Anup Barman's

CP-Arsenal

CODEFORCES
AnupBarman

AtCoder
AnupBarman

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Updated: January 6, 2026

Contents

1 Setup

1.1 Linux Build

```
{
  "cmd" : ["ulimit -s 268435456; g++ -std=c++20
      $file_name -o $file_base_name && timeout 4s
      ./$file_base_name < input.txt > output.txt"],
  "selector" : "source.c++",
  "shell" : true,
  "working_dir" : "$file_path"
}
```

1.2 Windows Build

```
{
  "cmd" : [ "g++.exe", "-std=c++20", "${file}", "-o",
      "${file_base_name}.exe", "&&",
      "${file_base_name}.exe<input.txt>output.txt" ],
  "selector" : "source.cpp",
  "shell" : true,
  "working_dir" : "$file_path"
}
```

2 DataStructures

2.1 Anti Hash Unordered Map

```
unordered_map<int, int> mp;
mp.reserve(1 << 20);       // about 1M buckets
mp.max_load_factor(0.7);
```

2.2 Co-Ordinate Compression

```
vector<int> pos;
sort(pos.begin(), pos.end());
pos.erase(unique(pos.begin(), pos.end()), pos.end());
// then lower_bound on this pos array to find the
   compressed co-ordinate
```

2.3 Is Sorted

```
is_sorted(first, last, comp);
```

2.4 PBDS

```
#include "ext/pb_ds/assoc_container.hpp"
#include "ext/pb_ds/tree_policy.hpp"
using namespace __gnu_pbds;
template <class T>
using oset = tree<T, null_type, less<T>, rb_tree_tag,
tree_order_statistics_node_update>;
Note: Use less_equal for multiset like behaviour
Usage:
st.find_by_order(k) :: returns iterator of the k-th
   smallest element
st.order_of_key(x) :: returns index of x (number of
   elements less than x)
```

2.5 Segment Tree(BSUA)

```
// CSES - 1749
const int MX = 2e5 + 10;
int n;
int arr[MX], st[MX << 2];
void assign(int i, int x, int u = 1, int s = 0, int e
   = n - 1) {
  if (s == e) {
    st[u] = x;
    return;
  }
  int v = u << 1, w = v | 1, m = (s + e) >> 1;
  if (i <= m) assign(i, x, v, s, m);
  else assign(i, x, w, m + 1, e);
  st[u] = st[v] + st[w];
}
int kth(int k, int u = 1, int s = 0, int e = n - 1) {
  if (st[u] < k) return -1;
  if (s == e) {
    return s;
  }
  int v = u << 1, w = v | 1, m = (s + e) >> 1;
  if (st[v] >= k) return kth(k, v, s, m);
  else return kth(k - st[v], w, m + 1, e);
}
void solve() {
  cin >> n;
  for (int i = 0; i < n; ++i) {
    cin >> arr[i];
  }
  for (int i = 0; i < n; ++i) {
    assign(i, 1);
  }
  for (int i = 0; i < n; ++i) {
    int x;
    cin >> x;
    int ind = kth(x);
    assign(ind, 0);
    cout << arr[ind] << " ";
  }
}
```

2.6 Segment Tree(LzP)

```
class stree {
  vector<ll> st, lazy;
 public:
  stree(int n) {
    st.assign((n << 2) + 10, 0);
    lazy.assign((n << 2) + 10, 0);
  }
  void push(int u, int s, int e) {
    if (!lazy[u]) return;
    st[u] += (e - s + 1) * 1LL * lazy[u];
    if (s != e) {
      int v = u << 1, w = v | 1, m = (s + e) >> 1;
      lazy[v] += lazy[u];
      lazy[w] += lazy[u];
    }
    lazy[u] = 0;
  }
  void build(int u, int s, int e, int arr[]) {
    if (s == e) {
      st[u] = arr[s];
      return;
    }
    int v = u << 1, w = v | 1, m = (s + e) >> 1;
    build(v, s, m, arr);
    build(w, m + 1, e, arr);
    st[u] = st[v] + st[w];
  }
  void update(int l, int r, int x, int u, int s, int
   e) {
    push(u, s, e);
    if (e < l or r < s) return;
    int v = u << 1, w = v | 1, m = (s + e) >> 1;
    if (l <= s and e <= r) {
      st[u] += (e - s + 1) * 1LL * x;
      if (s != e) {
        lazy[v] += x;
        lazy[w] += x;
      }
      return;
    }
    update(l, r, x, v, s, m);
    update(l, r, x, w, m + 1, e);
    st[u] = st[v] + st[w];
  }
```

```
  ll query(int l, int r, int u, int s, int e) {
    push(u, s, e);
    if (e < l or r < s) return 0;
    if (l <= s and e <= r) return st[u];
    int v = u << 1, w = v | 1, m = (s + e) >> 1;
    return query(l, r, v, s, m) + query(l, r, w, m +
    ↪ 1, e);
  }
};
```

2.7 Segment Tree

```
class stree {
  vector<ll> st;
 public:
  stree(int n) {
    st.assign((n << 2) + 10, 0);
  }
  void build(int u, int s, int e, int arr[]) {
    if (s == e) {
      st[u] = arr[s];
      return;
    }
    int v = u << 1, w = v | 1, m = (s + e) >> 1;
    build(v, s, m, arr);
    build(w, m + 1, e, arr);
    st[u] = st[v] + st[w];
  }
  ll query(int l, int r, int u, int s, int e) {
    if (e < l or r < s) return 0;
    if (l <= s and e <= r) return st[u];
    int v = u << 1, w = v | 1, m = (s + e) >> 1;
    return query(l, r, v, s, m) + query(l, r, w, m +
    ↪ 1, e);
  }
  void update(int i, int x, int u, int s, int e) {
    if (s == e) {
      st[u] = x;
      return;
    }
    int v = u << 1, w = v | 1, m = (s + e) >> 1;
    if (i <= m) update(i, x, v, s, m);
    else update(i, x, w, m + 1, e);
    st[u] = st[v] + st[w];
  }
};
```

