



Daffodil International University

DIU_DividedByZero

khun_, tasnim07, kazi_amir

Team Reference Document

Contents

1 Code

1.1

CP_Ubuntu

1.2

CP_Windows

1.3 StressTesting(check.sh)

1.4 StressTesting(gen.cpp)

1.5 Articulation Point

1.6 Binary Lifting using LCA

1.7 BridgeFinding

1.8 DAGCycleDetection

1.9 DSU

1.10 DSUOnTrees

1.11 Euler Tour

1.12 FloydWarshall

1.13 MST

1.14 Manacher_palindrome

1.15 MatExpo

1.16 Max Bipartite Matching [Hopcroft Karp]

1.17 Max Bipartite Matching [Kuhn's]

1.18 SOD_NOD

1.19 SparseTable

1.20 StrHash

1.21 StrHash_2

1.22 TopologicalSorting

1.23 UniquePF of all elements till MX

1.24 WeightedUnionFind

1.25 allInOneNT

1.26 customHash

1.27 divisorSieve

1.28 int128

1.29 lis

1.30 mergeSort

1.31 pbds

1.32 phi

1.33 polynomial_interpolation

1.34 segTree

1.35 sieve

1.36 spf

1.37 suffixArray

1.38 suffixAutomata

1.39 suffixAutomation

1.40 trie

2 Geometry

2.1 Convex Hull

3 Notes

3.1 Geometry

3.2 Binomial Coefficient

3.3 Fibonacci Number

3.4 Sums

3.5 Series

3.6 Pythagorean Triples

3.7 Number Theory

3.8 Permutations

3.9 Partitions and subsets

3.10 Coloring

3.11 General purpose numbers

3.12 Ballot Theorem 10
 3.13 Classical Problem 10
 3.14 Matching Formula 10
 3.15 Inequalities 10
 3.16 Games 10
 3.17 Tree Hashing 11
 3.18 Permutation 11
 3.19 String 11
 3.20 Bit 11
 3.21 Convolution 11

1 Code

1.1

CP_Ubuntu

```
{
  "cmd": ["ulimit -s 268435456; g++ -std=c++20
    $file name -o $file_base_name && timeout
    4s ./ $file_base_name < inputf.in >
    outputf.in"],
  "selector": "source.cpp",
  "shell": true,
  "working_dir": "$file_path"
}
```

1.2

CP_Windows

```
{
  "cmd": ["g++.exe", "-std=c++20", "${file}",
    "-o", "${file_base_name}.exe", "&&", "${f}
    ile_base_name}.exe<inputf.in>outputf.in"],
  "selector": "source.cpp",
  "shell": true,
  "working_dir": "$file_path"
}
```

1.3 StressTesting(check.sh)

```
// chmod u+x check.sh
// ./check.sh
set -e
g++ gen.cpp -o gen
g++ code.cpp -o code
g++ brute.cpp -o brute
for ((i = 1; ; ++i)); do
  echo "Passed on TestCase: " $i
  ./gen $i > in
  ./code < in > out1
  ./brute < in > out2
  diff -Z out1 out2 || break
done
echo -e "WA on the following test:"
cat in
echo -e "\nExpected:"
cat out2
echo -e "\nFound:"
cat out1
```

1.4 StressTesting(gen.cpp)

```
#include <bits/stdc++.h>
using namespace std;
using ll = long long;
mt19937_64 rng(chrono::steady_clock::now().time_
  _ since_epoch().count());
inline ll gen_random(ll l, ll r) {
  return uniform_int_distribution<ll>(l, r)(rng);
}
inline double gen_random_real(double l, double
  r) {
  return uniform_real_distribution<double>(l,
    r)(rng);
}
int main(int argc, char* args[]) {
  int _ = atoi(args[1]);
  rng.seed(_);
  int n = gen_random(1, 5);
  vector<int> per;
  for (int i = 0; i < n; ++i) {
    per.push_back(i + 1);
  }
  shuffle(per.begin(), per.end(), rng);
  return 0;
}
```

1.5 Articulation Point

```
int n; // number of nodes
vector<vector<int>> lst; // adjacency list of
  graph
vector<bool> vis;
vector<int> tin, low;
int timer;
void dfs(int u, int p = -1) {
  vis[u] = true;
  tin[u] = low[u] = timer++;
  int children = 0;
  for (int v : lst[u]) {
    if (v == p) continue;
    if (vis[v]) {
      low[u] = min(low[u], tin[v]);
    } else {
      dfs(v, u);
      low[u] = min(low[u], low[v]);
      if (low[v] >= tin[u] && p != -1)
        IS_CUTPOINT(u);
      ++children;
    }
  }
  // if no vertex below v can reach u or higher
  // removing u disconnects that subtree
  if (p == -1 && children > 1)
    IS_CUTPOINT(u);
}
void find_cutpoints() {
  timer = 0;
  vis.assign(n, false);
  tin.assign(n, -1);
  low.assign(n, -1);
  for (int i = 0; i < n; ++i) {
    if (!vis[i])
      dfs(i);
  }
}
```

```

}

1.6 Binary Lifting using LCA

int n, l;
vector<vector<int>> adj;
int timer;
vector<int> tin, tout;
vector<vector<int>> up;
void dfs(int v, int p) {
    tin[v] = ++timer;
    up[v][0] = p;
    for (int i = 1; i <= l; ++i)
        up[v][i] = up[up[v][i-1]][i-1];
    for (int u : adj[v]) {
        if (u != p)
            dfs(u, v);
    }
    tout[v] = ++timer;
}
bool is_ancestor(int u, int v) {
    return tin[u] <= tin[v] && tout[u] >= tout[v];
}
int lca(int u, int v) {
    if (is_ancestor(u, v))
        return u;
    if (is_ancestor(v, u))
        return v;
    for (int i = l; i >= 0; --i) {
        if (!is_ancestor(up[u][i], v))
            u = up[u][i];
    }
    return up[u][0];
}
void preprocess(int root) {
    tin.resize(n);
    tout.resize(n);
    timer = 0;
    l = ceil(log2(n));
    up.assign(n, vector<int>(l + 1));
    dfs(root, root);
}

```

1.7 BridgeFinding

```

const int MX = 1e5 + 10;
int n, m, timer = 0;
vector<int> adj[MX];
vector<int> tin(MX, -1), low(MX, -1);
vector<bool> vis(MX, false);
void is_bridge(int u, int v) {
    // do something with the edge
}
void dfs(int u, int p = -1) {
    vis[u] = true;
    tin[u] = low[u] = timer++;
    for (int v : adj[u]) {
        if (v == p) continue;
        if (vis[v]) {
            low[u] = min(low[u], tin[v]);
        } else {
            dfs(v, u);
            low[u] = min(low[u], low[v]);
            if (low[v] > tin[u]) {
                is_bridge(u, v);
            }
        }
    }
}

```

```

}

1.8 DAGCycleDetection

const int MX = 1e5 + 10;
bool vis[MX], pathVis[MX];
vector<int> lst[MX];
bool dfs(int u) {
    vis[u] = true;
    pathVis[u] = true;
    for (auto v : lst[u]) {
        if (!vis[v]) {
            if (dfs(v))
                return true;
        } else if (pathVis[v]) {
            return true;
        }
    }
    pathVis[u] = false;
    return false;
}
void solve() {
    // take graph input
    for (int i = 0; i < n; ++i) {
        if (!vis[i])
            dfs(i);
    }
}

```

1.9 DSU

```

const int MX = 1e5 + 10;
int par[MX], sz[MX];
void init() {
    for (int i = 1; i < MX; i++) {
        par[i] = i;
        sz[i] = 1;
    }
}
int findpar(int x) {
    if (par[x] == x) return x;
    return par[x] = findpar(par[x]);
}
void unite(int u, int v) {
    u = findpar(u);
    v = findpar(v);
    if (u != v) {
        if (sz[u] < sz[v]) {
            swap(u, v);
        }
        sz[u] += sz[v];
        par[v] = u;
    }
}

```

1.10 DSUOnTrees

```

int n, color[MX], ans[MX];
vector<int> g[MX];
set<int> bucket[MX];
int merge(int a, int b) {
    if (bucket[a].size() < bucket[b].size())
        swap(a, b);
    bucket[a].insert(bucket[b].begin(),
        bucket[b].end());
    bucket[b].clear();
    return a;
}

