DSC 462 Assignment 2

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Question 1

(a) The PMF of X is given by

$$p_i = P(X = i) = P(X_1 - X_2 = i).$$

The support of X is $\mathcal{S}_X = \{0, \pm 1, ..., \pm 5\}$. Then the PMF is given by

$$\begin{aligned}
p_0 &= P(X=0) = P((X_1,X_2) \in \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}) = 6/36 \\
p_1 &= P(X=1) = P((X_1,X_2) \in \{(2,1),(3,2),(4,3),(5,4),(6,5)\}) = 5/36 \\
p_{-1} &= P(X=-1) = P((X_1,X_2) \in \{(1,2),(2,3),(3,4),(4,5),(5,6)\}) = 5/36 \\
p_2 &= P(X=2) = P((X_1,X_2) \in \{((3,1),(4,2),(5,3),(6,4)\}) = 4/36 \\
p_{-2} &= P(X=-2) = P((X_1,X_2) \in \{(1,3),(2,4),(3,5),(4,6)\}) = 4/36 \\
p_3 &= P(X=3) = P((X_1,X_2) \in \{((4,1),(5,2),(6,3)\}) = 3/36 \\
p_{-3} &= P(X=-3) = P((X_1,X_2) \in \{(1,4),(2,5),(3,6)\}) = 3/36 \\
p_4 &= P(X=4) = P((X_1,X_2) \in \{((5,1),(6,2)\}) = 2/36 \\
p_{-4} &= P(X=-4) = P((X_1,X_2) \in \{((1,5),(2,6)\}) = 2/36 \\
p_5 &= P(X=5) = P((X_1,X_2) \in \{((6,1)\}) = 1/36 \\
p_{-5} &= P(X=-5) = P((X_1,X_2) \in \{((1,6)\}) = 1/36
\end{aligned}$$

(b) Similarly, The support of X is $\mathcal{S}_X = \{0, 1, ..., 6\}$. Then the PMF is given by

$$P(X=0) = P((X_1,X_2) \in \{(j,k) | j \neq k, 1 \leq j,k \leq 6\}) = 30/36$$

And for $1 \leq i \leq 6$, we have

$$P(X=i) = P((X_1,X_2) \in \{(i,i)\}) = 1/36.$$

Question 2

(a)We have

$$1 = \int_2^4 -c(x-2)(x-4)\ dx = \int_2^4 -c(x^2-6x+8)\ dx = -c[x^3/3 - 3x^2 + 8x] = c(4/3)$$

So, $c = 3/4$.

(b) For $x \leq 2$, we have $F_X(x) = 0$, and for $x \geq 4$, we have $F_X(x) = 1$. For $x \in [2,4]$ we have $F_X(x) = \int_2^x (3/4)(x-2)(x-4)\ dx = (3/4)(x^3/3 - 3x^2 + 8x - 20/3)$. So our CDF will be

$$F_X(x) = \begin{cases} 0 \text{ if } x \leq 2 \\ (3/4)(x^3/3 - 3x^2 + 8x - 20/3) \text{ if } x \in [2,4] \\ 1 \text{ if } x \geq 4 \end{cases}$$

Question 3

Let X denote the number of boys. So, $X \sim bin(10, 0.5)$.

(a) $P(X = 6) = \binom{10}{6}(0.5)^6(0.5)^4 = \binom{10}{6}(0.5)^{10} = 0.2050781.$

(b) $P(X \geq 8) = P(X = 8) + P(X = 9) + P(X = 10) = \left[\binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right] 0.5^{10} = 0.0546875.$

(c) Let A be the event that first six of the babies are boys. Then

$$P(A|X = 6) = \frac{P(A \cap \{X = 6\})}{P(X = 6)} = \frac{1}{\binom{10}{6}} = \frac{1}{210}.$$

Question 4

The height of a plant $X \sim N(49.2, 1.75^2)$. So the probability that a single plant has a height of no more than 48 inches is

$$\begin{aligned} p &= P(X \leq 48) \\ &= P(\frac{X - 49.2}{1.75} \leq \frac{48 - 49.2}{1.75}) \\ &\approx P(Z \leq -0.6857143) \qquad \approx 0.2464 \end{aligned}$$

where $Z \sim N(0, 1)$. The number of plants of no more than 48 inches height is a binomial distribution $Y \sim bin(20, p)$. So

$$P(Y \leq 3) = \binom{20}{0}(1-p)^{20} + \binom{20}{1}p(1-p)^{19} + \binom{20}{2}p^2(1-p)^{18} + \binom{20}{3}p^3(1-p)^{17}$$

$$\approx 0.2361194$$

Question 5

```
set.seed(12) # for reproducibility
f = function(n=10000){
    coordinates = matrix(runif(n*2),n,2)
    m = 0.0
    for(i in 1:n){
        x = coordinates[i,1]
        y = coordinates[i,2]
        if((x-.5)^2+(y-.5)^2<=.25) m = m+1
    }
    return(4*m/n)
}

# Some outputs
f()
```

```
## [1] 3.1276
```

```
f()
```

```
## [1] 3.1536
```

```
f()
```

```
## [1] 3.1576
```

```
f()
```

```
## [1] 3.1404
```

Question 6

Using the notation of hint, $T_1 = 1$ with probability 1. And for $j \geq 1$, $T_{j+1} - T_j \sim geom(\frac{m-j}{m})$. Let X be the number of coupon samples needed to have at least one of each type. So

$$X = T_m = T_m - T_{m-1} + T_{m-1} - T_{m-2} + \cdots + T_2 - T_1 + T_1$$

Therefore,

$$\begin{aligned}
E[X] &= \sum_{j=1}^{m-1} E[T_{j+1} - T_j] + E[T_1] \\
&= \sum_{j=1}^{m-1} \frac{m}{m-j} + 1 \\
&= 1 + m\left(1 + 1/2 + 1/3 + \cdots + 1/(m-1)\right) \\
&= m\left(1 + 1/2 + 1/3 + \cdots + 1/(m-1) + 1/m\right)
\end{aligned}$$