ICE Model - Analytic Solution For A Cylindrical Cavity

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In the ICE model, the mouth canal is assumed to be a simple cylinder closed at both ends by rigidly clamped circular membranes. We define (m, n) as the modes of the homogeneous cylindrical wave equation such that,

$$(m, n) \equiv cos(m\phi) J_m (\mu_{mn} r) \tag{1}$$

where J_m is the Bessel function of the first kind of order m and $J_m(\mu_{mn} a) = 0$. Here, $\mu_{mn} a$ is the n^{th} zero of J_m.

0.1 Internal Cavity

The propagation of a pressure disturbance p in the internal cavity is assumed to be defined by the following equation

$$\frac{1}{c^2}\frac{\partial^2 p(x,r,\phi,t)}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p(x,r,\phi,t)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p(x,r,\phi,t)}{\partial \phi^2} + \frac{\partial^2 p(x,r,\phi,t)}{\partial x^2} \tag{2}$$

i.e., the wave equation in cylindrical coordinates. c is the speed of sound in air. We can use a separation ansatz, $p(x, r, \phi, t) = f(x)g(r)h(\phi)e^{j\omega t}$ to find a particular solution to this equation; this solution is given by,

$$\begin{aligned} p(x,r,\phi,t) = \big[&\left(A^{+}_{qs} e^{j\zeta_{qs}x} + B^{+}_{qs} e^{-i\zeta_{qs}x}\right) e^{jq\phi} \\ &+ \left(A^{-}_{qs} e^{j\zeta_{qs}x} + B^{-}_{qs} e^{-i\zeta_{qs}x}\right) e^{-iq\phi}\big] J_p(\nu_{qs} r) e^{j\omega t} \end{aligned} \tag{3}$$

Here, $g(r) = J_p(\nu_{qs} r)$ is the order p Bessel function of the first kind wich satisfies the following second order linear ODE

$$\frac{\partial^2 g(r)}{\partial r^2} + \frac{1}{r}\frac{\partial g(r)}{\partial r} + \left(\nu_{qs}^2 - \frac{q^2}{r^2}\right) g(r) = 0 \tag{4}$$

where,

$$k = \omega / c \tag{5}$$
$$\zeta_{qs}^2 = k^2 - \nu_{qs}^2 \tag{6}$$

The velocity of air (physically, the velocity of the fluid particle) in the x-direction is given by

$$\rho \frac{\partial v_x}{\partial t} = -\nabla_x p \tag{7}$$

where ρ is the density of the air inside the cavity. Since (2) is a second order PDE in each of its variables, we require two boundary conditions for each of them to completely solve it, i.e. to determine all the coefficients in (3). We require the pressure and its derivative to be periodic in ϕ,

$$p(x, r, \phi, t) = p(x, r, \phi + 2\pi, t) \tag{8}$$

$$\partial_\phi p(x, r, \phi, t) = \partial_\phi p(x, r, \phi + 2\pi, t) \tag{9}$$

As a result, q is required to be an integer and (3) reduces to

$$p(x, r, \phi, t) = \left[A_{qs} e^{j\zeta_{qs}x} + B_{qs} e^{-i\zeta_{qs}x}\right] \cos(q\phi) J_q(\nu_{qs} r) e^{j\omega t} \tag{10}$$

Finally, we require that the velocity of the fluid particle normal to the cylindrical boundary at $r = a$ disappears. This is due to the requirement that the boundary is solid and the fluid does not penetrate it. This means that,

$$\left.\frac{\partial J_q(\nu_{qs} r)}{\partial r}\right|_{r=a} = 0 \tag{11}$$

As a result $\nu_{qs} = \widetilde{k}_{qs}/a$, i.e. ν_{qs} corresponds to the s^{th} zero of J'_q. The general solution is given by linear combinations of (10). We also note that there exists a plane wave solution to (2) which corresponds to $\nu_{00} = 0$. This is given by

$$p(x, r, \phi; t) = \left[A_{00} e^{jkx} + B_{00} e^{-jkx}\right] e^{j\omega t} \tag{12}$$

