SOLVING THE COMPRESSIBLE EULER EQUATIONS IN TIMEDEPENDENT GEOMETRIES

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ABSTRACT. In this paper we shall present a numerical algorithm to solve the compressible Euler equations in three dimensional geometries with a moving boundary. The numerical algorithm consists of a new cell–centered upwind finite volume scheme of higher order on a grid of simplices and the possibility to refine and coarse the grid locally appropriate to the approximated solution. We will describe a new criterion to control the refinement and coarsening process. Furthermore we will describe briefly the idea to model the moving boundary without lack of conservation in the flux quantities.

1. Flow Problem

We want to solve the compressible Euler equations in geometries with inlets and outlets (cylinder model of a two–stroke–engine).

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In this geometry consisting of tetrahedra a piston is moving up and down. The piston motion will cause very high jumps e. g. in the pressure, if an inlet or outlet opens. This motivates the use of an upwind scheme. More details to this article can be found in the author's PhD thesis.

2. Definition of a New Higher Order Scheme in Space

In this section we will present an upwind finite volume scheme of higher order with the new limiter function Ansatz C on an unstructured grid of simplices. This new limiter function is motivated by the constraints in [10] to prove convergence in the case of scalar conservation laws. We improved the result in [10] to the effect that the B–triangulation [4] is no longer necessary to have a L^∞–bound (maximum principle) for this scheme. The new limiter function Ansatz C is built such that this bound exists without any additional restriction onto the triangulation and is simpler than the one proposed in [11]. The numerical scheme in use is an explicit upwind cell–centered finite volume scheme working on simplices T_i (first order in time, dim space dimension, g_{ij} numerical flux):

$$u_i^{n+1} := u_i^n - \frac{\Delta t^n}{|T_i|} \sum_{j=1}^{dim+1} g_{ij}(L_i^n(z_{ij}), L_{ij}^n(z_{ij})) \tag{2.2}$$

with $L_i^n(z_{ij}) = u_i^n$ and $L_{ij}^n(z_{ij}) = u_{ij}^n$ for all j to have the first order scheme and $L_i^n(z_{ij})$, $L_{ij}^n(z_{ij})$ as in the following definition to get a second order scheme.

Definition 2.1. (the limiter function Ansatz C) Let I be the index set of simplices. Construct for each simplex T_i, $i \in I$, a linear reconstruction function of the form

$$F_i^n(x) = u_i^n + s_i^n \cdot (x - x_i)$$

(for example with the approach in [1], [5], [7] or [9]). A concrete example will be given in the following note.
Let $i \in I$, $n \in IN$ be fixed and $j = 1, \ldots, dim + 1$.
Let

$$u_i^{max} := max\{u_i^n, u_{ik}^n, \ldots, u_{idim+1}^n\} \text{ and } u_i^{min} := min\{u_i^n, u_{ik}^n, \ldots, u_{idim+1}^n\}$$

the maximum and minimum of the values on simplex T_i and its neighbours.
We then define the values $L_i^n(z_{ij})$ by

$$L_i^n(z_{ij}) := u_i^n + \beta \cdot term(j)$$

where β and $term(j)$ are defined by the following algorithm:

$$term(j) := \begin{cases} s_i^n \cdot (z_{ij} - x_i), & \text{if } s_i^n \cdot (z_{ij} - x_i) \cdot (u_i^n - u_{ij}^n) \leq 0 \\ 0, & \text{otherwise .} \end{cases}$$

$$J^+ := \{j, term(j) > 0\} \qquad \text{and} \qquad J^- := \{j, term(j) < 0\}$$

if $(\#J^+ = 0)$ or $(\#J^- = 0)$ then

$\beta = 0$
else {
if $(\#J^+ + \#J^- \neq dim + 1)$ then {
$sum^{\pm} := \sum_{j \in J^{\pm}} term(j)$
if $(-sum^- > sum^+)$ then
$\gamma := \frac{sum^+}{-sum^-}$
for all $j \in J^-$: $term(j) := \gamma \cdot term(j)$
else
$\gamma := \frac{-sum^-}{sum^+}$
for all $j \in J^+$: $term(j) := \gamma \cdot term(j)$
}
determine largest $\beta \in [0,1]$ such that $\forall j$ and a fixed $\alpha \in (\frac{1}{2}, 1]$
a) $\beta \cdot |term(j)| \leq C_1 h^{\alpha}$
and
b) $u_i^{min} \leq u_i^n + \beta \cdot term(j) \leq u_i^{max}$
}