```

```

}
int dfs(int u, int p = -1) {
    int cur = u;
    for (int v : g[u])
        if (v != p)
            cur = merge(cur, dfs(v, u));
    ans[u] = (int)bucket[cur].size();
    return cur;
}
void solve() {
    cin >> n;
    for (int i = 0; i < n; ++i) {
        cin >> color[i];
        bucket[i].insert(color[i]);
    }
    // graph input
    dfs(0);
    // print output
}

```

1.11 Euler Tour

```

const int MX = 2e5 + 10;
int timer = -1;
// s = start pos, e = end pos
int val[MX], s[MX], e[MX], flat[MX];
vector<int> lst[MX];
void dfs(int u, int p) {
    s[u] = ++timer;
    flat[timer] = val[u];
    for (auto v : lst[u]) {
        if (v != p)
            dfs(v, u);
    }
    e[u] = timer;
}

```

1.12 FloydWarshall

```

vector<vector<int>> d(n, vector<int>(n, INF));
// take graph input into d
for (int k = 0; k < n; ++k) {
    for (int i = 0; i < n; ++i) {
        for (int j = 0; j < n; ++j) {
            if (d[i][k] < INF && d[k][j] < INF)
                d[i][j] = min(d[i][j], d[i][k] +
                    d[k][j]);
        }
    }
}

```

1.13 MST

```

// DSU first
void solve() {
    int n, m;
    cin >> n >> m;
    vector<tuple<int, int, int>> edges;
    for (int i = 0; i < m; ++i) {
        int u, v, wt;
        cin >> u >> v >> wt;
        edges.push_back({wt, u, v});
    }
    sort(edges.begin(), edges.end());
    init(n);
}

```

```

int cost = 0;
for (tuple& [ wt, u, v ] : edges) {
    if (findpar(u) == findpar(v)) continue;
    unite(u, v);
    cost += wt;
}
cout << cost << endl;
}

```

1.14 Manacher_palindrome

```

// pal[1][i] = longest odd (half rounded down)
// palindrome around pos i and
// starts at i - pal[1][i] and ends at i +
// pal[1][i] pal[0][i] = half length of
// longest even palindrome around pos i, i + 1
// and starts at i - par[0][i] + 1
// and ends at i + pal[0][i]
const int N = 5e5 + 10;
int pal[2][N];
void manacher(string& s) {
    int n = s.size(), idx = 2;
    while (idx--> 0) {
        for (int l = -1, r = -1, i = 0; i < n - 1;
            ++i) {
            if (i > r)
                l = r = i;
            else {
                int k = min(r - i, pal[idx][l + r - i]);
                l = i - k, r = i + k;
            }
            while (l - idx >= 0 and r + 1 < n and s[l]
                == s[r + 1]) l--, r++;
            pal[idx][i] = r - i;
            // [l - 1 + idx : r] palindrome
        }
    }
}

```

1.15 MatExpo

```

const ll mod = 1e9;
vector<vector<ll>> matMul(vector<vector<ll>>& a,
    vector<vector<ll>>& b) {
    ll row1 = a.size(), col1 = a[0].size();
    ll row2 = b.size(), col2 = b[0].size();
    vector<vector<ll>> res(row1, vector<ll>(col2,
        0));
    for (ll i = 0; i < row1; i++) {
        for (ll j = 0; j < col1; j++) {
            for (ll k = 0; k < row2; k++) {
                res[i][j] = (res[i][j] + (1LL * a[i][k]
                    * b[k][j]) % mod) % mod;
            }
        }
    }
    return res;
}
vector<vector<ll>> matExpo(vector<vector<ll>>&
    Mat, ll exp) {
    ll row = Mat.size(), col = Mat[0].size();
    ll p = row;
    vector<vector<ll>> res(p, vector<ll>(p, 0));
    for (ll i = 0; i < p; i++) res[i][i] = 1;
    while (exp) {
        if (exp & 1) res = matMul(res, Mat);
        Mat = matMul(Mat, Mat);
    }
}

```

```

exp >= 1;
}
return res;
}
// b = (A(i), A(i-1), A(i-2), A(i-3))
// M = Magic matrix, nth = nth term, known =
// known value
ll get_nth(ll nth, ll known, vector<ll>& b,
    vector<vector<ll>>& M) {
    if (nth <= known) return b[nth - 1] % mod;
    reverse(b.begin(), b.end());
    vector<vector<ll>> me = matExpo(M, nth -
        known); // MAT^(nth-known)
    ll ans = 0;
    for (int i = 0; i < known; i++) {
        ans = (ans + (b[i] * me[i][0]) % mod) % mod;
    }
    return ans;
}

```

1.16 Max Bipartite Matching [Hopcroft Karp]

```

const int INF = 1e9;
void hopcroftCarp() {
    int n, m, e;
    cin >> n >> m >> e;
    vector<int> adj[n];
    for (int i = 0; i < e; ++i) {
        int u, v;
        cin >> u >> v;
        --u;
        --v;
        adj[u].push_back(v);
    }
    vector<int> ml(m, -1), mr(n, -1), dist(n);
    auto bfs = [&]() -> bool {
        queue<int> q;
        for (int u = 0; u < n; ++u) {
            if (mr[u] == -1) {
                dist[u] = 0;
                q.push(u);
            } else {
                dist[u] = INF;
            }
        }
        bool foundAugmenting = false;
        while (!q.empty()) {
            int u = q.front();
            q.pop();
            for (int v : adj[u]) {
                int pairedLeft = ml[v];
                if (pairedLeft == -1) {
                    foundAugmenting = true;
                } else if (dist[pairedLeft] == INF) {
                    dist[pairedLeft] = dist[u] + 1;
                    q.push(pairedLeft);
                }
            }
        }
        return foundAugmenting;
    };
    function<bool(int)> dfs = [&](int u) -> bool {
        for (int v : adj[u]) {
            int pairedLeft = ml[v];
            if (pairedLeft == -1 or (dist[pairedLeft]
                == dist[u] + 1 and dfs(pairedLeft))) {
                mr[u] = v;
            }
        }
    };
}

```

```

ml[v] = u;
return true;
}
}
dist[u] = INF;
return false;
};
int matching = 0;
while (bfs()) {
    for (int u = 0; u < n; ++u) {
        if (mr[u] == -1) {
            if (dfs(u)) matching++;
        }
    }
}
cout << matching << endl;
for (int u = 0; u < n; ++u) {
    if (mr[u] != -1) {
        cout << u << " " << mr[u] << endl;
    }
}
}
}

```

1.17 Max Bipartite Matching [Kuhn's]

```

// left set size, right set size, edge count
int n, k, m, visToken = 1;
vector<int> lst[MX];
int mr[MX], ml[MX], vis[MX];
bool try_kuhn(int u) {
    if (vis[u] == visToken)
        return false;
    vis[u] = visToken;
    for (auto v : lst[u]) {
        if (ml[v] == -1 or try_kuhn(ml[v])) {
            ml[v] = u;
            mr[u] = v;
            return true;
        }
    }
    return false;
}
void solve() {
    cin >> n >> k >> m;
    for (int i = 0; i < m; ++i) {
        int u, v;
        cin >> u >> v;
        --u;
        --v;
        lst[u].push_back(v);
    }
    fill(mr, mr + n, -1);
    fill(ml, ml + k, -1);
    int ans = 0;
    for (int u = 0; u < n; ++u) {
        for (auto v : lst[u]) {
            if (ml[v] == -1) {
                ml[v] = u;
                mr[u] = v;
                ans++;
                break;
            }
        }
    }
    for (int u = 0; u < n; ++u) {
        if (mr[u] != -1) continue;
        visToken++;
    }
}

```

```

    if (try kuhn(u))
        ans++;
}
cout << ans << el;
for (int v = 0; v < k; ++v) {
    if (ml[v] != -1) {
        cout << ml[v] + 1 << " " << v + 1 << el;
    }
}
}

