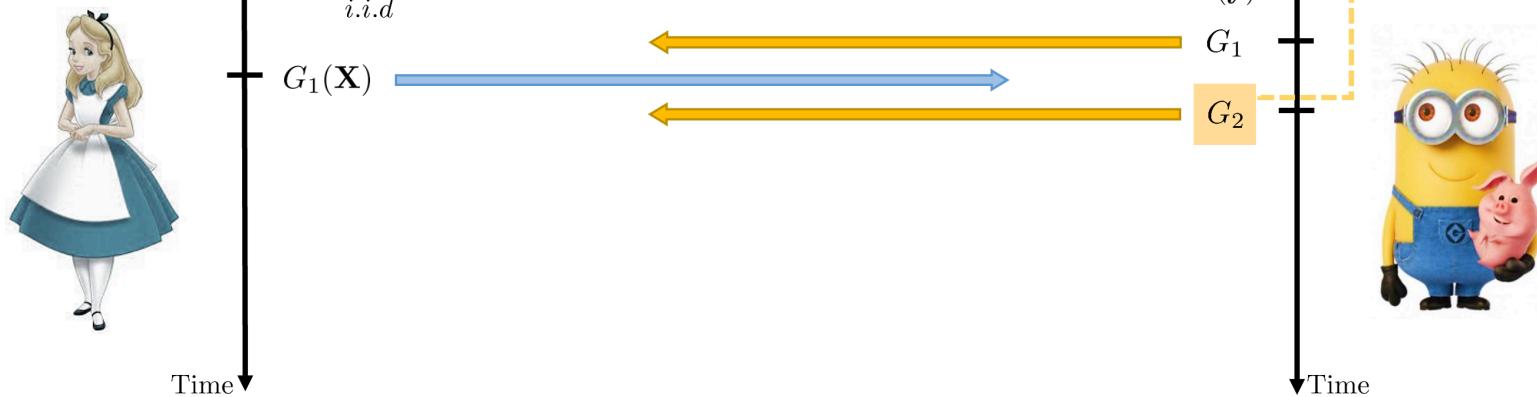
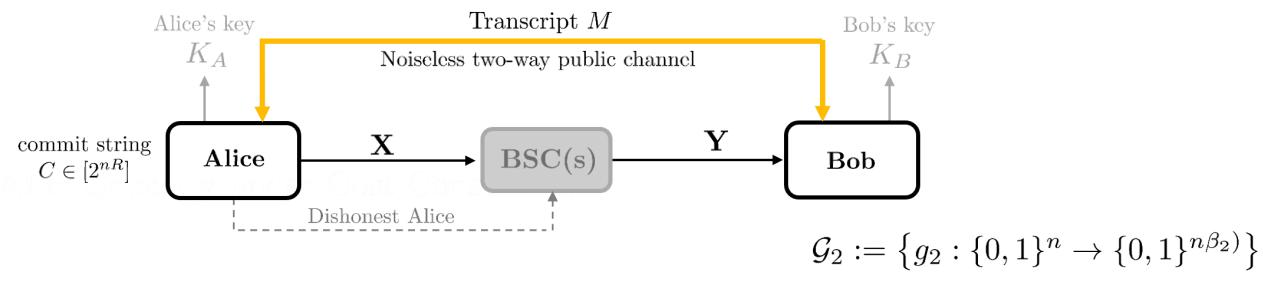
Time



