

Anup Barman's

CP-Arsenal



AnupBarman



AnupBarman



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1 Setup

1.1 Linux Build

```
{
  "cmd" : ["ulimit -s 268435456; g++ -std=c++20
           $file_name -o $file_base_name && timeout 4s
           ./ $file_base_name < input.txt > output.txt"],
  "selector" : "source.c++",
  "shell" : true,
  "working_dir" : "$file_path"
}
```

1.2 Windows Build

```
{
  "cmd" : [ "g++.exe", "-std=c++20", "${file}", "-o",
            "${file_base_name}.exe", "&&",
            "${file_base_name}.exe<input.txt>output.txt" ],
  "selector" : "source.cpp",
  "shell" : true,
  "working_dir" : "$file_path"
}
```

2 DataStructures

2.1 Anti Hash Unordered Map

```
unordered_map<int, int> mp;
mp.reserve(1 << 20); // about 1M buckets
mp.max_load_factor(0.7);
```

2.2 Co-Ordinate Compression

```
vector<int> pos;
sort(pos.begin(), pos.end());
pos.erase(unique(pos.begin(), pos.end()), pos.end());
// then lower_bound on this pos array to find the
// compressed co-ordinate
```

2.3 Lazy Propagation

```
class stree {
  vector<int> seg, lazy;
public:
  segtree(int n) {
    seg.resize(4 * n + 5);
    lazy.resize(4 * n + 5);
  }
  void propagate(int i, int low, int high) {
    if (lazy[i] != 0) {
      seg[i] += (high - low + 1) * lazy[i];
      if (low != high) {
        lazy[2 * i + 1] += lazy[i];
        lazy[2 * i + 2] += lazy[i];
      }
      lazy[i] = 0;
    }
  }
  void build(int i, int low, int high, int arr[]) {
    if (low == high) {
      seg[i] = arr[low];
      return;
    }
    int mid = (low + high) >> 1;
    build(2 * i + 1, low, mid, arr);
    build(2 * i + 2, mid + 1, high, arr);
    seg[i] = seg[2 * i + 1] + seg[2 * i + 2];
  }
  void update(int i, int low, int high, int l, int r,
             int val) {
    propagate(i, low, high);
    if (high < l or r < low) return;
    if (low >= l and high <= r) {
      seg[i] += (high - low + 1) * val;
      if (low != high) {
        lazy[2 * i + 1] += val;
        lazy[2 * i + 2] += val;
      }
      return;
    }
    int mid = (low + high) >> 1;
    update(2 * i + 1, low, mid, l, r, val);
    update(2 * i + 2, mid + 1, high, l, r, val);
    seg[i] = seg[2 * i + 1] + seg[2 * i + 2];
  }
};
```

```
seg[i] += (high - low + 1) * val;
if (low != high) {
  // has children
  lazy[2 * i + 1] += val;
  lazy[2 * i + 2] += val;
}
return;
}
int mid = (low + high) >> 1;
update(2 * i + 1, low, mid, l, r, val);
update(2 * i + 2, mid + 1, high, l, r, val);
seg[i] = seg[2 * i + 1] + seg[2 * i + 2];
}
int query(int i, int low, int high, int l, int r) {
  propagate(i, low, high);
  if (high < l or r < low) return 0;
  if (low >= l and high <= r) return seg[i];
  int mid = (low + high) >> 1;
  int left = query(2 * i + 1, low, mid, l, r);
  int right = query(2 * i + 2, mid + 1, high, l, r);
  return left + right;
}
};
```

2.4 PBDS

```
#include "ext/pb_ds/assoc_container.hpp"
#include "ext/pb_ds/tree_policy.hpp"
using namespace gnu_pbds;
template <class T>
using oset = tree<T, null type, less<T>, rb_tree_tag,
tree_order_statistics_node_update>;
Note: Use less_equal for multiset like behaviour
Usage:
st.find_by_order(k) :: returns iterator of the k-th
- smallest element
st.order_of_key(x) :: returns index of x (number of
- elements less than x)
```

2.5 Segment Tree :: Binary search on Unsorted array

```
// 1 based indexing
struct SegTree {
  int n;
  vector<int> seg;
  SegTree(int n) {
    n = n;
    seg.assign(4 * n + 5, 0);
  }
  void build(int node, int l, int r) {
    if (l == r) {
      seg[node] = 1;
      return;
    } // each position initially present
    int mid = (l + r) >> 1;
    build(node * 2, l, mid);
    build(node * 2 + 1, mid + 1, r);
    seg[node] = seg[node * 2] + seg[node * 2 + 1];
  }
  // set position pos to value val (0 or 1)
  void update(int node, int l, int r, int pos, int
             val) {
    if (l == r) {
      seg[node] = val;
      return;
    }
    int mid = (l + r) >> 1;
    if (pos <= mid)
      update(node * 2, l, mid, pos, val);
    else
      update(node * 2 + 1, mid + 1, r, pos, val);
    seg[node] = seg[node * 2] + seg[node * 2 + 1];
  }
};
```

```
// find index of k-th "present" element (1-based k)
int kth(int node, int l, int r, int k) {
    if (l == r) return l;
    int leftCnt = seg[node * 2];
    int mid = (l + r) >> 1;
    if (k <= leftCnt)
        return kth(node * 2, l, mid, k);
    else
        return kth(node * 2 + 1, mid + 1, r, k - leftCnt);
}

void solve() {
    int n;
    cin >> n;
    vector<ll> a(n + 1), p(n + 1);
    for (int i = 1; i <= n; ++i) {
        cin >> a[i];
    }
    for (int i = 1; i <= n; ++i) {
        cin >> p[i];
    }
    SegTree st(n);
    st.build(1, 1, n);
    // For each removal request p[i], find the p[i]-th
    // present element,
    // print it and mark that position as removed (set to
    // 0).
    for (int i = 1; i <= n; ++i) {
        int k = p[i];
        int idx = st.kth(1, 1, n, k); // index in original
        // array
        cout << a[idx] << (i == n ? '\n' : ' ');
        st.update(1, 1, n, idx, 0);
    }
}
```