```

1.18 SOD_NOD

```

// SOD = ((P^(x+1)-1)/(P-1)) *
// ((Q^(y+1)-1)/(Q-1)) * ((R^(z+1)-1)/(R-1))
// NOD = P^x * Q^y * R^z => here, P, Q, R are
// prime factors of x, y, z are
// powers NOD = (x + 1) (y + 1) (z + 1)
pair<int, int> SOD_NOD(int n) {
    int sod = 1, nod = 1;
    for (int i = 2; i * i <= n; ++i) {
        if (n % i == 0) {
            int pown = 1, pows = 0;
            while (n % i == 0) {
                pown *= i; // p^e
                pows++;
                n /= i;
            }
            pown *= i;
            sod *= (pown - 1) / (i - 1); // p^(e+1) - 1
            nod *= (pows + 1);
        }
    }
    if (n > 1) {
        sod *= (n + 1);
        nod *= 2;
    }
    return {sod, nod};
}

```

1.19 SparseTable

```

const int mxN = 1e5 + 10, M = 21;
int sparse[mxN][M];
void build_sparse(int n, vector<int>& v) {
    for (int i = 0; i < n; i++) sparse[i][0] = v[i];
    for (int k = 1; k < M; k++) {
        for (int i = 0; i + (1 << k) <= n; i++) {
            sparse[i][k] = max(sparse[i][k - 1],
                               sparse[i + (1 << (k - 1))][k - 1]);
        }
    }
}
int query(int l, int r) { // 0 based index
    if (l > r) swap(l, r);
    int b = bit_width(r - l + 1) - 1;
    return max(sparse[l][b], sparse[r - (1 << b) + 1][b]);
}

```

1.20 StrHash

```

const int mod1 = 911382323, mod2 = 972663749, b1 = 137, b2 = 139;
const int mxN = 5000010;

```

```

int pow_b1[mxN], pow_b2[mxN], inv_b1[mxN],
    inv_b2[mxN];
int binExp(int base, int power, int mod) {
    int res = 1;
    while (power) {
        if (power & 1) res = (1LL * res * base) % mod;
        base = (1LL * base * base) % mod;
        power >>= 1;
    }
    return res;
}
void pre() {
    pow_b1[0] = pow_b2[0] = 1;
    for (int i = 1; i < mxN; i++) {
        pow_b1[i] = (1LL * pow_b1[i - 1] * b1) % mod1;
        pow_b2[i] = (1LL * pow_b2[i - 1] * b2) % mod2;
    }
    inv_b1[mxN - 1] = binExp(pow_b1[mxN - 1], mod1 - 2, mod1);
    inv_b2[mxN - 1] = binExp(pow_b2[mxN - 1], mod2 - 2, mod2);
    for (int i = mxN - 2; i >= 0; i--) {
        inv_b1[i] = (1LL * inv_b1[i + 1] * b1) % mod1;
        inv_b2[i] = (1LL * inv_b2[i + 1] * b2) % mod2;
    }
}
vector<pair<int, int>> getPref(string& s) {
    int qq = s.size();
    vector<pair<int, int>> hsh(qq);
    for (int i = 0; i < qq; i++) {
        if (i == 0) {
            hsh[i].first = (1LL * s[i] * pow_b1[i]) % mod1;
            hsh[i].second = (1LL * s[i] * pow_b2[i]) % mod2;
        } else {
            hsh[i].first = (hsh[i - 1].first + (1LL * s[i] * pow_b1[i]) % mod1) % mod1;
            hsh[i].second = (hsh[i - 1].second + (1LL * s[i] * pow_b2[i]) % mod2) % mod2;
        }
    }
    return hsh;
}
pair<int, int> getHash(string& str) {
    int hsh1 = 0, hsh2 = 0, sz = str.size();
    for (int i = 0; i < sz; ++i) {
        hsh1 = (hsh1 + 1LL * str[i] * pow_b1[i] % mod1) % mod1;
    }
    for (int i = 0; i < sz; ++i) {
        hsh2 = (hsh2 + 1LL * str[i] * pow_b2[i] % mod2) % mod2;
    }
    return {hsh1, hsh2};
}
pair<int, int> getSub(int l, int r, vector<pair<int, int>>& v) {

```

```

    pair<int, int> q;
    if (l == 0) {
        q = {v[r].first, v[r].second};
    } else {
        int x = (1LL * ((v[r].first - v[l - 1].first + mod1) % mod1) * inv_b1[l]) % mod1;
        int y = (1LL * ((v[r].second - v[l - 1].second + mod2) % mod2) * inv_b2[l]) % mod2;
        q = {x, y};
    }
    return q;
}

```

1.21 StrHash_2

```

const int N = 1000010, MOD = 1e9 + 7;
const ll P[] = {97, 1000003};
ll bigMod(ll a, ll e) {
    if (e == -1) e = MOD - 2;
    ll ret = 1;
    while (e) {
        if (e & 1) ret = ret * a % MOD;
        a = a * a % MOD; e >>= 1;
    }
    return ret;
}
ll pwr[2][N], inv[2][N];
void initHash() {
    for (int it = 0; it < 2; ++it) {
        pwr[it][0] = inv[it][0] = 1;
        ll INV_P = bigMod(P[it], -1);
        for (int i = 1; i < N; ++i) {
            pwr[it][i] = pwr[it][i - 1] * P[it] % MOD;
            inv[it][i] = inv[it][i - 1] * INV_P % MOD;
        }
    }
}
struct RangeHash {
    vector<ll> h[2], rev[2];
    RangeHash(const string S, bool revFlag = 0) {
        for (int it = 0; it < 2; ++it) {
            h[it].resize(S.size() + 1, 0);
            for (int i = 0; i < S.size(); ++i) {
                h[it][i + 1] = (h[it][i] + pwr[it][i + 1] * (S[i] - 'a' + 1)) % MOD;
            }
            if (revFlag) {
                rev[it].resize(S.size() + 1, 0);
                for (int i = 0; i < S.size(); ++i) {
                    rev[it][i + 1] = (rev[it][i] + inv[it][i + 1] * (S[i] - 'a' + 1)) % MOD;
                }
            }
        }
    }
    inline ll get(int l, int r) {
        ll one = (h[0][r + 1] - h[0][l]) * inv[0][l + 1] % MOD;
        ll two = (h[1][r + 1] - h[1][l]) * inv[1][l + 1] % MOD;
        if (one < 0) one += MOD;
    }
}

```

```

    if (two < 0) two += MOD;
    return one << 31 | two;
}
inline ll getReverse(int l, int r) {
    ll one = (rev[0][r + 1] - rev[0][l]) *
        pwr[0][r + 1] % MOD;
    ll two = (rev[1][r + 1] - rev[1][l]) *
        pwr[1][r + 1] % MOD;
    if (one < 0) one += MOD;
    if (two < 0) two += MOD;
    return one << 31 | two;
}
};

```

1.22 TopologicalSorting

```

const int N = 1e5 + 10;
vector<int> g[N], indegree(N, 0);
vector<int> topSort(int n) {
    queue<int> q;
    vector<int> order;
    for (int i = 1; i <= n; i++) {
        if (indegree[i] == 0) {
            q.push(i);
        }
    }
    while (!q.empty()) {
        int u = q.front();
        q.pop();
        order.push_back(u);
        for (int v : g[u]) {
            indegree[v]--;
            if (indegree[v] == 0) {
                q.push(v);
            }
        }
    }
    return order;
}

```

1.23 UniquePF of all elements till MX

```

const int MX = 2e5 + 10;
vector<int> pfac[MX];
void factorize() {
    for (int i = 2; i < MX; i++) {
        if (!pfac[i].empty()) continue;
        for (int j = i; j < MX; j += i)
            pfac[j].push_back(i);
    }
}

```

1.24 WeightedUnionFind

```

const int MX = 2e5 + 10;
int par[MX], sz[MX];
ll d[MX];
void init() {
    for (int i = 0; i < MX; ++i) {
        par[i] = i;
        sz[i] = 1;
        d[i] = 0;
    }
}
int findpar(int x) {
    if (par[x] == x) return x;
    int p = par[x];

