SWITCHING BETWEEN MARKOV CHAINS AND TRAFFIC SIMULATION

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1. Introduction

This progress report is broadly divided into two themes, namely *switching between Markov chains* and *traffic simulation*, and our future aim is to link them.

In the *switching between Markov chains* theme, we consider finding conditions which leads to *weak ergodicity* of non-homogeneous Markov chain. The weak ergodicity of an infinite product of stochastic matrices corresponds to the asymptotic behavior of all rows being equal to each other, while strong ergodicity corresponds to weak ergodicity and existence of a limit for that infinite product of stochastic matrices. We investigate conditions on the switching which will guarantee ergodicity of the switched system. Many fundamental results on weak ergodicity and definitions for coefficient of ergodicity are derived in [W] and [S].

We consider three cases for weak ergodicity characterization of an infinite word $\{A_1 A_2 \dots\}$ where $A_i \in \mathcal{P} := \{P_1, \dots, P_n\}$, a finite collection of stochastic matrices. **IID** case where at each step, A_k is chosen IID according to a fixed probability distribution on $\mathcal{P}$. **Independent non-homogeneous** case when A_k is still chosen at each step according to a probability distribution on $\mathcal{P}$ which does not depend on the state or other variables, but the distribution changes over time. The result in this case requires an extension of the strong law. **Markov chain for strings** case where A_k is defined as a non-homogeneous Markov chain on $\mathcal{P}$. So at each step the distribution of A_{k+1} depends on the previous state A_k through a transition matrix, and the transition matrices change over time.

The *traffic simulation* part illustrates some stochastic policies that regulates the traffic flow by assigning vehicles to corresponding regions. The entry and exit probabilities for the vehicles are determined by some defined Markov process. We consider the cases which leads to desired balanced load ratio of number of vehicles in different regions. In particular, we analyze *expectation based* model and *estimated-duration based waiting-cars* model which are two cases of **prior-statistics based** model, where a probability distribution of estimated stay duration of vehicles is assumed to be known. Our traffic simulation models are inspired by the models in [SKCS].

2. Independent Switching

Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite collection of stochastic matrices. Let $\{q_i\}$ be a probability distribution on $\mathcal{P}$. Let $X_0, X_1, X_2, \dots$ be a stochastic process which evolves according to the following update rule:

$$\mathbb{P}(X_{k+1} = b \,|\, X_k = a) = \sum_{i=1}^{n} q_i \, P_i(a, b), \quad k \geq 0. \tag{1}$$

The interpretation of (1) is that at each step one of the matrices P_i is randomly chosen with probability q_i. These choices are independent and identically distributed (IID) at each step.

We are interested in the question of weak ergodicity for the process X_k. That is, we consider how the distribution of X_k depends on the initial state X_0, as $k \to \infty$.

2.1. **Method 1: equivalent ergodic chain.** We have

$$\mathbb{P}(X_k = b \,|\, X_0 = a) = \sum_{i_1,\dots,i_k} \prod_{j=1}^k q_{i_j} \sum_{b_1,\dots,b_{k-1}} P_{i_1}(a,b_1)\, P_{i_2}(b_1,b_2) \cdots P_{i_k}(b_{k_1}, b) \quad (2)$$
$$= \left(Q^k\right)(a,b) \quad (3)$$

where

$$Q = \sum_{i=1}^n q_i \, P_i \quad (4)$$

Therefore $\{X_k\}$ is a Markov chain with transition matrix Q. Now suppose that at least one of the matrices P_i is SIA. Then if $q_i > 0$ we can show that Q is also SIA, and therefore $\{X_k\}$ is ergodic. Hence in particular there is a probability distribution π such that

$$\mathbb{P}(X_k = b \,|\, X_0 = a) \to \pi_b \quad \text{for all } a \quad (5)$$

This implies independence of initial conditions, hence weak ergodicity.

2.2. **Method 2: pathwise analysis.** Suppose that there is a matrix $P_i \in \mathcal{P}$ such that $q_i > 0$ and $a(P_i) < 1$, where $a(\cdot)$ is the coefficient of ergodicity. Define the following random variables:

$$Y_k = \begin{cases} 1 & \text{if we choose matrix } P_i \text{ at step } k \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

It is clear that

$$\mathbb{E}[Y_k] = \mathbb{P}(Y_k = 1) = q_i \quad (7)$$

Furthermore since the matrices are chosen independently at each step, it follows that $\{Y_k\}$ are IID. So by the strong law of large numbers it follows that

$$\frac{Y_1 + \cdots Y_k}{k} \to \mathbb{E}[Y_k] = q_i \quad \text{a.s. as } k \to \infty \quad (8)$$

where 'a.s.' means almost surely. [More precisely, the IID choice of matrix at each step defines a probability measure on the space of all infinite sequences $\{c_1, c_2, \dots\}$, where $c_k \in \{1, \dots, n\}$ corresponds to the choice at step k (see [R], Sections 5.3, 7.1). With respect to this probability measure, the event $\left\{k^{-1} \sum_{j=1}^k Y_j \to q_i\right\}$ has probability 1]. It follows that a.s. the matrix P_i is chosen infinitely often. Therefore a.s. for any chosen product we have

$$a\left(P_{i_1} P_{i_2} \cdots\right) \leq \prod_{k=1}^{\infty} a(P_{i_k}) \quad (9)$$
$$\leq a(P_i)^{Y_1 + Y_2 + \cdots} \quad (10)$$
$$= 0 \quad (11)$$

3. Independent Non-Homogeneous Switching

Suppose that matrices from $\mathcal{P}$ are chosen independently at each step, but the probabilities *change over time*. So at step k we choose P_i with probability $q_i(k)$. The distribution at time k is $\{q_1(k), \dots, q_n(k)\}$, satisfying $\sum_i q_i(k) = 1$. In the special case $q_i(k) = q_i$ we regain the IID model described above.

3.1. **probability of selecting P_i changes over time.**

Lemma 3.1. *Let at each time step k, probability of selection of P_i is $q_i(k)$. The problem is same as before except the case that $q_i(k)$ is dependent on time step k. Since we are fixing i for P_i, we will be using $q_k := q_i(k)$ for brevity of notation.*
$\frac{Y_1+\cdots+Y_k}{k} \to \frac{q_1+\cdots+q_k}{k}$ *a.s. as $k \to \infty$.*

Proof. Define $S_k = \frac{1}{k}\sum\limits_{j=1}^{k}(Y_j - q_j)$. Then, we need to show that $S_k \to 0$ a.s. as $k \to \infty$. We will be using two results from the theory of convergence to proof the claim.

$$S_k \to 0 \text{ a.s. } \Leftrightarrow \forall \epsilon > 0,\ \mathbb{P}(|S_k| \geq \epsilon \textsf{ i.o.}) = 0 \tag{12}$$

and Borel-Cantelli Lemma: for events $A_1, A_2, \dots$

$$\sum_{j=1}^{\infty} \mathbb{P}(A_j) < \infty \Rightarrow \mathbb{P}(A_j \textsf{ i.o.}) = 0 \tag{13}$$

