Analysis of the ICE Model

Anupam Prasad Vedurmudi

München 2013

Analysis of the ICE Model

Anupam Prasad Vedurmudi

Dissertation
an der Fakultät für Physik
der Ludwig–Maximilians–Universität
München

vorgelegt von
Anupam Prasad Vedurmudi
aus Kolkata, Indien

München, den May 17, 2013

Erstgutachter: Prof. Dr. J. Leo van Hemmen
Zweitgutachter: Zweitgutachter
Tag der mündlichen Prüfung: Prüfungsdatum

Contents

List of Figures

List of Tables

Abstract

Hier steht eine maximal einseitige Zusammenfassung der Dissertation.

Dies ist ein neuer Absatz.

Chapter 1

Introduction

Chapter 2

The ICE Model

Our goal is to model the middle-ear of the vertebrates in question in the simplest possible way while ensuring an accurately replication of its main properties. The main components of such a system are the mouth-cavity, the two tympani and the two extracolumellar footplates (one on each tympanum). In general, the shape of the mouth-cavity is highly irregular and therefore not conducive to an analytical treatment. Moreover, the system corresponds to a pair of second-order PDE's with moving boundaries. For this reason we will need to make further approximations for the sake of expediency.

2.1 Description

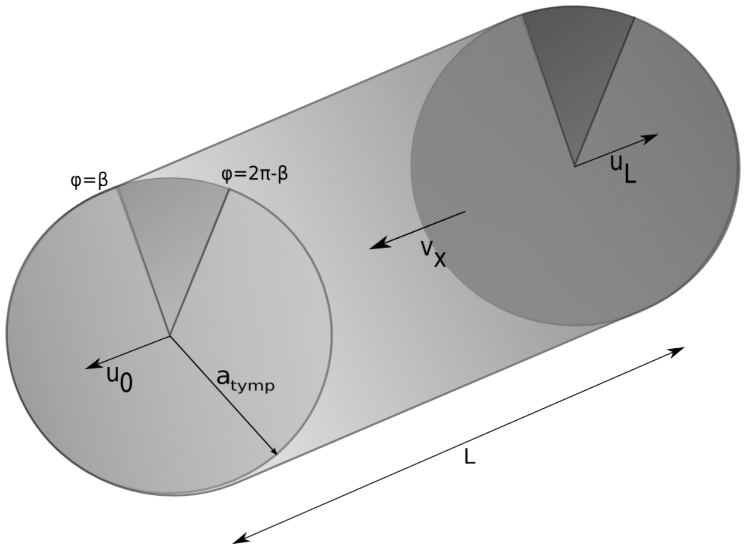
In the earlier treatment of the ICE model, the mouth canal is modelled as a simple cylinder closed at both ends by rigidly clamped (baffled) circular membranes; these model the tympanic membranes. As shown in [3] and [4], the length of the cylinder was chosen to be equal to the interaural distance and the radius of the model tympanum is determined from the typical area of the realistic tympanum.. The advantage of using a cylindrical cavity model for the mouth cavity is that the pressure distribution inside the cavity is easy to calculate - something that is even more important at higher frequencies as the pressure distribution inside the cavity is highly non-uniform.

The problem with this description is that, due to the small area of the tympani, the volume of the model cavity is an order of magnitude smaller than that of the actual mouth-cavity in the corresponding animal. In general, a smaller volume results in a stronger coupling - both in terms of an increased iTD and an increased iLD. For this reason, the earlier model overestimates the iTDs at low frequencies and the iLDs at high frequencies respectively.