```

```

    par[x] = findpar(p);
    d[x] += d[p];
    return par[x];
}
bool unite(int a, int b, ll w) {
    int ra = findpar(a);
    int rb = findpar(b);
    if (ra == rb) {
        return (d[b] - d[a] == w);
    }
    if (sz[ra] < sz[rb]) {
        swap(a, b);
        swap(ra, rb);
        w = -w;
    }
    par[rb] = ra;
    d[rb] = d[a] + w - d[b];
    sz[ra] += sz[rb];
    return true;
}
ll dist(int a, int b) {
    findpar(a), findpar(b);
    return d[b] - d[a];
}

```

1.25 allInOneNT

```

const int MAXN = 1e6 + 9;
typedef struct info {
    int lowest_prime = 0, greatest_prime = 0,
        distinct_prime = 0;
    int total_prime = 0, NOD = 0, SOD = 0;
} info;
info num[MAXN];
void preStore() {
    for (int i = 2; i < MAXN; i++) {
        int n = i;
        map<int, int> factors; // Key->Factor,
            Val->count
        int SOD = 1, NOD = 1, total_p_factor = 0;
        if (n % 2 == 0) {
            while (n % 2 == 0) {
                n /= 2;
                factors[2]++;
                total_p_factor++;
            }
            SOD *= (1 << (factors[2] + 1)) - 1;
            NOD *= (factors[2] + 1);
        }
        for (int i = 3; i * i <= n; i += 2) {
            if (n % i == 0) {
                while (n % i == 0) {
                    n /= i;
                    factors[i]++;
                    total_p_factor++;
                }
                SOD *= (pow(i, factors[i] + 1) - 1) / (i - 1);
                NOD *= (factors[i] + 1);
            }
        }
        if (n > 1) {
            factors[n]++;
            SOD *= (pow(n, 2) - 1) / (n - 1);
            NOD *= 2;
            total_p_factor++;
        }
    }
}

```

```

    num[i].distinct_prime = factors.size();
    num[i].total_prime = total_p_factor;
    num[i].NOD = NOD;
    num[i].SOD = SOD;
    auto lowest_prime = factors.begin();
    auto greatest_prime = factors.rbegin();
    num[i].lowest_prime = lowest_prime->first;
    num[i].greatest_prime =
        greatest_prime->first;
}
}

```

1.26 customHash

```

#include <ext/pb_ds/assoc_container.hpp>
using namespace gnu_pbds;
struct customHash {
    static uint64_t Meaw(uint64_t x) {
        x += 0x9e3779b97f4a7c15;
        x = (x ^ (x >> 30)) * 0xbf58476d1ce4e5b9;
        x = (x ^ (x >> 27)) * 0x94d049bb133111eb;
        return x ^ (x >> 31);
    }
    size_t operator()(uint64_t x) const {
        static const uint64_t FIXED_RANDOM =
            chrono::steady_clock::now().time_since_epoch().count();
        return Meaw(x + FIXED_RANDOM);
    }
}; // gp_hash_table<int, int> table;

```

1.27 divisorSieve

```

const int mxN = 1e5 + 10;
vector<int> divisors[mxN]; //
void divisorSeive() {
    for (int i = 1; i < mxN; i++) {
        for (int j = i; j < mxN; j += i) {
            divisors[j].push_back(i);
        }
    }
}

```

1.28 int128

```

__int128 read() {
    __int128 x = 0, f = 1;
    char ch = getchar();
    while (ch < '0' || ch > '9') {
        if (ch == '-') f = -1;
        ch = getchar();
    }
    while (ch >= '0' && ch <= '9') {
        x = x * 10 + ch - '0';
        ch = getchar();
    }
    return x * f;
}
void print(__int128 x) {
    if (x < 0) {
        putchar('-');
        x = -x;
    }
    if (x > 9) print(x / 10);
    putchar(x % 10 + '0');
}

```

1.29 lis

```
vector<int> lis(int n, vector<int>& v) {
    vector<int> parent(n, -1), ind(n);
    vector<int> lis;
    for (int i = 0; i < n; i++) {
        int it = lower_bound(lis.begin(), lis.end(),
            v[i]) - lis.begin();
        if (it == lis.size()) {
            lis.push_back(v[i]);
            ind[lis.size() - 1] = i;
            parent[i] = (lis.size() == 1 ? -1 : ind[it - 1]);
        } else {
            lis[it] = v[i];
            ind[it] = i;
            parent[i] = (it == 0 ? -1 : ind[it - 1]);
        }
    }
    vector<int> LIS;
    int it = ind[lis.size() - 1];
    LIS.push_back(lis.back());
    while (parent[it] != -1) {
        it = parent[it];
        LIS.push_back(v[it]);
    }
    return LIS;
}
```

1.30 mergeSort

```
// use array of elements, if multiple testcase
// make inv = 0 each time
// int inv = 0;
void merge(int vct[], int l, int m, int r) {
    int left = m - l + 1, right = r - m, lv[left],
        rv[right];
    for (int i = 0; i < left; i++) {
        lv[i] = vct[l + i];
    }
    for (int i = 0; i < right; i++) {
        rv[i] = vct[m + 1 + i];
    }
    int i = 0, j = 0, to = l;
    while (i < left && j < right) {
        if (lv[i] <= rv[j]) {
            vct[to] = lv[i];
            i++;
        } else {
            vct[to] = rv[j];
            j++;
        }
        // inversion count
        // int pore = left - i; inv += pore;
        to++;
    }
    while (i < left) {
        vct[to] = lv[i];
        i++;
        to++;
    }
    while (j < right) {
        vct[to] = rv[j];
        j++;
        to++;
    }
}
```

```
void merge_sort(int vct[], int l, int r) {
    if (r <= l) return;
    int m = l + ((r - l) / 2);
    merge_sort(vct, l, m);
    merge_sort(vct, m + 1, r);
    merge(vct, l, m, r);
}
```

1.31 pbds

```
#include <ext/pb_ds/assoc_container.hpp>
#include <ext/pb_ds/tree_policy.hpp>
#include <functional>
using namespace __gnu_pbds;
typedef tree<int, null_type, less<int>,
    rb_tree_tag,
    tree_order_statistics_node_update>
    ordered_set;
// s.order_of_key(x) = number of elements
// strictly less than x
// *s.find_by_order(i) = ith element in set (0
// index)
```

1.32 phi

```
int phi(int n) { // sqrt(n)
    int result = n;
    for (int i = 2; i * i <= n; i++) {
        if (n % i == 0) {
            while (n % i == 0) n /= i;
            result -= result / i;
        }
    }
    if (n > 1) result -= result / n;
    return result;
}
```

1.33 polynomial_interpolation

```
// P(x) = a0 + a1x + a2x^2 + ... + anx^n
// y[i] = P(i)
const int mod = 1e9 + 7;
ll BigMod(ll a, ll b) {
    ll res = 1;
    while (b) {
        if (b & 1) res = 1ll * res * a % mod;
        a = 1ll * a * a % mod;
        b >>= 1;
    }
    return res;
}
ll inv(ll x) {
    if (x < 0) x += mod;
    return BigMod(x, mod - 2);
}
ll add(ll& a, ll b) {
    a += b;
    if (a >= mod) a -= mod;
    return a;
}
ll eval(vector<ll> y, ll k) {
    int n = y.size() - 1;
    if (k <= n) {
        return y[k];
    }
    vector<ll> L(n + 1, 1);
    for (int x = 1; x <= n; ++x) {
```

```
L[0] = L[0] * (k - x) % mod;
L[0] = L[0] * inv(-x) % mod;
}
for (int x = 1; x <= n; ++x) {
    L[x] = L[x - 1] * inv(k - x) % mod * (k - (x - 1)) % mod;
    L[x] = L[x] * ((x - 1) - n + mod) % mod *
        inv(x) % mod;
}
ll yk = 0;
for (int x = 0; x <= n; ++x) {
    yk = add(yk, L[x] * y[x] % mod);
}
return yk;
}
```

