



Daffodil International University

DIU_DivideByZero

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Team Reference Document

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1 Code

1.1

CP_Ubuntu

```

1 "cmd": ["ulimit -s 268435456; g++ -std=c++20
2     $file_base_name && timeout 4s
3     ./${file_base_name} < inputf.in > outputf.in"],
4     "selector": "source.cpp",
5     "shell": true,
6     "working_dir": "$file_path"
7 }
```

1.2

CP_Windows

```

1 "cmd": ["g++.exe", "-std=c++20", "${file}", "-o",
2     "${file_base_name}.exe", "&&",
2     "${file_base_name}.exe<inputf.in>outputf.in"],
3     "selector": "source.cpp",
4     "shell": true,
5     "working_dir": "$file_path"
6 }
```

1.3 StressTesting(check.sh)

```

3 // chmod u+x check.sh
4 // ./check.sh
5 set -e
6 g++ gen.cpp -o gen
7 g++ code.cpp -o code
8 g++ brute.cpp -o brute
9 for ((i = 1; ; ++i)); do
10    echo "Passed on TestCase: " $i
11    ./gen $i > in
12    ./code < in > out1
13    ./brute < in > out2
14    diff -Z out1 out2 || break
15 done
16 echo -e "WA on the following test:"
17 cat in
18 echo -e "\nExpected:"
19 cat out2
20 echo -e "\nFound:"
21 cat out1
22 }
```

1.4 StressTesting(gen.cpp)

```

4 #include <bits/stdc++.h>
5 using namespace std;
6 using ll = long long;
7 mt19937_64 rng(chrono::steady_clock::now().time_since_
8 epoch().count());
9 inline ll gen_random(ll l, ll r) {
10    return uniform_int_distribution<ll>(l, r)(rng);
11 }
12 inline double gen_random_real(double l, double r) {
13    return uniform_real_distribution<double>(l, r)(rng);
14 }
15 int main(int argc, char* args[]) {
16    int _ = atoi(args[1]);
17    rng.seed(_);
18    int n = gen_random(1, 5);
19    vector<int> per;
20    for (int i = 0; i < n; ++i) {
21        per.push_back(i + 1);
22    }
23    shuffle(per.begin(), per.end(), rng);
24    return 0;
25 }
```

1.5 Manacher_palindrome

```

// pal[1][i] = longest odd (half rounded down)
// palindrome around pos i and starts at i - pal[1][i]
// and ends at i + pal[1][i]
// pal[0][i] = half length of longest even palindrome
// around pos i, i + 1 and starts at i - par[0][i] + 1
// and ends at i + pal[0][i]
const int N = 5e5+10;
int pal[2][N];
void manacher(string &s) {
    int n = s.size(), idx = 2;
    while (idx--) {
        for (int l=-1, r=-1, i=0; i<n-1; ++i){
            if (i > r) l = r = i;
            else {
                int k = min(r-i, pal[idx][l+r-i]);
                l = i - k, r = i + k;
            }
            while (l - idx >= 0 and r + 1 < n and s[l - idx]
                == s[r + 1]) l--, r++;
            pal[idx][i] = r - i;
            // [l - 1 + idx : r] palindrome
        }
    }
}
```

1.6 MatExpo

```

const ll mod = 1e9;
vector<vector<ll>> matMul(vector<vector<ll>> &a,
    vector<vector<ll>> &b){
    ll row1 = a.size(), col1 = a[0].size();
    ll row2 = b.size(), col2 = b[0].size();
    vector<vector<ll>> res(row1, vector<ll> (col2, 0));
    for(ll i = 0; i<row1; i++){
        for(ll j = 0; j<col1; j++){
            for(ll k = 0; k<row2; k++){
                res[i][j]=(res[i][j] + (1LL *
                    a[i][k]*b[k][j])%mod)%mod;
            }
        }
    }
    return res;
}
vector<vector<ll>> matExpo(vector<vector<ll>> &Mat, ll
    exp){
    ll row = Mat.size(), col = Mat[0].size(); ll p =
        row;
    vector<vector<ll>> res(p, vector<ll> (p, 0));
    for(ll i = 0; i<p; i++) res[i][i] = 1;
    while(exp){
        if(exp&1) res = matMul(res, Mat);
        Mat = matMul(Mat, Mat); exp>>=1;
    }
    return res;
}
//b = (A(i), A(i-1), A(i-2), A(i-3))
//M = Magic matrix, nth = nth term, known = known value
ll get_nth(ll nth, ll known, vector<ll> &b,
    vector<vector<ll>> &M){
    if(nth<known) return b[nth-1]%mod;
    reverse(b.begin(), b.end());
    vector<vector<ll>> me = matExpo(M, nth-known);
    ll ans = 0;
    for(int i = 0; i<known; i++){
        ans = (ans + (b[i] * me[i][0])) % mod);
    }
    return ans;
}
```

1.7 SOD_NOD

```
//SOD = ((P^(x+1)-1)/(P-1)) * ((Q^(y+1)-1)/(Q-1)) *
      ((R^(z+1)-1)/(R-1))
//NOD = P^x * Q^y * R^z => here, P, Q, R are prime
// factors & x, y, z are powers
//NOD = (x + 1) (y + 1) (z + 1)
pair<int, int> SOD_NOD(int n) {
    int sod = 1, nod = 1;
    for (int i = 2; i * i <= n; ++i) {
        if (n % i == 0) {
            int pown = 1, pows = 0;
            while (n % i == 0) {
                pown *= i; // p^e
                pows++; n /= i;
            }
            pown *= i; // p^e+1
            sod *= (pown - 1) / (i - 1); // (p^e+1)-1 /
                                         // p-1
            nod *= (pows + 1);
        }
    }
    if (n > 1){sod *= (n + 1); nod *= 2;}
    return {sod, nod};
}
```

