DSC 462 Assignment 1

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Question Q1. (a) $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) + P(A_1 \cap A_2 \cap A_3)$.
(b) Let A_1, A_2, A_3 be event $(i), (ii), (iii)$ respectively. So

$$P(A_1) = \frac{10}{216},\ P(A_2) = \frac{6}{216},\ P(A_3) = \frac{27}{216}, P(A_1 \cap A_2) = \frac{1}{216},\ P(A_2 \cap A_3) = \frac{3}{216},\ P(A_3 \cap A_1) = \frac{4}{216},$$

$$P(A_1 \cap A_2 \cap A_3) = \frac{1}{216}.$$

By the inclusion-exclusion principle,

$$P(A_1 \cup A_2 \cup A_3) = \frac{10 + 6 + 27 - 1 - 3 - 4 + 1}{216} = \frac{36}{216}.$$

Therefore the probability of no points is

$$P((A_1 \cap A_2 \cap A_3)^c) = 1 - \frac{36}{216} = \boxed{\frac{180}{216}}$$

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Question Q2. (a) I am assuming customers are indistinguishable. Total number of possible arrangements is

$$\binom{16}{5}.$$

Since there are 12 slots to fill up by 5 customers, the total number of arrangements where the customers have empty seats between them is

$$\binom{12}{5}.$$

So the probability of no two adjacent occupied stools is

$$\frac{\binom{12}{5}}{\binom{16}{5}} = \boxed{\frac{33}{182}}.$$

(b) The number of arrangements where all customers seat together is

$$\binom{12}{1}$$

. So the probability is

$$\frac{\binom{12}{1}}{\binom{16}{5}} = \boxed{\frac{1}{364}}$$

Question Q3. (a) Task 1: First choose the suit in 4 ways.
Task 2: For each suit, we can choose 10 cards in $\binom{13}{10}$ ways. So

$$P(\text{Only one suit}) = \frac{4\binom{13}{10}}{\binom{52}{10}} = \boxed{\frac{2}{27657385}}$$

(b) Task 1: First choose the two suits in $\binom{4}{2}$ ways.
Task 2: For each pair of suits, we can choose 5 cards for each pair in $\binom{13}{5}\binom{13}{5}$ ways. So

$$P(\text{Two suits with 5 cards each}) = \frac{\binom{4}{2}\binom{13}{5}\binom{13}{5}}{\binom{52}{10}} = \boxed{\frac{34749}{55314770}}$$

(c) Task 1: First choose ten ranks in $\binom{13}{10}$ ways.
Task 2: For each rank, there are 4 suits. So,

$$P(\text{No rank twice}) = \frac{\binom{13}{10}4^{10}}{\binom{52}{10}} = \boxed{\frac{524288}{27657385}}$$

Question Q4. Here is the offspring genotype distribution for parents with non-recessive trait.

		Offspring		
Mother	Father	rr	rR	RR
rR	rR	1/4	1/2	1/4
rR	RR	0	1/2	1/2
RR	rR	0	1/2	1/2
RR	RR	0	0	1

(a) We have, $P(R_f^c \cap R_m^c) = P(R_f^c)P(R_m^c) = (1-q^2)^2$ and
$P(R_1 \cap R_f^c \cap R_m^c) = P(R_1, G_f = rR, G_m = rR) = P(R_1|G_f = rR, G_m = rR)P(G_f = rR, G_m = rR) = \frac{1}{4}\,(2(1-q)q)^2 = (1-q)^2q^2$. Therefore,

$$P(R_1|R_f^c \cap R_m^c) = \frac{P(R_1 \cap R_f^c \cap R_m^c)}{P(R_f^c \cap R_m^c)} = \frac{(1-q)^2q^2}{(1-q^2)^2} = \boxed{\frac{q^2}{(1+q)^2}}$$

(b) Note that $R_1 \cap R_f^c \cap R_m^c$ is possible only if we have $\{G_f = rR,\ G_f = rR\}$. So

$$P(R_2|R_1 \cap R_f^c \cap R_m^c) = P(R_2|G_f = rR, G_m = rR) = \boxed{1/4}$$

(c) The probability doesn't depend on q since the condition gives us the information that $G_m = G_f = rR$. So the mating is not random.