1.34 segTree

```
const int N = 1e5 + 9;
int a[N];
struct ST {
    int t[4 * N];
    // static const int inf = 1e9;
    ST() {
        // memset(t, 0, sizeof t);
        t[0] = 0;
    }
    void build(int n, int b, int e) {
        if (b == e) {
            t[n] = a[b];
            return;
        }
        int mid = (b + e) >> 1, l = n << 1, r = l | 1;
        build(l, b, mid);
        build(r, mid + 1, e);
        t[n] = t[l] + t[r];
    }
    void update(int n, int b, int e, int i, int x) {
        if (b > i || e < i) return;
        if (b == e && b == i) {
            t[n] = x;
            return;
        }
        int mid = (b + e) >> 1, l = n << 1, r = l | 1;
        update(l, b, mid, i, x);
        update(r, mid + 1, e, i, x);
        t[n] = t[l] + t[r];
    }
    int query(int n, int b, int e, int i, int j) {
        if (b > j || e < i) return 0;
        if (b >= i && e <= j) return t[n];
        int mid = (b + e) >> 1, l = n << 1, r = l | 1;
        int L = query(l, b, mid, i, j);
        int R = query(r, mid + 1, e, i, j);
        return L + R;
    }
}; // Declare: ST sgt;
```

1.35 sieve

```

const ll MAXN = 1e7 + 10;
bool prime[MAXN];
vector<ll> prm;
void sieve() {
    prime[0] = prime[1] = true;
    for (ll i = 2; i < MAXN; i++) {
        if (!prime[i]) {
            prm.push_back(i);
            for (ll j = i + i; j < MAXN; j += i) {
                prime[j] = true;
            }
        }
    }
}

```

1.36 spf

```

const int MAXN = 1e6 + 2;
int spf[MAXN];
vector<int> prms;
void preStore() {
    for (int i = 1; i < MAXN; i++) spf[i] = i;
    for (int i = 2; i < MAXN; i++) {
        if (spf[i] == i) {
            prms.push_back(i);
            for (int j = i + i; j < MAXN; j += i) {
                spf[j] = min(spf[j], i);
            }
        }
    }
}

```

1.37 suffixArray

```

// fahimcp495
array<vector<int>, 2> get_sa(string& s, int lim
= 128) { // for integer, just change string
to vector<int> and minimum value of vector
must be >= 1
int n = s.size() + 1, k = 0, a, b;
vector<int> x(begin(s), end(s) + 1), y(n),
    sa(n), lcp(n), ws(max(n, lim)), rank(n);
x.back() = 0;
iota(begin(sa), end(sa), 0);
for (int j = 0, p = 0; p < n; j = max(1, j *
    2), lim = p) {
    p = j, iota(begin(y), end(y), n - j);
    for (int i = 0; i < n; ++i)
        if (sa[i] >= j) y[p++] = sa[i] - j;
    fill(begin(ws), end(ws), 0);
    for (int i = 0; i < n; ++i) ws[x[i]]++;
    for (int i = 1; i < lim; ++i) ws[i] += ws[i
        - 1];
    for (int i = n; i--;) sa[--ws[x[y[i]]]] =
        y[i];
    swap(x, y), p = 1, x[sa[0]] = 0;
    for (int i = 1; i < n; ++i) a = sa[i - 1], b
        = sa[i], x[b] = (y[a] == y[b] && y[a +
            1] == y[b + 1]) ? p - 1 : p++;
}
for (int i = 1; i < n; ++i) rank[sa[i]] = i;
for (int i = 0, j; i < n - 1; lcp[rank[i++]] =
    k)
    for (k && k--, j = sa[rank[i] - 1]; s[i + k]
        == s[j + k]; k++);
}

```

```

sa.erase(sa.begin()), lcp.erase(lcp.begin());
return {sa, lcp};
}

```

1.38 suffixAutomata

```

const int N = 2e5 + 10; // max string size
int len[N], lnk[N]{-1}, last, sz = 1;
unordered_map<char, int> to[N];
void add(char c) {
    int cur = sz++;
    len[cur] = len[last] + 1;
    int u = last;
    while (u != -1 and !to[u].count(c)) {
        to[u][c] = cur;
        u = lnk[u];
    }
    if (u == -1) {
        lnk[cur] = 0;
    } else {
        int v = to[u][c];
        if (len[v] == len[u] + 1) {
            lnk[cur] = v;
        } else {
            int w = sz++;
            len[w] = len[u] + 1, lnk[w] = lnk[v],
                to[w] = to[v];
            while (u != -1 and to[u][c] == v) {
                to[u][c] = w;
                u = lnk[u];
            }
            lnk[cur] = lnk[v] = w;
        }
    }
    last = cur;
}

```

1.39 suffixAutomation

```

int len[N], lnk[N]{-1}, last, sz = 1;
unordered_map<char, int> to[N];
void init() {
    while (sz) {
        sz--;
        to[sz].clear();
    }
    last = 0, sz = 1;
}
void add(char c) {
    int cur = sz++;
    int u = last;
    len[cur] = len[last] + 1;
    while (u != -1 and !to[u].count(c)) {
        to[u][c] = cur;
        u = lnk[u];
    }
    if (u == -1) {
        lnk[cur] = 0;
    } else {
        int v = to[u][c];
        if (len[v] == len[u] + 1) {
            lnk[cur] = v;
        } else {
            int w = sz++;
            len[w] = len[u] + 1, lnk[w] = lnk[v],
                to[w] = to[v];
            while (u != -1 and to[u][c] == v) {
                to[u][c] = w;
            }
        }
    }
}

```

```

    u = lnk[u];
    {lnk[cur] = lnk[v] = w;
    }
    last = cur;
}

```

1.40 trie

```

const ll N = 1e6 + 5, A = 26;
ll trie[N][A], cnt[N], tot = 1, root = 1;
void initTrie() {
    cnt[tot] = 0;
    root = 1;
}
void addStr(string& s) {
    ll u = 1;
    for (auto it : s) {
        ll n = it - 'a';
        if (trie[u][n] == 0) {
            trie[u][n] = ++tot;
        }
        u = trie[u][n];
        cnt[u]++;
    }
}
ll wordCount(string& s) {
    ll u = 1;
    for (auto it : s) {
        int n = it - 'a';
        if (trie[u][n] == 0) return 0;
        u = trie[u][n];
    }
    return cnt[u];
}

```

2 Geometry

2.1 Convex Hull

```

vector<PT> convexHull (vector<PT> p) {
    int n = p.size(), m = 0;
    if (n < 3) return p;
    vector<PT> hull(n + n);
    sort(p.begin(), p.end(), [&] (PT a, PT b) {
        return (a.x==b.x? a.y<b.y: a.x<b.x);
    });
    for (int i = 0; i < n; ++i) {
        while (m > 1 and cross(hull[m - 2] - p[i],
            hull[m - 1] - p[i]) <= 0) --m;
        hull[m++] = p[i];
    }
    for (int i = n - 2, j = m + 1; i >= 0; --i) {
        while (m >= j and cross(hull[m - 2] - p[i],
            hull[m - 1] - p[i]) <= 0) --m;
        hull[m++] = p[i];
    }
    hull.resize(m - 1); return hull;
}

```

3 Notes**3.1 Geometry****3.1.1 Triangles**

Circumradius: $R = \frac{abc}{4A}$, Inradius: $r = \frac{A}{s}$

The area of a triangle using two sides and the included angle can be given as:

$$A = \frac{1}{2}ab \sin \angle C$$

Length of median (divides triangle into two equal-area triangles): $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

Length of bisector (divides angles in two): $s_a = \sqrt{bc \left[1 - \left(\frac{a}{b+c} \right)^2 \right]}$

Law of tangents: $\frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}$

3.1.2 Quadrilaterals

With side lengths a, b, c, d , diagonals e, f , diagonals angle θ , area A and magic flux $F = b^2 + d^2 - a^2 - c^2$:

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is 180° , $ef = ac + bd$, and $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$.

3.1.3 Spherical coordinates

$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \arccos(z/\sqrt{x^2 + y^2 + z^2}) \\ z &= r \cos \theta & \phi &= \arctan 2(y, x) \end{aligned}$$

3.1.4 Pick's Theorem:

Given a lattice polygon with non-zero area, we define: S as the area of the polygon, I as the number of integer-coordinate points strictly inside the polygon, B as the number of integer-coordinate points on the boundary of the polygon. Then, Pick's Theorem states:

$$S = I + \frac{B}{2} - 1$$

The number of lattice points on segments (x1, y1) to (x2, y2) is: $\gcd(\text{abs}(x2 - x1), \text{abs}(y2 - y1)) + 1$

3.1.5 Polygon

For a regular polygon with n sides and side length a , the circumradius R is given by:

$$R = \frac{a}{2 \sin \left(\frac{\pi}{n} \right)}$$

3.1.6 Area of a Circular Segment

The area of a circular segment, which is the region enclosed by a chord and the corresponding arc, can be calculated using the formula:

$$A = \frac{R^2}{2} (\theta - \sin \theta)$$

where: R is the radius of the circle, θ is the central angle subtended by the chord, in radians.