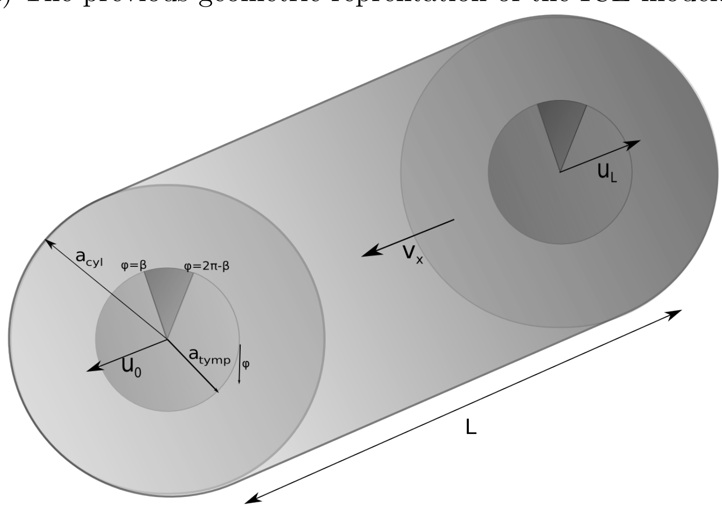
In order to get around this problem we make some slight modifications to the model. We basically maintain the cylindrical shape of the internal cavity but require it to have a volume which is equal to that of the realistic cavity. We maintain the same tympanum size and interaural distance and can therefore calculate the radius of the cylinder as,

$$a_{cyl} = \sqrt{\frac{V_{cav}}{\pi L}} \tag{2.1}$$

where a_{cyl} is the cylinder radius and L is the interaural distance. Simply put, the model



(a) The previous geometric reprentation of the ICE model.



(b) The representation of the new model.

Figure 2.1: The bold arrows represent the direction conventions. The darkly shaded region corresponds to the extracolumella.

consists of a cylindrical shell of radius a_{cyl} and length L with circular holes on either side with radius of the tympanic membrane, a_{tymp}. These holes are in turn closed by rigidly clamped circular membranes which will be subsequently described. The previous and current geometric representations of our model are shown in figures 2.1a and 2.1b.

2.1.1 Tympanic Membrane

The tympanic membrane or the eardrum is a thin membrane that effectively separates the outer ear and the middle ear. It is responsible for transmitting sound from the air.

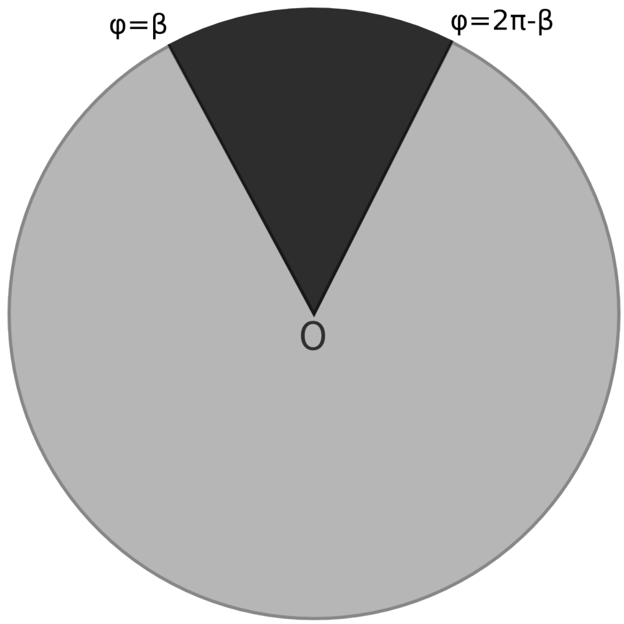


Figure 2.2: Model of the loaded tympanice membrane. The lightly shaded region is modelled as a linear elastic membrane whereas the darkly shaded region ($\beta < \phi < 2\pi - \beta$) represents the extracolumella.

2.1.2 Sound Input

2.2 Internal Cavity

We assume that the air inside the cavity obeys linear acoustics (briefly discussed in A). The pressure distribution inside the cavity is therefore given by the 3D acoustic wave-equation in cylindrical polar coordinates,

$$\frac{1}{c^2}\frac{\partial^2 p(x,r,\phi,t)}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p(x,r,\phi,t)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial p(x,r,\phi,t)}{\partial \phi^2} + \frac{\partial p(x,r,\phi,t)}{\partial x^2} \tag{2.2}$$

where c is the sound propagation velocity. The complete solution must take into account the boundary conditions at and within the cavity walls and the ones at the air-membrane interface. We also note that the above equation implies that the animal's mouth is closed, which is typical for a waiting animal. In order to solve (2.2) for a particular frequency f (angular frequency $\omega = 2\pi f$), we use the following separation ansatz