1.8 StrHash

```
const int mod1 = 911382323, mod2 = 972663749, b1 =
    137, b2 = 139;
const int mxN = 5000010;
int pow_b1[mxN], pow_b2[mxN], inv_b1[mxN], inv_b2[mxN];
int binExp(int base, int power, int mod){
    int res = 1;
    while(power){
        if(power&1) res = (1LL * res * base)%mod;
        base = (1LL * base * base) % mod; power>>=1;
    } return res;
}
void pre(){
    pow_b1[0] = pow_b2[0] = 1;
    for(int i = 1; i<mxN; i++){
        pow_b1[i] = (1LL * pow_b1[i-1]*b1)%mod1;
        pow_b2[i] = (1LL * pow_b2[i-1]*b2)%mod2;
    }
    inv_b1[mxN-1] = binExp(pow_b1[mxN-1], mod1-2,
                           mod1);
    inv_b2[mxN-1] = binExp(pow_b2[mxN-1], mod2-2,
                           mod2);
    for(int i = mxN-2; i>=0; i--){
        inv_b1[i] = (1LL * inv_b1[i+1] * b1)%mod1;
        inv_b2[i] = (1LL * inv_b2[i+1] * b2)%mod2;
    }
}
vector<pair<int, int>> getPref(string &s){
    int qq = s.size();
    vector<pair<int, int>> hsh(qq);
    for(int i = 0; i<qq; i++){
        if(i==0){
            hsh[i].first = (1LL * s[i] * pow_b1[i]) %
                           mod1;
            hsh[i].second = (1LL * s[i] * pow_b2[i]) %
                           mod2;
        } else{
            hsh[i].first = (hsh[i-1].first + (1LL *
                                              s[i] * pow_b1[i])%mod1)%mod1;
            hsh[i].second = (hsh[i-1].second + (1LL *
                                              s[i] * pow_b2[i])%mod2)%mod2;
        }
    }
    return hsh;
}
pair<int,int> getHash(string &str){
    int hsh1 = 0, hsh2 = 0, sz = str.size();
```

```
for(int i = 0; i < sz; ++i){
    hsh1 = (hsh1 + 1LL * str[i] * pow_b1[i] %
             mod1) % mod1;
}
for(int i = 0; i < sz; ++i){
    hsh2 = (hsh2 + 1LL * str[i] * pow_b2[i] %
             mod2) % mod2;
}
return {hsh1, hsh2};
```

1.9 StrHash_2

```
const int N = 1000010, MOD = 1e9 + 7;
const ll P[] = {97, 1000003};
ll bigMod (ll a, ll e) {
    if (e == -1) e = MOD - 2;
    ll ret = 1;
    while (e) {
        if (e & 1) ret = ret * a % MOD;
        a = a * a % MOD, e >>= 1;
    } return ret;
}
ll pwr[2][N], inv[2][N];
void initHash() {
    for (int it = 0; it < 2; ++it) {
        pwr[it][0] = inv[it][0] = 1;
        ll INV_P = bigMod(P[it], -1);
        for (int i = 1; i < N; ++i) {
            pwr[it][i] = pwr[it][i - 1] * P[it] % MOD;
            inv[it][i] = inv[it][i - 1] * INV_P % MOD;
        }
    }
}
struct RangeHash {
    vector<ll> h[2], rev[2];
    RangeHash (const string S, bool revFlag = 0) {
        for (int it = 0; it < 2; ++it) {
            h[it].resize(S.size() + 1, 0);
            for (int i = 0; i < S.size(); ++i) {
                h[it][i + 1] = (h[it][i] + pwr[it][i + 1] *
                                 (S[i] - 'a' + 1)) % MOD;
            }
        }
        if(revFlag){
            rev[it].resize(S.size() + 1, 0);
            for (int i = 0; i < S.size(); ++i) {
                rev[it][i + 1] = (rev[it][i] + inv[it][i + 1] *
                                 (S[i] - 'a' + 1)) % MOD;
            }
        }
    }
    inline ll get (int l, int r) {
        ll one = (h[0][r + 1] - h[0][l]) * inv[0][l + 1] %
                 MOD;
        ll two = (h[1][r + 1] - h[1][l]) * inv[1][l + 1] %
                 MOD;
        if (one < 0) one += MOD; if (two < 0) two += MOD;
        return one << 31 | two;
    }
    inline ll getReverse (int l, int r) {
```

```
ll one = (rev[0][r + 1] - rev[0][l]) * pwr[0][r +
                                             1] % MOD;
ll two = (rev[1][r + 1] - rev[1][l]) * pwr[1][r +
                                             1] % MOD;
if (one < 0) one += MOD; if (two < 0) two += MOD;
return one << 31 | two;
}
```

1.10 allInOneNT

```
const int MAXN = 1e6 + 9;
typedef struct info {
    int lowest_prime = 0, greatest_prime = 0,
        distinct_prime = 0;
    int total_prime = 0, NOD = 0, SOD = 0;
} info;
info num[MAXN];
void preStore() {
    for (int i = 2; i < MAXN; i++) {
        int n = i; map<int, int> factors; // Key->Factor,
                                         Val->count
        int SOD = 1, NOD = 1, total_p_factor = 0;
        if (n % 2 == 0) {
            while (n % 2 == 0) {n /= 2;
                factors[2]++;
                total_p_factor++;
            }
            SOD *= (1 << (factors[2] + 1)) - 1;
            NOD *= (factors[2] + 1);
        }
        for (int i = 3; i * i <= n; i += 2) {
            if (n % i == 0) {
                while (n % i == 0) {
                    n /= i; factors[i]++;
                    total_p_factor++;
                }
                SOD *= (pow(i, factors[i] + 1) - 1) / (i - 1);
                NOD *= (factors[i] + 1);
            }
        }
        if (n > 1) {factors[n]++;
            SOD *= (pow(n, 2) - 1) / (n - 1);
            NOD *= 2; total_p_factor++;
        }
    }
    num[i].distinct_prime = factors.size();
    num[i].total_prime = total_p_factor;
    num[i].NOD = NOD; num[i].SOD = SOD;
    auto lowest_prime = factors.begin();
    auto greatest_prime = factors.rbegin();
    num[i].lowest_prime = lowest_prime->first;
    num[i].greatest_prime = greatest_prime->first;
}
```

1.11 customHash

```
#include<ext/pb_ds/assoc_container.hpp>
using namespace __gnu_pbds;
struct customHash{
    static uint64_t Meaw(uint64_t x){
        x += 0x9e3779b97f4a7c15;
        x = (x^(x>>30)) * 0xbff58476d1ce4e5b9;
        x = (x^(x>>27)) * 0x94d0049bb13311leb;
        return x^(x>>31);
    }
    size_t operator()(uint64_t x) const{
        static const uint64_t FIXED_RANDOM =
            chrono::steady_clock::now().time_since_epoch()
            .count();
        return Meaw(x+FIXED_RANDOM);
    }
}; //gp_hash_table<int, int> table;
```