2.6 Segment Tree

```
class stree {
    vector<int> seg;
public:
    segtree(int n) {
        seg.assign(4 * n + 5, 0);
    }
    void build(int ind, int low, int high, int arr[]) {
        if (low == high) {
            seg[ind] = arr[low];
            return;
        }
        int mid = (low + high) >> 1;
        build(2 * ind + 1, low, mid, arr);
        build(2 * ind + 2, mid + 1, high, arr);
        seg[ind] = min(seg[2 * ind + 1], seg[2 * ind + 2]);
    }
    int query(int ind, int low, int high, int l, int r) {
        if (r < low or high < l) return INT_MAX;
        if (low >= l and high <= r) return seg[ind];
        int mid = (low + high) / 2;
        int left = query(2 * ind + 1, low, mid, l, r);
        int right = query(2 * ind + 2, mid + 1, high, l, r);
        return min(left, right);
    }
    void update(int ind, int low, int high, int i, int
        val) {
        if (low == high) {
            seg[ind] = val;
            return;
        }
        int mid = (low + high) / 2;
        if (i <= mid) update(2 * ind + 1, low, mid, i,
            val);
    }
}
```

```
else update(2 * ind + 2, mid + 1, high, i, val);
seg[ind] = min(seg[2 * ind + 1], seg[2 * ind + 2]);
};
```

2.7 Sort and Unique

```
sort(v.begin(), v.end());
v.erase(unique(v.begin(), v.end()), v.end());
```

2.8 Sorted check

```
is_sorted(first, last, comp);
```

2.9 Sparse Table

```
const int MX = 2e5 + 10;
int n, arr[MX], st[25][MX];
int log2Floor(int i) {
    return 31 - __builtin_clz(i);
}
void build() {
    int k = log2Floor(n);
    copy(arr, arr + n, st[0]);
    for (int i = 1; i <= k; ++i) {
        for (int j = 0; j + (1 << i) <= n; j++) {
            st[i][j] = min(st[i - 1][j], st[i - 1][j + (1 <<
                i - 1)]);
        }
    }
}
int query(int l, int r) {
    int i = log2Floor(r - l + 1);
    return min(st[i][l], st[i][r - (1 << i) + 1]);
}
```

3 Graphs

3.1 Articulation Point

```
int n; // number of nodes
vector<vector<int>> lst; // adjacency list of graph
vector<bool> vis;
vector<int> tin, low;
int timer;
void dfs(int u, int p = -1) {
    vis[u] = true;
    tin[u] = low[u] = timer++;
    int children = 0;
    for (int v : lst[u]) {
        if (v == p) continue;
        if (vis[v]) {
            low[u] = min(low[u], tin[v]);
        } else {
            dfs(v, u);
            low[u] = min(low[u], low[v]);
            if (low[v] >= tin[u] && p != -1)
                IS_CUTPOINT(u);
            ++children;
        }
    }
    // if no vertex below v can reach u or higher
    // removing u disconnects that subtree
    if (p == -1 && children > 1)
        IS_CUTPOINT(u);
}
void find_cutpoints() {
    timer = 0;
    vis.assign(n, false);
    tin.assign(n, -1);
    low.assign(n, -1);
    for (int i = 0; i < n; ++i) {
        if (!vis[i])
            dfs(i);
    }
}
```

```
}
```

3.2 Bridge Finding Algorithm

```
const int MX = 1e5 + 10;
int n, m, timer = 0;
vector<int> adj[MX];
vector<int> tin(MX, -1), low(MX, -1);
vector<bool> vis(MX, false);
void is_bridge(int u, int v) {
    // do something with the edge
}
void dfs(int u, int p = -1) {
    vis[u] = true;
    tin[u] = low[u] = timer++;
    for (int v : adj[u]) {
        if (v == p) continue;
        if (vis[v]) {
            low[u] = min(low[u], tin[v]);
        } else {
            dfs(v, u);
            low[u] = min(low[u], low[v]);
            if (low[v] > tin[u]) {
                is_bridge(u, v);
            }
        }
    }
}
```

3.3 Cycle Detection in DAG

```
const int MX = 1e5 + 10;
bool vis[MX], pathVis[MX];
vector<int> lst[MX];
bool dfs(int u) {
    vis[u] = true;
    pathVis[u] = true;
    for (auto v : lst[u]) {
        if (!vis[v]) {
            if (dfs(v))
                return true;
        } else if (pathVis[v]) {
            return true;
        }
    }
    pathVis[u] = false;
    return false;
}
void solve() {
    // take graph input
    for (int i = 0; i < n; ++i) {
        if (!vis[i])
            dfs(i);
    }
}
```

3.4 DSU on Trees

```
int n, color[MX], ans[MX];
vector<int> g[MX];
set<int> bucket[MX];
int merge(int a, int b) {
    if (bucket[a].size() < bucket[b].size()) swap(a, b);
    bucket[a].insert(bucket[b].begin(), bucket[b].end());
    bucket[b].clear();
    return a;
}
int dfs(int u, int p = -1) {
    int cur = u;
    for (int v : g[u])
        if (v != p)
            cur = merge(cur, dfs(v, u));
    return cur;
}
```

```

    cur = merge(cur, dfs(v, u));
    ans[u] = (int)bucket[cur].size();
    return cur;
}

void solve() {
    cin >> n;
    for (int i = 0; i < n; ++i) {
        cin >> color[i];
        bucket[i].insert(color[i]);
    }
    // graph input
    dfs(0);
    // print output
}

```

3.5 DSU

```

const int MX = 1e5 + 10;
int par[MX], sz[MX];
void init() {
    for (int i = 1; i < MX; i++) {
        par[i] = i;
        sz[i] = 1;
    }
}

int findpar(int x) {
    if (par[x] == x) return x;
    return par[x] = findpar(par[x]);
}

void unite(int u, int v) {
    u = findpar(u);
    v = findpar(v);
    if (u != v) {
        if (sz[u] < sz[v]) {
            swap(u, v);
        }
        sz[u] += sz[v];
        par[v] = u;
    }
}

