Analysis of the ICE Model

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Abstract

Hier steht eine maximal einseitige Zusammenfassung der Dissertation.

Dies ist ein neuer Absatz.

Chapter 1

Introduction

Chapter 2

The ICE Model

Our goal is to model the middle-ear of the vertebrates in question in the simplest possible way while ensuring an accurately replication of its main properties. The main components of such a system are the mouth-cavity, the two tympani and the two extracollumellar footplates (one on each tympanum). In general, the shape of the mouth-cavity is highly irregular and therefore not conducive to an analytical treatment. Moreover, the system corresponds to a pair of second-order PDE's with moving boundaries. For this reason we will need to make further approximations for the sake of expediency.

2.1 Description

In the earlier treatment of the ICE model, the mouth canal is modelled as a simple cylinder closed at both ends by rigidly clamped (baffled) circular membranes. As shown in [2], The length of the cylinder was chosen to be equal to the interaural distance. The advantage of using a cylindrical cavity model for the mouth cavity is that the pressure distribution inside the cavity is easy to calculate - something that is even more important at higher frequencies as the pressure distribution inside the cavity is highly non-uniform.

The problem with this description is the fact that the volume of the model's cavity is an order of magnitude smaller than that of the mouth-cavity in the corresponding animal with similar tympani and a the same interaural distance. In general, a smaller volume results in a stronger coupling - both in terms of an increased iTD and an increased iLD. For this reason, the earlier model overestimates the iTDs at low frequencies and the iLDs at high frequencies respectively.

2.2 Sound Input

2.3 Internal Cavity

We assume that the air inside the cavity obeys linear acoustics (briefly discussed in A). The pressure distribution inside the cavity is therefore given by the 3D acoustic wave-equation in cylindrical polar coordinates,

$$\frac{1}{c^2}\frac{\partial^2 p(x,r,\phi,t)}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p(x,r,\phi,t)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial p(x,r,\phi,t)}{\partial \phi^2} + \frac{\partial p(x,r,\phi,t)}{\partial x^2} \tag{2.1}$$

where c is the sound propagation velocity. The complete solution must take into account the boundary conditions at and within the cavity walls and the ones at the air-membrane interface. We also note that the above equation implies that the animal's mouth is closed, which is typical for a waiting animal. In order to solve (2.1) for a particular frequency f (angular frequency $\omega = 2\pi f$), we use the following separation ansatz

$$p(x,r,\phi,t) = f(x)g(r)h(\phi)e^{j\omega t} \tag{2.2}$$

which after substitution into the acoustic wave-equation leads to,

$$\begin{aligned} k^2 f(x)g(r)h(\phi) + f(x)h(\phi)&\left[\frac{\partial^2 g(r)}{\partial r^2} + \frac{1}{r}\frac{\partial g(r)}{\partial r}\right] \\ &+ f(x)g(r)\frac{1}{r^2}\frac{\partial h(\phi)}{\partial \phi} + \frac{\partial^2 f(x)}{\partial x^2} = 0 \end{aligned} \tag{2.3}$$

with $k := \omega/c$. This yields the following set of equations (ODEs):

$$\frac{d^2 f(x)}{dx^2} + \zeta^2 f(x) = 0 \tag{2.4}$$

$$\frac{d^2 h(\phi)}{d\phi^2} + q^2 h(\phi) = 0 \tag{2.5}$$

$$\frac{\partial^2 g(r)}{\partial r^2} + \frac{1}{r}\frac{\partial g(r)}{\partial r} + \left[\underbrace{(k^2 - \zeta^2)}_{=:\nu^2} - \frac{q^2}{r^2}\right] = 0 \tag{2.6}$$

with separation constants q and ζ. The last equation is the Bessel differential equation [1, p. 313] and its general solution is given by,

$$g(r) = C_{qs}J_q(\nu r) + D_{qs}Y_q(\nu r) \tag{2.7}$$

where J_q and Y_q are the order-q Bessel functions of the first and second kind respectively. The Bessel function of the second kind can be ignored as it diverges at $r = 0$. The solutions to the separated equations are therefore given by,

$$f(x) = e^{\pm\zeta x},\ h(\phi) = e^{\pm j\phi}, \text{and } g(r) = J_q(\nu r) \tag{2.8}$$

with a specific solution to (2.1) given by,

$$p(x,r,\phi;t) = \left[(A^+e^{jq\phi} + A^-e^{-jq\phi})e^{j\zeta x} + (B^+e^{jq\phi} + B^-e^{-jq\phi})e^{-j\zeta x}\right] J_q(\nu r)e^{j\omega t}. \quad (2.9)$$

The coefficients $A^\pm$, $B^\pm$, q, ζ and ν will be subsequently determined by the boundary conditions.

Pressure Boundary Conditions

There are three sets of boundary conditions -

- Continuity and smoothness in ϕ which is equivalent to $h(0) = h(2\pi)$ and $h'(0) = h'(2\pi)$ where, $h' = dh/d\phi$.
- Vanishing of the normal derivative at the cavity walls - $g'(a_{cyl}) = 0$ (a_{cyl} is the radius of the cylinder).
- Equating the membrane velocity to the air velocity at the membrane boundaries (to be discussed in the next section).