Commitment over REC- $[\gamma, \delta]$

Achievability: Protocol: Commit phase

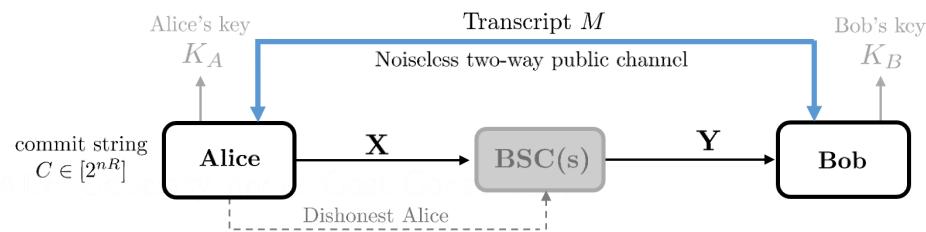
The Protocol



Commitment over REC- $[\gamma, \delta]$

Achievability: Protocol: Commit phase

The Protocol



Time



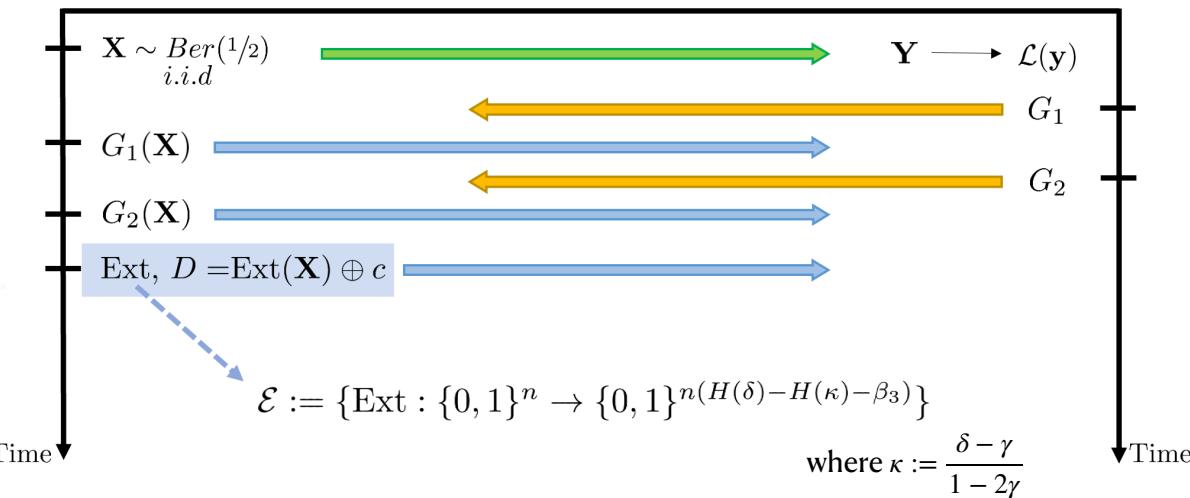
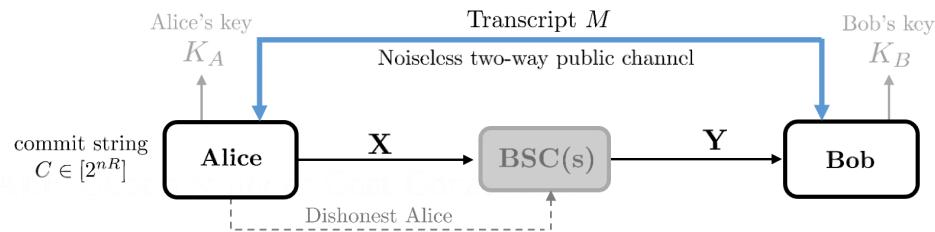
Time



Commitment over REC- $[\gamma, \delta]$

Achievability: Protocol: Commit phase

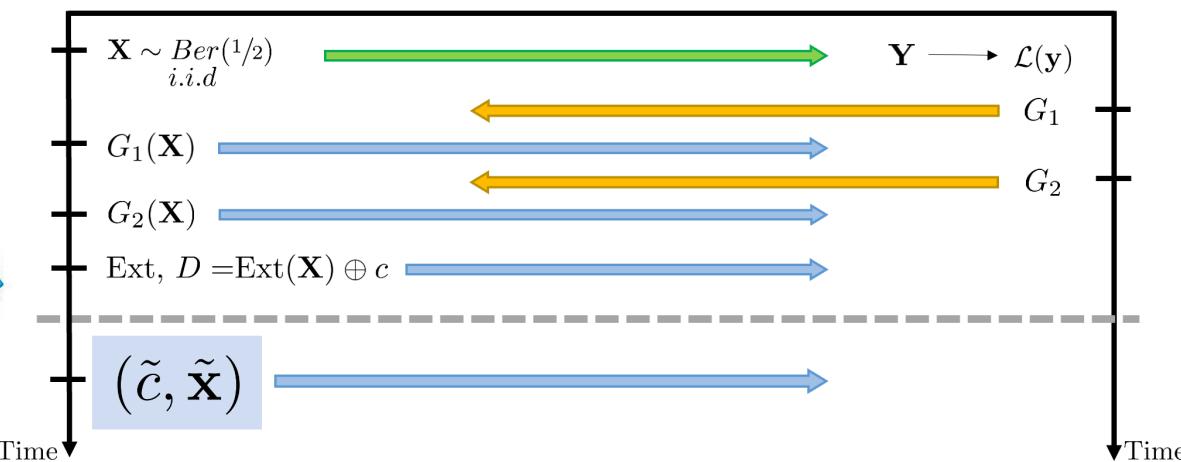
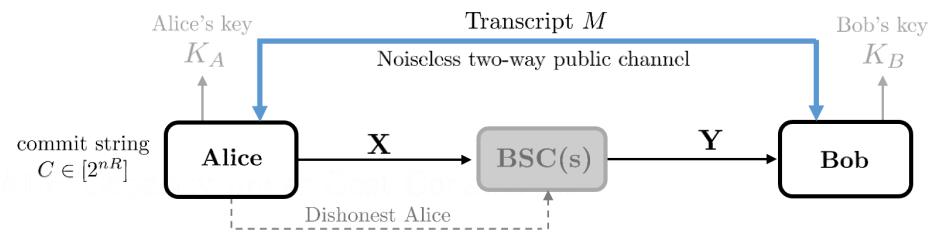
The Protocol