2.8 Sort and Unique

```
sort(v.begin(), v.end());
v.erase(unique(v.begin(), v.end()), v.end());
```

2.9 Sparse Table

```
const int MX = 2e5 + 10;
int n, arr[MX], st[25][MX];
int log2Floor(int i) {
  return 31 - __builtin_clz(i);
}
void build() {
  int k = log2Floor(n);
  copy(arr, arr + n, st[0]);
  for (int i = 1; i <= k; ++i) {
    for (int j = 0; j + (1 << i) <= n; j++) {
      st[i][j] = min(st[i - 1][j], st[i - 1][j + (1 <<
      ↪ (i - 1))]);
    }
  }
}
int query(int l, int r) {
  int i = log2Floor(r - l + 1);
  return min(st[i][l], st[i][r - (1 << i) + 1]);
}
```

3 Graphs

3.1 Articulation Point

```
int n;                    // number of nodes
vector<vector<int>> lst;  // adjacency list of graph
vector<bool> vis;
vector<int> tin, low;
int timer;
void dfs(int u, int p = -1) {
  vis[u] = true;
  tin[u] = low[u] = timer++;
  int children = 0;
  for (int v : lst[u]) {
    if (v == p) continue;
    if (vis[v]) {
      low[u] = min(low[u], tin[v]);
    } else {
      dfs(v, u);
      low[u] = min(low[u], low[v]);
      if (low[v] >= tin[u] && p != -1)
        IS_CUTPOINT(u);
      ++children;
    }
  }
  // if no vertex below v can reach u or higher
  ↪ removing u disconnects that subtree
  if (p == -1 && children > 1)
    IS_CUTPOINT(u);
}
void find_cutpoints() {
  timer = 0;
  vis.assign(n, false);
  tin.assign(n, -1);
  low.assign(n, -1);
  for (int i = 0; i < n; ++i) {
    if (!vis[i])
      dfs(i);
  }
}
```

3.2 Bridge Finding Algorithm

```
const int MX = 1e5 + 10;
int n, m, timer = 0;
vector<int> adj[MX];
vector<int> tin(MX, -1), low(MX, -1);
vector<bool> vis(MX, false);
void is_bridge(int u, int v) {
  // do something with the edge
}
void dfs(int u, int p = -1) {
  vis[u] = true;
  tin[u] = low[u] = timer++;
  for (int v : adj[u]) {
    if (v == p) continue;
    if (vis[v]) {
      low[u] = min(low[u], tin[v]);
    } else {
      dfs(v, u);
      low[u] = min(low[u], low[v]);
      if (low[v] > tin[u]) {
        is_bridge(u, v);
      }
    }
  }
}
```

3.3 Cycle Detection in DAG

```
const int MX = 1e5 + 10;
bool vis[MX], pathVis[MX];
vector<int> lst[MX];
bool dfs(int u) {
  vis[u] = true;
  pathVis[u] = true;
  for (auto v : lst[u]) {
    if (!vis[v]) {
      if (dfs(v))
        return true;
    } else if (pathVis[v]) {
      return true;
    }
  }
  pathVis[u] = false;
  return false;
}
void solve() {
  // take graph input
  for (int i = 0; i < n; ++i) {
    if (!vis[i])
      dfs(i);
  }
}
```

3.4 DSU on Trees

```
int n, color[MX], ans[MX];
vector<int> g[MX];
set<int> bucket[MX];
int merge(int a, int b) {
  if (bucket[a].size() < bucket[b].size()) swap(a, b);
  bucket[a].insert(bucket[b].begin(), bucket[b].end());
  bucket[b].clear();
  return a;
}
int dfs(int u, int p = -1) {
  int cur = u;
  for (int v : g[u])
    if (v != p)
      cur = merge(cur, dfs(v, u));
  ans[u] = (int)bucket[cur].size();
  return cur;
}
void solve() {
  cin >> n;
  for (int i = 0; i < n; ++i) {
    cin >> color[i];
    bucket[i].insert(color[i]);
  }
  // graph input
  dfs(0);
  // print output
}
```

3.5 DSU

```
const int MX = 1e5 + 10;
int par[MX], sz[MX];
void init() {
  for (int i = 1; i < MX; i++) {
    par[i] = i;
    sz[i] = 1;
  }
}
int findpar(int x) {
  if (par[x] == x) return x;
  return par[x] = findpar(par[x]);
}
void unite(int u, int v) {
  u = findpar(u);
  v = findpar(v);
  if (u != v) {
    if (sz[u] < sz[v]) {
      swap(u, v);
    }
```

```
    sz[u] += sz[v];
    par[v] = u;
  }
}
```

3.6 Euler Tour

```
const int MX = 2e5 + 10;
int timer = -1;
// s = start pos, e = end pos
int val[MX], s[MX], e[MX], flat[MX];
vector<int> lst[MX];
void dfs(int u, int p) {
  s[u] = ++timer;
  flat[timer] = val[u];
  for (auto v : lst[u]) {
    if (v != p)
      dfs(v, u);
  }
  e[u] = timer;
}
```

3.7 Floyd Warshall

```
vector<vector<int>> d(n, vector<int> (n, INF));
// take graph input into d
for (int k = 0; k < n; ++k) {
    for (int i = 0; i < n; ++i) {
        for (int j = 0; j < n; ++j) {
            if (d[i][k] < INF && d[k][j] < INF)
                d[i][j] = min(d[i][j], d[i][k] +
                 ↪ d[k][j]);
        }
    }
}
```

3.8 LCA using Binary Lifting

```
int n, l;
vector<vector<int>> adj;

int timer;
vector<int> tin, tout;
vector<vector<int>> up;

void dfs(int v, int p) {
  tin[v] = ++timer;
  up[v][0] = p;
  for (int i = 1; i <= l; ++i)
    up[v][i] = up[up[v][i - 1]][i - 1];

  for (int u : adj[v]) {
    if (u != p)
      dfs(u, v);
  }

  tout[v] = ++timer;
}

bool is_ancestor(int u, int v) {
  return tin[u] <= tin[v] && tout[u] >= tout[v];
}

int lca(int u, int v) {
  if (is_ancestor(u, v))
    return u;
  if (is_ancestor(v, u))
    return v;
  for (int i = l; i >= 0; --i) {
    if (!is_ancestor(up[u][i], v))
      u = up[u][i];
  }
  return up[u][0];
}

void preprocess(int root) {
  tin.resize(n);
  tout.resize(n);
  timer = 0;
  l = ceil(log2(n));
  up.assign(n, vector<int>(l + 1));
  dfs(root, root);
}
```

