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Let X_1, X_2, ... , X_n be a random sample of size n from a population having probability density function

$$f(x;\mu) = \begin{cases} e^{-(x-\mu)}, & \text{if } \mu \leq x < \infty \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $\mu \in \mathbb{R}$ is an unknown parameter. If $\hat{M}$ is the maximum likelihood estimator of the median of X_1, then which one of the following statements is true?

A) $\mathrm{P}(\hat{M} \leq 2) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

B) $\mathrm{P}(\hat{M} \leq 1) = 1 - e^{-n \log_e 2}$ if $\mu = 1$

C) $\mathrm{P}(\hat{M} \leq 3) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

D) $\mathrm{P}(\hat{M} \leq 4) = 1 - e^{-n(2\log_e 2-1)}$ if $\mu = 1$

Solution: For continuous random variable X, median M is such that,

$$P(X \leq M) = 0.5 \tag{2}$$

The pdf of X is given by,

$$p_X(x) = \begin{cases} e^{-(x-\mu)}, & \text{if } \mu \leq x < \infty \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

Hence, cdf is given by

$$F(x;\mu) = \int_{\mu}^{x} e^{-(t-\mu)} dt \tag{4}$$

$$= e^{\mu}[-e^{-x} + e^{-\mu}] \tag{5}$$

$$= 1 - e^{-(x-\mu)} \tag{6}$$

Now,

$$F(x;\mu) = 0.5 \tag{7}$$

$$\implies 1 - e^{-(M-\mu)} = 0.5 \tag{8}$$

$$\implies \hat{M} = \mu + \ln(2) \tag{9}$$

Definition 1. *L, the Maximum Likelihood Estimator of the distribution is given by,*

$$L = \prod e^{-(x-\mu)} \tag{10}$$
$$= e^{-(\sum x_i - n\mu)} \tag{11}$$
$$\tag{12}$$

For the Likelihood function to be maximum, $\sum x_i - n\mu$ should be minimum Hence,

$$X_i > \mu \tag{13}$$
$$\implies \sum x_i > n\mu \tag{14}$$
$$\sum x_i - n\mu > 0 \tag{15}$$
$$\implies \mu = \frac{\sum x_i}{n} \tag{16}$$

Substituting this in (9),

$$\hat{M} = \frac{\sum x_i}{n} + \ln(2) \tag{17}$$

Thus,

$$P(\hat{M} \leq y) = P(\frac{\sum x_i}{n} + \ln(2) \leq y) \tag{18}$$
$$= P(X + \ln(2) \leq y) \tag{19}$$
$$= P(X \leq y - \ln(2)) \tag{20}$$
$$= F(y - \ln(2)) \tag{21}$$

From (3), X_i follows Exp(1),
X_1 follows Exp($\sum X_i$)
X_1 follows Exp(n)
Hence,

$$f(X_i) = e^{-(x-\mu)} \tag{22}$$
$$f(X_1) = ne^{-n(x-\mu)} \tag{23}$$
$$\implies F(X_1) = 1 - e^{-n(x-\mu)} \tag{24}$$
$$\implies F(y - \ln(2)) = 1 - e^{-n(y-\ln(2)-\mu)} \tag{25}$$

When $y = 2$ and $\mu = 1$,

$$P(\hat{M} \leq 2) = 1 - e^{-n(1-\log_e 2)} \tag{26}$$

Therefore, option (A) is correct.