1.12 divisorSieve

```
const int mxN = 1e5+10;
vector<int> divisors[mxN]; //  
void divisorSeive(){
    for(int i = 1; i < mxN; i++) {
        for(int j = i; j < mxN; j += i) {
            divisors[j].push_back(i);
        }
    }
}
```

1.13 int128

```
_int128 read() {
    _int128 x = 0, f = 1;
    char ch = getchar();
    while (ch < '0' || ch > '9') {
        if (ch == '-') f = -1;
        ch = getchar();
    }
    while (ch >= '0' && ch <= '9') {
        x = x * 10 + ch - '0';
        ch = getchar();
    }
    return x * f;
}
void print(_int128 x) {
    if (x < 0) {
        putchar('-');
        x = -x;
    }
    if (x > 9) print(x / 10);
    putchar(x % 10 + '0');
}
```

1.14 lis

```
vector<int> lis(int n, vector<int> &v){
    vector<int> parent(n, -1), ind(n);
    vector<int> lis;
    for (int i = 0; i < n; i++) {
        int it = lower_bound(lis.begin(), lis.end(),
                            v[i]) - lis.begin();
        if (it == lis.size()) {
            lis.push_back(v[i]); ind[lis.size() - 1] =
                i;
            parent[i] = (lis.size() == 1 ? -1 : ind[it - 1]);
        } else {
            lis[it] = v[i]; ind[it] = i;
            parent[i] = (it == 0 ? -1 : ind[it - 1]);
        }
    }
    vector<int> LIS; int it = ind[lis.size() - 1];
    LIS.push_back(lis.back());
    while (parent[it] != -1) {
        it = parent[it]; LIS.push_back(v[it]);
    }
    return LIS;
}
```

1.15 mergeSort

```
//use array of elements, if multiple testcase make inv  
= 0 each time
//int inv = 0;
void merge(int vct[], int l, int m, int r){
    int left = m-l+1, right = r-m, lv[left], rv[right];
    for(int i = 0; i<left; i++){ lv[i] = vct[i+l]; }
    for(int i = 0; i<right; i++){ rv[i] = vct[m+i]; }
    int i = 0, j = 0, to = l;
    while(i<left && j<right){
        if(lv[i]<=rv[j]){
            vct[to] = lv[i]; i++;
        }
        else{
            vct[to] = rv[j]; j++;
            //inversion count
            //int pore = left-i; inv+=pore;
        }
        to++;
    }
    while(i<left){ vct[to] = lv[i]; i++; to++; }
    while(j<right){ vct[to] = rv[j]; j++; to++; }
}
```

```
else{
    vct[to] = rv[j]; j++;
    //inversion count
    //int pore = left-i; inv+=pore;
} to++;
}
void merge_sort(int vct[], int l, int r){
    if(r<l) return;
    int m = l+(r-l)/2;
    merge_sort(vct, l, m); merge_sort(vct, m+1, r);
    merge(vct, l, m, r);
}
```

1.16 pbds

```
#include<ext/pb_ds/assoc_container.hpp>
#include<ext/pb_ds/tree_policy.hpp>
#include<functional>
using namespace __gnu_pbds;
typedef tree<int, null_type, less<int>, rb_tree_tag,
    ~ tree_order_statistics_node_update> ordered_set;
// s.order_of_key(x) = number of elements strictly less
// than x
// *s.find_by_order(i) = ith element in set (0 index)
```

1.17 phi

```
int phi(int n) { //sqrt(n)
    int result = n;
    for (int i = 2; i * i <= n; i++) {
        if (n % i == 0) {
            while (n % i == 0) n /= i;
            result -= result / i;
        }
    }
    if (n > 1) result -= result / n;
    return result;
}
```

1.18 polynomial_interpolation

```
// P(x) = a0 + a1x + a2x^2 + ... + anx^n
// y[i] = P(i)
ll eval (vector<ll> y, ll k) {
    int n = y.size() - 1;
    if (k <= n) { return y[k]; }
    vector<ll> L(n + 1, 1);
    for (int x = 1; x <= n; ++x) {
        L[0] = L[0] * (k - x) % mod;
        L[0] = L[0] * inv(-x) % mod;
    }
    for (int x = 1; x <= n; ++x) {
        L[x] = L[x - 1] * inv(k - x) % mod * (k - (x - 1)) %
mod;
        L[x] = L[x] * ((x - 1) - n + mod) % mod * inv(x) %
mod;
    }
    ll yk = 0;
    for (int x = 0; x <= n; ++x){
        yk = add(yk, L[x] * y[x] % mod);
    }
}
```

1.19 segTree

```
const int N = 1e5 + 9;
int a[N];
struct ST {
    int t[4 * N];
    // static const int inf = 1e9;
    ST() {
        // memset(t, 0, sizeof t);
        t[0] = 0;
    }
}
```

```
void build(int n, int b, int e) {
    if (b == e) {
        t[n] = a[b];
        return;
    }
    int mid = (b + e) >> 1, l = n << 1, r = l | 1;
    build(l, b, mid);
    build(r, mid + 1, e);
    t[n] = t[l] + t[r];
}
void update(int n, int b, int e, int i, int x) {
    if (b > i || e < i) return;
    if (b == e && b == i) {
        t[n] = x;
        return;
    }
    int mid = (b + e) >> 1, l = n << 1, r = l | 1;
    update(l, b, mid, i, x);
    update(r, mid + 1, e, i, x);
    t[n] = t[l] + t[r];
}
int query(int n, int b, int e, int i, int j) {
    if (b > j || e < i) return 0;
    if (b >= i && e <= j) return t[n];
    int mid = (b + e) >> 1, l = n << 1, r = l | 1;
    int L = query(l, b, mid, i, j);
    int R = query(r, mid + 1, e, i, j);
    return L+R;
}
}; // Declare: ST sgt;
```

1.20 sieve

```
const ll MAXN = 1e7+10;
bool prime[MAXN];
vector<ll> prm;
void sieve(){
    prime[0] = prime[1] = true;
    for(ll i = 2; i<MAXN; i++){
        if(!prime[i]){
            prm.push_back(i);
            for(ll j = i+i; j<MAXN; j+=i){
                prime[j] = true;
            }
        }
    }
}
```

