1 Loaded Membrane

1.1 Loaded String Analogy

As an example of an extended vibrating object coupled to a rigid body we consider a vibrating string attached to a rigid rod of mass M_R at one end. The other ends of both the rod and the string are fixed and the rod can rotate about the axis through this end normal to the plane. The advantage of this system is that, under some reasonable physical assumptions, it can be solved exactly.

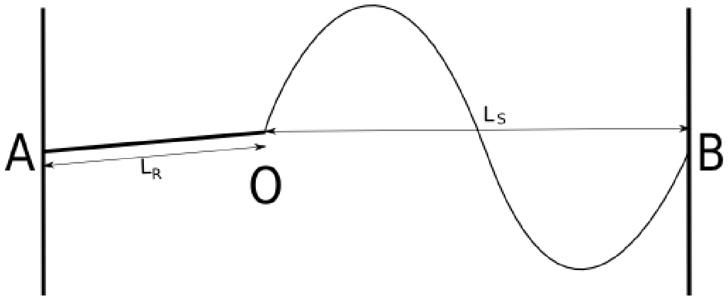

Figure 1: String Attached to a Rod of Finite Mass

In the above figure, we have a rigid rod of length L_R attached to a string of length L_S (at rest). For small vibration amplitudes, we can ignore the changes in the length of the rod. This means that the end corresponding to the point **O** moves along a straight line.

1.1.1 Modes of the Free String

The equation of motion of a freely vibrating string is given by,

$$\frac{\partial^2 z}{\partial t^2} = c_S^2 \frac{\partial^2 z}{\partial x^2} \tag{1}$$

Where, $z(x,t)$ is the displacement of the string at time t and at the position x. **O** corresponds to $x = 0$ and **B** to $x = L_S$. c_S is the propagation speed of transverse waves on the string. It is given in terms of the string tension, T and total mass, m_S as,

$$c_S = \sqrt{\frac{TL_S}{m_S}} \tag{2}$$

The reason for writing the above expression in this form is that the boundary condition at **O**, which in turn gives us a discretization condition for the modes can be written in terms of the ratio of the rod mass and string mass. Otherwise, it is traditionally written in terms of the mass per unit length as, $\sqrt{\frac{T}{\lambda}}$.

A specific solution to (1) is found by using separation of variables to be,

$$\begin{aligned} z(x,t) &= \left(Ae^{jkx} + Be^{-jkx}\right)e^{j\omega t} \\ \omega &= ck \end{aligned} \tag{3}$$

For any k. Applying the boundary condition $z(L_S,t) = 0$ gives us,

$$z(x,t) = \sin(k(L_S - x))e^{j\omega t} \tag{4}$$

Where we've ignored the coefficient A.

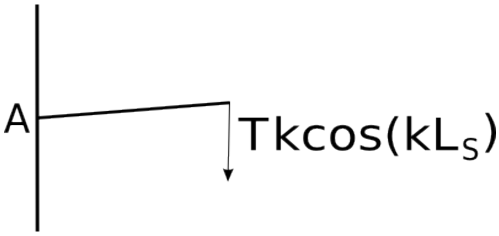

Figure 2: Motion of the Rod

2 Modelling the Loaded Membrane

We consider a tympanic membrane of radius a with the extra-columellar footplate situated between $0 < \phi < \beta$ and $2\pi - \beta \leq \phi \leq 2\pi$ (or equivalently between $-\beta < \phi < \beta$). The attached extracolummelar footplate is modelled by the following set of equations,

$$u_{0/L}(r, \phi, t) = \begin{cases} D_{0/L}\left(1 - \frac{r\cos\phi}{a\cos\beta}\right) & \text{if } 0 \leq r < a\cos\beta/\cos\phi \\ 0 & \text{if } a\cos\beta/\cos\phi \leq r \leq a \end{cases} \tag{5}$$

where, ϕ is in the region mentioned above. The above equations model the extracollumela as a triangular plate. The model is described in figure 3 - The triangle OAB hatches about the line AB and the striped section is rigid. The subscripts 0 and L denote the ipsi-lateral and contra-lateral membranes respectively. The coefficients $D_{0/L}$ will be determined later. This also requires the

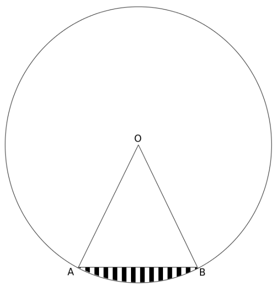

Figure 3: Loaded Membrane with Extracolumella situated in the sector OAB

membrane to have the same displacement for a given radius at the angles β and $2\pi - \beta$.

2.1 Infinite Mass Extracolumella

Analytically solving the system for a moving extracolumella is intractable. Hence, as an approximation, we assume that the Extracolumella has infinite mass, i.e., it remains stationary. This approximation gives us the advantage that the motion of the membrane is given as a linear combination of a new set of orthogonal modes,

$$f_{mn}(r, \phi) = sin[\mu(\phi - \beta)]J_\mu(k_{\mu n}r) \tag{6}$$

where $\mu = \frac{m\pi}{2(\pi-\beta)}$. This value arises because of the new membrane boundary condition which requires the membrane displacement to go to zero at $\phi = \beta, 2\pi - \beta$. We also make note of the fact that $\int dS f_{mn}(r, \phi) = 0$ for even m. This means that, analogous to the unloaded membrane, a uniform periodic pressure only excites the m-odd modes. The difference here is that the these modes are no longer circularly symmetric and their eigenfrequencies are more closely spaced than those of the unloaded circular membrane.