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Anupama Kulshreshtha
EE22BTECH11009

Let X_1, X_2, ... , X_n be a random sample of size n from a population having probability density function

$$f(x;\mu) = \begin{cases} e^{-(x-\mu)}, & \text{if } \mu \leq x < \infty \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $\mu \in \mathbb{R}$ is an unknown parameter. If $\hat{M}$ is the maximum likelihood estimator of the median of X_1, then which one of the following statements is true?

A) $\text{P}(\hat{M} \leq 2) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

B) $\text{P}(\hat{M} \leq 1) = 1 - e^{-n\log_e 2}$ if $\mu = 1$

C) $\text{P}(\hat{M} \leq 3) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

D) $\text{P}(\hat{M} \leq 4) = 1 - e^{-n(2\log_e 2-1)}$ if $\mu = 1$

Solution: Let $\hat{M}$ = Maximum likelihood estimator of median of X_1 = m
By definition, The median of a distribution is the value such that half of the probability mass of the distribution is below that value.
Thus, for a function $f(x;\mu)$,

$$\int_{\mu}^{m} f(x;\mu)\,dx = \frac{1}{2} \tag{2}$$

$$\implies \int_{\mu}^{m} e^{-(x-\mu)}\,dx = \frac{1}{2} \tag{3}$$

$$\implies -e^{-(m-\mu)} + 1 = \frac{1}{2} \tag{4}$$

$$\implies e^{-(m-\mu)} = \frac{1}{2} \tag{5}$$

$$\implies m = \mu + \log_e 2 \tag{6}$$

Now,

$$P(\hat{M} \leq y) = P(\mu < y - \log_e 2) \tag{7}$$

$$= F(y - \log_e 2) \tag{8}$$

Using cdf, $F(X_1) = 1 - e^{-n(x-\mu)}$

$$\implies P(\hat{M} \leq y) = F(y - \log_e 2) \tag{9}$$
$$= 1 - e^{-n(y - \log_e 2 - \mu)} \tag{10}$$

When $y = 2$ and $\mu = 1$,

$$P(\hat{M} \leq 2) = 1 - e^{-n(1 - \log_e 2)} \tag{11}$$

Therefore, option (A) is correct.