```

3.6 Euler Tour

```

const int MX = 2e5 + 10;
int timer = -1;
// s = start pos, e = end pos
int val[MX], s[MX], e[MX], flat[MX];
vector<int> lst[MX];
void dfs(int u, int p) {
    s[u] = ++timer;
    flat[timer] = val[u];
    for (auto v : lst[u]) {
        if (v != p)
            dfs(v, u);
    }
    e[u] = timer;
}

```

3.7 Floyd Warshall

```

vector<vector<int>> d(n, vector<int>(n, INF));
// take graph input into d
for (int k = 0; k < n; ++k) {
    for (int i = 0; i < n; ++i) {
        for (int j = 0; j < n; ++j) {
            if (d[i][k] < INF && d[k][j] < INF)
                d[i][j] = min(d[i][j], d[i][k] + d[k][j]);
        }
    }
}

```

3.8 LCA using Binary Lifting

```

int n, l;
vector<vector<int>> adj;
int timer;
vector<int> tin, tout;
vector<vector<int>> up;
void dfs(int v, int p) {
    tin[v] = ++timer;
    up[v][0] = p;
    for (int i = 1; i <= l; ++i)
        up[v][i] = up[up[v][i-1]][i-1];
    for (int u : adj[v]) {
        if (u != p)
            dfs(u, v);
    }
    tout[v] = ++timer;
}

bool is_ancestor(int u, int v) {
    return tin[u] <= tin[v] && tout[u] >= tout[v];
}

int lca(int u, int v) {
    if (is_ancestor(u, v))
        return u;
    if (is_ancestor(v, u))
        return v;
    for (int i = l; i >= 0; --i) {
        if (!is_ancestor(up[u][i], v))
            u = up[u][i];
    }
    return up[u][0];
}

void preprocess(int root) {
    tin.resize(n);
    tout.resize(n);
    timer = 0;
    l = ceil(log2(n));
    up.assign(n, vector<int>(l + 1));
    dfs(root, root);
}

```

3.9 MST

```

// DSU first
void solve() {
    int n, m;
    cin >> n >> m;
    vector<tuple<int, int, int>> edges;
    for (int i = 0; i < m; ++i) {
        int u, v, wt;
        cin >> u >> v >> wt;
        edges.push_back({wt, u, v});
    }
    sort(edges.begin(), edges.end());
    init(n);
    int cost = 0;
    for (tuple<int, int, int> &e : edges) {
        if (findpar(u) == findpar(v)) continue;
        unite(u, v);
        cost += wt;
    }
    cout << cost << endl;
}

```

3.10 Max Bipartite Matching [Hopcroft-Karp]

```

const int INF = 1e9;
void hopcroftCarp() {
    int n, m, e;
    cin >> n >> m >> e;
    vector<int> adj[n];
}

```

```

for (int i = 0; i < e; ++i) {
    int u, v;
    cin >> u >> v;
    --u;
    adj[u].push_back(v);
}
vector<int> ml(m, -1), mr(n, -1), dist(n);
auto bfs = [&]() -> bool {
    queue<int> q;
    for (int u = 0; u < n; ++u) {
        if (mr[u] == -1) {
            dist[u] = 0;
            q.push(u);
        } else {
            dist[u] = INF;
        }
    }

    bool foundAugmenting = false;
    while (!q.empty()) {
        int u = q.front();
        q.pop();
        for (int v : adj[u]) {
            int pairedLeft = ml[v];
            if (pairedLeft == -1) {
                foundAugmenting = true;
            } else if (dist[pairedLeft] == INF) {
                dist[pairedLeft] = dist[u] + 1;
                q.push(pairedLeft);
            }
        }
    }
    return foundAugmenting;
};

function<bool(int)> dfs = [&](int u) -> bool {
    for (int v : adj[u]) {
        int pairedLeft = ml[v];
        if (pairedLeft == -1 or (dist[pairedLeft] ==
            dist[u] + 1 and dfs(pairedLeft))) {
            mr[u] = v;
            ml[v] = u;
            return true;
        }
    }
    dist[u] = INF;
    return false;
};

int matching = 0;
while (bfs()) {
    for (int u = 0; u < n; ++u) {
        if (mr[u] == -1) {
            if (dfs(u)) matching++;
        }
    }
}
cout << matching << endl;
for (int u = 0; u < n; ++u) {
    if (mr[u] != -1) {
        cout << u << " " << mr[u] << endl;
    }
}
}

```

3.11 Max Bipartite Matching [Kuhn's]

```

// left set size, right set size, edge count
int n, k, m, visToken = 1;
vector<int> lst[MX];
int mr[MX], ml[MX], vis[MX];
bool try_kuhn(int u) {
    if (vis[u] == visToken)
        return false;
    vis[u] = visToken;
    for (auto v : lst[u]) {

```

```

    if (ml[v] == -1 or try_kuhn(ml[v])) {
        ml[v] = u;
        mr[u] = v;
        return true;
    }
}
return false;
}
void solve() {
    cin >> n >> k >> m;
    for (int i = 0; i < m; ++i) {
        int u, v;
        cin >> u >> v;
        --u --v;
        lst[u].push_back(v);
    }
    fill(mr, mr + n, -1);
    fill(ml, ml + k, -1);
    int ans = 0;
    for (int u = 0; u < n; ++u) {
        for (auto v : lst[u]) {
            if (ml[v] == -1) {
                ml[v] = u;
                mr[u] = v;
                ans++;
                break;
            }
        }
    }
    for (int u = 0; u < n; ++u) {
        if (mr[u] != -1) continue;
        visToken++;
        if (try_kuhn(u))
            ans++;
    }
    cout << ans << el;
    for (int v = 0; v < k; ++v) {
        if (ml[v] != -1) {
            cout << ml[v] + 1 << " " << v + 1 << el;
        }
    }
}
}

```

3.12 Topological Sorting

```

const int N = 1e5 + 10;
vector<int> g[N], indegree(N, 0);
vector<int> topSort(int n) {
    queue<int> q;
    vector<int> order;
    for (int i = 1; i <= n; i++) {
        if (indegree[i] == 0) {
            q.push(i);
        }
    }
    while (!q.empty()) {
        int u = q.front();
        q.pop();
        order.push_back(u);
        for (int v : g[u]) {
            indegree[v]--;
            if (indegree[v] == 0) {
                q.push(v);
            }
        }
    }
    return order;
}