3.2 Binomial Coefficient

• Factoring in: $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$

• Sum over k : $\sum_{k=0}^n \binom{n}{k} = 2^n$

• Alternating sum: $\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$

• Even and odd sum: $\sum_{k=0}^n \binom{n}{2k} = \sum_{k=0}^n \binom{n}{2k+1} = 2^{n-1}$

• The Hockey Stick Identity

- (Left to right) Sum over n and k : $\sum_{k=0}^m \binom{n+k}{k} = \binom{n+m+1}{m+1}$

- (Right to left) Sum over n : $\sum_{m=0}^n \binom{m}{k} = \binom{n+1}{k+1}$

• Sum of the squares: $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

• Weighted sum: $\sum_{k=1}^n k \binom{n}{k} = n 2^{n-1}$

• Connection with the fibonacci numbers: $\sum_{k=0}^n \binom{n-k}{k} = F_{n+1}$

• Vandermonde's Identity: $\sum_{i=0}^k \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}$

• If $f(n, k) = C(n, 0) + C(n, 1) + \dots + C(n, k)$, Then $f(n+1, k) = 2 * f(n, k) - C(n, k)$ [For multiple $f(n, k)$ queries, use Mo's algo]

Lucas Theorem

$$\binom{m}{n} \equiv \prod_{i=0}^k \binom{m_i}{n_i} \pmod{p}$$

• $\binom{m}{n}$ is divisible by p if and only if at least one of the base- p digits of n is greater than the corresponding base- p digit of m .

• The number of entries in the n th row of Pascal's triangle that are not divisible by $p = \prod_{i=0}^k (n_i + 1)$

• All entries in the $(p^k - 1)th$ row are not divisible by p .

• $\binom{n}{m} \equiv \lfloor \frac{n}{p} \rfloor \pmod{p}$

3.3 Fibonacci Number

1. $k = A - B, F_A F_B = F_{k+1} F_A^2 + F_k F_A F_{A-1}$

2. $\sum_{i=0}^n F_i^2 = F_{n+1} F_n$ 3. $\sum_{i=0}^n F_i F_{i+1} = F_{n+1}^2 - (-1)^n$

4. $\sum_{i=0}^n F_i F_{i+1} = F_{n+1}^2 - (-1)^n$ 5. $\sum_{i=0}^n F_i F_{i-1} = \sum_{i=0}^{n-1} F_i F_{i+1}$

6. $\gcd(F_m, F_n) = F_{\gcd(m, n)}$ 7. $\sum_{0 \leq k \leq n} \binom{n-k}{k} = F_{n+1}$

8. $\gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+2}) = \gcd(F_{n+1}, F_{n+2}) = 1$

3.4 Sums

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

$$\sum_{i=1}^n i^m = \frac{1}{m+1} \left[(n+1)^{m+1} - 1 - \sum_{i=1}^n ((i+1)^{m+1} - i^{m+1} - (m+1)i^m) \right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}$$

$$\sum_{k=0}^n k x^k = (x - (n+1)x^{n+1} + nx^{n+2}) / (x-1)^2$$

3.5 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty)$$

$$(x+a)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k a^{-n-k}$$

Generating Function

$$1/(1-x) = 1 + x + x^2 + x^3 + \dots$$

$$1/(1-ax) = 1 + ax + (ax)^2 + (ax)^3 + \dots$$

$$1/(1-x)^2 = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$1/(1-x)^3 = C(2, 2) + C(3, 2)x + C(4, 2)x^2 + C(5, 2)x^3 + \dots$$

$$1/(1-ax)^{k+1} = 1 + C(1+k, k)(ax) + C(2+k, k)(ax)^2 + C(3+k, k)(ax)^3 + \dots$$

$$x(x+1)(1-x)^{-3} = 1 + x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + \dots$$

$$e^x = 1 + x + (x^2)/2! + (x^3)/3! + (x^4)/4! + \dots$$

3.6 Pythagorean Triples

The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), \quad b = k \cdot (2mn), \quad c = k \cdot (m^2 + n^2),$$

with $m > n > 0$, $k > 0$, $m \perp n$, and either m or n even.

3.7 Number Theory

- HCN: 1e6(240), 1e9(1344), 1e12(6720), 1e14(17280), 1e15(26880), 1e16(41472)
- $gcd(a,b,c,d,...) = gcd(a,b-a,c-b,d-c,...)$
- $gcd(a+k,b+k,c+k,d+k,...) = gcd(a+k,b-a,c-b,d-c,...)$
- Primitive root exists iff $n = 1, 2, 4, p^k, 2 \times p^k$, where p is an odd prime.
- If primitive root exists, there are $\phi(\phi(n))$ primitive roots of n .
- The numbers from 1 to n have in total $O(n \log \log n)$ unique prime factors.
- $x \equiv r_1 \pmod{m_1}$ and $x \equiv r_2 \pmod{m_2}$ has a solution iff $gcd(m_1, m_2) | (r_1 - r_2)$ Solution of $x^2 \equiv a \pmod{p}$
- $ca \equiv cb \pmod{m} \iff a \equiv b \pmod{\frac{n}{gcd(n,c)}}$
- $ax \equiv b \pmod{m}$ has a solution $\iff gcd(a, m) | b$
- If $ax \equiv b \pmod{m}$ has a solution, then it has $gcd(a, m)$ solutions and they are separated by $\frac{m}{gcd(a, m)}$
- $ax \equiv 1 \pmod{m}$ has a solution or a is invertible $\pmod{m} \iff gcd(a, m) = 1$
- $x^2 \equiv 1 \pmod{p}$ then $x \equiv \pm 1 \pmod{p}$
- There are $\frac{p-1}{2}$ has no solution.
- There are $\frac{p-1}{2}$ has exactly two solutions.
- When $p \% 4 = 3$, $x \equiv \pm a^{\frac{p+1}{4}}$
- When $p \% 8 = 5$, $x \equiv a^{\frac{p+3}{8}}$ or $x \equiv 2^{\frac{p-1}{4}} a^{\frac{p+3}{8}}$

3.7.1 Primes

$p = 962592769$ is such that $2^{21} | p - 1$, which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than 1 000 000.

Primitive roots exist modulo any prime power p^a , except for $p = 2, a > 2$, and there are $\phi(\phi(p^a))$ many. For $p = 2, a > 2$, the group $\mathbb{Z}_{2^a}^\times$ is instead isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$.

3.7.2 Estimates

$\sum_{d|n} d = O(n \log \log n)$.

The number of divisors of n is at most around 100 for $n < 5e4$, 500 for $n < 1e7$, 2000 for $n < 1e10$, 200 000 for $n < 1e19$.

3.7.3 Perfect numbers

$n > 1$ is called perfect if it equals sum of its proper divisors and 1. Even n is perfect iff $n = 2^{p-1}(2^p - 1)$ and $2^p - 1$ is prime (Mersenne's). No odd perfect numbers are yet found.

3.7.4 Carmichael numbers

A positive composite n is a Carmichael number ($a^{n-1} \equiv 1 \pmod{n}$ for all a : $gcd(a, n) = 1$), iff n is square-free, and for all prime divisors p of n , $p - 1$ divides $n - 1$.