1.21 sparseTable

```
const int mxN = 1e5+10, M = 21;
int sparse[mxN][M];
void build_sparse(int n, vector<int> &v){
    for(int i = 0; i<n; i++) sparse[i][0] = v[i];
    for(int k = 1; k<M; k++){
        for(int i = 0; i+(1<<k)<=n; i++){
            sparse[i][k] = max(sparse[i][k-1],
                sparse[i+(1<<(k-1))][k-1]);
        }
    }
}
int query(int l, int r){ //0 based index
    if(l>r) swap(l, r); int b = __bit_width(r-l+1)-1;
    return max(sparse[l][b], sparse[r-(1<<b)+1][b]);
}
```

```

1.22 spf
const int MAXN = 1e6 + 2;
int spf[MAXN];
vector<int> prms;
void preStore() {
    for(int i = 1; i < MAXN; i++) spf[i] = i;
    for(int i = 2; i < MAXN; i++){
        if(spf[i]==i){
            prms.push_back(i);
            for(int j = i+i; j < MAXN; j+=i){
                spf[j] = min(spf[j], i);
            }
        }
    }
}

```

1.23 suffixArray

```

//fahimcp495
array<vector<int>, 2> get_sa(string& s, int lim=128) {
    // for integer, just change string to vector<int>
    // and minimum value of vector must be >= 1
    int n = s.size() + 1, k = 0, a, b;
    vector<int> x(begin(s), end(s)+1), y(n), sa(n),
    lcp(n), ws(max(n, lim)), rank(n);
    x.back() = 0;
    iota(begin(sa), end(sa), 0);
    for (int j = 0, p = 0; p < n; j = max(1, j * 2), lim
    = p) {
        p = j, iota(begin(y), end(y), n - j);
        for (int i = 0; i < n; ++i) if (sa[i] >= j) y[p++] =
        = sa[i] - j;
        fill(begin(ws), end(ws), 0);
        for (int i = 0; i < n; ++i) ws[x[i]]++;
        for (int i = 1; i < lim; ++i) ws[i] += ws[i - 1];
        for (int i = n; i--;) sa[-ws[x[y[i]]]] = y[i];
        swap(x, y), p = 1, x[sa[0]] = 0;
        for (int i = 1; i < n; ++i) a = sa[i - 1], b =
        sa[i], x[b] =
        (y[a] == y[b] && y[a + j] == y[b + j]) ? p - 1 :
        p++;
    }
    for (int i = 1; i < n; ++i) rank[sa[i]] = i;
    for (int i = 0, j; i < n - 1; lcp[rank[i++]] = k)
        for (k && k--, j = sa[rank[i] - 1]; s[i + k] ==
        s[j + k]; k++);
    sa.erase(sa.begin()), lcp.erase(lcp.begin());
    return {sa, lcp};
}

```

1.24 suffixAutomata

```

const int N = 2e5+10; // max string size
int len[N], lnk[N]{-1}, last, sz = 1;
unordered_map<char, int> to[N];
void add(char c) {
    int cur = sz++;
    len[cur] = len[last] + 1;
    int u = last;
    while (u != -1 and !to[u].count(c)) {
        to[u][c] = cur;
        u = lnk[u];
    }
    if (u == -1) {
        lnk[cur] = 0;
    } else {
        int v = to[u][c];
        if (len[v] == len[u] + 1) {
            lnk[cur] = v;
        } else {

```

```

            int w = sz++;
            len[w] = len[u] + 1, lnk[w] = lnk[v], to[w] =
            to[v];
            while (u != -1 and to[u][c] == v) {
                to[u][c] = w;
                u = lnk[u];
            }
            lnk[cur] = lnk[v] = w;
        }
        last = cur;
    }
}

```

1.25 suffixAutomation

```

int len[N], lnk[N]{-1}, last, sz = 1;
unordered_map<char, int> to[N];
void init() {
    while (sz) {
        sz--;
        to[sz].clear();
    }
    last = 0, sz = 1;
}
void add(char c) {
    int cur = sz++;
    int u = last;
    len[cur] = len[last] + 1;
    while (u != -1 and !to[u].count(c)) {
        to[u][c] = cur;
        u = lnk[u];
    }
    if (u == -1) { lnk[cur] = 0; }
    else {
        int v = to[u][c];
        if (len[v] == len[u] + 1) {
            lnk[cur] = v;
        } else {
            int w = sz++;
            len[w] = len[u] + 1, lnk[w] = lnk[v], to[w] =
            to[v];
            while (u != -1 and to[u][c] == v) {
                to[u][c] = w;
                u = lnk[u];
            }
            lnk[cur] = lnk[v] = w;
        }
        last = cur;
    }
}

```

1.26 trie

```

const ll N = 1e6+5, A = 26;
ll trie[N][A], cnt[N], tot = 1, root = 1;
void initTrie(){
    cnt[tot] = 0, root = 1;
}
void addStr(string &s){
    ll u = 1;
    for(auto it: s){
        ll n = it-'a';
        if(trie[u][n]==0){
            trie[u][n] = ++tot;
        }
        u = trie[u][n];
        cnt[u]++;
    }
}
ll wordCount(string &s){
    ll u = 1;
    for(auto it: s){
        int n = it-'a';
        if(trie[u][n]==0) return 0;
        u = trie[u][n];
    }
    return cnt[u];
}

```

2 Geometry

2.1 Convex Hull

```

vector<PT> convexHull (vector<PT> p) {
    int n = p.size(), m = 0;
    if (n < 3) return p;
}

```

```

vector<PT> hull(n + n);
sort(p.begin(), p.end(), [&] (PT a, PT b) {
    return (a.x==b.x? a.y<b.y: a.x<b.x);
});
for (int i = 0; i < n; ++i) {
    while (m > 1 and cross(hull[m - 2] - p[i], hull[m
    - 1] - p[i]) <= 0) --m;
    hull[m++] = p[i];
}
for (int i = n - 2, j = m + 1; i >= 0; --i) {
    while (m >= j and cross(hull[m - 2] - p[i], hull[m
    - 1] - p[i]) <= 0) --m;
    hull[m++] = p[i];
}
hull.resize(m - 1); return hull;
}

```

3 Notes

3.1 Geometry

3.1.1 Triangles

$$\text{Circumradius: } R = \frac{abc}{4A}, \text{ Inradius: } r = \frac{A}{s}$$

The area of a triangle using two sides and the included angle can be given as:

$$A = \frac{1}{2}ab \sin \angle C$$

Length of median (divides triangle into two equal-area triangles):
 $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

$$\text{Length of bisector (divides angles in two): } s_a = \sqrt{bc \left[1 - \left(\frac{a}{b+c} \right)^2 \right]}$$

$$\text{Law of tangents: } \frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}$$

3.1.2 Quadrilaterals

With side lengths a, b, c, d , diagonals e, f , diagonals angle θ , area A and magic flux $F = b^2 + d^2 - a^2 - c^2$:

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is 180° , $ef = ac + bd$, and $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$.

