ICE Model - Analytic Solution For Loaded Membrane

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In the ICE model, the mouth canal is assumed to be a simple cylinder closed at both ends by rigidly clamped circular membranes. We define (m,n) as the modes of the homogeneous cylindrical wave equation such that,

$$(m,n) \equiv cos(m\phi) J_m\left(\mu_{mn} r\right) \tag{1}$$

Where J_m is the Bessel function of the first kind of order m and $J_m\left(\mu_{mn} a\right) = 0$. Here, $\mu_{mn} a$ is the n^{th} zero of J_m.

0.1 Internal Cavity

The propagation of a pressure disturbance p in the internal cavity is assumed to be defined by the following equation,

$$\frac{1}{c^2}\frac{\partial^2 p(x,r,\phi,t)}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p(x,r,\phi,t)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p(x,r,\phi,t)}{\partial \phi^2} + \frac{\partial^2 p(x,r,\phi,t)}{\partial x^2} \tag{2}$$

i.e., the wave equation in cylindrical coordinates. c is the speed of sound in air. We can use a separation ansatz, $p(x,r,\phi,t) = f(x)g(r)h(\phi)e^{j\omega t}$ to find a particular solution to this equation; this solution is given by,

$$\begin{aligned} p(x,r,\phi,t) = \big[&\left(A^+_{qs} e^{j\zeta_{qs}} + B^+_{qs} e^{-i\zeta_{qs}}\right) e^{jq\phi} \\ &+ \left(A^-_{qs} e^{j\zeta_{qs}} + B^-_{qs} e^{-i\zeta_{qs}}\right) e^{-iq\phi}\big] J_p(\nu_{qs} r) e^{j\omega t} \end{aligned} \tag{3}$$

Here, $g(r) = J_p(\nu_{qs} r)$ is the order p Bessel function of the first kind wich satisfies the following second order linear ODE,

$$\frac{\partial^2 g(r)}{\partial r^2} + \frac{1}{r}\frac{\partial g(r)}{\partial r} + \left(k^2_{qs} - \frac{q^2}{r^2}\right) g(r) = 0 \tag{4}$$

Where,

$$k = \omega/c. \tag{5}$$

$$\zeta^2_{qs} = k^2 - k^2_{qs} \tag{6}$$

The velocity of air (physically, the velocity of the fluid particle) in the x-direction is given by,

$$\rho\frac{\partial v_x}{\partial t} = -\nabla_x p \tag{7}$$

Where, ρ is the density of the air inside the cavity. Since (2) is a second order PDE in each of its variables, we require two boundary conditions for each of them to completely solve it, i.e. to determine all the coefficients in (3). We require the pressure and its derivative to be periodic in ϕ,

$$p(x, r, \phi, t) = p(x, r, \phi + 2\pi, t) \tag{8}$$

$$\partial_\phi p(x, r, \phi, t) = \partial_\phi p(x, r, \phi + 2\pi, t) \tag{9}$$

As a result, q is required to be an integer and (3) reduces to

$$p(x, r, \phi, t) = \left[A_{qs} e^{j\zeta_{qs} x} + B_{qs} e^{-i\zeta_{qs} x}\right] \cos(q\phi) J_q(\nu_{qs} r) e^{j\omega t} \tag{10}$$

Finally, we require that the velocity of the fluid particle normal to the cylindrical boundary at $r = a$ disappears. This is due to the requirement that the boundary is solid and the fluid does not penetrate it. This means that,

$$\left.\frac{\partial J_q(\nu_{qs} r)}{\partial r}\right|_{r=a} = 0 \tag{11}$$

As a result $\nu_{qs} = \widetilde{k}_{qs}/a$, i.e. ν_{qs} corresponds to the s^{th} zero of J_q'. The general solution is given by linear combinations of (10). We also note that there exists a plane wave solution to (2) which corresponds to $\nu_{00} = 0$. This is given by,

$$p(x, r, \phi; t) = \left[A_{00} e^{jkx} + B_{00} e^{-jkx}\right] e^{j\omega t} \tag{12}$$

0.2 Vibration of a Circular Membrane

The free vibrations of a clamped membrane of radius a are governed by the 2D wave equation,

$$\frac{1}{c_M^2} \frac{\partial^2 u(r, \phi, t)}{\partial t^2} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u(r, \phi, t)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u(r, \phi, t)}{\partial \phi^2} \tag{13}$$