3.9 MST

```
// DSU first
void solve() {
  int n, m;
  cin >> n >> m;
  vector<tuple<int, int, int>> edges;
  for (int i = 0; i < m; ++i) {
    int u, v, wt;
    cin >> u >> v >> wt;
    edges.push_back({wt, u, v});
  }
  sort(edges.begin(), edges.end());
  init(n);
  int cost = 0;
  for (tuple& [ wt, u, v ] : edges) {
    if (findpar(u) == findpar(v)) continue;
    unite(u, v);
    cost += wt;
  }
  cout << cost << endl;
}
```

3.10 Max Bipartite Matching [Hopcroft-Karp]

```
const int INF = 1e9;
void hopcroftCarp() {
  int n, m, e;
  cin >> n >> m >> e;
  vector<int> adj[n];
  for (int i = 0; i < e; ++i) {
    int u, v;
    cin >> u >> v;
    --u;
    --v;
    adj[u].push_back(v);
  }
  vector<int> ml(m, -1), mr(n, -1), dist(n);
  auto bfs = [&]() -> bool {
    queue<int> q;
    for (int u = 0; u < n; ++u) {
      if (mr[u] == -1) {
        dist[u] = 0;
        q.push(u);
      } else {
        dist[u] = INF;
      }
    }
    bool foundAugmenting = false;
    while (!q.empty()) {
      int u = q.front();
      q.pop();
      for (int v : adj[u]) {
        int pairedLeft = ml[v];
        if (pairedLeft == -1) {
          foundAugmenting = true;
        } else if (dist[pairedLeft] == INF) {
          dist[pairedLeft] = dist[u] + 1;
          q.push(pairedLeft);
        }
      }
    }
    return foundAugmenting;
  };
  function<bool(int)> dfs = [&](int u) -> bool {
    for (int v : adj[u]) {
      int pairedLeft = ml[v];
      if (pairedLeft == -1 or (dist[pairedLeft] ==
       ↪ dist[u] + 1 and dfs(pairedLeft))) {
        mr[u] = v;
        ml[v] = u;
        return true;
      }
    }
    dist[u] = INF;
    return false;
  };
  int matching = 0;
  while (bfs()) {
    for (int u = 0; u < n; ++u) {
      if (mr[u] == -1) {
        if (dfs(u)) matching++;
      }
    }
  }
  cout << matching << el;
  for (int u = 0; u < n; ++u) {
    if (mr[u] != -1) {
      cout << u << " " << mr[u] << el;
    }
  }
}
```

3.11 Max Bipartite Matching [Kuhn's]

```
// left set size, right set size, edge count
int n, k, m, visToken = 1;
vector<int> lst[MX];
int mr[MX], ml[MX], vis[MX];
bool try_kuhn(int u) {
  if (vis[u] == visToken)
    return false;
  vis[u] = visToken;
  for (auto v : lst[u]) {
    if (ml[v] == -1 or try_kuhn(ml[v])) {
      ml[v] = u;
      mr[u] = v;
      return true;
    }
  }
  return false;
}
void solve() {
  cin >> n >> k >> m;
  for (int i = 0; i < m; ++i) {
    int u, v;
    cin >> u >> v;
    --u, --v;
    lst[u].push_back(v);
  }
  fill(mr, mr + n, -1);
  fill(ml, ml + k, -1);
  int ans = 0;
  for (int u = 0; u < n; ++u) {
    for (auto v : lst[u]) {
      if (ml[v] == -1) {
        ml[v] = u;
        mr[u] = v;
        ans++;
        break;
      }
    }
  }
  for (int u = 0; u < n; ++u) {
    if (mr[u] != -1) continue;
    visToken++;
    if (try_kuhn(u))
      ans++;
  }
  cout << ans << el;
  for (int v = 0; v < k; ++v) {
    if (ml[v] != -1) {
```

```
    cout << ml[v] + 1 << " " << v + 1 << el;
  }
 }
}
```

3.12 Topological Sorting

```
const int N = 1e5 + 10;
vector<int> g[N], indegree(N, 0);
vector<int> topSort(int n) {
  queue<int> q;
  vector<int> order;
  for (int i = 1; i <= n; i++) {
    if (indegree[i] == 0) {
      q.push(i);
    }
  }
  while (!q.empty()) {
    int u = q.front();
    q.pop();
    order.push_back(u);
    for (int v : g[u]) {
      indegree[v]--;
      if (indegree[v] == 0) {
        q.push(v);
      }
    }
  }
  return order;
}
```

3.13 Weighted Union Find

```
const int MX = 2e5 + 10;
int par[MX], sz[MX];
ll d[MX];
void init() {
  for (int i = 0; i < MX; ++i) {
    par[i] = i;
    sz[i] = 1;
    d[i] = 0;
  }
}
int findpar(int x) {
  if (par[x] == x) return x;
  int p = par[x];
  par[x] = findpar(p);
  d[x] += d[p];
  return par[x];
}
bool unite(int a, int b, ll w) {
  int ra = findpar(a);
  int rb = findpar(b);
  if (ra == rb) {
    return (d[b] - d[a] == w);
  }
  if (sz[ra] < sz[rb]) {
    swap(a, b);
    swap(ra, rb);
    w = -w;
  }
  par[rb] = ra;
  d[rb] = d[a] + w - d[b];
  sz[ra] += sz[rb];
  return true;
}
ll dist(int a, int b) {
  findpar(a), findpar(b);
  return d[b] - d[a];
}
```

4 NumberTheory

4.1 Unique Prime Factorization using Sieve

```
const int MX = 2e5 + 10;
vector<int> pfac[MX];
void factorize() {
    for (int i = 2; i < MX; i++){
        if (!pfac[i].empty()) continue;
        for (int j = i; j < MX; j += i)
            pfac[j].push_back(i);
    }
}
```