3.1.3 Spherical coordinates

$$x = r \sin \theta \cos \phi \quad r = \sqrt{x^2 + y^2 + z^2} \\ y = r \sin \theta \sin \phi \quad \theta = \arccos(z / \sqrt{x^2 + y^2 + z^2}) \\ z = r \cos \theta \quad \phi = \arctan(y/x)$$

3.1.4 Pick's Theorem:

Given a lattice polygon with non-zero area, we define: S as the area of the polygon, I as the number of integer-coordinate points strictly inside the polygon, B as the number of integer-coordinate points on the boundary of the polygon. Then, Pick's Theorem states:

$$S = I + \frac{B}{2} - 1$$

The number of lattice points on segments (x_1, y_1) to (x_2, y_2) is: $\gcd(\text{abs}(x_2 - x_1), \text{abs}(y_2 - y_1)) + 1$

3.1.5 Polygon

For a regular polygon with n sides and side length a , the circumradius R is given by:

$$R = \frac{a}{2 \sin\left(\frac{\pi}{n}\right)}$$

3.1.6 Area of a Circular Segment

The area of a circular segment, which is the region enclosed by a chord and the corresponding arc, can be calculated using the formula:

$$A = \frac{R^2}{2} (\theta - \sin\theta)$$

where: R is the radius of the circle, θ is the central angle subtended by the chord, in radians.

3.2 Binomial Coefficient

- Factoring in: $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$
- Sum over k : $\sum_{k=0}^n \binom{n}{k} = 2^n$
- Alternating sum: $\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$
- Even and odd sum: $\sum_{k=0}^n \binom{n}{2k} = \sum_{k=0}^n \binom{n}{2k+1} 2^{n-1}$
- The Hockey Stick Identity
 - (Left to right) Sum over n and k : $\sum_{k=0}^m \binom{n+k}{k} = \binom{n+m-1}{m}$
 - (Right to left) Sum over n : $\sum_{m=0}^n \binom{m}{k} = \binom{n+1}{k+1}$
- Sum of the squares: $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$
- Weighted sum: $\sum_{k=1}^n k \binom{n}{k} = n2^{n-1}$
- Connection with the fibonacci numbers: $\sum_{k=0}^n \binom{n-k}{k} = F_{n+1}$
- Vandermonde's Identity: $\sum_{i=0}^k \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$
- If $f(n, k) = C(n, 0) + C(n, 1) + \dots + C(n, k)$, Then $f(n+1, k) = 2 * f(n, k) - C(n, k)$ [For multiple $f(n, k)$ queries, use Mo's algo]

Lucas Theorem

$$\binom{m}{n} \equiv \prod_{i=0}^k \binom{m_i}{n_i} \pmod{p}$$

- $\binom{m}{n}$ is divisible by p if and only if at least one of the base- p digits of n is greater than the corresponding base- p digit of m .
- The number of entries in the n th row of Pascal's triangle that are not divisible by $p = \prod_{i=0}^k (n_i + 1)$
- All entries in the $(p^k - 1)$ th row are not divisible by p .
- $\binom{n}{m} \equiv \lfloor \frac{n}{p} \rfloor \pmod{p}$

3.3 Fibonacci Number

- $k = A - B, F_A F_B = F_{k+1} F_A^2 + F_k F_A F_{A-1}$
- $\sum_{i=0}^n F_i^2 = F_{n+1} F_n$
- $\sum_{i=0}^n F_i F_{i+1} = F_{n+1}^2 - (-1)^n$
- $\gcd(F_m, F_n) = F_{\gcd(m, n)}$
- $\gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+2}) = \gcd(F_n, F_{n+2}) = 1$
- $\sum_{i=0}^n F_i F_{i+1} = F_{n+1}^2 - (-1)^n$
- $\sum_{i=0}^n F_i F_{i+1} = \sum_{i=0}^{n-1} F_i F_{i+1}$
- $\gcd(F_m, F_n) = F_{\gcd(m, n)}$
- $7. \sum_{0 \leq k \leq n} \binom{n-k}{k} = F_{n+1}$
- $8. \gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+2}) = \gcd(F_{n+1}, F_{n+2}) = 1$

3.4 Sums

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$$\sum_{i=1}^n i^m = \frac{1}{m+1} \left[(n+1)^{m+1} - 1 - \sum_{i=1}^n ((i+1)^{m+1} - i^{m+1} - (m+1)i^m) \right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}$$

$$\sum_{k=0}^n kx^k = (x - (n+1)x^{n+1} + nx^{n+2})/(x-1)^2$$

3.5 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty)$$

$$(x+a)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k a^{-n-k}$$

Generating Function

$$1/(1-x) = 1 + x + x^2 + x^3 + \dots$$

$$1/(1-ax) = 1 + ax + (ax)^2 + (ax)^3 + \dots$$

$$1/(1-x)^2 = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$1/(1-x)^3 = C(2, 2) + C(3, 2)x + C(4, 2)x^2 + C(5, 2)x^3 + \dots$$

$$1/(1-ax)^{(k+1)} = 1 + C(1+k, k)(ax) + C(2+k, k)(ax)^2 + C(3+k, k)(ax)^3 + \dots$$

$$x(x+1)(1-x)^{-3} = 1 + x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + \dots$$

$$e^x = 1 + x + (x^2)/2! + (x^3)/3! + (x^4)/4! + \dots$$

3.6 Pythagorean Triples

The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), b = k \cdot (2mn), c = k \cdot (m^2 + n^2),$$

with $m > n > 0, k > 0, m \perp n$, and either m or n even.