$$p(x,r,\phi,t) = f(x)g(r)h(\phi)e^{j\omega t} \tag{2.3}$$

which after substitution into the acoustic wave-equation leads to,

$$\begin{aligned} k^2 f(x)g(r)h(\phi)+f(x)h(\phi)&\left[\frac{\partial^2 g(r)}{\partial r^2}+\frac{1}{r}\frac{\partial g(r)}{\partial r}\right] \\ &+ f(x)g(r)\frac{1}{r^2}\frac{\partial h(\phi)}{\partial \phi}+\frac{\partial^2 f(x)}{\partial x^2}=0 \end{aligned} \tag{2.4}$$

with $k := \omega/c$. This yields the following set of equations (ODEs):

$$\frac{d^2 f(x)}{dx^2}+\zeta^2 f(x)=0 \tag{2.5}$$

$$\frac{d^2 h(\phi)}{d\phi^2}+q^2 h(\phi)=0 \tag{2.6}$$

$$\frac{\partial^2 g(r)}{\partial r^2}+\frac{1}{r}\frac{\partial g(r)}{\partial r}+\left[\underbrace{(k^2-\zeta^2)}_{=:\nu^2}-\frac{q^2}{r^2}\right]g(r)=0 \tag{2.7}$$

with separation constants q and ζ. The last equation is the Bessel differential equation [1, p. 313] and its general solution is given by,

$$g(r) = C_{qs}J_q(\nu r) + D_{qs}Y_q(\nu r). \tag{2.8}$$

J_q and Y_q are the order-q Bessel functions of the first and second kind respectively. The Bessel function of the second kind can be ignored as it diverges at $r = 0$. The solutions to the separated equations are therefore given by,

$$f(x) = e^{\pm\zeta x},\ h(\phi) = e^{\pm j\phi}, \text{and } g(r) = J_q(\nu r) \tag{2.9}$$

with a specific solution to (2.2) given by,

$$p(x, r, \phi; t) = \left[(A^+e^{jq\phi} + A^-e^{-jq\phi})e^{j\zeta x} + (B^+e^{jq\phi} + B^-e^{-jq\phi})e^{-j\zeta x}\right] J_q(\nu r)e^{j\omega t}. \tag{2.10}$$

The coefficients $A^\pm$, $B^\pm$, q, ζ and ν will be subsequently determined by the boundary conditions.

In general, the time component of the pressure also has a *backward-moving* component, i.e. $e^{-j\omega t}$. By making the ansatz in (2.3), we have implicitly made use of the fact that the form of the input constrains the pressure to only have a *forward-moving* component, i.e. $e^{j\omega t}$.

Pressure Boundary Conditions

There are three sets of boundary conditions -

- Continuity and smoothness in ϕ which is equivalent to $h(0) = h(2\pi)$ and $h'(0) = h'(2\pi)$ where, $h' = dh/d\phi$.

- Vanishing of the normal derivative at the cavity walls - $g'(a_C) = 0$ (a_C is the radius of the cylinder).

- Equating the membrane velocity to the air velocity at the membrane boundaries (to be discussed in the next section).

The first set of requirements is obvious. This reduces (2.10) to

$$p(x, r, \phi; t) = \left[Ae^{j\zeta x} + Be^{-j\zeta x}\right] \cos q\phi J_q(\nu r) e^{j\omega t}. \tag{2.11}$$

With q constrained to be an integer.

The second and third are a result of the so called "no-penetration" boundary-condition of fluid-mechanics. It arises from the from the fact that the cavity wall is an impermeable boundary. This translates into the requirement that the normal fluid-particle velocity should vanish ([2, p. 111]). The fluid-particle velocity ($\mathbf{v}$) is related to the pressure by,

$$-\rho\frac{\partial \mathbf{v}}{\partial t} = \nabla p \tag{2.12}$$

At the cylindrical cavity wall, the normal velocity is in the radial direction. Substituting the expression for pressure in (2.10) in the above equation leads to a Neumann boundary condition for the pressure,

$$\begin{aligned} v_r = -\frac{1}{j\rho\omega}\frac{\partial p(x, r, \phi; t)}{\partial r}\bigg|_{r=a_C} &= 0 \\ \Rightarrow \frac{\partial J_q(\nu r)}{\partial r}\bigg|_{r=a_C} &= 0 \end{aligned} \tag{2.13}$$