The first set of requirements is obvious. This reduces (2.9) to

$$p(x,r,\phi;t) = \left[Ae^{j\zeta x} + Be^{-j\zeta x}\right] \cos q\phi J_q(\nu r)e^{j\omega t}. \quad (2.10)$$

With q constrained to be an integer.

The second and third are a result of the so called "no-penetration" boundary-condition of fluid-mechanics. It arises from the from the fact that the cavity wall is an impermeable boundary. This translates into the requirement that the normal fluid-particle velocity should vanish. The fluid-particle velocity ($\mathbf{v}$) is related to the pressure by,

$$-\rho\frac{\partial \mathbf{v}}{\partial t} = \nabla p \quad (2.11)$$

At the cylindrical cavity wall, the normal velocity is in the radial direction. Substituting the expression for pressure in (2.9) in the above equation leads to a Neumann boundary condition for the pressure,

$$\begin{aligned} v_r = -\frac{1}{j\rho\omega}\frac{\partial p(x,r,\phi;t)}{\partial r}\bigg|_{r=a} &= 0 \\ \Rightarrow \frac{\partial J_q(\nu r)}{\partial r}\bigg|_{r=a} &= 0 \end{aligned} \quad (2.12)$$

This constrains ν to a discrete set of values which correspond to the local minima and maxima of J_q. We can therefore index ν by q and $s = 0, 1, 2, 3, \ldots$ with $\nu_{qs} = z_{qs}/a$: z_{qs} being the s^{th} extremum of the order-q Bessel function of the first kind. This results in a discrete set of modes that satisfy (2.1) which are given by,

$$p(x,r,\phi;t) = \left[A_{qs}e^{j\zeta_{qs}x} + B_q s e^{-j\zeta_{qs}x}\right] \cos q\phi J_q(\nu_{qs}r)e^{j\omega t}. \quad (2.13)$$

where we have added the subscripts q and s to ζ. Effectively, the modes are 3D waves propagating with wave numbers ζ_{qs} in the x-direction and ν_{qs} in the radial direction. The first of these modes (corresponding to $q = 0, s = 0$) is of particular importance. Since the first maximum of J_0 occurs at $r = 0$, we have $\nu_{00} = 0$. This leads to the first mode being a plane-wave which is constant in r and ϕ and only varies in x.

We can therefore write the general solution to (2.1) as a linear combination of the specific solutions given in (2.13),

$$p(x, r, \phi; t) = \sum_{q=0}^{\infty} \sum_{s=0}^{\infty} \left(A_{qs} e^{j\zeta_{qs}x} + B_{qs} e^{-j\zeta_{qs}x} \right) \cos(m\phi) J_q(\nu_{qs} r) e^{j\omega t} \tag{2.14}$$

The remaining coefficients, A_{qs} and B_{qs}, will be determined by equating the fluid-particle velocity to the membrane velocity at both ends of the cylinder. To do so, we will first need to find an expression for the membrane vibrations and subsequently make use of some simplifying approximations.

2.4 Vibration of the Membrane

As a preliminary exercise, we will first derive expressions for the free and force-driven vibrations of a circular membrane. We will then use our results to move on to the sectoral membrane which corresponds to the tympanum loaded by the extracollumella. Physically, this means we have assumed the extracollumella to have infinite mass.

2.4.1 Circular Membrane

The equation of motion for the vibration of a rigidly clamped circular membrane of radius a solves for the membrane displacement u at a point (r, ϕ) with $r < a$ and $0 < \phi < 2\pi$. It is given by,

$$-\frac{\partial^2 u(r,\phi,t)}{\partial t^2} - 2\alpha \frac{\partial u(r,\phi,t)}{\partial t} + c_m^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u(r,\phi,t)}{\partial r} \right) + \frac{1}{r^2}\frac{\partial u(r,\phi,t)}{\partial \phi^2} \right] = \frac{1}{\rho_m d}\Psi(r,\phi,t) \tag{2.15}$$

subject to the boundary condition $u(r, \phi, t)|_{r=a} = 0.$. We've defined the following membrane material properties,

- c_m - propagation speed of vibrations.
- α - the damping coefficient.
- ρ_m - density.
- d - thickness.

$\Psi(r, \phi; t)$ is the pressure on the membrane surface at (r, ϕ). In our discussion we are only concerned with periodic and uniform pressure acting on the membrane surface. This is justified by the fact that for typical hearing ranges of these animals, the wavelength of sound is much greater than the membrane size and any spatial variation can be neglected.

Free Undamped Vibrations

We first determine the eigenmodes of a circular membrane by solving (2.15) for $\alpha = 0,\ \Psi = 0$. To do this we make a separation ansatz just as we did in (2.2),

$$u(r, \phi, t) = f(r)g(\phi)h(t) \tag{2.16}$$

Forced Vibrations

2.4.2 Sectoral Membrane

Chapter 3

Analysis of the ICE–Model

Chapter 4

Conclusion

Appendix A

Acoustic Theory

Hier steht der erste Anhang.

Appendix B

Second Appendix Chapter

Hier kommt der zweite Anhang.

Bibliography

[1] E.T. Copson: *Introduction to the Theory of Functions of a Complex Variable*. Clarendon Press, Oxford, 1973.

[2] C. Voßen: *Auditory Information Processing in Systems with Internally Coupled Ears*. Technische Universität München, Dissertation, 2010.

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Danke.