4.2 nCr

```
const int MX = 1e6 + 10;
const int M = 1e9 + 7;
int fact[MX], inv_fact[MX];
int modPow(int a, int b) {
  int ans = 1;
  while (b) {
    if (b & 1) ans = (1LL * ans * a) % M;
    a = (1LL * a * a) % M;
    b >>= 1;
  }
  return ans;
}
void precalFact() {
  fact[0] = inv_fact[0] = 1;
  for (int i = 1; i < MX; i++) {
    fact[i] = (1LL * fact[i - 1] * i) % M;
  }
  inv_fact[MX - 1] = modPow(fact[MX - 1], M - 2);
  for (int i = MX - 2; i >= 1; i--) {
    inv_fact[i] = (1LL * inv_fact[i + 1] * (i + 1)) %
    ↪ M;
  }
}
int nCr(int n, int r) {
  if (r < 0 or r > n) return 0;
  return 1LL * fact[n] * inv_fact[r] % M * inv_fact[n
  ↪ - r] % M;
}
```

5 String

5.1 Manachar

```
// pal[1][i] = longest odd (half rounded down)
↪ palindrome around pos i and
// starts at i - pal[1][i] and ends at i + pal[1][i]
↪ pal[0][i] = half length of
// longest even palindrome around pos i, i + 1 and
↪ starts at i - par[0][i] + 1
// and ends at i + pal[0][i]
const int N = 5e5 + 10;
int pal[2][N];
void manacher(string& s) {
  int n = s.size(), idx = 2;
  while (idx--) {
    for (int l = -1, r = -1, i = 0; i < n - 1; ++i) {
      if (i > r)
        l = r = i;
      else {
        int k = min(r - i, pal[idx][l + r - i]);
        l = i - k, r = i + k;
      }
      while (l - idx >= 0 and r + 1 < n and s[l - idx]
      ↪ == s[r + 1]) l--, r++;
      pal[idx][i] = r - i;
      // [l - 1 + idx : r] palindrome
    }
  }
}
```

6 Notes

6.1 Geometry

6.1.1 Triangles

Circumradius: $R = \dfrac{abc}{4A}$, Inradius: $r = \dfrac{A}{s}$

The area of a triangle using two sides and the included angle can be given as:

$$A = \frac{1}{2}ab\sin\angle C$$

Length of median (divides triangle into two equal-area triangles): $m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

Length of bisector (divides angles in two): $s_a = \sqrt{bc\left[1-\left(\frac{a}{b+c}\right)^2\right]}$

Law of tangents: $\dfrac{a+b}{a-b} = \dfrac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}}$

6.1.2 Quadrilaterals

With side lengths a,b,c,d, diagonals e,f, diagonals angle θ, area A and magic flux $F = b^2+d^2-a^2-c^2$:

$$4A = 2ef\cdot\sin\theta = F\tan\theta = \sqrt{4e^2f^2-F^2}$$

For cyclic quadrilaterals the sum of opposite angles is 180°, $ef = ac+bd$, and $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$.

6.1.3 Spherical coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi & r &= \sqrt{x^2+y^2+z^2} \\ y &= r\sin\theta\sin\phi & \theta &= \operatorname{acos}(z/\sqrt{x^2+y^2+z^2}) \\ z &= r\cos\theta & \phi &= \operatorname{atan2}(y,x) \end{aligned}$$

6.1.4 Pick's Theorem:

Given a lattice polygon with non-zero area, we define: S as the area of the polygon, I as the number of integer-coordinate points strictly inside the polygon, B as the number of integer-coordinate points on the boundary of the polygon. Then, Pick's Theorem states:

$$S = I + \frac{B}{2} - 1$$

The number of lattice points on segments (x1, y1) to (x2 , y2) is: gcd(abs(x2 - x1) , abs(y2 - y1)) + 1

6.1.5 Polygon

For a regular polygon with n sides and side length a, the circumradius R is given by:

$$R = \frac{a}{2\sin\left(\frac{\pi}{n}\right)}$$

6.1.6 Area of a Circular Segment

The area of a circular segment, which is the region enclosed by a chord and the corresponding arc, can be calculated using the formula:

$$A = \frac{R^2}{2}(\theta - \sin\theta)$$

where: R is the radius of the circle, θ is the central angle subtended by the chord, in radians.

6.2 Binomial Coefficent

- Factoring in: $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$
- Sum over k: $\sum_{k=0}^{n}\binom{n}{k} = 2^n$
- Alternating sum: $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$
- Even and odd sum: $\sum_{k=0}^{n}\binom{n}{2k} = \sum_{k=0}^{n}\binom{n}{2k+1}2^{n-1}$
- The Hockey Stick Identity
  - (Left to right) Sum over n and k: $\sum_{k=0}^{m}\binom{n+k}{k} = \binom{n+m-1}{m}$
  - (Right to left) Sum over n: $\sum_{m=0}^{n}\binom{m}{k} = \binom{n+1}{k+1}$
- Sum of the squares: $\sum_{k=0}^{n}(\binom{n}{k})^2 = \binom{2n}{n}$
- Weighted sum: $\sum_{k=1}^{n}k\binom{n}{k} = n2^{n-1}$
- Connection with the fibonacci numbers: $\sum_{k=0}\binom{n-k}{k} = F_{n+1}$
- Vandermonde's Identity: $\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$
- If $f(n,k) = C(n,0)+C(n,1)+...+C(n,k)$, Then $f(n+1,k) = 2*f(n,k)-C(n,k)$ [For multiple $f(n,k)$ queries, use Mo's algo]

Lucas Theorem

$$\binom{m}{n} \equiv \prod_{i=0}^{k}\binom{m_i}{n_i}(\mod p)$$

- $\binom{m}{n}$ is divisible by p if and only if at least one of the base-p digits of n is greater than the corresponding base-p digit of m.
- The number of entries in the nth row of Pascal's triangle that are not divisible by $p = \prod_{i=0}^{k}(n_i+1)$
- All entries in the $(p^k-1)th$ row are not divisible by p.
- $\binom{n}{m} \equiv \lfloor\frac{n}{p}\rfloor(\mod p)$