This constrains ν to a discrete set of values which correspond to the local minima and maxima of J_q. We can therefore index ν by q and $s = 0, 1, 2, 3, \ldots$ with $\nu_{qs} = z_{qs}/a_C$: z_{qs} being the s^{th} extremum of the order-q Bessel function of the first kind. This results in a discrete set of modes that satisfy (2.2) which are given by,

$$p_{qs}(x, r, \phi; t) = \left[A_{qs}e^{j\zeta_{qs}x} + B_q s e^{-j\zeta_{qs}x}\right] f_{qs}(r, \phi) e^{j\omega t} \tag{2.14}$$

where we have added the subscripts q and s to ζ and denoted the (r, ϕ) part of (2.11) by $f_{qs}(r, \phi)$. Effectively, the modes are 3D waves propagating with wave numbers ζ_{qs} in the x-direction and ν_{qs} in the radial direction. The first of these modes (corresponding to $q = 0, s = 0$) is of particular importance. Since the first maximum of J_0 occurs at $r = 0$, we have $\nu_{00} = 0$. This leads to the first mode being a plane-wave which is constant in r and ϕ and only varies in x.

A very useful property of the above modes is their orthgonality, i.e.

$$\int_{\Omega} dV p_{q_1 s_1} p_{q_2 s_2} = 0,\ if\ q_1 \neq q_2\ or\ s_1 \neq s_2 \tag{2.15}$$

the integral is over the volume of the cylinder. This is a consequence of the fact that for different q's the cosine parts of the modes are orthogonal whereas for a given q the Bessel parts are orthogonal for different s's or expressed as an equation,

$$\int dS f_{q_1 s_1} f_{q_2 s_2} = 0,\ q_1 \neq q_2\ or\ s_1 \neq s_2 \tag{2.16}$$

where $dS = rdrd\phi$ and the integral being taken over the disk of radius a_C. We can therefore write the general solution to (2.2) as a linear combination of the orthogonal modes given in (2.14),

$$p(x, r, \phi; t) = \sum_{q=0}^{\infty} \sum_{s=0}^{\infty} \left(A_{qs} e^{j\zeta_{qs} x} + B_{qs} e^{-j\zeta_{qs} x} \right) f_{qs}(r, \phi) e^{j\omega t} \tag{2.17}$$

The remaining coefficients, A_{qs} and B_{qs}, will be determined by equating the fluid-particle velocity to the membrane velocity at both ends of the cylinder. To do so, we will first need to find an expression for the membrane vibrations and subsequently make use of some simplifying approximations.

2.3 Vibration of the Membrane

As a preliminary exercise, we will first derive expressions for the free and force-driven vibrations of a circular membrane. We will then use our results to move on to the sectoral membrane which corresponds to the tympanum loaded by the extracolumella. Thi corresponds to the approximating the extracolumella to have infinite mass.

2.3.1 Circular Membrane

The equation of motion for the vibration of a rigidly clamped circular membrane of radius a_M solves for the membrane displacement u at a point (r, ϕ) with $r < a$ and $0 < \phi < 2\pi$. It is given by,

$$\begin{aligned} -\frac{\partial^2 u(r, \phi; t)}{\partial t^2} - 2\alpha \frac{\partial u(r, \phi; t)}{\partial t} + c_M^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u(r, \phi, t)}{\partial r} \right) + \frac{1}{r^2}\ \frac{\partial u(r, \phi, t)}{\partial \phi^2} \right] \\ = \frac{1}{\rho_M d} \Psi(r, \phi; t) \end{aligned} \tag{2.18}$$

subject to the boundary condition $u(r, \phi; t)|_{r=a_M} = 0.$. We've defined the following membrane material properties,

- c_M - propagation speed of vibrations.
- $\alpha (> 0)$ - the damping coefficient.
- ρ_M - density.

- d - thickness.

$\Psi(r, \phi; t)$ is the pressure on the membrane surface at (r, ϕ). In our discussion we are only concerned with periodic and uniform pressure acting on the membrane surface. This is justified by the fact that for typical hearing ranges of these animals, the wavelength of sound is much greater than the membrane size and any spatial variation can be neglected.