Using the above two results, it is enough to show that,

$$\sum_{k=1}^{\infty} \mathbb{P}(|S_k| \geq \epsilon) < \infty \tag{14}$$

Using Markov's inequality,

$$\mathbb{P}(|X-\mu| \geq \epsilon) = \mathbb{P}(|X-\mu|^4 \geq \epsilon^4) \leq \frac{\mathbb{E}[(X-\mu)^4]}{\epsilon^4} \Rightarrow \mathbb{P}(|S_k| \geq \epsilon) \leq \frac{\mathbb{E}[S_k^4]}{\epsilon^4} \tag{15}$$

For Bernoulli random variable,

$$\mathbb{E}[Y_k] = q_k \Rightarrow \mathbb{E}[Y_k - q_k] = 0 \tag{16}$$

$$\mathbb{E}[S_k^4] = \frac{1}{k^4}\sum_{j,l,m,n=1}^{k} \mathbb{E}[(Y_j - q_j)(Y_l - q_l)(Y_m - q_m)(Y_n - q_n)] \tag{17}$$

This sum can be partitioned into five parts, based on uniqueness of j, l, m, n. These five cases are, (1) when all j, l, m, n are equal, (2) when 2 sets of 2 indices are equal, but all 4 are not equal, (3) when 3 out of 4 of these are equal, (4) when 2 out of 4 are equal, and rest 2 are distinct and distinct from first 2, and (5) when all 4 indices are distinct.

$$k^4\mathbb{E}[S_k^4] = \sum_{j,l,m,n=1}^{k} \mathbb{E}[(Y_j - q_j)(Y_l - q_l)(Y_m - q_m)(Y_n - q_n)] \tag{18}$$

$$k^4\mathbb{E}[S_k^4] = \sum_{j=1}^{k}\mathbb{E}[(Y_j-q_j)^4] + {}^4C_2 \sum_{\substack{j,l=1\\ j\neq l}}^{k}\mathbb{E}[(Y_j-q_j)^2(Y_l-q_l)^2] \tag{19}$$

$$+ {}^4C_3 \sum_{\substack{j,l=1\\ j\neq l}}^{k}\mathbb{E}[(Y_j-q_j)^3(Y_l-q_l)] + {}^4C_2 \sum_{\substack{j,l,m=1\\ j\neq l\neq m}}^{k}\mathbb{E}[(Y_j-q_j)^2(Y_l-q_l)(Y_m-q_m)] \tag{20}$$

$$+ \sum_{\substack{j,l,m,n=1\\ j\neq l\neq m\neq n}}^{k}\mathbb{E}[(Y_j-q_j)(Y_l-q_l)(Y_m-q_m)(Y_n-q_n)] \tag{21}$$

Since $\mathbb{E}[Y_j - q_j] = 0 \ \forall\ j$ and expectation is linear in scalar, last three terms will be zero as each term is a multiple of $\sum_{j=1}^{k}\mathbb{E}[Y_j - q_j] = 0$. Hence,

$$k^4\mathbb{E}[S_k^4] = \sum_{j=1}^{k}\mathbb{E}[(Y_j-q_j)^4] + 6\left(\sum_{j=1}^{k}\mathbb{E}[(Y_j-q_j)^2]\right)^2 \tag{22}$$

For random variable X with Bernoulli distribution, with expectation $\mathbb{E}[X] = \mu = p$, we can calculate that

$$\mathbb{E}[(X-\mu)^4] = p(1-p)(3p^2-3p+1) \leq 1 \ \forall p \in [0,1] \tag{23}$$

since $p(1-p) \leq \frac{1}{4}$ and $(3p^2-3p+1) \leq 4 \ \forall p \leq 1$. Also,

$$\mathbb{E}[(X-\mu)^2] = p(1-p) \leq 4 \tag{24}$$

Substituting these inequality results in above equation gives:

$$k^4\mathbb{E}[S_k^4] \leq k + 6\left(\frac{1}{4}k\right)^2 \leq k + k^2 \Rightarrow \mathbb{E}[S_k^4] \leq \frac{1}{k^3} + \frac{1}{k^2} \tag{25}$$

This implies the following and completes the proof:

$$\sum_{k=1}^{\infty}\mathbb{P}(|S_k| \geq \epsilon) \leq \frac{1}{\epsilon^4}\sum_{k=1}^{\infty}\mathbb{E}[S_k^4] \leq \frac{1}{\epsilon^4}\sum_{k=1}^{\infty}\frac{1}{k^2} + \frac{1}{k^3} < \infty \ \forall\ \epsilon > 0 \tag{26}$$

□

3.2. power of P_i is scrambling.

Lemma 3.2. *Let $\mathcal{P} := \{P_1, \ldots, P_N\}$ be a collection of stochastic matrices. Let $a(\cdot)$ be the coefficient of ergodicity. Consider a stochastic process where at each step j, $A_j \in \mathcal{P}$ is chosen independently with a probability distribution $\{q_i(j)\}$ on $\mathcal{P}$. Let $P := P_i$ and $q(j) := q_i(j)$. Suppose there exists a least $m \in \mathbb{Z}_+$ such that P^m is scrambling, i.e. $a(P^m) < 1$, but $a(P^{m-1}) = 1$. Let's fix $m = 2$. Let $A_1, A_2, \cdots$ be an infinite word generated using such a stochastic process. We define*

$$q_2(j) := q(j)q(j+1) \tag{27}$$

$$s(n) := \frac{1}{n}\sum_{t=1}^{n-1} q_2(t) \tag{28}$$

$$\bar{s} := \lim_{n\to\infty} s(n) \tag{29}$$

Assuming that $\bar{s}$ exists, if $\bar{s} > 0$, then $a(A_1A_2\cdots) = 0$ a.s.

Proof. We define the following notations:

$$Y_2(j) := \begin{cases} 1 & \text{if } A_j = A_{j+1} = P \\ 0 & \text{else} \end{cases} \tag{30}$$

$$\mathcal{J}(n) := \left\{(j_1, j_2, \dots, j_l) : l \leq \frac{n}{2}, 1 \leq j_1 \leq \dots \leq j_l \leq n-1, j_{t+1} - j_t \geq 2\right\} \tag{31}$$

$$\tau_n(J) := \frac{1}{n} \sum_{t=1}^{|J|} Y_2(j_t) \text{ for } J = (j_1, j_2, \dots, j_l) \in \mathcal{J}(n) \tag{32}$$

$$T_n := \sup_{J \in \mathcal{J}(n)} \tau_n(J),\ T_+ := \limsup T_n,\ T_- := \liminf T_n \tag{33}$$

$$s_e(n) := \frac{1}{n} \sum_{t=1}^{n-1} q_2(t)\theta_e(t) \tag{34}$$

$$s_o(n) := \frac{1}{n} \sum_{t=1}^{n-1} q_2(t)\theta_o(t) \tag{35}$$

where $\theta_e(t) = 1$ if t is even, 0 if t is odd, and $\theta_o(t) = 0$ if t is even, 1 if t is odd.
Note that for all $J \in \mathcal{J}(n)$ we have,

$$\tau_n(J) \leq \frac{1}{n} \sum_{t=1}^{n-1} Y_2(t) \Rightarrow T_n \leq \frac{1}{n} \sum_{t=1}^{n-1} Y_2(t) \text{ for all } n \geq 1 \tag{36}$$