6.3 Fibonacci Number

1. $k = A-B, F_AF_B = F_{k+1}F_A^2 + F_kF_AF_{A-1}$
2.$\sum_{i=0}^{n}F_i^2 = F_{n+1}F_n$ **3.**$\sum_{i=0}^{n}F_iF_{i+1} = F_{n+1}^2-(-1)^n$
4.$\sum_{i=0}^{n}F_iF_{i+1} = F_{n+1}^2-(-1)^n$ **5.**$\sum_{i=0}^{n}F_iF_{i-1} = \sum_{i=0}^{n-1}F_iF_{i+1}$
6.$\gcd(F_m,F_n) = F_{\gcd(m,n)}$ **7.**$\sum_{0\le k\le n}\left(\binom{n-k}{k}\right) = F_{n+1}$
8.$\gcd(F_n,F_{n+1}) = \gcd(F_n,F_{n+2}) = \gcd(F_{n+1},F_{n+2}) = 1$

6.4 Sums

$1^2+2^2+3^2+\cdots+n^2 = \frac{n(2n+1)(n+1)}{6}$

$1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$

$1^4+2^4+3^4+\cdots+n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

$\sum_{i=1}^{n}i^m = \frac{1}{m+1}\left[(n+1)^{m+1}-1-\sum_{i=1}^{n}\left((i+1)^{m+1}-i^{m+1}-(m+1)i^m\right)\right]$

$\sum_{i=1}^{n-1}i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_kn^{m+1-k}$

$\sum_{k=0}^{n}kx^k = (x-(n+1)x^{n+1}+nx^{n+2})/(x-1)^2$

6.5 Series

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots,(-\infty<x<\infty)$$

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots,(-1<x\le1)$$

$$\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+\frac{2x^3}{32}-\frac{5x^4}{128}+\ldots,(-1\le x\le1)$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots,(-\infty<x<\infty)$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots,(-\infty<x<\infty)$$

$$(x+a)^{-n} = \sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}x^ka^{-n-k}$$

Generating Function

$$1/(1-x) = 1+x+x^2+x^3+\ldots$$

$$1/(1-ax) = 1+ax+(ax)^2+(ax)^3+\ldots$$

$$1/(1-x)^2 = 1+2x+3x^2+4x^3+\ldots$$

$$1/(1-x)^3 = C(2,2)+C(3,2)x+C(4,2)x^2+C(5,2)x^3+\ldots$$

$$1/(1-ax)^(k+1) = 1+C(1+k,k)(ax)+C(2+k,k)(ax)^2+C(3+k,k)(ax)^3+\ldots$$

$$x(x+1)(1-x)^-3 = 1+x+4x^2+9x^3+16x^4+25x^5+\ldots$$

$$e^x = 1+x+(x^2)/2!+(x^3)/3!+(x^4)/4!+\ldots$$

6.6 Pythagorean Triples

The Pythagorean triples are uniquely generated by

$$a = k\cdot(m^2-n^2),\ b = k\cdot(2mn),\ c = k\cdot(m^2+n^2),$$

with $m>n>0$, $k>0$, $m\perp n$, and either m or n even.

6.7 Number Theory

- HCN: 1e6(240), 1e9(1344), 1e12(6720), 1e14(17280), 1e15(26880), 1e16(41472)
- $gcd(a,b,c,d,\ldots) = gcd(a,b-a,c-b,d-c,\ldots)$
- $gcd(a+k,b+k,c+k,d+k,\ldots) = gcd(a+k,b-a,c-b,d-c,\ldots)$
- Primitive root exists iff $n = 1,2,4,p^k,2\times p^k$, where p is an odd prime.
- If primtive root exists, there are $\phi(\phi(n))$ primtive roots of n.
- The numbers from 1 to n have in total $O(n\log\log n)$ unique prime factors.
- $x \equiv r_1 \mod m1$ and $x \equiv r_2 \mod m2$ has a solution iff $\gcd(m_1,m_2)|(r_1-r_2)$ Solution of $x^2 \equiv a(\mod p)$
- $ca \equiv cb \pmod m \iff a \equiv b \pmod{\frac{n}{\gcd(n,c)}}$
- $ax \equiv b \pmod m$ has a solution $\iff \gcd(a,m)|b$
- If $ax \equiv b \pmod m$ has a solution, then it has $gcd(a,m)$ solutions and they are separated by $\frac{m}{\gcd(a,m)}$
- $ax \equiv 1 \pmod m$ has a solution or a is invertible $\pmod m \iff \gcd(a,m) = 1$
- $x^2 \equiv 1 \pmod p$ then $x \equiv \pm1 \pmod p$
- There are $\frac{p-1}{2}$ has no solution.
- There are $\frac{p-1}{2}$ has exaclty two solutions.
- When $p\%4 = 3$, $x \equiv \pm a^{\frac{p+1}{4}}$
- When $p\%8 = 5$, $x \equiv a^{\frac{p+3}{8}}$ or $x \equiv 2^{\frac{p-1}{4}}a^{\frac{p+3}{8}}$

6.7.1 Primes

$p = 962592769$ is such that $2^{21} \mid p-1$, which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than 1 000 000.
Primitive roots exist modulo any prime power p^a, except for $p = 2, a > 2$, and there are $\phi(\phi(p^a))$ many. For $p = 2, a > 2$, the group $\mathbb{Z}_{2^a}^\times$ is instead isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$.

6.7.2 Estimates

$\sum_{d|n} d = O(n\log\log n)$.
The number of divisors of n is at most around 100 for $n < 5e4$, 500 for $n < 1e7$, 2000 for $n < 1e10$, 200 000 for $n < 1e19$.

6.7.3 Perfect numbers

$n > 1$ is called perfect if it equals sum of its proper divisors and 1. Even n is perfect iff $n = 2^{p-1}(2^p-1)$ and 2^p-1 is prime (Mersenne's). No odd perfect numbers are yet found.

6.7.4 Carmichael numbers

A positive composite n is a Carmichael number ($a^{n-1} \equiv 1 \pmod n$ for all a: $\gcd(a,n) = 1$), iff n is square-free, and for all prime divisors p of n, $p-1$ divides $n-1$.