0.2 Vibration of Coupled Unloaded Membranes

We first treat the case in which we have two circular coupled by a cylindrical cavity. The forcing on both tympani is given by

$$p_0 = p e^{-jk\frac{L}{2}\sin\theta} \tag{13}$$

$$p_L = p e^{jk\frac{L}{2}\sin\theta} \tag{14}$$

In this example we attempt to model the case of the animal to a free field stimulus. This means that the sound on both tympani has the same amplitude but differ in phase by, $kL\sin\theta$ where, θ is the angle the sound source makes with the central axis of the head. We can assume that the instantaneous pressure is constant over the tympanum as its dimensions are much smaller than the wavelength of the sound wave. The pressure inside the cavity is given by,

$$p(x, r, \phi, t) = \sum_{q,s} \left[A_{qs} e^{j\zeta_{qs}x} + B_{qs} e^{-j\zeta_{qs}x}\right] \cos(q\phi) J_p(\nu_{qs} r) e^{j\omega t} \tag{15}$$

The equation of motion of the membranes gives us,

$$\sum_{m,n} \left(\omega^2 - 2j\alpha\omega - \omega_{mn}^2\right) C_{mn}^0 \cos(m\phi) J_m(\mu_{mn} r) = \left[p_0 - p(0, r, \phi, t)\right] / (\rho_M d) \tag{16}$$

$$\sum_{m,n} \left(\omega^2 - 2j\alpha\omega - \omega_{mn}^2\right) C_{mn}^L \cos(m\phi) J_m(\mu_{mn} r) = \left[p_L - p(L, r, \phi, t)\right] / (\rho_M d) \tag{17}$$

where as usual, 0 and L refer to the ipsi- and contralateral membranes respectively. We then equate the the velocity of air given in (7) to the velocity of the membrane surface at that point.

$$\rho\omega^2 \sum_{m,n} C^0_{mn} \cos(m\phi) J_m(\mu_{mn} r) = \sum_{q,s} j\zeta_{qs} \left[A_{qs} - B_{qs}\right] \cos(q\phi) J_p(\nu_{qs} r) \tag{18}$$

$$\rho\omega^2 \sum_{m,n} C^L_{mn} \cos(m\phi) J_m(\mu_{mn} r) = -\sum_{q,s} j\zeta_{qs} \left[A_{qs} e^{j\zeta_{qs}L} - B_{qs} e^{-j\zeta_{qs}L}\right] \cos(q\phi) J_p(\nu_{qs} r) \tag{19}$$

This is equivalent (11) as we've set the relative velocity of the air normal to the membrane surface to zero. We note that we have also used the direction conventions mentioned in the previous section.

First we use (18) and (19) to express the cavity coefficients in terms of the membrane coefficients. This gives us,

$$A_{ms} = \frac{\rho\omega^2}{2\zeta_{ms}\sin(\zeta_{ms}L)} \sum_l \left[C^0_{ml} e^{-j\zeta_{ms}L} + C^L_{ml}\right] \lambda_{msl} \tag{20}$$

$$B_{ms} = \frac{\rho\omega^2}{2\zeta_{ms}\sin(\zeta_{ms}L)} \sum_l \left[C^0_{ml} e^{j\zeta_{ms}L} + C^L_{ml}\right] \lambda_{msl} \tag{21}$$

where we've introduced,

$$\lambda_{msl} = \frac{\int_0^a r dr J_m(\nu_{ms} r) J(\mu_{ml} r)}{\int_0^a r dr J^2_m(\nu_{ms} r)}$$

We have also made use of the fact that $\int_0^{2\pi} \cos(m\phi)\cos(k\phi) = 0$ for $m \neq k$. This tells us that the J_m modes on the membrane only couple to the J_m modes in the cavity for a given m and vice versa. Also, in the above summations, l ranges from 1 to ∞ while n can also be 0. We now substitute the above expressions into (16) and (17) to get,

$$\sum_{m,n} \Omega_{mn} C^0_{mn} \cos(m\phi) J_m(\mu_{mn} r) = p_0 - \sum_{q,s} \sum_l \left[\Lambda_{qs} C^0_{ql} + \Gamma_{qs} C^L_{ql}\right] \lambda_{qsl} f_{qs}(r,\phi) \tag{22}$$

$$\sum_{m,n} \Omega_{mn} C^L_{mn} \cos(m\phi) J_m(\mu_{mn} r) = p_L - \sum_{q,s} \sum_l \left[\Gamma_{qs} C^0_{ql} + \Lambda_{qs} C^L_{ql}\right] \lambda_{qsl} f_{qs}(r,\phi) \tag{23}$$