Subject to the boundary condition, $u(a, \phi, t) = 0$. Here, c_M is the propagation velocity of waves on the surface of the membrane. It is expressed in terms of the membrane density ρ_m, thickness d and tension T as, $c_M = \sqrt{T/\rho_m d}$. The solution follows in a similar way to that of (2). The general solution is given by,

$$u(r, \phi, t) = \sum_{m,n} \left(M_{mn}^{+} e^{jm\phi} + M_{mn}^{-} e^{-im\phi} \right) J_m(\mu_{mn} r) e^{j\omega_{mn} t} \tag{14}$$

Using arguments similar to those in (8) and (9), the above equation reduces to,

$$u(r, \phi, t) = \sum_{m,n} C_{mn} \cos(m\phi) J_m(\mu_{mn} r) e^{j\omega_{mn} t} \tag{15}$$

μ_{mn} corresponds to the n^{th} zero of J_m, i.e., $\mu_{mn} = k_{mn}/a$ and $\omega_{mn} = c_M \mu_{mn}$.

0.2.1 Forced and Damped Vibration

The forced vibrations of a damped circular membrane is governed by the equation,

$$-\frac{1}{c_M^2} \frac{\partial^2 u(r, \phi, t)}{\partial t^2} + 2\alpha \frac{\partial u(r, \phi, t)}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u(r, \phi, t)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u(r, \phi, t)}{\partial \phi^2} = \frac{1}{\rho_m d} p e^{j\omega t} \tag{16}$$

Here, the membrane is forced by a periodic pressure $pe^{j\omega t}$ uniformly over its surface and α is the damping coefficient. We look for solutions of the form,

$$u(r,\phi,t) = \sum_{m,n} C_{mn} \cos(m\phi) J_m(\mu_{mn} r) e^{j\omega t} \tag{17}$$

0.2.2 Conventions

0.3 Vibration of Coupled Unloaded Membranes

We first treat the case in which we have two circular coupled by a cylindrical cavity. The forcing on both tympani is given by,

$$p_0 = pe^{-jkL\sin\theta} \tag{18}$$

$$p_L = pe^{jkL\sin\theta} \tag{19}$$

In this example we attempt to model the case of the animal to a free field stimulus. This means that the sound on both tympani has the same amplitude but differ in phase by, $kL\sin\theta$ where, θ is the angle the sound source makes with the central axis of the head. We can assume that the instantaneous pressure is constant over the tympanum as its dimensions are much smaller than the wavelength of the sound wave. The pressure inside the cavity is given by,

$$p(x,r,\phi,t) = \sum_{q,s} \left[A_{qs} e^{j\zeta_{qs} x} + B_{qs} e^{-j\zeta_{qs} x}\right] \cos(q\phi) J_p(\nu_{qs} r) e^{j\omega t} \tag{20}$$

The equation of motion of the membranes gives us,

$$\sum_{m,n} \left(\omega^2 - 2j\alpha\omega - \omega_{mn}^2\right) C_{mn}^0 \cos(m\phi) J_m(\mu_{mn} r) = \left[p_0 - p(0,r,\phi,t)\right] / (\rho_m d) \tag{21}$$

$$\sum_{m,n} \left(\omega^2 - 2j\alpha\omega - \omega_{mn}^2\right) C_{mn}^L \cos(m\phi) J_m(\mu_{mn} r) = \left[p_L - p(L,r,\phi,t)\right] / (\rho_m d) \tag{22}$$

Where as usual, 0 and L refer to the ipsi- and contralateral membranes respectively. We then equate the the velocity of air given in (7) to the velocity of the membrane surface at that point.

$$\rho\omega^2 \sum_{m,n} C_{mn}^0 \cos(m\phi) J_m(\mu_{mn} r) = -\sum_{q,s} j\zeta_{qs} \left[A_{qs} - B_{qs}\right] \cos(q\phi) J_p(\nu_{qs} r) \tag{23}$$

$$\rho\omega^2 \sum_{m,n} C_{mn}^L \cos(m\phi) J_m(\mu_{mn} r) = \sum_{q,s} j\zeta_{qs} \left[A_{qs} e^{j\zeta_{qs} L} - B_{qs} e^{-j\zeta_{qs} L}\right] \cos(q\phi) J_p(\nu_{qs} r) \tag{24}$$

This is the same as setting the component of the velocity of air normal to the membrane to zero. We note that we have also used the direction conventions mentioned in the previous section.