6.7.5 Totient

- If p is a prime $(p^k) = p^k - p^{k-1}$
- If a b are relatively prime, $\phi(ab) = \phi(a)\phi(b)$
- $\phi(n) = n(1-\frac{1}{p_1})(1-\frac{1}{p_2})(1-\frac{1}{p_3})\ldots(1-\frac{1}{p_k})$
- Sum of coprime to $n = n*\frac{\phi(n)}{2}$
- If $n = 2^k, \phi(n) = 2^{k-1} = \frac{n}{2}$
- For a b, $\phi(ab) = \phi(a)\phi(b)\frac{d}{\phi(d)}$
- $\phi(ip) = p\phi(i)$ whenever p is a prime and it divides i
- The number of $a(1 <= a <= N)$ such that $gcd(a,N) = d$ is $\phi(\frac{n}{d})$
- If $n > 2$, $\phi(n)$ is always even
- Sum of gcd, $\sum_{i=1}^{n}gcd(i,n) = \sum_{d|n}d\phi(\frac{n}{d})$
- Sum of lcm, $\sum_{i=1}^{n}lcm(i,n) = \frac{n}{2}(\sum_{d|n}(d\phi(d))+1)$
- $\phi(1) = 1$ and $\phi(2) = 1$ which two are only odd ϕ
- $\phi(3) = 2$ and $\phi(4) = 2$ and $\phi(6) = 2$ which three are only prime ϕ
- Find minimum n such that $\frac{\phi(n)}{n}$ is maximum- Multiple of small primes- $2*3*5*7*11*13*\ldots$

6.7.6 Mobius function

$\mu(1)=1$. $\mu(n)=0$, if n is not squarefree. $\mu(n)=(-1)^s$, if n is the product of s distinct primes. Let f, F be functions on positive integers. If for all $n\in N$, $F(n)=\sum_{d|n} f(d)$, then $f(n)=\sum_{d|n}\mu(d)F(\frac{n}{d})$, and vice versa. $\phi(n)=\sum_{d|n}\mu(d)\frac{n}{d}$. $\sum_{d|n}\mu(d)=1$.
If f is multiplicative, then $\sum_{d|n}\mu(d)f(d)=\prod_{p|n}(1-f(p))$, $\sum_{d|n}\mu(d)^2 f(d)=\prod_{p|n}(1+f(p))$.

$$\sum_{i=1}^{n}\sum_{j=1}^{n}[gcd(i,j)=1]=\sum_{k=1}^{n}\mu(k)\lfloor\frac{n}{k}\rfloor^2$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}gcd(i,j)=\sum_{k=1}^{n}k\sum_{l=1}^{\lfloor\frac{n}{k}\rfloor}\mu(l)\lfloor\frac{n}{kl}\rfloor^2$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}gcd(i,j)=\sum_{k=1}^{n}(\frac{\lfloor\frac{n}{k}\rfloor(1+\lfloor\frac{n}{k}\rfloor)}{2})^2\sum_{d|k}\mu(d)kd$$

6.7.7 Legendre symbol

If p is an odd prime, $a\in\mathbb{Z}$, then $\left(\frac{a}{p}\right)$ equals 0, if $p|a$; 1 if a is a quadratic residue modulo p; and -1 otherwise. Euler's criterion: $\left(\frac{a}{p}\right)=a^{\left(\frac{p-1}{2}\right)} \pmod p$.

6.7.8 Jacobi symbol

If $n=p_1^{a_1}\cdots p_k^{a_k}$ is odd, then $\left(\frac{a}{n}\right)=\prod_{i=1}^{k}\left(\frac{a}{p_i}\right)^{k_i}$.

6.7.9 Primitive roots

If the order of g modulo m (min $n>0$: $g^n\equiv 1 \pmod m$) is $\phi(m)$, then g is called a primitive root. If Z_m has a primitive root, then it has $\phi(\phi(m))$ distinct primitive roots. Z_m has a primitive root iff m is one of 2, 4, p^k, $2p^k$, where p is an odd prime. If Z_m has a primitive root g, then for all a coprime to m, there exists unique integer $i=\text{ind}_g(a)$ modulo $\phi(m)$, such that $g^i\equiv a \pmod m$. $\text{ind}_g(a)$ has logarithm-like properties: $\text{ind}(1)=0$, $\text{ind}(ab)=\text{ind}(a)+\text{ind}(b)$.
If p is prime and a is not divisible by p, then congruence $x^n\equiv a \pmod p$ has $\gcd(n,p-1)$ solutions if $a^{(p-1)/\gcd(n,p-1)}\equiv 1 \pmod p$, and no solutions otherwise. (Proof sketch: let g be a primitive root, and $g^i\equiv a \pmod p$, $g^u\equiv x \pmod p$. $x^n\equiv a \pmod p$ iff $g^{nu}\equiv g^i \pmod p$ iff $nu\equiv i \pmod p$.)

6.7.10 Discrete logarithm problem

Find x from $a^x\equiv b \pmod m$. Can be solved in $O(\sqrt{m})$ time and space with a meet-in-the-middle trick. Let $n=\lceil\sqrt{m}\rceil$, and $x=ny-z$. Equation becomes $a^{ny}\equiv ba^z \pmod m$. Precompute all values that the RHS can take for $z=0,1,\dots,n-1$, and brute force y on the LHS, each time checking whether there's a corresponding value for RHS.

6.7.11 Pythagorean triples

Integer solutions of $x^2+y^2=z^2$ All relatively prime triples are given by: $x=2mn, y=m^2-n^2, z=m^2+n^2$ where $m>n, \gcd(m,n)=1$ and $m\not\equiv n \pmod 2$. All other triples are multiples of these. Equation $x^2+y^2=2z^2$ is equivalent to $(\frac{x+y}{2})^2+(\frac{x-y}{2})^2=z^2$.

6.7.12 Postage stamps/McNuggets problem

Let a, b be relatively-prime integers. There are exactly $\frac{1}{2}(a-1)(b-1)$ numbers *not* of form $ax+by$ $(x,y\geq 0)$, and the largest is $(a-1)(b-1)-1=ab-a-b$.