Free Vibrations

Undamped Membrane: We first determine the eigenmodes of an undamped circular membrane by solving (2.18) for $\alpha = 0$, $\Psi = 0$. To do this we make a separation ansatz just as we did in (2.3),

$$u(r, \phi; t) = f(r)g(\phi)h(t) \tag{2.19}$$

This gives us the following set of equations

$$\frac{\partial^2 f(r)}{\partial r^2} + \frac{1}{r}\frac{\partial f(r)}{\partial r} + \left[\mu^2 - \frac{m^2}{r^2}\right] f(r) = 0 \tag{2.20}$$

$$\frac{d^2 g(\phi)}{d\phi^2} + m^2 g(\phi) = 0 \tag{2.21}$$

$$\frac{d^2 h(t)}{dt^2} + c_M^2 \mu^2 h(t) = 0 \tag{2.22}$$

with separation constants μ and m. The solution of the first of these equations should already be familiar to us from the previous section - $J_m(\mu r)$, the order-m Bessel function of the first kind. The boundary conditions in ϕ direction remain the same resulting in,

$$u(r, \phi; t) = \left[\left(M^+ e^{jm\phi} + M^- e^{-jm\phi}\right) e^{jc_M \mu t} + (N^+ e^{jm\phi} + N^- e^{-jm\phi})e^{-jc_M \mu t}\right] J_m(\mu r) \tag{2.23}$$

Unlike in the case of the internal cavity, we require u to vanish at the boundary so we have a Dirichlet boundary condition which effectively requires: $J_m(\mu a_M) = 0$. This constrains μ to a discrete set of values which correspond to the zeros of J_m. The eigenmodes of a the circular membrane are therefore given by,

$$\begin{aligned} u_{mn}(r, \phi; t) = \big[(M_{mn}^+ e^{jm\phi} + M_{mn}^- e^{-jm\phi})\, e^{j\omega_{mn} t} + \\ (N_{mn}^+ e^{jm\phi} + N_{mn}^- e^{-jm\phi}) e^{-j\omega_{mn} t}\big]\, J_m(\mu_{mn} r) \end{aligned} \tag{2.24}$$

where $\mu_{mn} = z_{mn}/a_M$, z_{mn} being the n^{th} zero of J_m and, $\omega_{mn} = c_M \mu_{mn}$ is the eigenfrequency of the (m, n) eigenmode. At this point m can take any positive real value – a fact that will help us solve the sectoral membrane problem. However, in the case of a full circular membrane – as in the case of the pressure inside a cylindrical cavity – requirements of continuity and smoothness in ϕ reduce (2.24) to,

$$u_{mn}(r, \phi; t) = \cos m\phi J_m(\mu_{mn} r) \left[M_{mn} e^{j\omega_{mn} t} + N_{mn} e^{-j\omega_{mn} t}\right] \tag{2.25}$$

with $m = 0, 1, 2, \ldots$ with the (m, n) eigenmodes forming an orthogonal set. For later convenience we denote the spatial part of the above mode by $u_{mn}(r, \phi)$. We note that unlike in the case of the internal cavity, the free membrane has components that are both forward- and backward-moving in time. The presence of a driving force, however, will simplify the expressions.

Damped Membrane: For a damped membrane, i.e. $\alpha \neq 0$, the spatial part of the above eigenmodes remains the same. The form of the time-dependent part is given by the solution of the equation,

$$\frac{d^2 h_{mn}(t)}{dt^2} + 2\alpha \frac{dh_{mn}(t)}{dt} + \omega_{mn}^2 h_{mn}(t) = 0. \tag{2.26}$$

Assuming h_{mn} takes the form $e^{j\widetilde{\omega}_{mn}}$ leads to a quadratic equation with solutions,

$$\widetilde{\omega}_{mn} = j\alpha \pm \omega_{mn}^* \tag{2.27}$$

$$\omega_{mn}^* = \sqrt{\alpha^2 + \omega_{mn}^2} \tag{2.28}$$

We require the membrane displacement to remain finite as $t \to \infty$. We can therefore neglect the $e^{-j\widetilde{\omega}_{mn}}$ terms leading to,

$$\widetilde{u}_{mn}(r, \phi; t) = \cos m\phi J_m(\mu_{mn} r) \left[M_{mn} e^{j\omega_{mn}^* t} + N_{mn} e^{-j\omega_{mn}^* t} \right] e^{-\alpha t} \tag{2.29}$$

The general solution is given by a linear combination of the above and the coefficients are determined by initial conditions – for example, membrane displacement and velocity at $t = 0$.