```

3.13 Weighted Union Find

```

const int MX = 2e5 + 10;
int par[MX], sz[MX];
ll d[MX];
void init() {

```

```

    for (int i = 0; i < MX; ++i) {
        par[i] = i;
        sz[i] = 1;
        d[i] = 0;
    }
}
int findpar(int x) {
    if (par[x] == x) return x;
    int p = par[x];
    par[x] = findpar(p);
    d[x] += d[p];
    return par[x];
}
bool unite(int a, int b, ll w) {
    int ra = findpar(a);
    int rb = findpar(b);
    if (ra == rb) {
        return (d[b] - d[a] == w);
    }
    if (sz[ra] < sz[rb]) {
        swap(a, b);
        swap(ra, rb);
        w = -w;
    }
    par[rb] = ra;
    d[rb] = d[a] + w - d[b];
    sz[ra] += sz[rb];
    return true;
}
ll dist(int a, int b) {
    findpar(a), findpar(b);
    return d[b] - d[a];
}
}

```

4 NumberTheory

4.1 Unique Prime Factorization using Sieve

```

const int MX = 2e5 + 10;
vector<int> pfac[MX];
void factorize() {
    for (int i = 2; i < MX; i++) {
        if (!pfac[i].empty()) continue;
        for (int j = i; j < MX; j += i)
            pfac[j].push_back(i);
    }
}

```

4.2 nCr

```

const int MX = 1e6 + 10;
const int M = 1e9 + 7;
int fact[MX], inv_fact[MX];
int modPow(int a, int b) {
    int ans = 1;
    while (b) {
        if (b & 1) ans = (1LL * ans * a) % M;
        a = (1LL * a * a) % M;
        b >>= 1;
    }
    return ans;
}
void precalFact() {
    fact[0] = inv_fact[0] = 1;
    for (int i = 1; i < MX; i++) {
        fact[i] = (1LL * fact[i - 1] * i) % M;
    }
    inv_fact[MX - 1] = modPow(fact[MX - 1], M - 2);
    for (int i = MX - 2; i >= 1; i--) {
        inv_fact[i] = (1LL * inv_fact[i + 1] * (i + 1)) %
            M;
    }
}
int nCr(int n, int r) {
    if (r < 0 or r > n) return 0;

```

```

    return 1LL * fact[n] * inv_fact[r] % M * inv_fact[n
        - r] % M;
}

```

5 Notes

5.1 Geometry

5.1.1 Triangles

Circumradius: $R = \frac{abc}{4A}$, Inradius: $r = \frac{A}{s}$

The area of a triangle using two sides and the included angle can be given as:

$$A = \frac{1}{2}ab \sin \angle C$$

Length of median (divides triangle into two equal-area triangles):

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

Length of bisector (divides angles in two): $s_a = \sqrt{bc \left[1 - \left(\frac{a}{b+c} \right)^2 \right]}$

Law of tangents: $\frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}$

5.1.2 Quadrilaterals

With side lengths a, b, c, d , diagonals e, f , diagonals angle θ , area A and magic flux $F = b^2 + d^2 - a^2 - c^2$:

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is 180° , $ef = ac + bd$, and $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$.

5.1.3 Spherical coordinates

$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \arccos(z / \sqrt{x^2 + y^2 + z^2}) \\ z &= r \cos \theta & \phi &= \operatorname{atan2}(y, x) \end{aligned}$$

5.1.4 Pick's Theorem:

Given a lattice polygon with non-zero area, we define: S as the area of the polygon, I as the number of integer-coordinate points strictly inside the polygon, B as the number of integer-coordinate points on the boundary of the polygon. Then, Pick's Theorem states:

$$S = I + \frac{B}{2} - 1$$

The number of lattice points on segments (x_1, y_1) to (x_2, y_2) is: $\gcd(\operatorname{abs}(x_2 - x_1), \operatorname{abs}(y_2 - y_1)) + 1$

5.1.5 Polygon

For a regular polygon with n sides and side length a , the circumradius R is given by:

$$R = \frac{a}{2 \sin \left(\frac{\pi}{n} \right)}$$

5.1.6 Area of a Circular Segment

The area of a circular segment, which is the region enclosed by a chord and the corresponding arc, can be calculated using the formula:

$$A = \frac{R^2}{2} (\theta - \sin \theta)$$

where: R is the radius of the circle, θ is the central angle subtended by the chord, in radians.

5.2 Binomial Coefficient

- Factoring in: $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$
- Sum over k : $\sum_{k=0}^n \binom{n}{k} = 2^n$
- Alternating sum: $\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$
- Even and odd sum: $\sum_{k=0}^n \binom{n}{2k} = \sum_{k=0}^n \binom{n}{2k+1} = 2^{n-1}$
- The Hockey Stick Identity
 - (Left to right) Sum over n and k : $\sum_{k=0}^m \binom{n+k}{k} = \binom{n+m-1}{m}$
 - (Right to left) Sum over n : $\sum_{m=0}^n \binom{m}{k} = \binom{n+1}{k+1}$
- Sum of the squares: $\sum_{k=0}^n ((\binom{n}{k})^2) = \binom{2n}{n}$
- Weighted sum: $\sum_{k=1}^n k \binom{n}{k} = n2^{n-1}$
- Connection with the fibonacci numbers: $\sum_{k=0}^n \binom{n-k}{k} = F_{n+1}$
- Vandermonde's Identity: $\sum_{i=0}^k \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$
- If $f(n, k) = C(n, 0) + C(n, 1) + \dots + C(n, k)$, Then $f(n+1, k) = 2 * f(n, k) - C(n, k)$ [For multiple $f(n, k)$ queries, use Mo's algo]

Lucas Theorem

$$\binom{m}{n} \equiv \prod_{i=0}^k \binom{m_i}{n_i} \pmod{p}$$

- $\binom{m}{n}$ is divisible by p if and only if at least one of the base- p digits of n is greater than the corresponding base- p digit of m .
- The number of entries in the n th row of Pascal's triangle that are not divisible by $p = \prod_{i=0}^k (n_i + 1)$
- All entries in the $(p^k - 1)th$ row are not divisible by p .
- $\binom{n}{m} \equiv \lfloor \frac{n}{p} \rfloor! \pmod{p}$