3.7 Number Theory

- HCN: $1\text{e}6(240), 1\text{e}9(1344), 1\text{e}12(6720), 1\text{e}14(17280), 1\text{e}15(26880), 1\text{e}16(41472)$
- $\gcd(a, b, c, d, \dots) = \gcd(a, b-a, c-b, d-c, \dots)$
- $\gcd(a+k, b+k, c+k, d+k, \dots) = \gcd(a+k, b-a, c-b, d-c, \dots)$
- Primitive root exists iff $n = 1, 2, 4, p^k, 2 \times p^k$, where p is an odd prime.
- If primitive root exists, there are $\phi(\phi(n))$ primitive roots of n .
- The numbers from 1 to n have in total $O(n \log \log n)$ unique prime factors.
- $x \equiv r_1 \pmod{m_1}$ and $x \equiv r_2 \pmod{m_2}$ has a solution iff $\gcd(m_1, m_2)|(r_1 - r_2)$ Solution of $x^2 \equiv a \pmod{p}$
- $ca \equiv cb \pmod{m} \iff a \equiv b \pmod{\frac{n}{\gcd(n, c)}}$
- $ax \equiv b \pmod{m}$ has a solution $\iff \gcd(a, m)|b$
- If $ax \equiv b \pmod{m}$ has a solution, then it has $\gcd(a, m)$ solutions and they are separated by $\frac{m}{\gcd(a, m)}$
- $ax \equiv 1 \pmod{m}$ has a solution or a is invertible $\pmod{m} \iff \gcd(a, m) = 1$
- $x^2 \equiv 1 \pmod{p}$ then $x \equiv \pm 1 \pmod{p}$
- There are $\frac{p-1}{2}$ has no solution.
- There are $\frac{p-1}{2}$ has exactly two solutions.
- When $p \% 4 = 3, x \equiv \pm a^{\frac{p+1}{4}}$
- When $p \% 8 = 5, x \equiv a^{\frac{p+3}{8}}$ or $x \equiv 2^{\frac{p-1}{4}} a^{\frac{p+3}{8}}$

3.7.1 Primes

$p = 962592769$ is such that $2^{21} \mid p-1$, which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703050456749 (52-bit). There are 78498 primes less than 1 000 000.

Primitive roots exist modulo any prime power p^a , except for $p = 2, a > 2$, and there are $\phi(\phi(p^a))$ many. For $p = 2, a > 2$, the group $\mathbb{Z}_{2^a}^\times$ is instead isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$.

3.7.2 Estimates

$$\sum_{d \mid n} d = O(n \log \log n).$$

The number of divisors of n is at most around 100 for $n < 5e4$, 500 for $n < 1e7$, 2000 for $n < 1e10$, 200 000 for $n < 1e19$.

3.7.3 Perfect numbers

$n > 1$ is called perfect if it equals sum of its proper divisors and 1. Even n is perfect iff $n = 2^{p-1}(2^p - 1)$ and $2^p - 1$ is prime (Mersenne's). No odd perfect numbers are yet found.

3.7.4 Carmichael numbers

A positive composite n is a Carmichael number ($a^{n-1} \equiv 1 \pmod{n}$ for all $a: \gcd(a, n) = 1$), iff n is square-free, and for all prime divisors p of n , $p-1$ divides $n-1$.

3.7.5 Totient

- If p is a prime ($p^k = p^k - p^{k-1}$)
- If a, b are relatively prime, $\phi(ab) = \phi(a)\phi(b)$
- $\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})(1 - \frac{1}{p_3}) \dots (1 - \frac{1}{p_k})$
- Sum of coprime to $n = n * \frac{\phi(n)}{2}$
- If $n = 2^k, \phi(n) = 2^{k-1} = \frac{n}{2}$
- For $a, b, \phi(ab) = \phi(a)\phi(b)\frac{d}{\phi(d)}$
- $\phi(ip) = p\phi(i)$ whenever p is a prime and it divides i
- The number of $a (1 \leq a \leq N)$ such that $\gcd(a, N) = d$ is $\phi(\frac{n}{d})$
- If $n > 2, \phi(n)$ is always even
- Sum of gcd, $\sum_{i=1}^n \gcd(i, n) = \sum_{d|n} d\phi(\frac{n}{d})$
- Sum of lcm, $\sum_{i=1}^n nlcm(i, n) = \frac{n}{2}(\sum_{d|n} d\phi(d)) + 1$
- $\phi(1) = 1$ and $\phi(2) = 1$ which two are only odd ϕ
- $\phi(3) = 2$ and $\phi(4) = 2$ and $\phi(6) = 2$ which three are only prime ϕ
- Find minimum n such that $\frac{\phi(n)}{n}$ is maximum- Multiple of small primes- $2 * 3 * 5 * 7 * 11 * 13 * \dots$

3.7.6 Möbius function

$\mu(1) = 1, \mu(n) = 0$, if n is not squarefree. $\mu(n) = (-1)^s$, if n is the product of s distinct primes. Let f, F be functions on positive integers. If for all $n \in N, F(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} \mu(d)F(\frac{n}{d})$, and vice versa. $\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}, \sum_{d|n} \mu(d) = 1$.

If f is multiplicative, then $\sum_{d|n} \mu(d)f(d) = \prod_{p|n} (1 - f(p)), \sum_{d|n} \mu(d)^2 f(d) = \prod_{p|n} (1 + f(p))$.

$$\sum_{i=1}^n \sum_{j=1}^n [\gcd(i, j) = 1] = \sum_{k=1}^n \mu(k) \lfloor \frac{n}{k} \rfloor^2$$

$$\sum_{i=1}^n \sum_{j=1}^n \gcd(i, j) = \sum_{k=1}^n k \sum_{l=1}^{\lfloor \frac{n}{k} \rfloor} \mu(l) \lfloor \frac{n}{kl} \rfloor^2$$

$$\sum_{i=1}^n \sum_{j=1}^n \gcd(i, j) = \sum_{k=1}^n (\frac{\lfloor \frac{n}{k} \rfloor (\lfloor \frac{n}{k} \rfloor + 1)}{2})^2 \sum_{d|k} \mu(d)kd$$

3.7.7 Legendre symbol

If p is an odd prime, $a \in \mathbb{Z}$, then $\left(\frac{a}{p}\right)$ equals 0, if $p|a$; 1 if a is a quadratic residue modulo p ; and -1 otherwise. Euler's criterion: $\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}} \pmod{p}$.

3.7.8 Jacobi symbol

If $n = p_1^{a_1} \cdots p_k^{a_k}$ is odd, then $\left(\frac{a}{n}\right) = \prod_{i=1}^k \left(\frac{a}{p_i}\right)^{k_i}$.