6.7.13 Fermat's two-squares theorem

Odd prime p can be represented as a sum of two squares iff $p\equiv 1 \pmod 4$. A product of two sums of two squares is a sum of two squares. Thus, n is a sum of two squares iff every prime of form $p=4k+3$ occurs an even number of times in n's factorization.

6.8 Permutations

6.8.1 Factorial

n	1 2 3	4	5	6	7	8	9	10
$n!$	1 2 6	24	120	720	5040	40320	362880	3628800
n	11	12	13	14	15	16	17	
$n!$	4.0e7	4.8e8	6.2e9	8.7e10	1.3e12	2.1e13	3.6e14	
n	20	25	30	40	50	100	150	171
$n!$	2e18	2e25	3e32	8e47	3e64	9e157	6e262	>DBL_MAX

6.8.2 Cycles

Let $g_S(n)$ be the number of n-permutations whose cycle lengths all belong to the set S. Then

$$\sum_{n=0}^{\infty}g_S(n)\frac{x^n}{n!}=\exp\left(\sum_{n\in S}\frac{x^n}{n}\right)$$

6.8.3 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n)=(n-1)(D(n-1)+D(n-2))=nD(n-1)+(-1)^n=\left\lfloor\frac{n!}{e}\right\rceil$$

6.8.4 Burnside's lemma

Given a group G of symmetries and a set X, the number of elements of X *up to symmetry* equals

$$\frac{1}{|G|}\sum_{g\in G}|X^g|,$$

where X^g are the elements fixed by g $(g.x=x)$.
If $f(n)$ counts "configurations" (of some sort) of length n, we can ignore rotational symmetry using $G=\mathbb{Z}_n$ to get

$$g(n)=\frac{1}{n}\sum_{k=0}^{n-1}f(\gcd(n,k))=\frac{1}{n}\sum_{k|n}f(k)\phi(n/k)$$

6.9 Partitions and subsets

6.9.1 Partition function

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0)=1,\ p(n)=\sum_{k\in\mathbb{Z}\setminus\{0\}}(-1)^{k+1}p(n-k(3k-1)/2)$$

$$p(n)\sim 0.145/n\cdot\exp(2.56\sqrt{n})$$

n	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	~2e5	~2e8

6.9.2 Partition Number

- Time Complexity: $O(n\sqrt{n})$

```
for (int i = 1; i <= n; ++i) {
  pent[2 * i - 1] = i * (3 * i - 1) / 2;
  pent[2 * i] = i * (3 * i + 1) / 2;
}
p[0] = 1;
for (int i = 1; i <= n; ++i) {
  p[i] = 0;
  for (int j = 1, k = 0; pent[j] <= i; ++j) {
    if (k < 2) p[i] = add(p[i], p[i - pent[j]]);
    else p[i] = sub(p[i], p[i - pent[j]]); ++k, k &= 3;
  }
}
```

- The number of partitions of a positive integer n into exactly k parts equals the number of partitions of n whose largest part equals k

$$p_k(n)=p_k(n-k)+p_{k-1}(n-1)$$

6.9.3 2nd Kaplansky's Lemma

The number of ways of selecting k objects, no two consecutive,from n labelled objects arrayed in a circle is $\frac{n}{k}\binom{n-k-1}{k-1}=\frac{n}{n-k}\binom{n-k}{k}$

6.9.4 Distinct Objects into Distinct Bins

- n distinct objects into r distinct bins $=r^n$
- Among n distinct objects, exactly k of them into r distincts bins $=\binom{n}{k}r^k$
- n distinct objects into r distinct bins such that each bin contains at least one object $=\sum_{i=0}^{r}(-1)^i\binom{r}{i}(r-i)^n$

6.10 Coloring

The number of labeled undirected graphs with n vertices, $G_n=2^{\binom{n}{2}}$

The number of labeled directed graphs with n vertices, $G_n=2^{n(n1)}$

The number of connected labeled undirected graphs with n vertices, $C_n=2^{\binom{n}{2}}-\frac{1}{n}\sum_{k=1}^{n-1}k\binom{n}{k}2^{\binom{n-k}{2}}C_k=2^{\binom{n}{2}}-\sum_{k=1}^{n-1}\binom{n-1}{k-1}2^{\binom{n-k}{2}}C_k$

The number of k-connected labeled undirected graphs with n vertices, $D[n][k]=\sum_{s=1}^{n}\binom{n-1}{s-1}C_sD[n-s][k-1]$

Cayley's formula: the number of trees on n labeled vertices = the number of spanning trees of a complete graph with n labeled vertices = n^{n-2}

Number of ways to color a graph using k color such that no two adjacent nodes have same color
Complete graph = $k(k-1)(k-2)...(k-n+1)$
Tree = $k(k-1)^{n-1}$
Cycle = $(k-1)^n+(-1)^n(k-1)$

Number of trees with n labeled nodes: n^{n-2}

6.11 General purpose numbers

6.11.1 Eulerian numbers

Number of permutations $\pi\in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j)>\pi(j+1)$, $k+1$ j:s s.t. $\pi(j)\geq j$, k j:s s.t. $\pi(j)>j$.

$$E(n,k)=(n-k)E(n-1,k-1)+(k+1)E(n-1,k)$$

$$E(n,0)=E(n,n-1)=1$$

$$E(n,k)=\sum_{j=0}^{k}(-1)^j\binom{n+1}{j}(k+1-j)^n$$

6.11.2 Bell numbers

Total number of partitions of n distinct elements. $B(n)=1,1,2,5,15,52,203,877,4140,21147,\dots$. For p prime,

$$B(p^m+n)\equiv mB(n)+B(n+1) \pmod p$$

6.11.3 Bernoulli numbers

$\sum_{j=0}^{m}\binom{m+1}{j}B_j=0$. $B_0=1, B_1=-\frac{1}{2}$. $B_n=0$, for all odd $n\neq 1$.