where we have defined

$$\Omega_{mn} = \rho_M d \left(\omega^2 - 2j\alpha\omega - \omega^2_{mn}\right), \tag{24}$$

$$\Lambda_{qs} = \rho\omega^2 \frac{\cot(\zeta_{qs}L)}{\zeta_{qs}}, \tag{25}$$

$$\Gamma_{qs} = \frac{\rho\omega^2}{\zeta_{qs}\sin(\zeta_{qs}L)}, \tag{26}$$

$$f_{qs}(r,\phi) = \cos(q\phi) J_q(\nu_{qs} r) \tag{27}$$

We now multiply both sides of (22) and (23) with the membrane modes and integrate over the circle to get,

$$\Omega_{mn}C^0_{mn} = p_0 K_{mn} - \sum_{s,l}\left[\Lambda_{ms}C^0_{ml} + \Gamma_{ms}C^L_{ml}\right]\lambda_{msl}\widetilde{\lambda}_{msn} \tag{28}$$

$$\Omega_{mn}C^L_{mn} = p_L K_{mn} - \sum_{s,l}\left[\Gamma_{ms}C^0_{ml} + \Lambda_{ms}C^L_{ml}\right]\lambda_{msl}\widetilde{\lambda}_{msn} \tag{29}$$

where, as usual, we have made the following definitions for aesthetic reasons,

$$K_{mn} = \frac{\int_0^a dS cos(m\phi)J_m(\mu_{mn}r)}{\int_0^a dS cos^2(m\phi)J^2_m(\mu_{mn}r)} \tag{30}$$

$$\widetilde{\lambda}_{msn} = \frac{\int_0^a rdr J_m(\nu_{ms}r)J(\mu_{mn}r)}{\int_0^a rdr J^2_m(\mu_{mn}r)} \tag{31}$$

In order to get a more symmetric looking expression, we can redefine λ and $\widetilde{\lambda}$ in the following way,

$$\lambda_{msl}\widetilde{\lambda}_{msn} = \widehat{\lambda}_{msl}\widehat{\lambda}_{msn} \tag{32}$$

$$\widehat{\lambda}_{msl} = \frac{\int_0^a rdr J_m(\nu_{ms}r)J_m(\mu_{ml}r)}{\sqrt{\int_0^a rdr J^2_m(\mu_{ml}r)\int_0^a rdr J^2_m(\nu_{ms}r)}} \tag{33}$$

We see that I_{mn} vanishes for $m \geq 1$. This means that we can ignore these modes. We, therefore neglect the terms which have $m \neq 0$ to get,

$$\Omega_{0n}C^0_{0n} = p_0 K_{0n} - \sum_{s,l}\left[\Lambda_{0s}C^0_{0l} + \Gamma_{0s}C^L_{0l}\right]\widehat{\lambda}_{0sl}\widehat{\lambda}_{0sn} \tag{34}$$

$$\Omega_{0n}C^L_{0n} = p_L K_{0n} - \sum_{s,l}\left[\Gamma_{0s}C^0_{0l} + \Lambda_{0s}C^L_{0l}\right]\widehat{\lambda}_{0sl}\widehat{\lambda}_{0sn} \tag{35}$$

We now define a new set of variables, $C^+_n = C^L_{0n} + C^0_{0n}$ and $C^-_n = C^L_{0n} - C^0_{0n}$ and add and subtract the above equations to get a system in terms of the newly defined variables,

$$\Omega_{0n}C^+_n + \sum_{s,l}\left[\Lambda_{0s} + \Gamma_{0s}\right]C^+_l\widehat{\lambda}_{0sl}\widehat{\lambda}_{0sn} = (p_L + p_0)K_{0n} \tag{36}$$

$$\Omega_{0n}C^-_n + \sum_{s,l}\left[\Lambda_{0s} - \Gamma_{0s}\right]C^-_l\widehat{\lambda}_{0sl}\widehat{\lambda}_{0sn} = (p_L - p_0)K_{0n} \tag{37}$$

0.3 Comparison

We now want to compare our analysis with the previous one where the cavity was said to have an arbitrary shape. Since we reduced both systems to the same form, we can go ahead and directly compare the respective coefficients.