First we use (23) and (24) to express the cavity coefficients in terms of the membrane coefficients. This gives us,

$$A_{ms} = -\frac{\rho\omega^2}{2\zeta_{ms}\sin(\zeta_{ms}L)} \sum_l \left[C^0_{ml} e^{-j\zeta_{ms}L} + C^L_{ml}\right] \lambda_{msl} \tag{25}$$

$$B_{ms} = -\frac{\rho\omega^2}{2\zeta_{ms}\sin(\zeta_{ms}L)} \sum_l \left[C^0_{ml} e^{j\zeta_{ms}L} + C^L_{ml}\right] \lambda_{msl} \tag{26}$$

Where we've made the definition,

$$\lambda_{mnl} = \frac{\int_0^a rdr J_m(\nu_{mn}r)J(\mu_{ml}r)}{\int_0^a rdr J_m^2(\nu_{mn}r)}$$

We have also made use of the fact that $\int_0^{2\pi} \cos(m\phi)\cos(k\phi) = 0$ for $m \neq k$. This tells us that the J_m modes on the membrane only affect the J_m modes in the cavity for a given m and vice versa. Also, in the above summations, l ranges from 1 to ∞ while n can also be 0. We now substitute the above expressions into (21) and (22) to get,

$$\begin{aligned}&\sum_{m,n} \Omega_{mn} C^0_{mn} \cos(m\phi) J_m(\mu_{mn}r) \\ &\qquad = p_0 + \sum_{q,s}\sum_l \left[\Lambda_{qs} C^0_{ql} + K_{qs} C^L_{ql}\right] \lambda_{qsl} f_{qs}(r,\phi)\end{aligned} \tag{27}$$

$$\begin{aligned}&\sum_{m,n} \Omega_{mn} C^L_{mn} \cos(m\phi) J_m(\mu_{mn}r) \\ &\qquad = p_L + \sum_{q,s}\sum_l \left[K_{qs} C^0_{ql} + \Lambda_{qs} C^L_{ql}\right] \lambda_{qsl} f_{qs}(r,\phi)\end{aligned} \tag{28}$$

Where we have defined,

$$\Omega_{mn} = \rho_m d\left(\omega^2 - 2j\alpha\omega - \omega^2_{mn}\right) \tag{29}$$

$$\Lambda_{qs} = \rho\omega^2 \frac{\cot(\zeta_{qs}L)}{\zeta_{qs}} \tag{30}$$

$$K_{qs} = \frac{\rho\omega^2}{\zeta_{qs}\sin(\zeta_{qs}L)} \tag{31}$$

$$f_{qs}(r,\phi) = \cos(q\phi) J_q(\nu_{qs}r) \tag{32}$$

We now multiply both sides of (27) and (28) with the membrane modes and integrate over the circle to get,

$$\Omega_{mn} C^0_{mn} = p_0 I_{mn} + \sum_{s,l} \left[\Lambda_{ms} C^0_{ml} + K_{ms} C^L_{ml}\right] \lambda_{msl} \widetilde{\lambda}_{msn} \tag{33}$$

$$\Omega_{mn} C^L_{mn} = p_L I_{mn} + \sum_{s,l} \left[K_{ms} C^0_{ml} + \Lambda_{ms} C^L_{ml}\right] \lambda_{msl} \widetilde{\lambda}_{msn} \tag{34}$$

Where, as usual, we have made the following definitions for aesthetic reasons,

$$I_{mn} = \frac{\int_0^a rdr J_m(\mu_{mn}r)}{\int_0^a rdr J_m^2(\mu_{mn}r)} \tag{35}$$

$$\widetilde{\lambda}_{msn} = \frac{\int_0^a rdr J_m(\nu_{ms}r)J(\mu_{mn}r)}{\int_0^a rdr J_m^2(\mu_{mn}r)} \tag{36}$$