Commitment over REC- $[\gamma, \delta]$

Achievability: Protocol: Reveal phase

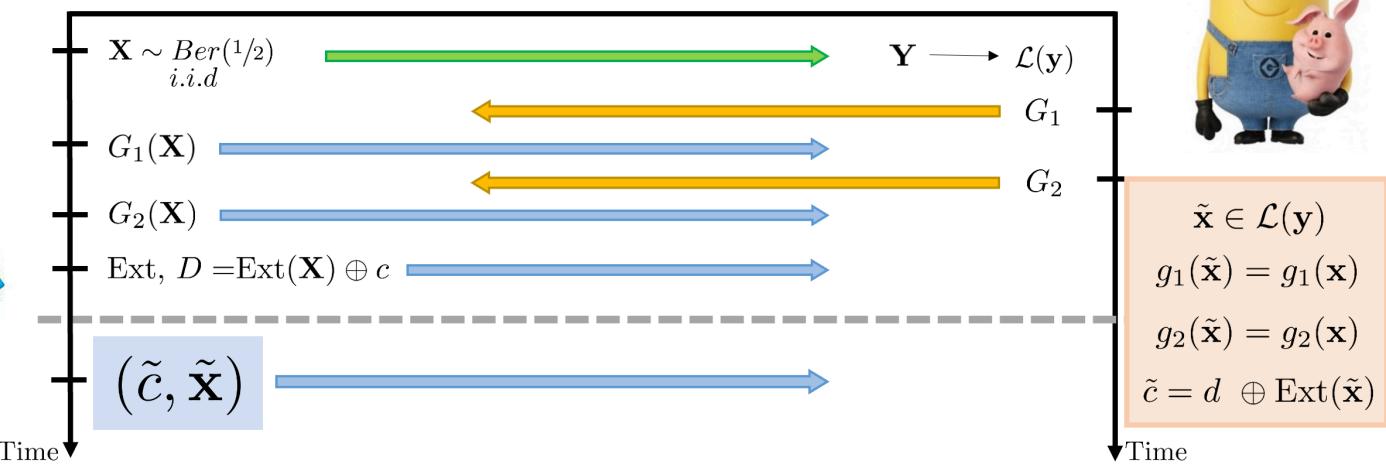
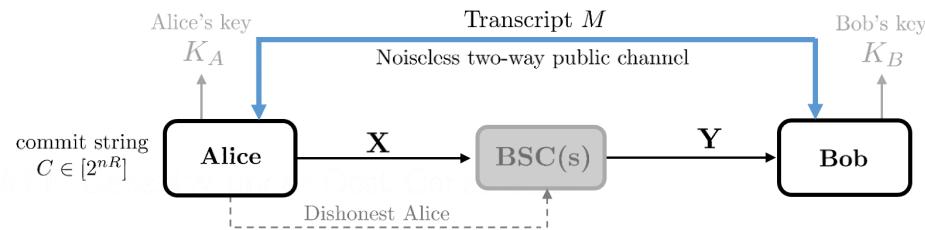
The Protocol