By the SLLN (non-homogeneous version), we have:

$$\frac{1}{n} \sum_{t=1}^{n-1} Y_2(t) = \frac{1}{2} \left(\frac{1}{\frac{n}{2}} \sum_{t=1}^{n-1} Y_2(t)\theta_e(t) + \frac{1}{\frac{n}{2}} \sum_{t=1}^{n-1} Y_2(t)\theta_o(t) \right) \to \frac{1}{2}(2(\lim_{n\to\infty}(s_e(n)+s_o(n)))) \text{ a.s.} \tag{37}$$

This implies that:

$$\frac{1}{n} \sum_{t=1}^{n-1} Y_2(t) \to \bar{s} \text{ a.s.} \tag{38}$$

Using (36), we have,

$$T_+ = \limsup T_n \leq \limsup \frac{1}{n} \sum_{t=1}^{n-1} Y_2(t) = \bar{s} \quad \text{a.s.} \tag{39}$$

Similarly, by the SLLN (non-homogeneous version):

$$T_n \geq \frac{1}{n} \sum_{t=1}^{n-1} Y_2(t)\theta_e(t) \to \lim_{n\to\infty} s_e(n) \text{ a.s.} \tag{40}$$

This implies

$$T_- = \liminf T_n \geq \lim_{n\to\infty} s_e(n) \text{ a.s.} \tag{41}$$

and similarly

$$T_- \geq \lim_{n\to\infty} s_o(n) \tag{42}$$

Hence, we have

$$T_- \geq \max(\lim_{n\to\infty} s_e(n), \lim_{n\to\infty} s_o(n)) \geq \frac{1}{2}\bar{s} \quad \text{a.s.} \tag{43}$$

$$\frac{1}{2}\bar{s} \leq T_- \leq T_+ \leq \bar{s} \quad \text{a.s.} \tag{44}$$

Since coefficient of ergodicity of a word is less than equal to product of coefficient of ergodicity of its constituent sub-words, we have

$$a(A_1 \cdots A_n) \le a(P^2)^{\sum_{j_t \in J} Y_2(j_t)} \text{ for all } J \in \mathcal{J}(n) \tag{45}$$

$$\implies a(A_1 \cdots A_n) \le a(P^2)^{n \sup_{J \in \mathcal{J}(n)} \frac{1}{n} \sum_{j_t \in J} Y_2(j_t)} = a(P^2)^{nT_n} \tag{46}$$

$$\implies \lim_{n\to\infty} a(A_1 \cdots A_n) \le \limsup a(P^2)^{nT_n} = a(P^2)^{\liminf nT_n} \tag{47}$$

$$\implies a(A_1 A_2 \cdots) \le a(P^2)^{(\lim_{n\to\infty} n)\cdot T_-} \le a(P^2)^{(\lim_{n\to\infty} n)\cdot \bar{s}/2} \text{ a.s.} \tag{48}$$

Hence, if $\bar{s}$ exists and $\bar{s} > 0$, then $a(A_1 A_2 \cdots) = 0$ a.s.. ∎

4. Non-homogeneous Markov Chain for Strings

Let $\mathcal{P} := \{P_1, \dots, P_m\}$ be a collection of stochastic $k \times k$ matrices. For each $n \ge 1$, let Q_n be a stochastic $m \times m$ matrix. Define a non-homogeneous discrete Markov Chain $A_0, A_1, A_2, \dots$ on the state space $\mathcal{P}$ with transition probabilities

$$\mathbb{P}(A_{n+1} = P_j | A_n = P_i) = Q_n(i,j), \text{ for } n = 0, 1, 2, \dots \tag{49}$$

We consider the investigations of conditions under which $a(A_0 A_1 \cdots A_n) \to 0$ as $n \to \infty$.

Lemma 4.1. *Let $\mathcal{P} := \{P_1, \dots, P_m\}$ be a collection of stochastic $k \times k$ matrices. For each $n \ge 1$, let Q_n be a stochastic $m \times m$ matrix selected from the set $\mathcal{Q} := \{Q_1, \dots, Q_l\}$. Define a non-homogeneous discrete Markov Chain $A_0, A_1, A_2, \dots$ on the state space $\mathcal{P}$ with transition probabilities*

$$\mathbb{P}(A_{n+1} = P_j | A_n = P_i) = Q_n(i,j), \text{ for } n = 0, 1, 2, \dots \tag{50}$$

Let $P := P_i$ be a scrambling matrix for some i. Let $\{Q(n)\}$ satisfy strong ergodicity, i.e.

$$\lim_{n\to\infty} Q(m,n) = \mathbf{1}\pi^T \text{ where } Q(m,n) := \prod_{j=m}^{n-1} Q(j) \tag{51}$$

If $\pi_i \ne 0$, then $a(A_0 A_1 \cdots A_n) \to 0$ a.s. as $n \to \infty$.

Proof. Since $\{Q(n)\}$ satisfy strong ergodicity, it also satisfy weak ergodicity. This implies that asymptotically all rows are equal, and hence probability of selecting from $\mathcal{P}$ in the next step is independent of selection from $\mathcal{P}$ in the previous step. This satisfies independence of $\{A_n\}$ asymptotically.

Assuming $\pi_i \ne 0$ implies $\pi_i > 0$. Consider a value $\epsilon > 0$ such that $\pi_i - \epsilon > 0$. The strong condition ergodicity condition implies

$$\lim_{n\to\infty} Q(m,n)(j,i) = \mathbf{1}\pi^T(j,i) = \pi_i \ \forall j \tag{52}$$

which implies that for any initial distribution ν_0, we have

$$\lim_{n\to\infty} \nu_0 Q(1,n)(j,i) = \mathbf{1}\pi^T(j,i) = \pi_i \ \forall j \tag{53}$$

This follows since for any stochastic matrix A,

$$A\mathbf{1}\pi^T = \mathbf{1}\pi^T \tag{54}$$

Hence, there exists $N \in \mathbb{Z}_+$ such that

$$\mathbb{P}(A_n = P_i) = \nu_0 Q(1,n)(j,i) > \pi_i - \epsilon \text{ for all } n \ge N \tag{55}$$

This implies that

$$\sum_{n=N}^{\infty} \mathbb{P}(A_n = P_i) = \sum_{n=N}^{\infty} (\pi_i - \epsilon) = \infty \tag{56}$$

since $(\pi_i - \epsilon) > 0$.

Hence, using Borel Cantelli Lemma, $\mathbb{P}(\limsup A_n = P_i) = 1$. Using P_i being scrambling matrix along with sub-multiplicity of coefficient of Ergodicity $a(\cdot)$ implies $a(A_0 A_1 \cdots A_n) \to 0$ a.s. as $n \to \infty$. □

Proposition 4.2. *Let $\mathcal{P} := \{P_1, \ldots, P_m\}$ be a collection of stochastic $k \times k$ matrices. For each $n \geq 1$, let Q_n be a stochastic $m \times m$ matrix selected from the set $\mathcal{Q} := \{Q_1, \ldots, Q_l\}$. Define a non-homogeneous discrete Markov Chain $A_0, A_1, A_2, \ldots$ on the state space $\mathcal{P}$ with transition probabilities*

$$\mathbb{P}(A_{n+1} = P_j | A_n = P_i) = Q(n)(i,j), \text{ for } n = 0, 1, 2, \ldots \tag{57}$$

Let $P := P_i$ be a scrambling matrix for some i. Let Q_l commute with every other Q_j, and Q_l is ergodic, i.e. there exists a vector q such that

$$\lim_{n\to\infty} \left\| Q_l^n - \mathbf{1}q^T \right\|_1 = 0 \tag{58}$$

Let $\mathbb{P}(\limsup A_i = Q_l) = 1$. If $q_i \neq 0$, then $a(A_0 A_1 \cdots A_n) \to 0$ a.s. as $n \to \infty$.