In order to get a more symmetric looking expression, we can redefine λ and $\widetilde{\lambda}$ in the following way,

$$\lambda_{msl}\widetilde{\lambda}_{msn} = \widehat{\lambda}_{msl}\widehat{\lambda}_{msn} \tag{37}$$

$$\widehat{\lambda}_{msl} = \frac{\int_0^a rdrJ_m(\nu_{ms}r)J(\mu_{ml}r)}{\sqrt{\int_0^a rdrJ_m^2(\mu_{ml}r)\int_0^a rdrJ_m^2(\nu_{ms}r)}} \tag{38}$$

We see that I_{mn} vanishes for $m \geq 1$. This means that we can ignore these modes. We can therefore omit the m-index, and implicitly assume $m = 0$ to rewrite the above system of equations as,

$$\Omega_n C_n^0 = p_0 I_n + \sum_{s,l}\left[\Lambda_s C_l^0 + K_s C_l^L\right]\lambda_{sl}\widetilde{\lambda}_{sn} \tag{39}$$

$$\Omega_n C_n^L = p_L I_n + \sum_{s,l}\left[K_s C_l^0 + \Lambda_s C_l^L\right]\lambda_{sl}\widetilde{\lambda}_{sn} \tag{40}$$

We now define a new set of variables, $C^+ = C^L + C^0$ and $C^- = C^L - C^0$ and add and subtract the above equations to get a system in terms of the newly defined variables,

$$\Omega_n C_n^+ - \sum_{s,l}\left[\Lambda_s + K_s\right]C_l^+\widehat{\lambda}_{sl}\widehat{\lambda}_{sn} = (p_L + p_0)I_n \tag{41}$$

$$\Omega_n C_n^- - \sum_{s,l}\left[\Lambda_s - K_s\right]C_l^-\widehat{\lambda}_{sl}\widehat{\lambda}_{sn} = (p_L - p_0)I_n \tag{42}$$

0.4 Loaded Membrane

We consider a tympanic membrane of radius a with the extra-columellar footplate situated between $0 < \phi < \beta$ and $2\pi - \beta \leq \phi \leq 2\pi$ (or equivalently between $-\beta < \phi < \beta$). The attached extracolummelar footplate is modelled by the following set of equations,

$$u_{0/L}(r,\phi,t) = \begin{cases} D_{0/L}\left(1 - \frac{r\cos\phi}{a\cos\beta}\right) & \text{if } 0 \leq r < a\cos\beta/\cos\phi \\ 0 & \text{if } a\cos\beta/\cos\phi \leq r \leq a \end{cases} \tag{43}$$

where, ϕ is in the region mentioned above. The above equations model the extracollumela as a triangular plate. The model is described in figure 1 - The triangle OAB hatches about the line AB and the striped section is rigid. The subscripts 0 and L denote the ipsi-lateral and contra-lateral membranes respectively. The coefficients $D_{0/L}$ will be determined later. This also requires the membrane to have the same displacement for a given radius at the angles β and $2\pi - \beta$. As a result, the possible number of modes are reduced and from (15) we get,

$$u_{0/L}(r,\phi,t) = \sum_{m,n} C_{mn}^{0/L}\cos(m\phi)J_m(\mu_{mn}r)e^{j\omega t} \tag{44}$$

0.4.1 Orthogonality

We define $f_{mn}(r,\phi) = \cos(m\phi)J_m(\mu_{mn}r)$. These modes satisy the orthogonality condition,

$$\int dSf_{mn}f_{kl} = \delta_{m,k}\delta_{n,l}\frac{\pi}{2}J_{m+1}^2(\mu_{mn}a) \tag{45}$$

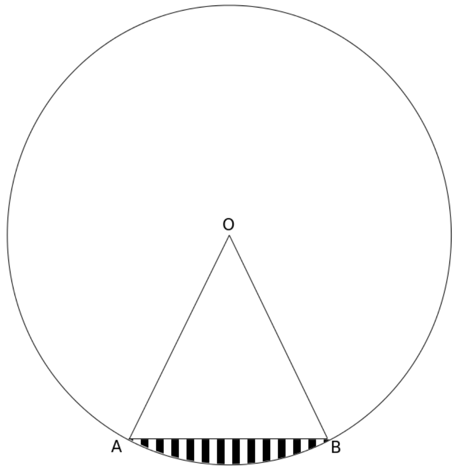

Figure 1: Loaded Membrane with Extracolumella situated in the sector OAB

Where, $dS = rdrd\phi$. This condition relies on the orthogonality of $J_m(\mu_{mn}r)$ for different n.

