Vibration of Cavity Backed Membranes

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October 26, 2012

We first consider a rigidly clamped circular membrane of radius a backed by an air cavity of volume V_0.

Equation of Motion

$$-\ddot{u} - 2j\alpha\dot{u} + c_M^2 \Delta u = \left[p_{force} - p_{cav}\right]/(\rho_m d) \qquad (1)$$

- p_{cav} is the reaction pressure of the cavity
- p_{force} - external pressure acting on the membrane

For a given configuration of the membrane, the change in volume of the container is given by,

$$\Delta V(t) = -\int_S u(r, \phi, t) dS \tag{2}$$

- This quantity is small compared to the volume of the cavity.

If the changes in pressure inside the cavity are fast enough, we can assume an adiabatic equation of state,

$$P_0 V_0^\gamma = (P_0 + \Delta P)(V_0 + \Delta V)^\gamma \tag{3}$$

$$\Rightarrow \Delta P \approx -\gamma \frac{P_0}{V_0} \Delta V \tag{4}$$

- Linearization in ΔV

Suppose we have periodic pressure, $p_{force} = pe^{j\omega t}$ on the membrane outside the cavity. As a possible solution to (1) we try the steady-state ansatz,

$$u(r, \phi, t) = \sum_{m,n} C_{mn} \cos(m\phi) J_m(\mu_{mn} r) e^{j\omega t} \tag{5}$$

This gives us,

$$\int u(r,\phi,t)dS = \sum_n 2\pi e^{j\omega t} C_{0n} \int r J_0(\mu_{0n} r) dr \tag{6}$$

- Cavity pressure has no effect on the non-axisymmetric modes, i.e J_m for $m \geq 1$
- We can ignore the higher modes

Substitution into (1) gives us,

$$\sum_n (\omega^2 - 2j\alpha\omega - \omega_n^2) C_n J_0(\mu_n r) = \left[p - \gamma \frac{P_0}{V_0} \sum_n C_n I_n \right] / (\rho_m d) \tag{7}$$

$$C_n + \gamma \frac{P_0}{V_0} \frac{I_n}{\Omega_n \rho_m d} \sum_k C_k I_k = p I_n / (\Omega_n \rho_m d) \tag{8}$$

- System of infinite linear equations

We find an approximate solution by cutting off the summation at some point.

$$\widetilde{C}_n + \gamma \frac{P_0}{V_0} \frac{I_n}{\Omega_n \rho_m d} \sum_{k}^{K} \widetilde{C}_k I_k = p I_n / (\Omega_n \rho_m d) \tag{9}$$

$$[\mathbb{I} + \mathbf{D}] \, \underline{\widetilde{C}} = \underline{P} \tag{10}$$

where,

$$\mathbf{D}_{mn} = \gamma \frac{P_0}{V_0} I_m I_n / (\Omega_m \rho_m d)$$

Coupled Membranes

- Arbitrary cavity with two membranes placed some distance apart.
- Solution reduces to the same form as for the single membrane case

Suppose we have a periodic external pressure on both membranes differing by an angle dependent phase. These are given by,

$$p_0 e^{i\omega t} = p e^{i\beta \sin(\theta)/2} e^{i\omega t} \tag{11}$$

$$p_L e^{i\omega t} = p e^{-i\beta \sin(\theta)/2} e^{i\omega t} \tag{12}$$

We can expand the solution of the membrane equations as,

$$u_{0/L}(r, \phi, t) = \sum_{m,n} C_{mn}^{0/L} \cos(m\phi) J_m(\mu_{mn} r) e^{j\omega t} \tag{13}$$

Following (2), the change in volume of the container at a given instant of time is given by,

$$\Delta V(t) = -\int_S (u_0 + u_L) dS \tag{14}$$

Substiting into (1) gives,

$$\sum_n (\omega^2 - 2j\alpha\omega - \omega_n^2) C_n^0 \cos(m\phi) J_m(\mu_n r) = \left[p_0 - \gamma \frac{P_0}{V_0} \sum_n \left(C_n^0 + C_n^L \right) I_n \right] / (\rho_m d) \tag{15}$$

$$\sum_n (\omega^2 - 2j\alpha\omega - \omega_n^2) C_n^L \cos(m\phi) J_m(\mu_n r) = \left[p_L - \gamma \frac{P_0}{V_0} \sum_n \left(C_n^0 + C_n^L \right) I_n \right] / (\rho_m d) \tag{16}$$

We define $C^+ = C^L + C^0$ and $C^- = C^L - C^0$. This results in,

$$\widetilde{C}_n^+ + 2\gamma \frac{P_0}{V_0} \frac{I_n}{\Omega_n \rho_m d} \sum_k^K \widetilde{C}_k^+ I_k = (p_L + p_0) I_n / (\Omega_n \rho_m d) \tag{17}$$

$$[\mathbb{I} + 2\mathbf{D}]\, \underline{\widetilde{\mathrm{C}}}^+ = \underline{\mathrm{P}}_L + \underline{\mathrm{P}}_0 \tag{18}$$

$$\tag{19}$$

C^- can be determined exactly,

$$C_n^- = (p_L - p_0)/(\Omega_n \rho_m d) \tag{20}$$

The approximate coefficients for the membrane modes are given by,

$$\widetilde{C}_n^0 = (\widetilde{C}_n^+ - C_n^-)/2 \quad (21)$$

$$\widetilde{C}_n^L = (\widetilde{C}_n^+ + C_n^-)/2 \quad (22)$$

Cylindrical Cavity

- In this section we consider the vibrations of two rigidly clamped circular membranes on either end of a cylindrical tube of length L.
- Our goal is to show that this is equivalent to the earlier formulation for a certain parameter range.