Forced Vibrations

For a periodically driven membrane, there are two components of the full solution for forced vibrations. The first of these is the steady state solution which oscillates with the same frequency as the input and does not depend on the initial conditions - u_{ss}. The second of these is the transient solution that depends on the initial conditions but not on the driving pressure - u_t.

Steady State Solution: The steady state solution is expressed as a linear combination of the spatial part of the above eigenmodes and is given by,

$$u_{ss}(r, \phi; t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} C_{mn} \cos m\phi J_m(\mu_{mn} r) e^{j\omega t}. \tag{2.30}$$

Substituting this expression in (2.18) with $\Psi = pe^{j\omega t}$ gives,

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \Omega_{mn} C_{mn} \cos m\phi J_m(\mu_{mn} r) e^{j\omega t} = pe^{j\omega t} \tag{2.31}$$

$$\text{where, } \Omega_{mn} = \rho_M d(\omega^2 - 2j\alpha\omega - \omega_m^2 n). \tag{2.32}$$

Using the orthogonality of the eigenmodes, the coefficients C_{mn} can be calculated,

$$C_{mn} = \frac{p \int dS u_{mn}}{\Omega_{mn} \int dS u_{mn}^2} \tag{2.33}$$

with the integral this time being taken over the circular disk of radius a_M.

Transient Solution: The transient solution is effectively a solution of the free damped membrane, i.e. a linear combination of the eigenmodes given in (2.29). Therefore,

$$u_t(r, \phi; t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \cos m\phi J_m(\mu_{mn} r) \left[M_{mn} e^{j\omega^*_{mn} t} + N_{mn} e^{-j\omega^*_{mn} t} \right] e^{-\alpha t}. \tag{2.34}$$

The complete solution is given by $u = u_t + u_{ss}$ and the coefficients M_{mn} and N_{mn} are determined by the initial conditions (at $t = 0$).

The damping coefficient α is usually given in terms of the membrane fundamental frequency and a quality factor Q as $\alpha = \omega_{01}/2Q$. For the tympani we will be concerned with, $\alpha \sim 4000 s^{-1}$ for the larger lizards and $\alpha \sim 8000 s^{-1}$ for the smaller ones. Therefore, the transient response can be safely neglected for their hearing ranges.

2.3.2 Sectoral Membrane

The eigenmodes of the sectoral membrane proceeds from (2.24) onwards. We now have a new set of boundary conditions in ϕ. The extracolumella is modelled as a triangular plate of infinite mass which constrains the membrane displacement to go to zero at $phi = \beta$ and $\phi = 2\pi - \beta$. This results in the following set of eigenmodes,

$$u_{mn}(r, \phi; t) = \sin \kappa(\phi - \beta) J_\kappa(\mu_{mn} r) \left[M_{mn} e^{j\omega_{mn} t} + N_{mn} e^{-j\omega_{mn} t} \right] \tag{2.35}$$

where $\kappa[m] = \frac{m\pi}{2(\pi - \beta)}$, $m = 1, 2, 3, \ldots$. We see that the r part of the above mode is given by the order-κ Bessel function of the first kind; μ_{mn} being its n^{th} zero, as before. It is apparent from the form of the above modes that, unlike in the case of the circular membrane eigenmodes, these modes are no longer circularly symmetric. We plot the first few of these modes in increasing order of frequency.

Chapter 3

Analysis of the ICE–Model

Chapter 4

Conclusion

Appendix A

Acoustic Theory

Hier steht der erste Anhang.

Appendix B

Second Appendix Chapter

Hier kommt der zweite Anhang.

Bibliography

[1] E.T. Copson: *Introduction to the Theory of Functions of a Complex Variable*. Clarendon Press, Oxford, 1973.

[2] C. Pozrikidis: *Fluid Dynamics: Theory, Computation, and Numerical Simulation*. Springer, 2009.

[3] C. Voßen: *Auditory Information Processing in Systems with Internally Coupled Ears*. Technische Universität München, Dissertation, 2010.

[4] C. Voßen, J. Christensen-Dalsgaard und J.L. van Hemmen, *Journal of the Acoustical Society of America* **128** (2010), 909.

Acknowledgements

Danke.