Question Q5. (a)

(i) By axiom 3, it is straightforward that $P(\emptyset) = \sum_{i=1}^{\infty} P(\emptyset)$. Now

$$\begin{aligned}
1 &= P(S), \text{ by axiom 2} \\
&= P(S \cup (\cup_{i=1}^{\infty}\emptyset)) \\
&= P(S) + \sum_{i=1}^{\infty} P(\emptyset) \text{ by axiom 3} \\
&= P(S) + P(\emptyset) = 1 + P(\emptyset)
\end{aligned}$$

So clearly, $P(\emptyset) = 0$.

(ii) Similarly,

$$\begin{aligned}
P(A \cup B) &= P(A \cup B \cup (\cup_{i=1}^{\infty} \emptyset)) \\
&= P(A) + P(B) + \sum_{i=1}^{\infty} P(\emptyset), \text{ by axim 3} \\
&= P(A) + P(B) + P(\cup_{i=1}^{\infty} \emptyset), \text{ by axim 3} \\
&= P(A) + P(B) + 0, \text{ by (i)} \\
&= P(A) + P(B)
\end{aligned}$$

(iii) Since $P(A \cup A^c) = P(S) = 1$ and $A \cap A^c = \emptyset$, we clearly have,

$$P(A^c) = 1 - P(A).$$

(iv) By (ii), $P(B) = P(A) + P(B - A) \geq P(A)$, since $P(B - A) \geq 0$ by axiom 1.

(b) (i) $\bar{E}_1 \subset E_1$ is clear. For $i \geq 2$, we have $\bar{E}_i = E_i \cap (E_{i-1} \cup \cdots \cup E_1)^c \subset E_i$.

(ii) This is straightforward beacause from the construction we can see that, for $i \geq 2$, we have $\bar{E}_i = E_i - (E_{i-1} \cup E_{i-2} \cup \cdots E_1)$. That means we are making $\bar{E}_i$ by subtracting all previous sets from E_i. So for two distinct i, j, $\bar{E}_i$ and $\bar{E}_j$ must be mutually disjoint.

(iii) $\supset$ is clear by (i). To prove $\subset$, consider an arbitrary element $x \in \cup_{i=1}^{\infty} E_i$. Let j be the lowest subscript such that $x \in E_j$. So $x \in E_j - (E_{j-1} \cup E_{j-2} \cup \cdots E_1) = \bar{E}_i \subset \cup_{i=1}^{\infty} \bar{E}_i$.

(c)

$$\begin{aligned}
P(\cup_{i=1}^{\infty} E_i) &= P(\cup_{i=1}^{\infty} \bar{E}_i) \text{ ,by b(iii)} \\
&= \sum_{i=1}^{\infty} P(\bar{E}_i) \text{ ,by axiom 3} \\
&\leq \sum_{i=1}^{\infty} P(E_i) \text{ ,by a(iv)}
\end{aligned}$$

(d) First note that $\sum_{i=1}^{\infty} P(E_i) < \infty$ implies that the tail of this infinite sum converges to zero. That is $\lim_{i \to \infty} \sum_{j=i}^{\infty} P(E_j) = 0$. Also by Boole's inequality and axiom 1, $0 \leq P(\cup_{j=i}^{\infty} E_j) \leq \sum_{j=i}^{\infty} P(E_j)$ which implies $\lim_{i \to \infty} P(\cup_{j=i}^{\infty} E_j) = 0$. Now we have

$$Q_i = \{\text{From year } i \text{ onwards, the current world record is never broken}\}$$

$$= E_i^c \cap E_{i+1}^c \cap \cdots$$

$$= (\cup_{j=i}^{\infty} E_j)^c$$

So $P(Q_i) = 1 - P(\cup_{j=i}^{\infty} E_j)$. Taking limit on both sides we get,

$$\lim_{i \to \infty} P(Q_i) = 1 - \lim_{i \to \infty} P(\cup_{j=i}^{\infty} E_j) = 1 - 0 = 1.$$