3.7.5 Totient

- If p is a prime $(p^k) = p^k - p^{k-1}$
- If a, b are relatively prime, $\phi(ab) = \phi(a)\phi(b)$
- $\phi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2})(1 - \frac{1}{p_3}) \dots (1 - \frac{1}{p_k})$
- Sum of coprime to $n = n * \frac{\phi(n)}{2}$
- If $n = 2^k$, $\phi(n) = 2^{k-1} = \frac{n}{2}$
- For a, b , $\phi(ab) = \phi(a)\phi(b) \frac{d}{\phi(d)}$
- $\phi(ip) = p\phi(i)$ whenever p is a prime and it divides i
- The number of $a(1 < a \leq N)$ such that $gcd(a, N) = d$ is $\phi(\frac{n}{d})$
- If $n > 2$, $\phi(n)$ is always even
- Sum of gcd, $\sum_{i=1}^n gcd(i, n) = \sum_{d|n} d\phi(\frac{n}{d})$
- Sum of lcm, $\sum_{i=1}^n nlcm(i, n) = \frac{n}{2}(\sum_{d|n} (d\phi(d)) + 1)$
- $\phi(1) = 1$ and $\phi(2) = 1$ which two are only odd ϕ
- $\phi(3) = 2$ and $\phi(4) = 2$ and $\phi(6) = 2$ which three are only prime ϕ
- Find minimum n such that $\frac{\phi(n)}{n}$ is maximum- Multiple of small primes- $2 * 3 * 5 * 7 * 11 * 13 * \dots$

3.7.6 Mobius function

$\mu(1) = 1$. $\mu(n) = 0$, if n is not squarefree. $\mu(n) = (-1)^s$, if n is the product of s distinct primes. Let f, F be functions on positive integers. If for all $n \in N$, $F(n) = \sum_{d|n} f(d)$, then $f(n) = \sum_{d|n} \mu(d)F(\frac{n}{d})$, and vice versa. $\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$. $\sum_{d|n} \mu(d) = 1$.

If f is multiplicative, then $\sum_{d|n} \mu(d)f(d) = \prod_{p|n} (1 - f(p))$, $\sum_{d|n} \mu(d)^2 f(d) = \prod_{p|n} (1 + f(p))$.

$$\sum_{i=1}^n \sum_{j=1}^n [gcd(i, j) = 1] = \sum_{k=1}^n \mu(k) \lfloor \frac{n}{k} \rfloor^2$$

$$\sum_{i=1}^n \sum_{j=1}^n gcd(i, j) = \sum_{k=1}^n k \sum_{l=1}^{\lfloor \frac{n}{k} \rfloor} \mu(l) \lfloor \frac{n}{kl} \rfloor^2$$

$$\sum_{i=1}^n \sum_{j=1}^n gcd(i, j) = \sum_{k=1}^n \left(\frac{\lfloor \frac{n}{k} \rfloor (1 + \lfloor \frac{n}{k} \rfloor)}{2} \right)^2 \sum_{d|k} \mu(d) k d$$

3.7.7 Legendre symbol

If p is an odd prime, $a \in \mathbb{Z}$, then $\left(\frac{a}{p}\right)$ equals 0, if $p|a$; 1 if a is a quadratic residue modulo p ; and -1 otherwise. Euler's criterion: $\left(\frac{a}{p}\right) = a^{\left(\frac{p-1}{2}\right)} \pmod{p}$.

3.7.8 Jacobi symbol

If $n = p_1^{a_1} \dots p_k^{a_k}$ is odd, then $\left(\frac{a}{n}\right) = \prod_{i=1}^k \left(\frac{a}{p_i}\right)^{a_i}$.

3.7.9 Primitive roots

If the order of g modulo m (min $n > 0$: $g^n \equiv 1 \pmod{m}$) is $\phi(m)$, then g is called a primitive root. If Z_m has a primitive root, then it has $\phi(\phi(m))$ distinct primitive roots. Z_m has a primitive root iff m is one of $2, 4, p^k, 2p^k$, where p is an odd prime. If Z_m has a primitive root g , then for all a coprime to m , there exists unique integer $i = \text{ind}_g(a)$ modulo $\phi(m)$, such that $g^i \equiv a \pmod{m}$. $\text{ind}_g(a)$ has logarithm-like properties: $\text{ind}(1) = 0$, $\text{ind}(ab) = \text{ind}(a) + \text{ind}(b)$.

If p is prime and a is not divisible by p , then congruence $x^n \equiv a \pmod{p}$ has $\gcd(n, p-1)$ solutions if $a^{(p-1)/\gcd(n, p-1)} \equiv 1 \pmod{p}$, and no solutions otherwise. (Proof sketch: let g be a primitive root, and $g^i \equiv a \pmod{p}$, $g^u \equiv x \pmod{p}$. $x^n \equiv a \pmod{p}$ iff $g^{nu} \equiv g^i \pmod{p}$ iff $nu \equiv i \pmod{p}$.)

3.7.10 Discrete logarithm problem

Find x from $a^x \equiv b \pmod{m}$. Can be solved in $O(\sqrt{m})$ time and space with a meet-in-the-middle trick. Let $n = \lceil \sqrt{m} \rceil$, and $x = ny - z$. Equation becomes $a^{ny} \equiv ba^z \pmod{m}$. Precompute all values that the RHS can take for $z = 0, 1, \dots, n-1$, and brute force y on the LHS, each time checking whether there's a corresponding value for RHS.

3.7.11 Pythagorean triples

Integer solutions of $x^2 + y^2 = z^2$ All relatively prime triples are given by: $x = 2mn, y = m^2 - n^2, z = m^2 + n^2$ where $m > n, \gcd(m, n) = 1$ and $m \not\equiv n \pmod{2}$. All other triples are multiples of these. Equation $x^2 + y^2 = 2z^2$ is equivalent to $(\frac{x+y}{2})^2 + (\frac{x-y}{2})^2 = z^2$.

3.7.12 Postage stamps/McNuggets problem

Let a, b be relatively-prime integers. There are exactly $\frac{1}{2}(a-1)(b-1)$ numbers *not* of form $ax + by$ ($x, y \geq 0$), and the largest is $(a-1)(b-1) - 1 = ab - a - b$.

3.7.13 Fermat's two-squares theorem

Odd prime p can be represented as a sum of two squares iff $p \equiv 1 \pmod{4}$. A product of two sums of two squares is a sum of two squares. Thus, n is a sum of two squares iff every prime of form $p = 4k + 3$ occurs an even number of times in n 's factorization.

3.8 Permutations

3.8.1 Factorial

n	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800
$\frac{n}{n!}$		11	12	13	14	15	16	17		
$\frac{n!}{n}$	4.0e7	4.8e8	6.2e9	8.7e10	1.3e12	2.1e13	3.6e14			
$\frac{n}{n!}$	20	25	30	40	50	100	150	171		
$\frac{n!}{n!}$	2e18	2e25	3e32	8e47	3e64	9e157	6e262	>DBL_MAX		

3.8.2 Cycles

Let $g_S(n)$ be the number of n -permutations whose cycle lengths all belong to the set S . Then

$$\sum_{n=0}^{\infty} g_S(n) \frac{x^n}{n!} = \exp \left(\sum_{n \in S} \frac{x^n}{n} \right)$$

3.8.3 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1) + D(n-2)) = nD(n-1) + (-1)^n = \left\lfloor \frac{n!}{e} \right\rfloor$$

3.8.4 Burnside's lemma

Given a group G of symmetries and a set X , the number of elements of X up to symmetry equals

$$\frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where X^g are the elements fixed by g ($g.x = x$).