Proof. Since Q_l commute with other Q_j,

$$Q(m,n) = \prod_{j=m}^{n-1} Q(j) = Q'(m,n) Q_l^{n_l} \tag{59}$$

where $Q'(m,n)$ is $Q(m,n)$ with Q_l replaced by identity matrix, n_l is number of times Q_l appearing in product $Q(m,n)$. Since $B\mathbf{1}q^T = \mathbf{1}q^T$ for any stochastic matrix B, and

$$\lim_{n\to\infty} \|Q'(m,n)\|_1 \leq k \tag{60}$$

since $Q'(m,n)$ is a stochastic matrix being product of stochastic matrices and

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |q_{ij}| \tag{61}$$

which is maximum absolute column sum of matrix, we have

$$\lim_{n\to\infty} \left\| Q(m,n) - \mathbf{1}q^T \right\|_1 = \lim_{n\to\infty} \left\| Q'(m,n) Q_l^{n_l} - Q'(m,n)\mathbf{1}q^T \right\|_1 \tag{62}$$

$$= \lim_{n\to\infty} \left\| Q'(m,n)(Q_l^{n_l} - \mathbf{1}q^T) \right\|_1 \tag{63}$$

$$\leq \lim_{n\to\infty} \|Q'(m,n)\|_1 \lim_{n_l\to\infty} \left\| Q_l^{n_l} - \mathbf{1}q^T \right\|_1, \text{ since } \mathbb{P}(A_i = Q_l \ i.o.) = 1 \tag{64}$$

$$\leq k \lim_{n_l\to\infty} \left\| Q_l^{n_l} - \mathbf{1}q^T \right\|_1 = 0, \text{ since } \lim_{n\to\infty} \left\| Q_l^{n} - \mathbf{1}q^T \right\|_1 = 0 \tag{65}$$

Hence,

$$\lim_{n\to\infty} \left\| Q(m,n) - \mathbf{1}q^T \right\|_1 = 0 \tag{66}$$

As $q_i \neq 0$, this satisfies assumptions from *Lemma 1*, and hence implies the statement. □

Lemma 4.3. *Let $\mathcal{P} := \{P_1, P_2\}$ be a collection of stochastic $k \times k$ matrices. For each $n \geq 1$, let Q_n be a stochastic 2×2 matrix. Define a non-homogeneous discrete Markov Chain $A_0, A_1, A_2, \ldots$ on the state space $\mathcal{P}$ with transition probabilities*

$$\mathbb{P}(A_{n+1} = P_j | A_n = P_i) = Q_n(i,j), \;\; \textit{for } n = 0, 1, 2, \ldots \tag{67}$$

Let $P := P_1$ be a scrambling matrix. If for all $N \geq 1$,

$$\lim_{M\to\infty} \prod_{n=N+1}^{M} Q_n(2,2) = 0 \tag{68}$$

then $a(A_0 A_1 \cdots A_n) \to 0$ a.s. as $n \to \infty$.

Proof. From the sub-multiplicity of coefficient of ergodicity, it's enough to show that $\mathbb{P}(A_n = P_1 \; i.o.) = 1$. From the equivalence

$$\begin{aligned}
\mathbb{P}(A_n = P_1 \; i.o.) = 1 &\iff \mathbb{P}(A_n = P_1 \text{ finitely often}) = 0 && (69)\\
&\iff \mathbb{P}(A_n = P_2 \text{ eventually}) = 0 && (70)
\end{aligned}$$

and

$$\{A_n = P_2 \text{ eventually}\} = \cup_{N\geq 1}\{A_n = P_2 \; \forall n \geq N\} \tag{71}$$

To prove

$$\mathbb{P}(A_n = P_2 \text{ eventually}) \leq \sum_{N=1}^{\infty} \mathbb{P}(A_n = P_2 \; \forall n \geq N) \tag{72}$$

it is enough to show that

$$\mathbb{P}(A_n = P_2 \; \forall n \geq N) = 0 \; \forall N \tag{73}$$

Define events

$$E_{N,M} := \{A_n = P_2 \; \forall N \leq n \leq M\} \tag{74}$$

For a fixed N, $E_{N,M}$ are decreasing events, i.e. $E_{N,M+1} \subset E_{N,M}$. This implies that

$$\mathbb{P}(\cap_{M\geq N} E_{N,M}) = \lim_{M\to\infty} \mathbb{P}(E_{N,M}) \tag{75}$$

Hence,

$$\begin{aligned}
\mathbb{P}(A_n = P_2 \; \forall n \geq N) = \mathbb{P}(\cap_{M\geq N} E_{N,M}) &= \lim_{M\to\infty} \mathbb{P}(E_{N,M}) && (76)\\
&= \lim_{M\to\infty} \mathbb{P}(A_n = P_2 \; \forall N \leq n \leq M) && (77)\\
&= \lim_{M\to\infty} \mathbb{P}(A_N = P_2) \prod_{n=N+1}^{M} Q_n(2,2) && (78)\\
&= \mathbb{P}(A_N = P_2) \lim_{M\to\infty} \prod_{n=N+1}^{M} Q_n(2,2) = 0 \text{ using assumption} && (79)
\end{aligned}$$

This proves the lemma. ☐

To mention a case where $\lim_{M\to\infty} \prod_{n=N+1}^{M} Q_n(2,2) = 0$ holds, consider when $Q_n(2,2) \leq \delta < 1$ for infinitely many n.

Theorem 4.4. *Let $\mathcal{P} := \{P_1, \ldots, P_m\}$ be a collection of stochastic $k \times k$ matrices. For each $n \geq 1$, let Q_n be a stochastic $m \times m$ matrix. Define a non-homogeneous discrete Markov Chain $A_0, A_1, A_2, \ldots$ on the state space $\mathcal{P}$ with transition probabilities*

$$\mathbb{P}(A_{n+1} = P_j | A_n = P_i) = Q_n(i,j), \text{ for } n = 0, 1, 2, \ldots$$

Let $P := P_1$ be a scrambling matrix. Let $Q_n|_{[2:m,2:m]}$ be Q_n restricted to 2nd to m rows and 2nd to m columns. If for all $N \geq 1$

$$\lim_{M \to \infty} \prod_{n=N+1}^{M} Q_n|_{[2:m,2:m]} = \mathbf{0}_{m-1 \times m-1} \tag{80}$$

then $a(A_0 A_1 \cdots A_n) \to 0$ a.s. as $n \to \infty$.