5.3 Fibonacci Number

- $k = A - B, F_A F_B = F_{k+1} F_A^2 + F_k F_A F_{A-1}$
- $\sum_{i=0}^n F_i^2 = F_{n+1} F_n$
- $\sum_{i=0}^n F_i F_{i+1} = F_{n+1}^2 - (-1)^n$
- $\sum_{i=0}^n F_i F_{i+1} = F_{n+1}^2 - (-1)^n$
- $\sum_{i=0}^n F_i F_{i-1} = \sum_{i=0}^{n-1} F_i F_{i+1}$
- $\gcd(F_m, F_n) = F_{\gcd(m, n)}$
- $\sum_{0 \leq k \leq n} \binom{n-k}{k} = F_{n+1}$
- $\gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+2}) = \gcd(F_{n+1}, F_{n+2}) = 1$

5.4 Sums

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$$\sum_{i=1}^n i^m = \frac{1}{m+1} \left[(n+1)^{m+1} - 1 - \sum_{i=1}^n ((i+1)^{m+1} - i^{m+1} - (m+1)i^m) \right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}$$

$$\sum_{k=0}^n kx^k = (x - (n+1)x^{n+1} + nx^{n+2})/(x-1)^2$$

5.5 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty)$$

$$(x+a)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k a^{-n-k}$$

Generating Function

$$1/(1-x) = 1 + x + x^2 + x^3 + \dots$$

$$1/(1-ax) = 1 + ax + (ax)^2 + (ax)^3 + \dots$$

$$1/(1-x)^2 = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$1/(1-x)^3 = C(2,2) + C(3,2)x + C(4,2)x^2 + C(5,2)x^3 + \dots$$

$$1/(1-ax)^k = 1 + C(1+k,k)(ax) + C(2+k,k)(ax)^2 + C(3+k,k)(ax)^3 + \dots$$

$$x(x+1)(1-x)^{-3} = 1 + x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + \dots$$

$$e^x = 1 + x + (x^2)/2! + (x^3)/3! + (x^4)/4! + \dots$$

5.6 Pythagorean Triples

The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), \quad b = k \cdot (2mn), \quad c = k \cdot (m^2 + n^2),$$

with $m > n > 0, k > 0, m \perp n$, and either m or n even.

5.7 Number Theory

- HCN: 1e6(240), 1e9(1344), 1e12(6720), 1e14(17280), 1e15(26880), 1e16(41472)

$$\gcd(a, b, c, d, \dots) = \gcd(a, b-a, c-b, d-c, \dots)$$

$$\gcd(a+k, b+k, c+k, d+k, \dots) = \gcd(a+k, b-a, c-b, d-c, \dots)$$

- Primitive root exists iff $n = 1, 2, 4, p^k, 2 \times p^k$, where p is an odd prime.

- If primitive root exists, there are $\phi(\phi(n))$ primitive roots of n .

- The numbers from 1 to n have in total $O(n \log \log n)$ unique prime factors.

- $x \equiv r_1 \pmod{m_1}$ and $x \equiv r_2 \pmod{m_2}$ has a solution iff $\gcd(m_1, m_2) | (r_1 - r_2)$ Solution of $x^2 \equiv a \pmod{p}$

$$ca \equiv cb \pmod{m} \iff a \equiv b \pmod{\frac{n}{\gcd(n, c)}}$$

$$ax \equiv b \pmod{m} \text{ has a solution } \iff \gcd(a, m) | b$$

- If $ax \equiv b \pmod{m}$ has a solution, then it has $\gcd(a, m)$ solutions and they are separated by $\frac{m}{\gcd(a, m)}$

- $ax \equiv 1 \pmod{m}$ has a solution or a is invertible $\pmod{m} \iff \gcd(a, m) = 1$

$$x^2 \equiv 1 \pmod{p} \text{ then } x \equiv \pm 1 \pmod{p}$$

- There are $\frac{p-1}{2}$ has no solution.

- There are $\frac{p-1}{2}$ has exactly two solutions.

$$\text{When } p \% 4 = 3, x \equiv \pm a^{\frac{p+1}{4}}$$

$$\text{When } p \% 8 = 5, x \equiv a^{\frac{p+3}{8}} \text{ or } x \equiv 2^{\frac{p-1}{4}} a^{\frac{p+3}{8}}$$

5.7.1 Primes

$p = 962592769$ is such that $2^{21} \mid p-1$, which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than 1 000 000.

Primitive roots exist modulo any prime power p^a , except for $p=2, a>2$, and there are $\phi(\phi(p^a))$ many. For $p=2, a>2$, the group $\mathbb{Z}_{2^a}^\times$ is instead isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$.

5.7.2 Estimates

$$\sum_{d|n} d = O(n \log \log n).$$

The number of divisors of n is at most around 100 for $n < 5e4$, 500 for $n < 1e7$, 2000 for $n < 1e10$, 200 000 for $n < 1e19$.

5.7.3 Perfect numbers

$n > 1$ is called perfect if it equals sum of its proper divisors and 1. Even n is perfect iff $n = 2^{p-1}(2^p - 1)$ and $2^p - 1$ is prime (Mersenne's). No odd perfect numbers are yet found.

5.7.4 Carmichael numbers

A positive composite n is a Carmichael number ($a^{n-1} \equiv 1 \pmod{n}$ for all a : $\gcd(a, n) = 1$), iff n is square-free, and for all prime divisors p of n , $p-1$ divides $n-1$.