If $f(n)$ counts “configurations” (of some sort) of length n , we can ignore rotational symmetry using $G = \mathbb{Z}_n$ to get

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n, k)) = \frac{1}{n} \sum_{k|n} f(k) \phi(n/k)$$

3.9 Partitions and subsets

3.9.1 Partition function

Number of ways of writing n as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k - 1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

$\frac{n}{p(n)}$	0	1	2	3	4	5	6	7	8	9	20	50	100
	1	1	2	3	5	7	11	15	22	30	627	~2e5	~2e8

3.9.2 Partition Number

- Time Complexity: $O(n\sqrt{n})$

```
for (int i = 1; i <= n; ++i) {
    pent[2 * i - 1] = i * (3 * i - 1) / 2;
    pent[2 * i] = i * (3 * i + 1) / 2;
}
p[0] = 1;
for (int i = 1; i <= n; ++i) {
    p[i] = 0;
    for (int j = 1, k = 0; pent[j] <= i; ++j) {
        if (k < 2) p[i] = add(p[i], p[i - pent[j]]);
        else p[i] = sub(p[i], p[i - pent[j]]); ++k, k
    }
}
```

- The number of partitions of a positive integer n into exactly k parts equals the number of partitions of n whose largest part equals k

$$p_k(n) = p_k(n - k) + p_{k-1}(n - 1)$$

3.9.3 2nd Kaplansky's Lemma

The number of ways of selecting k objects, no two consecutive, from n labelled objects arrayed in a circle is $\frac{n}{k} \binom{n-k-1}{k-1} = \frac{n}{n-k} \binom{n-k}{k}$

3.9.4 Distinct Objects into Distinct Bins

- n distinct objects into r distinct bins $= r^n$

- Among n distinct objects, exactly k of them into r distinct bins $= \binom{n}{k} r^k$

- n distinct objects into r distinct bins such that each bin contains at least one object $= \sum_{i=0}^r (-1)^i \binom{r}{i} (r-i)^n$

3.10 Coloring

The number of labeled undirected graphs with n vertices, $G_n = 2^{\binom{n}{2}}$

The number of labeled directed graphs with n vertices, $G_n = 2^{n(n-1)}$

The number of connected labeled undirected graphs with n vertices, $C_n = 2^{\binom{n}{2}} - \frac{1}{n} \sum_{k=1}^{n-1} k \binom{n}{k} 2^{\binom{n-k}{2}} C_k = 2^{\binom{n}{2}} - \sum_{k=1}^{n-1} \frac{n-1}{k-1} 2^{\binom{n-k}{2}} C_k$

The number of k -connected labeled undirected graphs with n vertices, $D[n][k] = \sum_{s=1}^n \binom{n-1}{s-1} C_s D[n-s][k-1]$

Cayley's formula: the number of trees on n labeled vertices = the number of spanning trees of a complete graph with n labeled vertices $= n^{n-2}$

Number of ways to color a graph using k color such that no two adjacent nodes have same color

Complete graph $= k(k-1)(k-2) \dots (k-n+1)$

Tree $= k(k-1)^{n-1}$

Cycle $= (k-1)^n + (-1)^n (k-1)$

Number of trees with n labeled nodes: n^{n-2}

3.11 General purpose numbers

3.11.1 Eulerian numbers

Number of permutations $\pi \in S_n$ in which exactly k elements are greater than the previous element. k j :s s.t. $\pi(j) > \pi(j+1)$, $k+1$ j :s s.t. $\pi(j) \geq j$, k j :s s.t. $\pi(j) > j$.

$$E(n, k) = (n - k)E(n - 1, k - 1) + (k + 1)E(n - 1, k)$$

$$E(n, 0) = E(n, n - 1) = 1$$

$$E(n, k) = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n$$

3.11.2 Bell numbers

Total number of partitions of n distinct elements. $B(n) = 1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, \dots$ For p prime,

$$B(p^m + n) \equiv mB(n) + B(n + 1) \pmod{p}$$

3.11.3 Bernoulli numbers

$$\sum_{j=0}^m \binom{m+1}{j} B_j = 0. \quad B_0 = 1, B_1 = -\frac{1}{2}. B_n = 0, \text{ for all odd } n \neq 1.$$

3.11.4 Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} = \frac{(2n)!}{(n+1)!n!}$$

$$C_0 = 1, C_{n+1} = \frac{2(2n+1)}{n+2} C_n, C_{n+1} = \sum C_i C_{n-i}$$

- $C_n = 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots$
- sub-diagonal monotone paths in an $n \times n$ grid.
- strings with n pairs of parenthesis, correctly nested.
- binary trees with $n+1$ leaves (0 or 2 children).
- ordered trees with $n+1$ vertices.
- ways a convex polygon with $n+2$ sides can be cut into triangles by connecting vertices with straight lines.
- permutations of $[n]$ with no 3-term increasing subseq.

- Find the count of balanced parentheses sequences consisting of $n+k$ pairs of parentheses where the first k symbols are open brackets.

$$C_n^{(k)} = \frac{k+1}{n+k+1} \binom{2n+k}{n}$$

- Recursive formula of Catalan Numbers:

$$C_n^{(k)} = \frac{(2n+k-1) \cdot (2n+k)}{n \cdot (n+k+1)} C_{n-1}^{(k)}$$

3.11.5 Lucas Number

Number of edge cover of a cycle graph C_n is L_n

$$L(n) = L(n-1) + L(n-2); L(0) = 2, L(1) = 1$$

3.12 Ballot Theorem

Suppose that in an election, candidate A receives a votes and candidate B receives b votes, where $a > b$ for some positive integer k . Compute the number of ways the ballots can be ordered so that A maintains more than k times as many votes as B throughout the counting of the ballots.

The solution to the ballot problem is $\frac{a-kb}{a+b} \times C(a+b, a)$

3.13 Classical Problem

$F(n, k)$ = number of ways to color n objects using exactly k colors

Let $G(n, k)$ be the number of ways to color n objects using no more than k colors.

Then, $F(n, k) = G(n, k) - C(k, 1) * G(n, k-1) + C(k, 2) * G(n, k-2) - C(k, 3) * G(n, k-3) \dots$

Determining $G(n, k)$:

Suppose, we are given a $1 * n$ grid. Any two adjacent cells can not have same color. Then, $G(n, k) = k * ((k-1)^{n-1})$

If no such condition on adjacent cells. Then, $G(n, k) = k^n$

3.14 Matching Formula

3.14.1 Normal Graph

MM + MEC = n (exculding vertex), IS + VC = G , MIS + MVC = G

3.14.2 Bipartite Graph

MIS = n - MBM, MVC = MBM, MEC = n - MBM

3.15 Inequalities

3.15.1 Titu's Lemma

For positive reals a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n ,

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n} \geq \frac{a_1 + a_2 + \dots + a_n^2}{b_1 + b_2 + \dots + b_n}$$

Equality holds if and only if $a_i = kb_i$ for a non-zero real constant k .

3.16 Games

3.16.1 Grundy numbers

For a two-player, normal-play (last to move wins) game on a graph (V, E) : $G(x) = \text{mex}(\{G(y) : (x, y) \in E\})$, where $\text{mex}(S) = \min\{n \geq 0 : n \notin S\}$. x is losing iff $G(x) = 0$.

3.16.2 Sums of games

- *Player chooses a game and makes a move in it* Grundy number of a position is xor of grundy numbers of positions in summed games.
- *Player chooses a non-empty subset of games (possibly, all) and makes moves in all of them* A position is losing iff each game is in a losing position.
- *Player chooses a proper subset of games (not empty and not all), and makes moves in all chosen ones.* A position is losing iff grundy numbers of all games are equal.
- *Player must move in all games, and loses if can't move in some game* A position is losing if any of the games is in a losing position.

3.16.3 Misère Nim

A position with pile sizes $a_1, a_2, \dots, a_n \geq 1$, not all equal to 1, is losing iff $a_1 \oplus a_2 \oplus \dots \oplus a_n = 0$ (like in normal nim.) A position with n piles of size 1 is losing iff n is odd.

3.17 Tree Hashing

$f(u) = sz[u] * \sum_{i=0} f(v) * p^i$; $f(v)$ are sorted $f(child) = 1$

3.18 Permutation

To maximize the sum of adjacent differences of a permutation, it is necessary and sufficient to place the smallest half numbers in odd position and the greatest half numbers in even position. Or, vice versa.

3.19 String

- If the sum of length of some strings is N , there can be at most \sqrt{N} distinct length.
- A Text can have at most $O(N \times \sqrt{N})$ distinct substrings that match with given patterns where the sum of the length of the given patterns is N .
- $\text{Period} = n \% (n - \text{pi.back()} == 0) ? n - \text{pi.back}() : n$
- The first (*period*) cyclic rotations of a string are distinct. Further cyclic rotations repeat the previous strings.
- S is a palindrome if and only if it's period is a palindrome.
- If S and T are palindromes, then the periods of $S + T$ are same if and only if $S + T$ is a palindrome.

3.20 Bit

- $(a \text{ xor } b)$ and $(a + b)$ has the same parity
- $(a + b) = (a \text{ xor } b) + 2(a \text{ \& } b)$
- $\text{gcd}(a, b) \leq a - b \leq \text{xor}(a, b)$

3.21 Convolution

- Hamming Distance: Replace 0 with -1 - SQRT Decomposition: Find block size, $B = \sqrt{8 * n}$