Commitment over REC- $[\gamma, \delta]$

Achievability: Protocol: Reveal phase

The Protocol

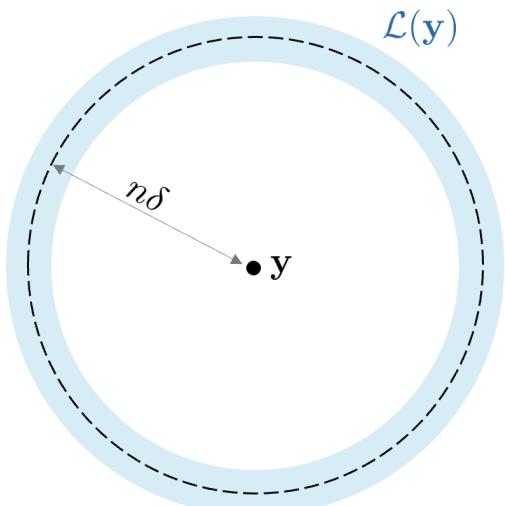


Commitment over REC- $[\gamma, \delta]$

Security Guarantees: ϵ -soundness

Bob prepares a list

$$\mathcal{L}(\mathbf{y}) = \left\{ \mathbf{x} \in \{0, 1\}^n : d_H(\mathbf{x}, \mathbf{y}) \in [n(\delta - \epsilon), n(\delta + \epsilon)] \right\}$$



Protocol is sound if $\mathbf{X} \in \mathcal{L}(\mathbf{y})$ with high probability

Using the Chernoff Bound, and the fact that \mathbf{X} and \mathbf{Y} are connected via a BSC(δ), we can show:

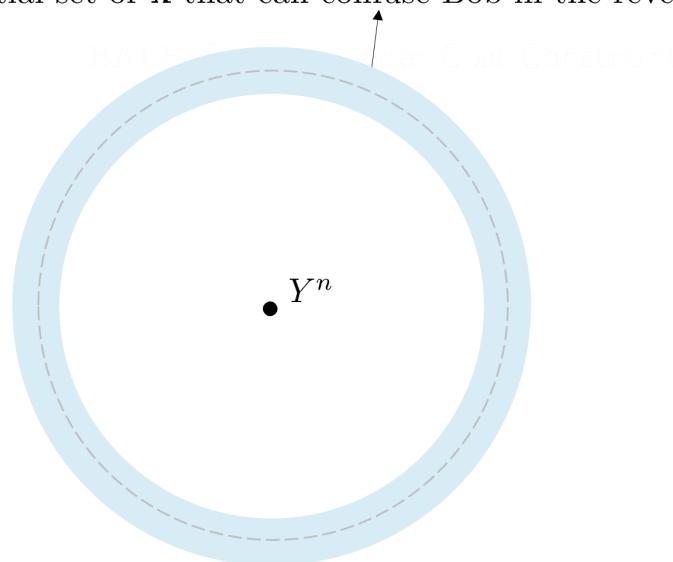
$$P(\mathbf{X} \notin \mathcal{L}(\mathbf{y})) \leq \epsilon'(n)$$