5.7.5 Totient

$$\text{If } p \text{ is a prime } (p^k) = p^k - p^{k-1}$$

$$\text{If } a, b \text{ are relatively prime, } \phi(ab) = \phi(a)\phi(b)$$

$$\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})(1 - \frac{1}{p_3}) \dots (1 - \frac{1}{p_k})$$

$$\text{Sum of coprime to } n = n * \frac{\phi(n)}{2}$$

$$\text{If } n = 2^k, \phi(n) = 2^{k-1} = \frac{n}{2}$$

$$\text{For } a, b, \phi(ab) = \phi(a)\phi(b) \frac{d}{\phi(d)}$$

$$\phi(ip) = p\phi(i) \text{ whenever } p \text{ is a prime and it divides } i$$

$$\text{The number of } a (1 \leq a \leq N) \text{ such that } \gcd(a, N) = d \text{ is } \phi(\frac{N}{d})$$

$$\text{If } n > 2, \phi(n) \text{ is always even}$$

$$\text{Sum of gcd, } \sum_{i=1}^n \gcd(i, n) = \sum_{d|n} d \phi(\frac{n}{d})$$

$$\text{Sum of lcm, } \sum_{i=1}^n \text{lcm}(i, n) = \frac{n^2}{2} (\sum_{d|n} (d \phi(d)) + 1)$$

$$\phi(1) = 1 \text{ and } \phi(2) = 1 \text{ which two are only odd } \phi$$

$$\phi(3) = 2 \text{ and } \phi(4) = 2 \text{ and } \phi(6) = 2 \text{ which three are only prime } \phi$$

$$\text{Find minimum } n \text{ such that } \frac{\phi(n)}{n} \text{ is maximum- Multiple of small primes- } 2 * 3 * 5 * 7 * 11 * 13 * \dots$$

5.7.6 Mobius function

$\mu(1) = 1$. $\mu(n) = 0$, if n is not squarefree. $\mu(n) = (-1)^s$, if n is the product of s distinct primes. Let f, F be functions on positive integers. If for all $n \in N$, $F(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} \mu(d)F(\frac{n}{d})$, and vice versa.

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}. \quad \sum_{d|n} \mu(d) = 1.$$

If f is multiplicative, then $\sum_{d|n} \mu(d)f(d) = \prod_{p|n} (1 - f(p))$,

$$\sum_{d|n} \mu(d)^2 f(d) = \prod_{p|n} (1 + f(p)).$$

$$\sum_{i=1}^n \sum_{j=1}^n [gcd(i, j) = 1] = \sum_{k=1}^n \mu(k) \left\lfloor \frac{n}{k} \right\rfloor^2$$

$$\sum_{i=1}^n \sum_{j=1}^n gcd(i, j) = \sum_{k=1}^n k \sum_{l=1}^{\lfloor \frac{n}{k} \rfloor} \mu(l) \left\lfloor \frac{n}{kl} \right\rfloor^2$$

$$\sum_{i=1}^n \sum_{j=1}^n gcd(i, j) = \sum_{k=1}^n \left(\frac{\lfloor \frac{n}{k} \rfloor (\lfloor \frac{n}{k} \rfloor + 1)}{2} \right)^2 \sum_{d|k} \mu(d) k d$$

5.7.7 Legendre symbol

If p is an odd prime, $a \in \mathbb{Z}$, then $\left(\frac{a}{p}\right)$ equals 0, if $p|a$; 1 if a is a quadratic residue modulo p ; and -1 otherwise. Euler's criterion:

$$\left(\frac{a}{p}\right) = a^{\left(\frac{p-1}{2}\right)} \pmod{p}.$$

5.7.8 Jacobi symbol

If $n = p_1^{a_1} \cdots p_k^{a_k}$ is odd, then $\left(\frac{a}{n}\right) = \prod_{i=1}^k \left(\frac{a}{p_i}\right)^{a_i}$.

5.7.9 Primitive roots

If the order of g modulo m ($\min n > 0: g^n \equiv 1 \pmod{m}$) is $\phi(m)$, then g is called a primitive root. If Z_m has a primitive root, then it has $\phi(\phi(m))$ distinct primitive roots. Z_m has a primitive root iff m is one of 2, 4, p^k , $2p^k$, where p is an odd prime. If Z_m has a primitive root g , then for all a coprime to m , there exists unique integer $i = \text{ind}_g(a)$ modulo $\phi(m)$, such that $g^i \equiv a \pmod{m}$. $\text{ind}_g(a)$ has logarithm-like properties: $\text{ind}(1) = 0$, $\text{ind}(ab) = \text{ind}(a) + \text{ind}(b)$.

If p is prime and a is not divisible by p , then congruence $x^n \equiv a \pmod{p}$ has $\gcd(n, p-1)$ solutions if $a^{(p-1)/\gcd(n, p-1)} \equiv 1 \pmod{p}$, and no solutions otherwise. (Proof sketch: let g be a primitive root, and $g^i \equiv a \pmod{p}$, $g^u \equiv x \pmod{p}$. $x^n \equiv a \pmod{p}$ iff $g^{nu} \equiv g^i \pmod{p}$ iff $nu \equiv i \pmod{p}$.)

5.7.10 Discrete logarithm problem

Find x from $a^x \equiv b \pmod{m}$. Can be solved in $O(\sqrt{m})$ time and space with a meet-in-the-middle trick. Let $n = \lceil \sqrt{m} \rceil$, and $x = ny - z$. Equation becomes $a^{ny} \equiv ba^z \pmod{m}$. Precompute all values that the RHS can take for $z = 0, 1, \dots, n-1$, and brute force y on the LHS, each time checking whether there's a corresponding value for RHS.

5.7.11 Pythagorean triples

Integer solutions of $x^2 + y^2 = z^2$ All relatively prime triples are given by: $x = 2mn$, $y = m^2 - n^2$, $z = m^2 + n^2$ where $m > n$, $\gcd(m, n) = 1$ and $m \not\equiv n \pmod{2}$. All other triples are multiples of these. Equation $x^2 + y^2 = 2z^2$ is equivalent to $(\frac{x+y}{2})^2 + (\frac{x-y}{2})^2 = z^2$.

5.7.12 Postage stamps/McNuggets problem

Let a, b be relatively-prime integers. There are exactly $\frac{1}{2}(a-1)(b-1)$ numbers *not* of form $ax + by$ ($x, y \geq 0$), and the largest is $(a-1)(b-1) - 1 = ab - a - b$.

5.7.13 Fermat's two-squares theorem

Odd prime p can be represented as a sum of two squares iff $p \equiv 1 \pmod{4}$. A product of two sums of two squares is a sum of two squares. Thus, n is a sum of two squares iff every prime of form $p = 4k + 3$ occurs an even number of times in n 's factorization.