Proof. The proof follows the proof of the previous lemma very closely. Let $\mathcal{P}_{-1} := \{P_2, \ldots, P_m\}$. Replacing $A_n = P_2$ by $A_n \in \mathcal{P}_{-1}$ and $\prod_{n=N+1}^{M} Q_n(2,2)$ by $\prod_{n=N+1}^{M} Q_n|_{[2:m,2:m]}$ in the proof of previous lemma proves the theorem. □

Proposition 4.5. *Let the condition that the sum of each row of $Q_n|_{[2:m,2:m]}$ is less than δ for some fixed δ such that $0 \leq \delta < 1$ appears for infinitely many Q_n. Then*

$$\lim_{M \to \infty} \prod_{n=N+1}^{M} Q_n|_{[2:m,2:m]} = 0 \tag{81}$$

Proof. We will use matrix norm $\|A\|_\infty$ and sub-multiplicity of norm

$$\|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty \tag{82}$$

to prove the result. Notice that

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \tag{83}$$

which is equivalent to max absolute row-sum. Using the assumption, and sub-multiplicity of norm, we have

$$\left\| \prod_{n=N+1}^{M} Q_n|_{[2:m,2:m]} \right\|_\infty \leq \prod_{n=N+1}^{M} \left\| Q_n|_{[2:m,2:m]} \right\|_\infty \leq \delta^t \tag{84}$$

where $t \leq M - N - 1$ is number of times Q_n occurs such that sum of each of its row is less that δ. Since $\delta^t \to 0$ as $M \to \infty$, and as $\|A\|_\infty = 0 \iff A = 0$, the proof concludes. □

5. Markov Switching in Traffic Regulation

The Markov switching problem arises from a model of distributed resource allocation. In order to motivate the problem we look at a concrete example involving traffic regulation in a city. There are several regions in a city and several routes which leads to them. Vehicles enter and exit through these routes. The city wants to regulate the traffic flow so that traffic density in the city remains balanced according to some metric. However the city does not directly route the traffic, but instead uses feedback to the vehicles (incentives and penalties) to indirectly influence the traffic flow. A similar situation can be thought for parking spaces.

5.1. **A Simple Model.**

5.1.1. *Some Assumptions about the Model.* Let us consider a model with two regions, R_1, R_2 having two entry and two exits along the same routes. At time step k of the Markov chain, let random variables $N_1(k), N_2(k)$ be number of vehicles in region 1 and 2. Let $X(k), Y(k) \in \{1, 2\}$ corresponds to regions of entry and exit of a vehicle. We want to balance the load of the cars in two regions, say we want $N_1^* : N_2^* = \alpha : \beta = \lambda$. Here N_i^* is representing $N_i(k)$ for large k values.

5.1.2. *Entry, Exit probabilities.* Under the assumptions of the model, let us consider two models with different entry, exit probabilities which leads to balance the load of the cars.

Model 1: Let entry probabilities of vehicles in two regions be $\underbrace{\begin{bmatrix} \frac{\alpha}{N_1(k-1)Z} & \frac{\beta}{N_2(k-1)Z} \end{bmatrix}}_{\text{probability } X(k)=1,\ X(k)=2}$ where Z is a normalizing factor. Let exit probabilities be $\underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}}_{Y(k)=1,\ Y(k)=2}$. Then, we claim that this leads to the ratio $N_1^* : N_2^*$ oscillating around $\alpha : \beta$.

Model 2: Consider another model with entry and exit probabilities are defined as:

$$\text{Entry probabilities in region 1 and 2 } = \underbrace{\begin{bmatrix} \frac{\alpha}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \end{bmatrix}}_{\text{probability } X(k)=1,\ X(k)=2} \tag{85}$$

$$\text{Exit probabilities in region 1 and 2 } = \frac{1}{Z} \underbrace{\begin{bmatrix} N_1(k-1) & N_2(k-1) \end{bmatrix}}_{Y(k)=1,\ Y(k)=2} \tag{86}$$

where Z is normalization factor.
We claim that this leads to the ratio $N_1^* : N_2^*$ oscillating around $\alpha : \beta$.

5.1.3. *Analysis and Extension.* We provide a heuristic explanation of why Model 2 leads to balance in the load of the cars. The heuristic explanation for Model 1 follows from similar arguments. Later we will provide a theorem for Model 2 which capture this result and extensions of this model.

Heuristic Explanation of why Model 2 works: Assuming the balanced load is reached, the entry probability distribution is equal to the exit probability distribution since exit probability distribution is then $\begin{bmatrix} \frac{\alpha}{Z} & \frac{\beta}{Z} \end{bmatrix}$ where Z is the normalization factor.
For unbalanced ratio at time step k, there are two cases:

$$\frac{N_1(k)}{N_2(k)} < \frac{\alpha}{\beta} \text{ or } \frac{N_1(k)}{N_2(k)} > \frac{\alpha}{\beta} \tag{87}$$

Consider the case when $\frac{N_1(k)}{N_2(k)} < \frac{\alpha}{\beta}$. Then exit probability from region 1 is lower than entry probability from region 1. Hence, this will lead to slight increase in ratio $\frac{N_1(k)}{N_2(k)}$ towards the balanced ratio. A similar explanation holds for another case.

Extension to Fluid-limit Time-Delay model: There can be a time-delay between vehicles entering the city and changing the number of vehicles in the region. This in the fluid model leads to Delay-Differential equations. Let τ_i be the time-delay of vehicles reaching to region i after taking the route i. Since it is challenging to analyse the stochastic model in full detail, so we begin with the analysis of a simplified deterministic model which describes the so-called fluid limit. Let the arrival process is Poisson with arrival rate η. This model should apply in the case where the arrival rate and the capacities of regions are very large. In this limit the discrete model is replaced by a continuous model, and we can view the traffic as a fluid which flows into and out of the regions. The traffic enters and leaves the regions as a steady stream. The evolution of this deterministic fluid model is described by a delay differential equation.

$$\frac{dN_1(t)}{dt} = \eta\frac{\alpha}{\alpha+\beta} - \eta\frac{N_1(t-\tau_1)}{\sum_{i=1}^{2} N_i(t-\tau_1)} \tag{88}$$

$$\frac{dN_2(t)}{dt} = \eta\frac{\beta}{\alpha+\beta} - \eta\frac{N_2(t-\tau_2)}{\sum_{i=1}^{2} N_i(t-\tau_2)} \tag{89}$$

Extension to multiple regions: Notice that these models can be naturally extended to n regions. The entry, exit probabilities, heuristic explanation and extension to fluid-limit time-delay model can naturally extended too. We will study these extensions in subsequent models.

5.2. **Prior Statistics based Models.** There are real life situations where some resources are comparatively easily accessible. E.g. the parking spaces near arrival and departure of airport are comparatively easy to access than farther ones. Another example would be lower level parking spaces compared to higher level ones. For these cases, one may want to increase traffic in comparatively easier accessible resources while simultaneously balancing the load. Prior Statistics based models achieve this goal where the system have access to probability distribution of estimated stay duration of vehicles, hence the name *prior statistics*.

We will consider two prior-statistics based models namely expectation-based model and estimated-duration based waiting-cars model. The models entry process is very similar as we will see later. Though the exit process of *expectation-based* model is based of expected values of estimated duration of partitioning its probability density function (pdf), and the exit process of *estimated-duration* based model is based directly on each individual vehicle's estimated stay duration in the region.