Note 2.1. i) Since the constant C_1 can not be adjusted automatically by the scheme for every individual problem the constraint a) in Definition 2.1 is dropped (see [14]).

ii) (example for the implementation of Ansatz C)
Determine values $\tilde{u}_{ij}^n, j = 1, \ldots, dim + 1$ at the vertices P_{ij} of the simplex T_i by averaging the piecewise constant values on the simplices with the vertex P_{ij}. This can be done for example with

$$\tilde{u}_{ij}^n = \frac{\sum_{k \in I, P_{ij} \in T_k} \frac{u_k^n}{r_k}}{\sum_{k \in I, P_{ij} \in T_k} \frac{1}{r_k}} \quad \text{and} \quad r_k = |P_{ij} - x_k|.$$

Let $\tilde{u}_{ij+1}^n, ..\tilde{u}_{ij+dim}^n$ be the values at the vertices of the j^{th} edge (2D) or surface (3D) of T_i. Then define

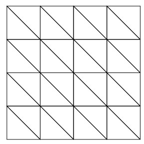
$$F_i^n(z_{ij}) := u_i^n + \frac{\frac{1}{dim} \sum_{m=1}^{dim} \tilde{u}_{ij+m}^n - \tilde{u}_{ij}^n}{dim + 1}. \tag{2.3}$$

This is the reconstruction suggested in [7].

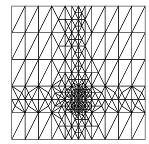
2.1. Numerical Order of Convergence. To value the new limiter function Ansatz C we compared it in [14] with other higher order approaches. Denote by u_h the approximated solution of the first order scheme, by u_h^C, u_h^S, u_h^F the approximated solutions of the second order schemes using Ansatz C, the superbee approach in [5] and the unlimited interpolation approach described in Note 2.1 ii). We made calculations on different kinds of triangulations for problems with smooth solutions and those containing discontinuities. We determined the "experimental order of convergence" (EOC). For the calculations of the Burgers equation we use the numerical flux defined by Engquist–Osher [6] and for the system of Euler equations the one of Steger–Warming [13].

2.1.1. *Approximation of a smooth solution of the Burgers equation.* Since there is a dependence of the order of consistency of the scheme on the form of the grid the calculations are performed on different grids with the following macro triangulations:

Grid 1:



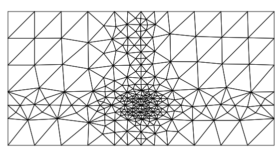
Grid 2:



On the structured grid 1 we obtain an EOC of one for the first order scheme and two for all three second order schemes. The results on the unstructured grid 2 are presented in the following table. For the second order schemes we used also a second order approximation in time as presented in [12].

h	$\|u - u_h\|_{L^1}$	EOC	$\|u - u_h^C\|_{L^1}$	EOC	$\|u - u_h^S\|_{L^1}$	EOC	$\|u - u_h^F\|_{L^1}$	EOC
0.22	0.0219		0.0065		0.0085		0.0061	
0.11	0.0075	1.54	0.0020	1.70	0.0033	1.36	0.0018	1.77
0.06	0.0031	1.28	0.0006	1.62	0.0013	1.37	0.0005	1.71
0.03	0.0015	1.03	0.0002	1.55	0.0005	1.50	0.0002	1.68

