Question 12.13.3.1

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For a loaded die, the probabilities of outcomes are given as under: $\Pr(1) = \Pr(2) = 0.2, \Pr(3) = \Pr(5) = \Pr(6) = 0.1$ and $\Pr(4) = 0.3$

The die is thrown two times. Let A and B be the events, 'same number each time', and 'a total score is 10 or more' ,respectively. Determine whether or not A and B are independent.

Solution: Let X, Y and Z be random variables with definition given as under:

X	Number appearing on dice the first time
Y	Number appearing on dice the second time
Z	Sum of the numbers appearing on the dice
W	Difference of the numbers appearing on the dice

Table 1: Definition of Random variables.

$$W = X - Y \tag{1}$$

W can take values ranging from {-5 to 5}.

$$p_X(k) = \begin{cases} 0.2, & k = 1, 2 \\ 0.1, & k = 3, 5, 6 \\ 0.3, & k = 4 \end{cases} \tag{2}$$

$$p_X(k) = p_Y(k) \tag{3}$$

PMF of W using z-transform:

applying the z-transform on both the sides

$$z\{W\} = z\{X - Y\} \tag{4}$$

$$M_W(z) = M_{X-Y}(z) \tag{5}$$

Using the expectation operator:

$$E[z^{-W}] = E[z^{-X+Y}] \tag{6}$$

$$= E[z^{-X}] \cdot E[z^{Y}] \tag{7}$$

$$= \left(\sum_{i=1}^{6} p_X(i) \cdot z^{-i} \right) \cdot \left(\sum_{j=1}^{6} p_Y(j) \cdot z^{j} \right) \tag{8}$$

Extracting the PMF by considering the defenition of z-transform

$$
\begin{aligned}
M_W(z) &= p_W(0) + p_W(1)z + ... + p_W(k)z^k + ... && (9) \\
&= 0.01(2z^{-5} + 4z^{-4} + 9z^{-3} + 12z^{-2} + 13z^{-1} + 20 \\
&\quad + 13z^1 + 12z^2 + 9z^3 + 4z^4 + 2z^5) && (10)
\end{aligned}
$$

defined for all the values of $-5 \leq k \leq 5$

$$
p_W(k) = \frac{1}{k!}\left(\frac{d^{|k|}}{dz^{|k|}} M_W(z)\right)_{z=0} \tag{11}
$$

Now,

$$
Z = X + Y \tag{12}
$$

Z can take values ranging from $\{2 \text{ to } 12\}$.

PMF of Z using z-transform:

applying the z-transform on both the sides

$$
z\{Z\} = z\{X + Y\} \tag{13}
$$

$$
M_Z(z) = M_{X+Y}(z) \tag{14}
$$

Using the expectation operator:

$$
\begin{aligned}
E[z^{-Z}] &= E[z^{-X-Y}] && (15) \\
&= E[z^{-X}] \cdot E[z^{-Y}] && (16) \\
&= \left(\sum_{i=1}^{6} p_X(i) \cdot z^{-i}\right)^2 && (17)
\end{aligned}
$$

Extracting the PMF by considering the defenition of z-transform

$$
\begin{aligned}
M_W(z) &= p_W(0) + p_W(1)z + ... + p_W(k)z^k + ... && (18) \\
&= (0.1z^{-6} + 0.1z^{-5} + 0.3z^{-4} + 0.1z^{-3} + 0.2z^{-2} + 0.2z^{-1})^2 && (19)
\end{aligned}
$$

defined for all the values of $2 \leq k \leq 12$

$$
p_W(k) = \frac{1}{k!}\left(\frac{d^{|k|}}{dz^{|k|}} M_W(z)\right)_{z=0} \tag{20}
$$

For event A,

Finding the probability for $W = 0$

$$
p_W(0) = 0.2 \tag{21}
$$

For event B,

Finding the probability for $Z \geq 10$

$$p_Z(10) = 0.07 \tag{22}$$
$$p_Z(11) = 0.02 \tag{23}$$
$$p_Z(12) = 0.01 \tag{24}$$

Hence,

$$\begin{aligned} \Pr(B) &= \Pr(Z = 10) + \Pr(Z = 11) + \Pr(Z = 12) && (25) \\ &= 0.1 && (26) \end{aligned}$$

Now, A and B will be independent if,

$$\begin{aligned} \Pr(A \cap B) &= \Pr(A) \Pr(B) && (27) \\ \Pr(A \cap B) &= 0.1 \times 0.1 + 0.1 \times 0.1 && (28) \\ &= 0.02 && (29) \\ &= \Pr(A) \Pr(B) && (30) \end{aligned}$$

Hence, events A and B are independent.