5.2.1. *Expectation-based model.* Let V denote the random variable for estimate duration probability distribution of vehicles. We want to find the critical-point c such that the following holds:

$$\frac{\mathbb{P}(V \le c)}{\mathbb{P}(V \ge c)} = \frac{\alpha/\mathbb{E}(VI_{\{V\le c\}})}{\beta/\mathbb{E}(VI_{\{V\ge c\}})} \tag{90}$$

There are two intuitive ideas behind defining the equation (90): maintaining the load ratio in region 1 and region 2 to ratio $\frac{\alpha}{\beta}$, while increasing traffic in region 1 compared to region, i.e. traffic ratio in region 1 to region 2 should be greater than $\frac{\alpha}{\beta}$.

Notice that calculation of c does not depend on the prior probability distribution and hence the model is not depending on it. Instead of using a standard probability distribution, one can also use real-time data from history of provided estimate duration as probability distribution. The process is a Markov chain since even we adapt to local history for finding c, we can restrict to finite past duration for the calculation of prior probability distribution.

Entry and Exit probabilities: we define entry and exit probabilities as:

$$\text{Entry probabilities in region 1 and 2} = \underbrace{\left[\mathbb{P}(V \leq c), \quad \mathbb{P}(V \geq c)\right]}_{\text{probability } X(k)=1,\ X(k)=2} \tag{91}$$

$$\text{Exit probabilities in region 1 and 2} = \frac{1}{Z(k)} \underbrace{\left[N_1(k)/\mathbb{E}(VI_{\{V\leq c\}}), \quad N_2(k)/\mathbb{E}(VI_{\{V\leq c\}})\right]}_{Y(k)=1,\ Y(k)=2} \tag{92}$$

where $Z(k)$ is the normalization factor.

Heuristic Explanation of why load ratio is balanced to $\frac{\alpha}{\beta}$: Notice that entry probability should be equal to exit probability when $N_1^*/N_2^* = \alpha/\beta$ where α/β is the desired balanced load ratio. For unbalanced ratio at time step k, there are two cases: $\frac{N_1(k)}{N_2(k)} < \frac{\alpha}{\beta}$ or $\frac{N_1(k)}{N_2(k)} > \frac{\alpha}{\beta}$.
Consider the case when $\frac{N_1(k)}{N_2(k)} < \frac{\alpha}{\beta}$ implying $N_1(k) < N_1^*$.
Using equation (90), $\frac{\mathbb{P}(V\leq c)}{\mathbb{P}(V\geq c)} = \frac{\alpha/\mathbb{E}(VI_{\{V\leq c\}})}{\beta/\mathbb{E}(VI_{\{V\geq c\}})} > \frac{N_1(k)/\mathbb{E}(VI_{\{V\leq c\}})}{N_2(k)/\mathbb{E}(VI_{\{V\leq c\}})}$. Hence, exit probability for region 1 is lower than its entry probability, and there is a higher probability of leading towards a balanced ratio in the next step. A similar explanation holds for another case.

Traffic ratio of region 1 to region 2 is greater than its load ratio: Notice that the traffic ratio is same as entry probability ratio $\frac{\mathbb{P}(V\leq c)}{\mathbb{P}(V\geq c)}$ since the model leads to a balanced load ratio $\frac{\alpha}{\beta}$. The traffic ratio $\geq$ load ratio of region 1 to region 2 holds using equation (90) and that $\mathbb{E}(VI_{\{V\leq c\}}) \leq \mathbb{E}(VI_{\{V\geq c\}})$.

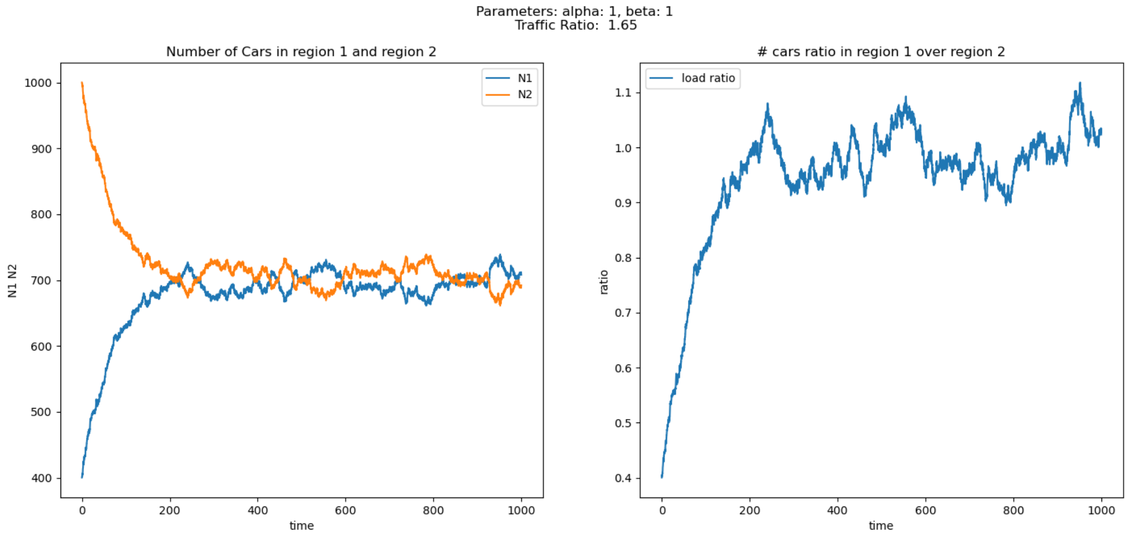
Uniqueness of solution for c: If $f(t)$ is the pdf of V, then equation (90) reduces to

$$\frac{\alpha}{\beta} = \frac{\int_0^c tf(t)dt}{\int_c^\infty tf(t)dt} \tag{93}$$

Since LHS is fixed, and numerator and denominator of RHS is monotonic in c, the above equation has a unique solution regardless of the probability distribution.

Consider the next plot obtained using the *expectation-based model*, where the LHS plot corresponds to the number of cars in region 1 and region 2 at different times and the RHS plot corresponds to the load ratio. The arrival probability distribution of vehicles follows Poisson distribution with average rate 20. The total number vehicles in the regions is 1400. We are considering the case where the load ratio $\alpha : \beta = 1 : 1$. We are assuming that prior probability distribution of estimated stay duration follows exponential distribution with location (starting point) 20, and scale $\lambda = 10$.