2.1.2. *Solution of the two–dimensional shock tube problem of the system of the Euler equations with one–dimensional structured solution.* In the next table the numerical errors in the density are presented corresponding to the calculations on the following grid.



h	$\|\rho - \rho_h\|_{L^1}$	EOC	$\|\rho - \rho_h^C\|_{L^1}$	EOC	$\|\rho - \rho_h^S\|_{L^1}$	EOC
0.15	0.3254		0.1934		0.2704	
0.08	0.2460	0.40	0.0995	0.96	0.1429	0.92
0.04	0.1713	0.52	0.0502	0.99	0.0752	0.93
0.02	0.1138	0.59	0.0267	0.91	0.0394	0.93

3. Local Adaption

We are using the local adaption algorithm on a grid of tetrahedra proposed in [2]. With this technique of bisection of tetrahedra it is very easy to project the data between two grids under the constraint of conservation. As criterion to mark regions for refinement and coarsening we used expressions of local residuals.

Definition 3.1. (of the **adaption criterion** used) Denote by L_i^n the linear reconstruction function to each conservative variable on the tetrahedron T_i determined with the unlimited reconstruction given in (2.3) and with $\mathcal{T}$ the grid of simplices. The approximated solution $\{u_i^n, T_i \in \mathcal{T}\}$ should be denoted by u_h^n. Set

$$S_i^n := \text{largest face of } T_i, \qquad r_i^n := \frac{u_i^{n+1} - u_i^n}{\Delta t^n} + \sum_{j=1}^{3} f_j'(u_i^n)\partial_j L_i^n(x_i)$$

with f_j the flux vector in the system of the Euler equations. Denote by $(r_i^n)_l$ the l^{th} component of the local residual r_i^n. Define

$$\eta_i(u_h^n) := \sqrt{|S_i^n|}|T_i| \sum_{l=1}^{5} |(r_i^n)_l| \qquad \text{and} \qquad \bar{\eta}^n := \frac{1}{\#\mathcal{T}} \sum_{i=1}^{\#\mathcal{T}} \eta_i(u_h^n).$$

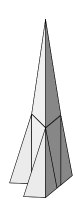
We then mark the tetrahedron T_i for refinement or coarsening as follows:

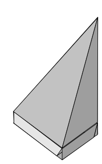
$if\,(\eta_i(u_h^n) > ref^n \cdot \bar{\eta}^n)$
  mark T_i for refinement
$if\,(\eta_i(u_h^n) < coa \cdot \bar{\eta}^n)$
  mark T_i for coarsening

The additional power of h ($\sqrt{|S_i^n|}$) is motivated in [14] to get the same scale for $||r_i^n||_{L^1(T_i)}$ and $||u(t^n) - u_h^n||_{L^1(T_i)}$. ref^n ($ref^0 = 0.9$) and $coa = 0.1$ are parameters to controll the adaption process.

4. Modelling of the Piston Motion in a Conservative Way

A lot of effort would be necessary to administrate the cell connectivities at the cell faces between cylinder and ports while the points of an unstructured grid are moving. Furthermore the conservation property of the scheme is complicated to handle because of the interpolation between different meshes and because the degeneration of the tetrahedra must be avoided by a grid smoother. Therefore we propose a new approach on a grid of tetrahedra: the piston motion is modelled by cutting the piston at a position depending on the time out of a fixed grid. Connected with this approach are two problems: First of all the cutting of tetrahedra can cause very small cells. The second problem appears while the piston is moving down: cells of the fixed grid in the previous timestep completely lying inside the piston can be exposed in the next timestep. This means that we have no new values in the new timestep for those exposed cells. In [3] (using the same idea of merging cells on structured grids) these two problems are called the Vanishing–Cell-Problem and the Newborn–Cell–Problem. A solution to these two problems are the **collected cells**, a union of cut and uncut tetrahedra, as shown the following examples.