0.5 Matching Boundary Conditions

We denote the pressure on the outside of either membrane as $p_0e^{j\omega t}$ and $p_Le^{j\omega t}$. This means that the tympanic membrane is stimulated by a single tone of a given frequency. Using the membrane equations of motion, (16), on both sides we get,

$$\sum_{m,n} \rho_m d \left(\omega^2 - 2i\alpha\omega - \omega_{mn}^2\right) C_{mn}^0 \cos(m\phi) J_m(\mu_{mn}r)e^{j\omega t}$$
$$= p_0 e^{j\omega t} - p(0, r, \phi, t) \tag{46}$$
$$\sum_{m,n} \rho_m d \left(\omega^2 - 2i\alpha\omega - \omega_{mn}^2\right) C_{mn}^L \cos(m\phi) J_m(\mu_{mn}r)e^{j\omega t}$$
$$= p_L e^{j\omega t} - p(L, r, \phi, t) \tag{47}$$

Next, we match the velocity of the membrane as given in (7) surface to the velocity the air (fluid elements) on both sides,

$$\dot{u}_0 = -\sum_{m,n} \omega^2 C^0_{mn} \cos(m\phi) J_m(\mu_{mn} r) e^{j\omega t} = \frac{1}{\rho} \nabla_x p(x, r, \phi, t)|_{x=0} \tag{48}$$

$$\dot{u}_L = -\sum_{m,n} \omega^2 C^L_{mn} \cos(m\phi) J_m(\mu_{mn} r) e^{j\omega t} = -\frac{1}{\rho} \nabla_x p(x, r, \phi, t)|_{x=L} \tag{49}$$

Since the above equations are valid for one full revolution, i.e., $0 < \beta < 2\pi$, we can use the orthogonality of the individual modes to express the coefficients in the expansion of the pressure in terms of $C^{0/L}_{mn}$,

$$A_{mn} = -\frac{\rho\omega^2}{2k_x \sin k_x L} \left(C^0_{mn} e^{-ik_x L} + C^L_{mn} \right) \tag{50}$$

$$B_{mn} = -\frac{\rho\omega^2}{2k_x \sin k_x L} \left(C^0_{mn} e^{jk_x L} + C^L_{mn} \right) \tag{51}$$

We now substitute the above into (46) and (47) to get,

$$\sum_{m,n} \left(\Lambda_{mn} C^0_{mn} - K_{mn} C^L_{mn} \right) \cos(m\phi) J_m(\mu_{mn} r) = p_0 \tag{52}$$

$$\sum_{m,n} \left(\Lambda_{mn} C^L_{mn} - K_{mn} C^0_{mn} \right) \cos(m\phi) J_m(\mu_{mn} r) = p_L \tag{53}$$

Where,

$$\Lambda_{mn} = \rho_m d \left(\omega^2 - 2i\alpha\omega - \omega^2_{mn} \right) - \rho L \omega^2 \frac{\cot k_x L}{k_x L}$$

$$K_{mn} = \frac{\rho L \omega^2}{k_x L \sin k_x L}$$

The equations (52) and (53) are valid only between $\beta < \phi < 2\pi - \beta$. It is therefore not possible to directly calculate $C^{0/L}_{mn}$ using the orthogonality relation (45) as we did in (50) and (51). k_x and μ_{mn} are related according to (6).

For low frequencies, i.e. when $\omega < \omega_{mn}$, k_x is imaginary. As a result, $K_{mn} \to e^{-|k_x L|}/|k_x L]$

Choosing a Cutoff

In the case of an unloaded membrane, the strength of the coefficients of the individual modes decrease as a function of their frequency as $1/(\omega^2 - 2i\alpha\omega + \omega^2_{mn})$. As a result, the modes with frequencies closest to ω have the strongest contribution in the expansion (44). We expect a similar behaviour in the loaded case as well and although we cannot directly calculate the coefficient of the individual modes, we can find the coefficients that give the best possible fits to equations (52) and (53) with the boundary condition, (43).

Extracolumella Boundary Condition

After restricting ourselves to a combination particular number of modes, we need to find the coefficients $C^{0/L}_{mn}$ and $D^{0/L}$ that best satisfy the condition (43).

First we note that after setting r to 0 we get,

$$D^{0/L} = \sum_n C_{0n}^{0/L} \tag{54}$$