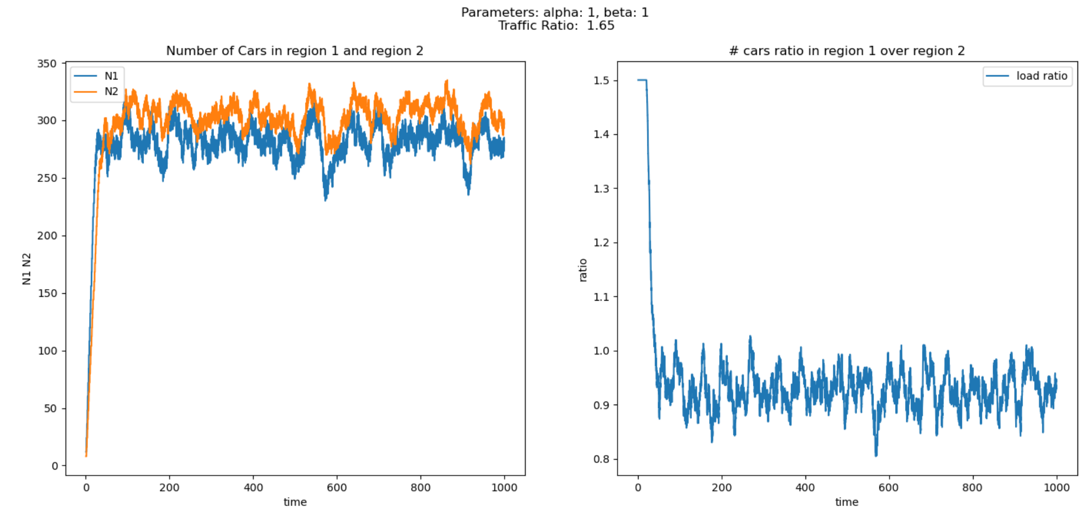
We have generated below plots for uniform, chi square and exponential probability distribution, with various load ratio to begin with. All generated plots are very similar to this plot.



5.2.2. *Estimated-duration based waiting-cars model.* This model mimics many assumptions from the *Expectation-based* model, but the entry and exit policies are different, and leaned more towards simulations. The controller wait for n (we took $n \in \{20, 30, 60\}$ during simulation) cars to enter the region. Then it sort the cars based on the estimated stay duration. Then it assign $\mathbb{P}(V \leq c) * n$ vehicles to region 1, and rest to region 2. For the exit policy, we assumed that the vehicles left the region exactly after the reported estimated duration.
The load ratio for this simulation oscillates about α/β, which is equal to *expectation-based* model as expected, which affirms our choice of entry exit policy of for the *expectation-based* model case. Notice that if we change the entry policy to assigning vehicles with estimated duration less than c to region 1, and to region 2 otherwise while keeping the exit policy same; it will lead to a very similar model. The reason for considering n waiting cars is that it provides a good reason for considering time-delay model, which we hope to address in future research work.

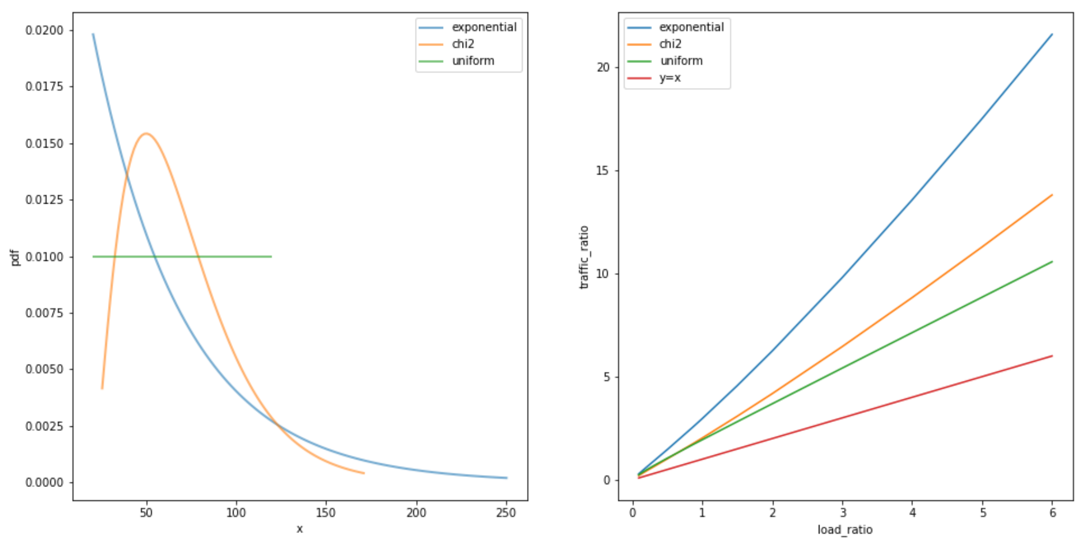
Notice that by assumptions, the traffic ratio of region 1 to region 2 is same for both cases.

Consider the next plot obtained using the *estimated-duration based waiting cars model*, where the LHS plot corresponds to the number of cars in region 1 and region 2 at different times and the RHS plot corresponds to the load ratio. Again, The arrival probability distribution of vehicles follows Poisson distribution with average rate 20. The total number vehicles in the regions is 1400. We are considering the case where the load ratio $\alpha : \beta = 1 : 1$. We are assuming that prior probability distribution of estimated stay duration follows exponential distribution with location (starting point) 20, and scale $\lambda = 10$. Additionally, we assume that number of waiting cars $n = 20$. As expected, the generated plots for both models are quite similar in nature.



Similar to the last model, we have generated above plots for uniform, chi square and exponential probability distribution, with various load ratio to begin with. All generated plots are very similar to this plot.

Traffic ratio vs Load ratio plot for different probability distributions: the RHS plot of the next figure shows traffic ratio vs load ratio for several values for corresponding exponential, chi square and uniform probability distribution whose pdf is plotted on the LHS of the figure. For a comparison, the plot $y = x$ is also added.



5.2.3. *Extension of expectation-based prior statistics model:* Let V be the random variable representing probability distribution of estimated stay duration of cars. Let there be n regions. Let $a[i]$ for $i \in \{0, 1, \ldots, n-1\}$ represents the load ratio of these regions, i.e. $a[i-1]/a[i]$ is load ratio of region i to $i+1$. Define $c[0] = -\infty$ and $c[n] = \infty$. Let $c[i]$ for $i \in \{1, \ldots, n-1\}$ be the critical points corresponding to the following relations:

$$\frac{\mathbb{P}(c[i-1] \le V \le c[i])}{\mathbb{P}(c[i] \le V \le c[i+1])} = \frac{a[i-1]/\mathbb{E}[VI_{\{c[i-1] \le V \le c[i]\}}]}{a[i]/\mathbb{E}[VI_{\{c[i] \le V \le c[i+1]\}}]} \tag{94}$$

Algorithm for solving for $c[i]$: we can solve for $c[i]$ iteratively as following: solve for

$$\frac{\mathbb{P}(c[i-1] \le V \le c[i])}{\mathbb{P}(c[i] \le V)} = \frac{a[i-1]/\mathbb{E}[VI_{\{c[i-1] \le V \le c[i]\}}]}{\sum\limits_{j \ge i} a[j]/\mathbb{E}[VI_{c[i] \le V}\}]} \tag{95}$$

Solving above equation amounts to solving the following equation iteratively for $c[i]$:

$$a[i-1] * \int\limits_{c[i]}^{c[n]} tf(t)dt - \sum_{j \ge i} a[j] * \int\limits_{c[i-1]}^{c[i]} tf(t)dt \tag{96}$$

where $f(t)$ is pdf of V.
Notice that **traffic ratio**, call it $tr[i]$ is given by

$$tr[i] = \frac{\mathbb{P}(c[i-1] \le V \le c[i])}{\mathbb{P}(c[i] \le V \le c[i+1])} \tag{97}$$

Uniqueness of solution for $c[i]$ holds using the similar argument as for 2 regions case, since the algorithm is about solving for $c[i]$ iteratively using two regions case.