5. Numerical Results

a) Forward facing step problem in 3D

b) Considering the vector field on some cutting planes in the cylinder model of a two–stroke–engine: the cutting plane in the left figure is near the symmetry plane of the geometry

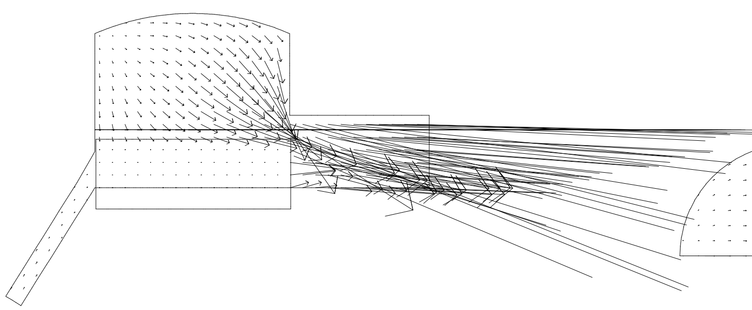


Figure 1. Piston motion down

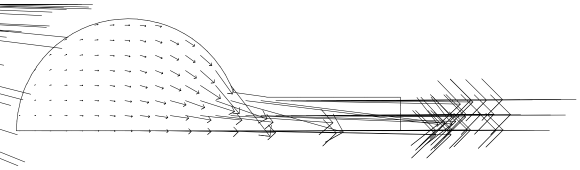


Figure 2. Orthogonal plane at z= 0.25

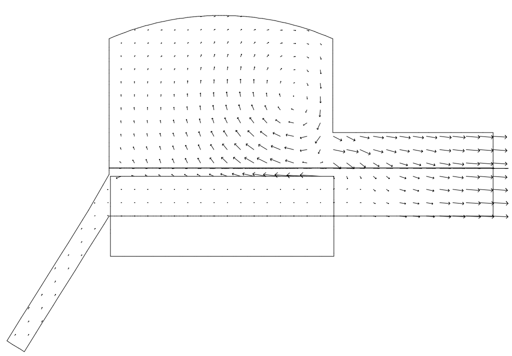


Figure 3. Piston motion up

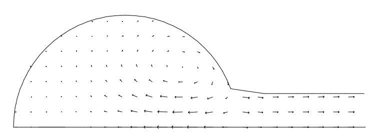


Figure 4. Orthogonal plane at z= 0.18

The little balls lying on the particle traces in the following figure correspond to the position of the particles at the moment where the piston is completely down. The same holds for the depicted velocity field. The particles starting in the upper right corner of the cylinder clarify that this geometry is not optimal with respect to the exchange of "old" and "new" gas.

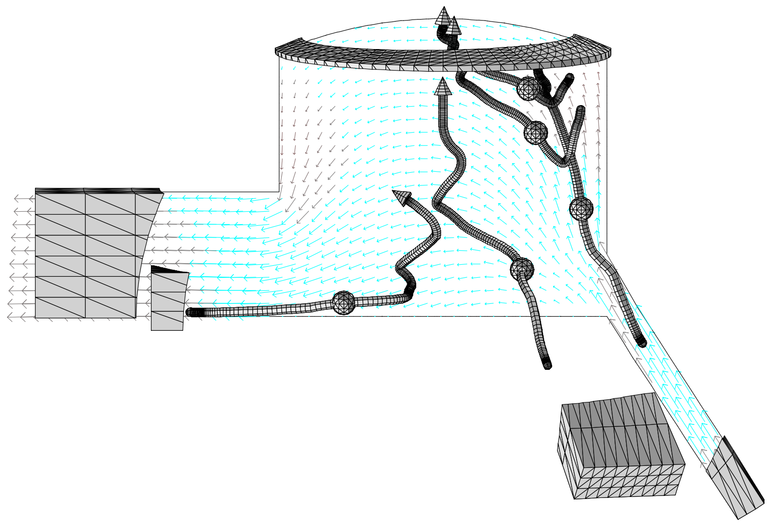


FIGURE 5. Particles during the movement of the piston down and up

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