6.11.4 Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1,\ C_{n+1} = \frac{2(2n+1)}{n+2}C_n,\ C_{n+1} = \sum C_i C_{n-i}$$

- $C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$
- sub-diagonal monotone paths in an $n \times n$ grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with with $n+1$ leaves (0 or 2 children).
- ordered trees with $n+1$ vertices.
- ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.
- permutations of $[n]$ with no 3-term increasing subseq.
- Find the count of balanced parentheses sequences consisting of $n+k$ pairs of parentheses where the first k symbols are open brackets.

$$C_n^{(k)} = \frac{k+1}{n+k+1}\binom{2n+k}{n}$$

- Recursive formula of Catalan Numbers:

$$C_n^{(k)} = \frac{(2n+k-1)\cdot(2n+k)}{n\cdot(n+k+1)} C_{n-1}^{(k)}$$

6.11.5 Lucas Number

Number of edge cover of a cycle graph C_n is L_n

$$L(n) = L(n-1) + L(n-2); L(0) = 2, L(1) = 1$$

6.12 Ballot Theorem

Suppose that in an election, candidate A receives a votes and candidate B receives b votes, where a kb for some positive integer k. Compute the number of ways the ballots can be ordered so that A maintains more than k times as many votes as B throughout the counting of the ballots.

The solution to the ballot problem is $\frac{a-kb}{a+b} \times C(a+b, a)$

6.13 Classical Problem

$F(n,k)$ = number of ways to color n objects using exactly k colors

Let $G(n,k)$ be the number of ways to color n objects using no more than k colors.

Then, $F(n,k) = G(n,k) - C(k,1)*G(n,k-1) + C(k,2)*G(n,k-2) - C(k,3)*G(n,k-3)\ldots$

Determining G(n, k) :

Suppose, we are given a 1 * n grid. Any two adjacent cells can not have same color. Then, $G(n,k) = k*((k-1)^(n-1))$

If no such condition on adjacent cells. Then, $G(n,k) = k^n$

6.14 Matching Formula

6.14.1 Normal Graph

MM + MEC = n (exculding vertex), IS + VC = G, MIS + MVC = G

6.14.2 Bipartite Graph

MIS = n - MBM, MVC = MBM, MEC = n - MBM

6.15 Inequalities

6.15.1 Titu's Lemma

For positive reals $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$,

$$\frac{{a_1}^2}{b_1} + \frac{{a_2}^2}{b_2} + \ldots + \frac{{a_n}^2}{b_n} \geq \frac{a_1 + a_2 + \ldots + {a_n}^2}{b_1 + b_2 + \ldots + b_n}$$

Equality holds if and only if $a_i = kb_i$ for a non-zero real constant k.

6.16 Games

6.16.1 Grundy numbers

For a two-player, normal-play (last to move wins) game on a graph (V,E): $G(x) = \text{mex}(\{G(y) : (x,y) \in E\})$, where $\text{mex}(S) = \min\{n \geq 0 : n \notin S\}$. x is losing iff $G(x) = 0$.

6.16.2 Sums of games

- *Player chooses a game and makes a move in it* Grundy number of a position is xor of grundy numbers of positions in summed games.
- *Player chooses a non-empty subset of games (possibly, all) and makes moves in all of them* A position is losing iff each game is in a losing position.
- *Player chooses a proper subset of games (not empty and not all), and makes moves in all chosen ones.* A position is losing iff grundy numbers of all games are equal.
- *Player must move in all games, and loses if can't move in some game* A position is losing if any of the games is in a losing position.

6.16.3 Misère Nim

A position with pile sizes $a_1, a_2, \ldots, a_n \geq 1$, not all equal to 1, is losing iff $a_1 \oplus a_2 \oplus \cdots \oplus a_n = 0$ (like in normal nim.) A position with n piles of size 1 is losing iff n is *odd*.

6.17 Tree Hashing

$f(u) = sz[u] * \sum_{i=0} f(v) * p^i$; $f(v)$ are sorted $f(child) = 1$

6.18 Permutation

To maximize the sum of adjacent differences of a permuation, it is necessary and sufficient to place the smallest half numbers in odd position and the greatest half numbers in even position. Or, vice versa.

6.19 String

- If the sum of length of some strings is N, there can be at most $\sqrt{N}$ distinct length.
- A Text can have at most $O(N \times \sqrt{N})$ distinct substrings that match with given patterns where the sum of the length of the given patterns is N.
- Period = n % (n - pi.back() == 0)? n - pi.back(): n
- The first $(period)$ cyclic rotations of a string are distinct. Further cyclic rotations repeat the previous strings.
- S is a palindrome if and only if it's period is a palindrome.
- If S and T are palindromes, then the periods of S T are same if and only if $S + T$ is a palindrome.

6.20 Bit

- (a xor b) and (a + b) has the same parity
- (a + b) = (a xor b) + 2 (a b)
- gcd(a, b) <= a - b <= xor(a, b)

6.21 Convolution

- Hamming Distance: Replace 0 with -1 - SQRT Decomposition: Find block size, B = $\sqrt{(8*n)}$

6.22 Matrix Rotation

6.22.1 Anti-Clockwise Rotation

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

6.22.2 Clockwise Rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

6.23 Common Formulas

6.23.1 Permutation

$${}^nP_r = \frac{n!}{(n-r)!}$$

6.23.2 Combination

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

6.24 Logarithms

6.24.1 Change of Base Formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

6.25 Common Series Sums

6.25.1 Sum of first n positive integers

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

6.25.2 Sum of first n odd positive integers

$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

6.25.3 Sum of first n even positive integers

$$2 + 4 + 6 + \cdots + 2n = n(n+1)$$

6.25.4 Sum of first n squares

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

6.25.5 Sum of first n cubes

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

6.26 Progressions

6.26.1 Arithmetic Progression

- **Sequence:** $a, a+d, a+2d, \ldots, a+(n-1)d$
- **Sum of first n terms:** $S_n = \frac{n}{2}[2a + (n-1)d]$

6.26.2 Geometric Progression

- **Sequence:** $a, ar, ar^2, \ldots, ar^{n-1}$
- **Sum (for $r > 1$):** $S_n = \frac{a(r^n - 1)}{r-1}$
- **Sum (for $r < 1$):** $S_n = \frac{a(1-r^n)}{1-r}$