6. Asymptotic Load-Equilibrium

This section formalizes the heuristic explanation of balanced load ratio to formal results. We first consider a general theorem.

Theorem 6.1. *Let $N_1(n), N_2(n)$ be the numbers of vehicles in region 1, 2 of city at time n. Let at each time step n, one vehicle enters and one vehicle exits from any of the two regions. Let the vehicle entry probability in region 1, region 2 be*

$$\left(\frac{\lambda}{\lambda+\gamma}, \frac{\gamma}{\lambda+\gamma}\right) \tag{98}$$

and let vehicle exit probability in region 1, region 2 be

$$\left(\frac{N_1(n)}{N_1(n)+N_2(n)\gamma}, \frac{N_2(n)\gamma}{N_1(n)+N_2(n)\gamma}\right) \tag{99}$$

Consider the fraction $\frac{N_1(n)}{N_2(n)+1}$ which is a 'regularized' version of the instantaneous load $\frac{N_1(n)}{N_2(n)}$. Let

$$V_m = \frac{1}{m} \sum_{n=0}^{m-1} \frac{N_1(n)}{N_2(n)+1} \tag{100}$$

Then for large number of total cars, $V_m \to \lambda$ a.s.

Proof. Due to the constraint of vehicle entry and exit at each time step, total number of vehicles N in both region is always constant. Then, $N = N_1(0) + N_2(0) = N_n(0) + N_n(0)\ \forall n$ and the state is in equilibrium. Let $\Omega = \{0, 1, \dots, N\}$ be the state space for $N_1(n)$ at time n. Then,

$N_2(n) = N - N_1(n)$. Then, this model describes a Markov chain on the state space Ω which is very similar to the well-known 'Ehrenfest model'. The transition probabilities of the model are

$$p'_k = \mathbb{P}(N_1(n+1) = k+1 | N_1(n) = k) = \frac{\lambda}{\lambda+\gamma} \cdot \frac{(N-k)\gamma}{k+(N-k)\gamma} \tag{101}$$

$$q'_k = \mathbb{P}(N_1(n+1) = k-1 | N_1(n) = k) = \frac{\gamma}{\lambda+\gamma} \cdot \frac{k\gamma}{k+(N-k)\gamma} \tag{102}$$

$$r'_k = \mathbb{P}(N_1(n+1) = k | N_1(n) = k) = 1 - p'_k - q'_k \tag{103}$$

This chain is positive recurrent since it is irreducible and has finite space. The chain is reversible since the stationary probability satisfying $\pi(n)p_n = \pi(n+1)q_{n+1}\ \forall n$ has the following solution.

$$\pi(k) = \pi(0)\lambda^k \binom{N}{k} \frac{k+(N-k)\gamma}{N\gamma} \tag{104}$$

This gives the normalization condition

$$\pi(0)^{-1} = (1+\lambda)^{N-1}\left(1 + \frac{1}{\gamma}\lambda\right) \tag{105}$$

The Markov chain being finite, reversible and irreducible implies existence of unique invariant probability measure π.
Now we will use the Ergodic theorem. The constraints of Ergodic theorem are satisfied due to existence of the unique invariant probability measure π and the chain being irreducible and positive recurrent. According to Ergodic theorem, for a non-negative function $g : \Omega \to \mathbb{R}$, integrable with respect to π: $\int |g| d\pi < \infty$, for all $k \in \Omega$,

$$\frac{1}{m}\sum_{n=0}^{m-1} g(N_1(n)) \to \mathbb{E}_\pi(g(N_1)), \qquad P_k - a.s. \tag{106}$$

Using equation (104) and (105), for $g(N_1(n)) = \frac{N_1(n)}{N_2(n)+1}$ we get

$$\mathbb{E}_\pi(g(N_1)) = \sum_{k=0}^{N} \pi(k)\frac{k}{N-k+1} = \lambda - \lambda\frac{\lambda^N}{(1+\lambda)^{N-1}(\gamma+\lambda)} + \frac{\lambda(1-\gamma)}{N(\lambda+\gamma)}\left(1 - \left(\frac{\lambda}{1+\lambda}\right)^N\right)$$

Using equation (106), we get for all $k \in \Omega$,

$$V_m \to \mathbb{E}_\pi(g(N_1)) = \lambda - \lambda\frac{\lambda^N}{(1+\lambda)^{N-1}(\gamma+\lambda)} + \frac{\lambda(1-\gamma)}{N(\lambda+\gamma)}\left(1 - \left(\frac{\lambda}{1+\lambda}\right)^N\right), \qquad P_k - a.s. \tag{107}$$

Notice that the error term

$$\lambda\frac{\lambda^N}{(1+\lambda)^{N-1}(\gamma+\lambda)} + \frac{\lambda(1-\gamma)}{N(\lambda+\gamma)}\left(1 - \left(\frac{\lambda}{1+\lambda}\right)^N\right) \tag{108}$$

converges to zero as $N \to \infty$. □

Corollary 6.2. *The case $\gamma = 1$ and $\lambda = \frac{\alpha}{\beta}$ corresponds to the equilibrium of the balanced load ratio for the entry, exit probabilities* (85), (86) *of Model 2 of* A Simple Model *case.*

Proof. Substituting $\gamma = 1$ in equation (107) leads to

$$V_m \to \mathbb{E}_\pi(g(N_1)) = \sum_{k=0}^{N} \pi(k)\frac{k}{N-k+1} = \lambda - \lambda\frac{\lambda^N}{(1+\lambda)^N} \tag{109}$$

where the error term $\lambda\frac{\lambda^N}{(1+\lambda)^N} \to 0$ as $N \to \infty$. Substituting λ, we get $V_m \to \frac{\alpha}{\beta}$ P_k-a.s. for all $k \in \Omega$. □

Corollary 6.3. *The case*

$$\gamma = \frac{\mathbb{E}(VI_{\{V\leq c\}})}{\mathbb{E}(VI_{\{V\geq c\}})} \text{ and } \lambda = \frac{\alpha}{\beta} \tag{110}$$

corresponds to the entry, exit probabilities (91)*,* (92) *of* Expectation-based *model.*

Proof. The proof follows from the theorem 6.1 and checking that substituting γ and λ corresponds to the desired case. □

7. Future Plan

Although the traffic simulation models we have considered is a Markov chain, and non-homogeneous since exit probabilities are depending $N_i(k)$, it's not yet connected to the weak ergodicity framework we worked with earlier. Our aim is to develop traffic models and link it to the earlier considered weak ergodicity theory.

We would like to extend our results to include the concepts from mixing time of Markov chains for traffic simulation model, and to add time-delay into our model and check its effect on stability of the model. More specifically, we would like to check division of traffic, oscillation around load-ratio equilibrium if delay is large, and fluid model in the limit of high arrival rate of traffic.

Our next aim is to add weaker compliance model for traffic simulation. The control signal of assigning regions (entry probability) to vehicles is direct. We want to use control theory to add a feedback loop to control the traffic indirectly. The aim is still to balance load ratio and efficient allocation of resources.

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