5.8 Permutations**5.8.1 Factorial**

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----------------|-------|-------|-------|--------|--------|--------|--------|----------|--------|---------|
| $n!$ | 1 | 2 | 6 | 24 | 120 | 720 | 5040 | 40320 | 362880 | 3628800 |
| $\frac{n!}{n}$ | | 1 | 12 | 12 | 13 | 14 | 15 | 16 | 17 | |
| $\frac{n!}{n}$ | 4.0e7 | 4.8e8 | 6.2e9 | 8.7e10 | 1.3e12 | 2.1e13 | 3.6e14 | | | |
| $\frac{n!}{n}$ | 20 | 25 | 30 | 40 | 50 | 100 | 150 | 171 | | |
| $\frac{n!}{n!}$ | 2e18 | 2e25 | 3e32 | 8e47 | 3e64 | 9e157 | 6e262 | >DBL_MAX | | |

5.8.2 Cycles

Let $g_S(n)$ be the number of n -permutations whose cycle lengths all belong to the set S . Then

$$\sum_{n=0}^{\infty} g_S(n) \frac{x^n}{n!} = \exp \left(\sum_{n \in S} \frac{x^n}{n} \right)$$

5.8.3 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1) + D(n-2)) = nD(n-1) + (-1)^n = \left\lfloor \frac{n!}{e} \right\rfloor$$

5.8.4 Burnside's lemma

Given a group G of symmetries and a set X , the number of elements of X up to symmetry equals

$$\frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where X^g are the elements fixed by g ($g.x = x$).

If $f(n)$ counts "configurations" (of some sort) of length n , we can ignore rotational symmetry using $G = \mathbb{Z}_n$ to get

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n, k)) = \frac{1}{n} \sum_{k|n} f(k) \phi(n/k)$$

5.9 Partitions and subsets**5.9.1 Partition function**

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, \quad p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k-1)/2)$$

$$p(n) \sim 0.145n \cdot \exp(2.56\sqrt{n})$$

| $\frac{n}{p(n)}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 20 | 50 | 100 |
|------------------|---|---|---|---|---|---|----|----|----|----|-----|------|------|
| | 1 | 1 | 2 | 3 | 5 | 7 | 11 | 15 | 22 | 30 | 627 | ~2e5 | ~2e8 |

5.9.2 Partition Number

- Time Complexity: $O(n\sqrt{n})$

```
for (int i = 1; i <= n; ++i) {
    pent[2 * i - 1] = i * (3 * i - 1) / 2;
    pent[2 * i] = i * (3 * i + 1) / 2;
}
p[0] = 1;
for (int i = 1; i <= n; ++i) {
    p[i] = 0;
    for (int j = 1, k = 0; pent[j] <= i; ++j) {
        if (k < 2) p[i] = add(p[i], p[i - pent[j]]);
        else p[i] = sub(p[i], p[i - pent[j]]); ++k, k &= 3;
    }
}
```

- The number of partitions of a positive integer n into exactly k parts equals the number of partitions of n whose largest part equals k

$$p_k(n) = p_k(n-k) + p_{k-1}(n-1)$$

5.9.3 2nd Kaplansky's Lemma

The number of ways of selecting k objects, no two consecutive, from n labelled objects arrayed in a circle is $\frac{n}{k} \binom{n-k-1}{k-1} = \frac{n}{n-k} \binom{n-k}{k}$

5.9.4 Distinct Objects into Distinct Bins

- n distinct objects into r distinct bins $= r^n$

- Among n distinct objects, exactly k of them into r distinct bins $= \binom{n}{k} r^k$

- n distinct objects into r distinct bins such that each bin contains at least one object $= \sum_{i=0}^r (-1)^i \binom{r}{i} (r-i)^n$

5.10 Coloring

The number of labeled undirected graphs with n vertices, $G_n = 2^{\binom{n}{2}}$

The number of labeled directed graphs with n vertices, $G_n = 2^{n(n-1)}$

The number of connected labeled undirected graphs with n vertices, $C_n = 2^{\binom{n}{2}} - \frac{1}{n} \sum_{k=1}^{n-1} k \binom{n}{k} 2^{\binom{n-k}{2}} C_k = 2^{\binom{n}{2}} - \sum_{k=1}^{n-1} \binom{n-1}{k-1} 2^{\binom{n-k}{2}} C_k$

The number of k -connected labeled undirected graphs with n vertices, $D[n][k] = \sum_{s=1}^n \binom{n-1}{s-1} C_s D[n-s][k-1]$

Cayley's formula: the number of trees on n labeled vertices = the number of spanning trees of a complete graph with n labeled vertices $= n^{n-2}$

Number of ways to color a graph using k color such that no two adjacent nodes have same color

Complete graph $= k(k-1)(k-2)\dots(k-n+1)$

Tree $= k(k-1)^{n-1}$

Cycle $= (k-1)^n + (-1)^n(k-1)$

Number of trees with n labeled nodes: n^{n-2}

5.11 General purpose numbers**5.11.1 Eulerian numbers**

Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j :s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j :s s.t. $\pi(j) \geq j$, k j :s s.t. $\pi(j) > j$.

$$E(n, k) = (n-k)E(n-1, k-1) + (k+1)E(n-1, k)$$

$$E(n, 0) = E(n, n-1) = 1$$

$$E(n, k) = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n$$

5.11.2 Bell numbers

Total number of partitions of n distinct elements. $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \dots$. For p prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

5.11.3 Bernoulli numbers

$\sum_{j=0}^m \binom{m+1}{j} B_j = 0$. $B_0 = 1, B_1 = -\frac{1}{2}$. $B_n = 0$, for all odd $n \neq 1$.