$\epsilon'(n) \rightarrow 0$ as $n \rightarrow \infty$

Commitment over REC- $[\gamma, \delta]$

Security Guarantees: ϵ -bindingness

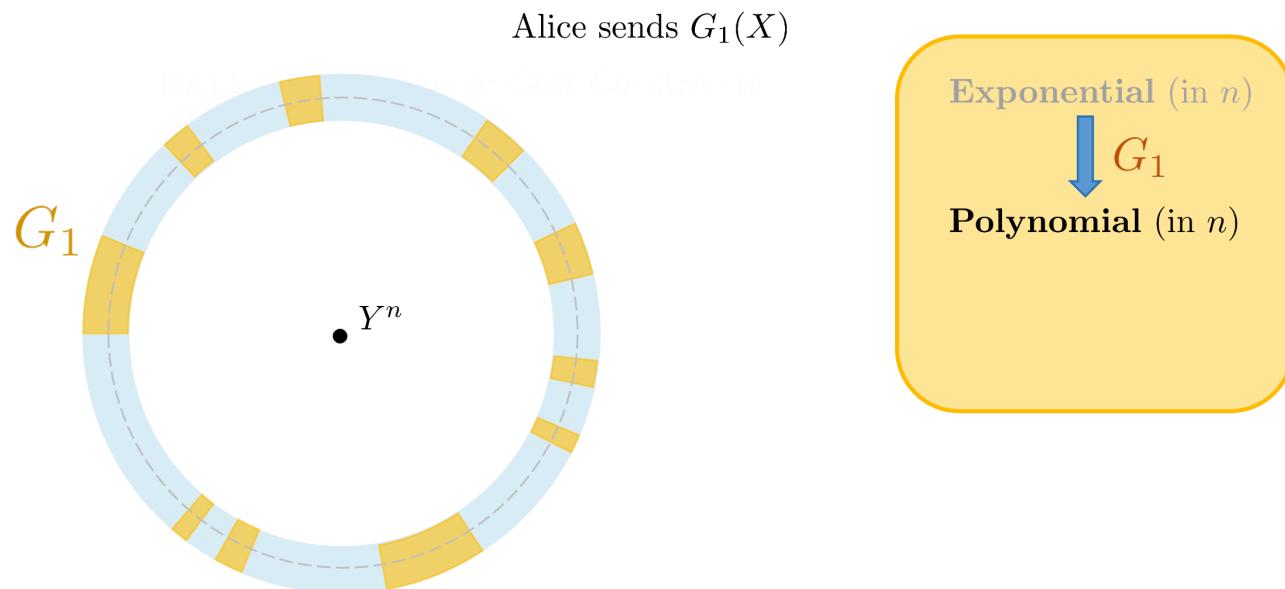
Initial set of \mathbf{x} that can confuse Bob in the reveal phase.



Exponential (in n)