3.7.9 Primitive roots

If the order of g modulo m ($\min n > 0: g^n \equiv 1 \pmod{m}$) is $\phi(m)$, then g is called a primitive root. If Z_m has a primitive root, then it has $\phi(\phi(m))$ distinct primitive roots. Z_m has a primitive root iff m is one of 2, 4, p^k , $2p^k$, where p is an odd prime. If Z_m has a primitive root g , then for all a coprime to m , there exists unique integer $i = \text{ind}_g(a)$ modulo $\phi(m)$, such that $g^i \equiv a \pmod{m}$. $\text{ind}_g(a)$ has logarithm-like properties: $\text{ind}(1) = 0$, $\text{ind}(ab) = \text{ind}(a) + \text{ind}(b)$.

If p is prime and a is not divisible by p , then congruence $x^n \equiv a \pmod{p}$ has $\gcd(n, p-1)$ solutions if $a^{(p-1)/\gcd(n, p-1)} \equiv 1 \pmod{p}$, and no solutions otherwise. (Proof sketch: let g be a primitive root, and $g^i \equiv a \pmod{p}$, $g^u \equiv x \pmod{p}$. $x^n \equiv a \pmod{p}$ iff $g^{nu} \equiv g^i \pmod{p}$ iff $nu \equiv i \pmod{p}$.)

3.7.10 Discrete logarithm problem

Find x from $a^x \equiv b \pmod{m}$. Can be solved in $O(\sqrt{m})$ time and space with a meet-in-the-middle trick. Let $n = \lceil \sqrt{m} \rceil$, and $x = ny - z$. Equation becomes $a^{ny} \equiv ba^z \pmod{m}$. Precompute all values that the RHS can take for $z = 0, 1, \dots, n-1$, and brute force y on the LHS, each time checking whether there's a corresponding value for RHS.

3.7.11 Pythagorean triples

Integer solutions of $x^2 + y^2 = z^2$ All relatively prime triples are given by: $x = 2mn, y = m^2 - n^2, z = m^2 + n^2$ where $m > n, \gcd(m, n) = 1$ and $m \not\equiv n \pmod{2}$. All other triples are multiples of these. Equation $x^2 + y^2 = 2z^2$ is equivalent to $(\frac{x+y}{2})^2 + (\frac{x-y}{2})^2 = z^2$.

3.7.12 Postage stamps/McNuggets problem

Let a, b be relatively-prime integers. There are exactly $\frac{1}{2}(a-1)(b-1)$ numbers not of form $ax + by$ ($x, y \geq 0$), and the largest is $(a-1)(b-1) - 1 = ab - a - b$.

3.7.13 Fermat's two-squares theorem

Odd prime p can be represented as a sum of two squares iff $p \equiv 1 \pmod{4}$. A product of two sums of two squares is a sum of two squares. Thus, n is a sum of two squares iff every prime of form $p = 4k + 3$ occurs an even number of times in n 's factorization.

3.8 Permutations**3.8.1 Factorial**

$n!$	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800
$n!$	11	12	13	14	15	16	17			
$n!$	4.0e7	4.8e8	6.2e9	8.7e10	1.3e12	2.1e13	3.6e14			
$n!$	20	25	30	40	50	100	150	171		
$n!$	2e18	2e25	3e32	8e47	3e64	9e157	6e262	>DBL_MAX		

3.8.2 Cycles

Let $g_S(n)$ be the number of n -permutations whose cycle lengths all belong to the set S . Then

$$\sum_{n=0}^{\infty} g_S(n) \frac{x^n}{n!} = \exp \left(\sum_{n \in S} \frac{x^n}{n} \right)$$

3.8.3 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1) + D(n-2)) = nD(n-1) + (-1)^n = \left\lfloor \frac{n!}{e} \right\rfloor$$

3.8.4 Burnside's lemma

Given a group G of symmetries and a set X , the number of elements of X up to symmetry equals

$$\frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where X^g are the elements fixed by g ($g \cdot x = x$).

If $f(n)$ counts "configurations" (of some sort) of length n , we can ignore rotational symmetry using $G = \mathbb{Z}_n$ to get

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n, k)) = \frac{1}{n} \sum_{k|n} f(k)\phi(n/k)$$

3.9 Partitions and subsets**3.9.1 Partition function**

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k-1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

n	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	~2e5	~2e8

3.9.2 Partition Number

- Time Complexity: $O(n\sqrt{n})$

```
for (int i = 1; i <= n; ++i) {
    pent[2 * i - 1] = i * (3 * i - 1) / 2;
    pent[2 * i] = i * (3 * i + 1) / 2;
}
p[0] = 1;
for (int i = 1; i <= n; ++i) {
    p[i] = 0;
    for (int j = 1, k = 0; pent[j] <= i; ++j) {
        if (k < 2) p[i] = add(p[i], p[i - pent[j]]);
        else p[i] = sub(p[i], p[i - pent[j]]); ++k, k &= 3;
    }
}
```

- The number of partitions of a positive integer n into exactly k parts equals the number of partitions of n whose largest part equals k

$$p_k(n) = p_k(n-k) + p_{k-1}(n-1)$$

3.9.3 2nd Kaplansky's Lemma

The number of ways of selecting k objects, no two consecutive, from n labelled objects arrayed in a circle is $\frac{n}{k} \binom{n-k-1}{k-1} = \frac{n}{n-k} \binom{n}{k}$

3.9.4 Distinct Objects into Distinct Bins

- n distinct objects into r distinct bins = r^n
- Among n distinct objects, exactly k of them into r distinct bins = $\binom{n}{k} r^k$
- n distinct objects into r distinct bins such that each bin contains at least one object = $\sum_{i=0}^r (-1)^i \binom{r}{i} (r-i)^n$

3.10 Coloring

The number of labeled undirected graphs with n vertices, $G_n = 2^{\binom{n}{2}}$

The number of labeled directed graphs with n vertices, $G_n = 2^{n(n-1)}$

The number of connected labeled undirected graphs with n vertices, $C_n = 2^{\binom{n}{2}} - \frac{1}{n} \sum_{k=1}^{n-1} k \binom{n}{k} 2^{\binom{n-k}{2}} C_k = 2^{\binom{n}{2}} - \sum_{k=1}^{n-1} \binom{n-1}{k-1} 2^{\binom{n-k}{2}} C_k$