5.11.4 Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1, C_{n+1} = \frac{2(2n+1)}{n+2} C_n, C_{n+1} = \sum C_i C_{n-i}$$

- $C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$
- sub-diagonal monotone paths in an $n \times n$ grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with with $n+1$ leaves (0 or 2 children).
- ordered trees with $n+1$ vertices.
- ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.
- permutations of $[n]$ with no 3-term increasing subseq.
- Find the count of balanced parentheses sequences consisting of $n+k$ pairs of parentheses where the first k symbols are open brackets.

$$C_n^{(k)} = \frac{k+1}{n+k+1} \binom{2n+k}{n}$$

- Recursive formula of Catalan Numbers:

$$C_n^{(k)} = \frac{(2n+k-1) \cdot (2n+k)}{n \cdot (n+k+1)} C_{n-1}^{(k)}$$

5.11.5 Lucas Number

Number of edge cover of a cycle graph C_n is L_n

$$L(n) = L(n-1) + L(n-2); L(0) = 2, L(1) = 1$$

5.12 Ballot Theorem

Suppose that in an election, candidate A receives a votes and candidate B receives b votes, where $a > b$ for some positive integer k . Compute the number of ways the ballots can be ordered so that A maintains more than k times as many votes as B throughout the counting of the ballots.

The solution to the ballot problem is $\frac{a-b}{a+b} \times C(a+b, a)$

5.13 Classical Problem

$F(n, k)$ = number of ways to color n objects using exactly k colors

Let $G(n, k)$ be the number of ways to color n objects using no more than k colors.

Then, $F(n, k) = G(n, k) - C(k, 1) * G(n, k-1) + C(k, 2) * G(n, k-2) - C(k, 3) * G(n, k-3) \dots$

Determining $G(n, k)$:

Suppose, we are given a $1 * n$ grid. Any two adjacent cells can not have same color. Then, $G(n, k) = k * ((k-1)^{n-1})$

If no such condition on adjacent cells. Then, $G(n, k) = k^n$

5.14 Matching Formula**5.14.1 Normal Graph**

$MM + MEC = n$ (excluding vertex), $IS + VC = G$, $MIS + MVC = G$

5.14.2 Bipartite Graph

$MIS = n - MBM$, $MVC = MBM$, $MEC = n - MBM$

5.15 Inequalities**5.15.1 Titu's Lemma**

For positive reals a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n ,

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geq \frac{a_1 + a_2 + \dots + a_n^2}{b_1 + b_2 + \dots + b_n}$$

Equality holds if and only if $a_i = k b_i$ for a non-zero real constant k .

5.16 Games**5.16.1 Grundy numbers**

For a two-player, normal-play (last to move wins) game on a graph (V, E) : $G(x) = \text{mex}(\{G(y) : (x, y) \in E\})$, where $\text{mex}(S) = \min\{n \geq 0 : n \notin S\}$. x is losing iff $G(x) = 0$.

5.16.2 Sums of games

- *Player chooses a game and makes a move in it* Grundy number of a position is xor of grundy numbers of positions in summed games.
- *Player chooses a non-empty subset of games (possibly, all) and makes moves in all of them* A position is losing iff each game is in a losing position.
- *Player chooses a proper subset of games (not empty and not all), and makes moves in all chosen ones.* A position is losing iff grundy numbers of all games are equal.
- *Player must move in all games, and loses if can't move in some game* A position is losing if any of the games is in a losing position.

5.16.3 Misère Nim

A position with pile sizes $a_1, a_2, \dots, a_n \geq 1$, not all equal to 1, is losing iff $a_1 \oplus a_2 \oplus \dots \oplus a_n = 0$ (like in normal nim.) A position with n piles of size 1 is losing iff n is odd.

5.17 Tree Hashing

$f(u) = sz[u] * \sum_{i=0} f(v) * p^i$; $f(v)$ are sorted $f(child) = 1$

5.18 Permutation

To maximize the sum of adjacent differences of a permutation, it is necessary and sufficient to place the smallest half numbers in odd position and the greatest half numbers in even position. Or, vice versa.

5.19 String

- If the sum of length of some strings is N , there can be at most \sqrt{N} distinct length.
- A Text can have at most $O(N \times \sqrt{N})$ distinct substrings that match with given patterns where the sum of the length of the given patterns is N .
- Period = $n \% (n - \text{pi.back}() == 0) ? n - \text{pi.back}() : n$
- The first (*period*) cyclic rotations of a string are distinct. Further cyclic rotations repeat the previous strings.
- S is a palindrome if and only if it's period is a palindrome.
- If S and T are palindromes, then the periods of $S + T$ are same if and only if $S + T$ is a palindrome.

5.20 Bit

- $(a \text{ xor } b)$ and $(a + b)$ has the same parity

- $(a + b) = (a \text{ xor } b) + 2(a \& b)$

- $\text{gcd}(a, b) \leq a - b \leq \text{xor}(a, b)$

5.21 Convolution

- Hamming Distance: Replace 0 with -1 - SQRT Decomposition: Find block size, $B = \sqrt{8 * n}$

5.22 Matrix Rotation**5.22.1 Anti-Clockwise Rotation**

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

5.22.2 Clockwise Rotation

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

5.23 Common Formulas**5.23.1 Permutation**

$${}_n P_r = \frac{n!}{(n-r)!}$$

5.23.2 Combination

$${}_n C_r = \frac{n!}{r!(n-r)!}$$

5.24 Logarithms**5.24.1 Change of Base Formula**

$$\log_a x = \frac{\log_b x}{\log_b a}$$

5.25 Common Series Sums**5.25.1 Sum of first n positive integers**

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

5.25.2 Sum of first n odd positive integers

$$1 + 3 + 5 + \dots + (2n-1) = n^2$$

5.25.3 Sum of first n even positive integers

$$2 + 4 + 6 + \dots + 2n = n(n+1)$$

5.25.4 Sum of first n squares

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

5.25.5 Sum of first n cubes

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

5.26 Progressions**5.26.1 Arithmetic Progression**

- **Sequence:** $a, a+d, a+2d, \dots, a+(n-1)d$
- **Sum of first n terms:** $S_n = \frac{n}{2}[2a + (n-1)d]$

5.26.2 Geometric Progression

- **Sequence:** $a, ar, ar^2, \dots, ar^{n-1}$
- **Sum (for $r > 1$):** $S_n = \frac{a(r^n - 1)}{r - 1}$
- **Sum (for $r < 1$):** $S_n = \frac{a(1 - r^n)}{1 - r}$