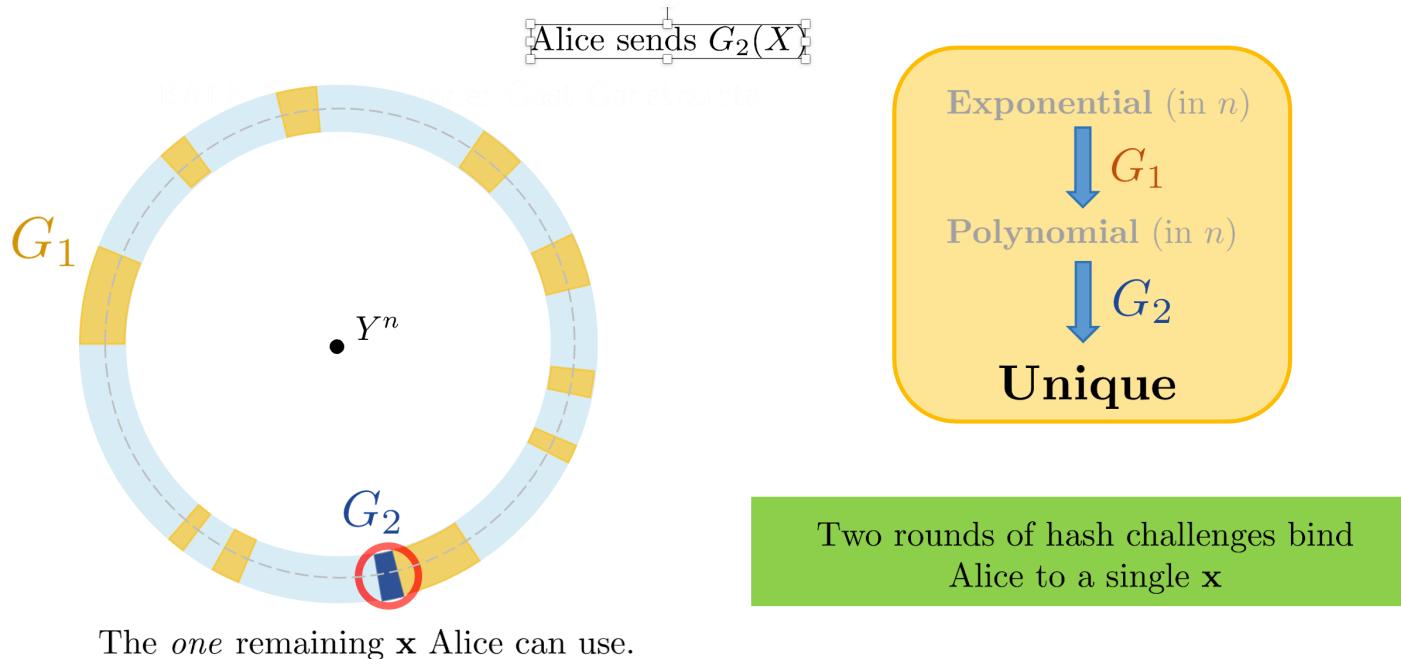
Commitment over REC- $[\gamma, \delta]$

Security Guarantees: ϵ -bindingness



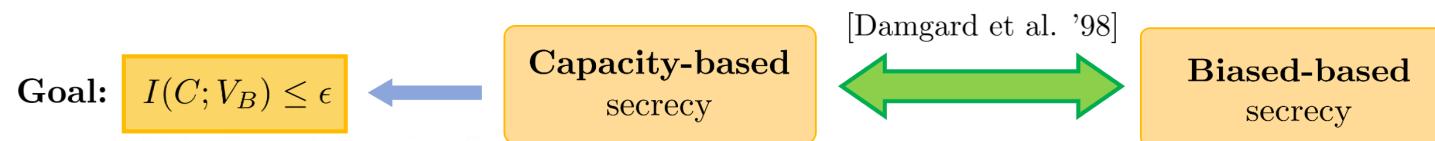
Commitment over REC- $[\gamma, \delta]$

Security Guarantees: ϵ -bindingness



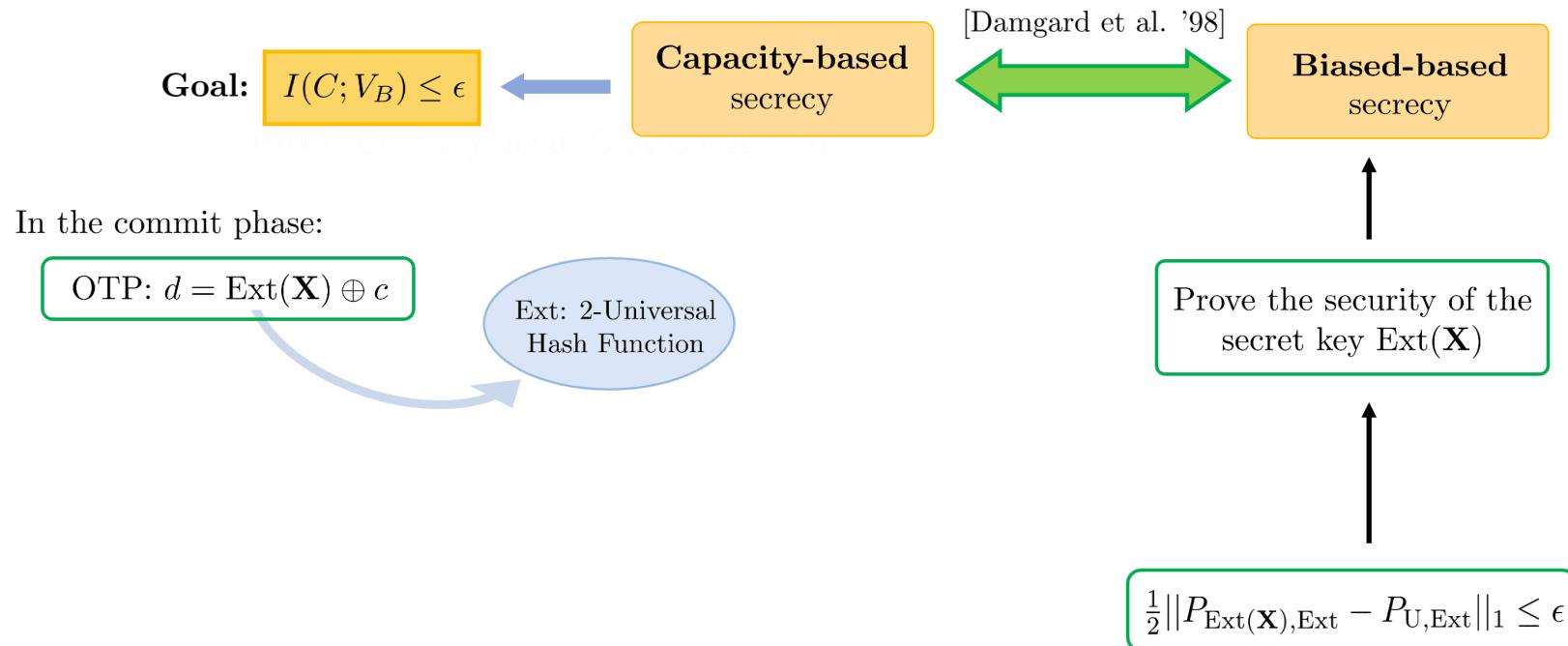
Commitment over REC- $[\gamma, \delta]$

Security Guarantees: ϵ -concealment



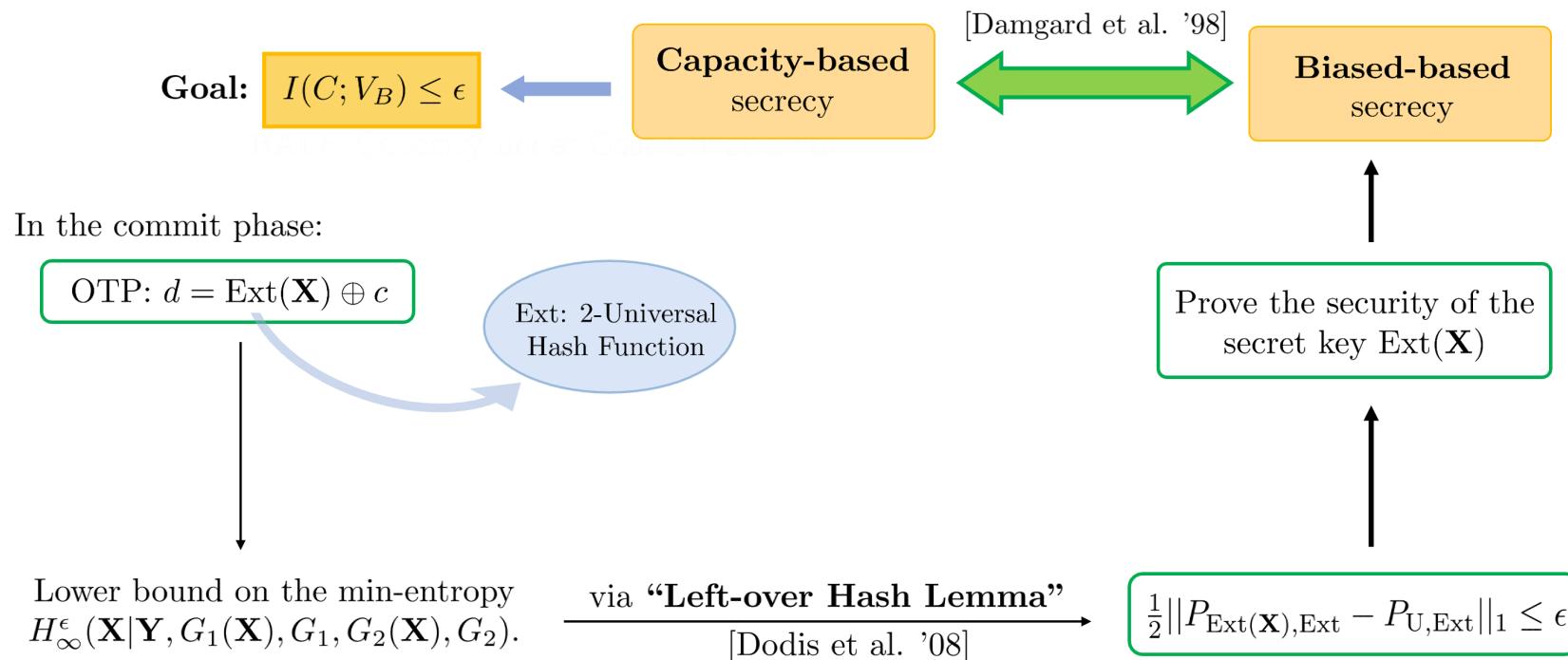
Commitment over REC- $[\gamma, \delta]$

Security Guarantees: ϵ -concealment



Commitment over REC- $[\gamma, \delta]$

Security Guarantees: ϵ -concealment



Commitment over Noisy Channels

Other Interesting Results:

- Commitment Capacity of AWGN channels is “Infinite”. [**Nascimento et.al (Trans. IT ‘08)**]
- UNC version of Gaussian Channels may have finite capacity. It has zero commitment capacity if $\delta^2 \geq 2\gamma^2$, even under infinite input power. and other results.. [**BJMY (ISIT ‘23)**]
- Bit commitment over Multiple-access channels. [**Chou and Bloch (Allerton ‘22)**]