The number of k -connected labeled undirected graphs with n vertices, $D[n][k] = \sum_{s=1}^n \binom{n-1}{s-1} C_s D[n-s][k-1]$

Cayley's formula: the number of trees on n labeled vertices = the number of spanning trees of a complete graph with n labeled vertices = n^{n-2}

Number of ways to color a graph using k colors such that no two adjacent nodes have same color

Complete graph = $k(k-1)(k-2) \dots (k-n+1)$

Tree = $k(k-1)^{n-1}$

Cycle = $(k-1)^n + (-1)^n (k-1)$

Number of trees with n labeled nodes: n^{n-2}

3.11 General purpose numbers

3.11.1 Eulerian numbers

Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j:s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j:s s.t. $\pi(j) \geq j$, k j:s s.t. $\pi(j) > j$.

$$E(n, k) = (n - k)E(n - 1, k - 1) + (k + 1)E(n - 1, k)$$

$$E(n, 0) = E(n, n - 1) = 1$$

$$E(n, k) = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n$$

3.11.2 Bell numbers

Total number of partitions of n distinct elements. $1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \dots$. For p prime,

$$B(p^m + n) \equiv mB(n) + B(n+1) \pmod{p}$$

3.11.3 Bernoulli numbers

$$\sum_{j=0}^m \binom{m+1}{j} B_j = 0. \quad B_0 = 1, B_1 = -\frac{1}{2}. \quad B_n = 0, \text{ for all odd } n \neq 1.$$

3.11.4 Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1, C_{n+1} = \frac{2(2n+1)}{n+2} C_n, C_{n+1} = \sum C_i C_{n-i}$$

- $C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$
- sub-diagonal monotone paths in an $n \times n$ grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with $n+1$ leaves (0 or 2 children).
- ordered trees with $n+1$ vertices.
- ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.
- permutations of $[n]$ with no 3-term increasing subseq.
- Find the count of balanced parentheses sequences consisting of $n+k$ pairs of parentheses where the first k symbols are open brackets.

$$C_n^{(k)} = \frac{k+1}{n+k+1} \binom{2n+k}{n}$$

- Recursive formula of Catalan Numbers:

$$C_n^{(k)} = \frac{(2n+k-1) \cdot (2n+k)}{n \cdot (n+k+1)} C_{n-1}^{(k)}$$

3.11.5 Lucas Number

Number of edge cover of a cycle graph C_n is L_n

$$L(n) = L(n-1) + L(n-2); L(0) = 2, L(1) = 1$$

3.12 Ballot Theorem

Suppose that in an election, candidate A receives a votes and candidate B receives b votes, where $a \geq b$ for some positive integer k . Compute the number of ways the ballots can be ordered so that A maintains more than k times as many votes as B throughout the counting of the ballots.

The solution to the ballot problem is $\frac{a-kb}{a+b} \times C(a+b, a)$

3.13 Classical Problem

$F(n, k)$ = number of ways to color n objects using exactly k colors
Let $G(n, k)$ be the number of ways to color n objects using no more than k colors.
Then, $F(n, k) = G(n, k) - C(k, 1)*G(n, k-1) + C(k, 2)*G(n, k-2) - C(k, 3)*G(n, k-3) \dots$

Determining $G(n, k)$:

Suppose, we are given a $1 \times n$ grid. Any two adjacent cells can not have same color. Then, $G(n, k) = k * ((k-1)^{n-1})$
If no such condition on adjacent cells. Then, $G(n, k) = k^n$

3.14 Matching Formula

3.14.1 Normal Graph

$MM + MEC = n$ (excluding vertex), $IS + VC = G$, $MIS + MVC = G$

3.14.2 Bipartite Graph

$MIS = n - MBM$, $MVC = MBM$, $MEC = n - MBM$

3.15 Inequalities

3.15.1 Titu's Lemma

For positive reals a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n ,

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geq \frac{a_1 + a_2 + \dots + a_n}{b_1 + b_2 + \dots + b_n}$$

Equality holds if and only if $a_i = kb_i$ for a non-zero real constant k .

3.16 Games

3.16.1 Grundy numbers

For a two-player, normal-play (last to move wins) game on a graph (V, E) : $G(x) = \text{mex}(\{G(y) : (x, y) \in E\})$, where $\text{mex}(S) = \min\{n \geq 0 : n \notin S\}$. x is losing iff $G(x) = 0$.

3.16.2 Sums of games

- Player chooses a game and makes a move in it Grundy number of a position is xor of grundy numbers of positions in summed games.
- Player chooses a non-empty subset of games (possibly, all) and makes moves in all of them A position is losing iff each game is in a losing position.
- Player chooses a proper subset of games (not empty and not all), and makes moves in all chosen ones. A position is losing iff grundy numbers of all games are equal.
- Player must move in all games, and loses if can't move in some game A position is losing if any of the games is in a losing position.

3.16.3 Misère Nim

A position with pile sizes $a_1, a_2, \dots, a_n \geq 1$, not all equal to 1, is losing iff $a_1 \oplus a_2 \oplus \dots \oplus a_n = 0$ (like in normal nim.) A position with n piles of size 1 is losing iff n is odd.

3.17 Tree Hashing

$$f(u) = sz[u] * \sum_{i=0}^n f(v) * p^i; f(v) \text{ are sorted } f(\text{child}) = 1$$

3.18 Permutation

To maximize the sum of adjacent differences of a permutation, it is necessary and sufficient to place the smallest half numbers in odd position and the greatest half numbers in even position. Or, vice versa.

3.19 String

- If the sum of length of some strings is N , there can be at most \sqrt{N} distinct length.
- A Text can have at most $O(N \times \sqrt{N})$ distinct substrings that match with given patterns where the sum of the length of the given patterns is N .
- Period = $n \% (n - pi.back() == 0)? n - pi.back(): n$

- The first (*period*) cyclic rotations of a string are distinct. Further cyclic rotations repeat the previous strings.

- S is a palindrome if and only if its period is a palindrome.

- If S and T are palindromes, then the periods of S and T are same if and only if $S + T$ is a palindrome.

3.20 Bit

- (a xor b) and (a + b) has the same parity
- $(a + b) = (a \text{ xor } b) + 2(a \text{ and } b)$
- $\text{gcd}(a, b) \leq a - b \leq \text{xor}(a, b)$

3.21 Convolution

- Hamming Distance: Replace 0 with -1 - SQRT Decomposition: Find block size, $B = \sqrt{(8 * n)}$