Question 39.2023

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Let X_1, X_2, ... , X_n be a random sample of size n from a population having probability density function

$$f(x;\mu) = \begin{cases} e^{-(x-\mu)}, & \text{if } \mu \leq x < \infty \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $\mu \in \mathbb{R}$ is an unknown parameter. If $\hat{M}$ is the maximum likelihood estimator of the median of X_1, then which one of the following statements is true?

A) $\mathrm{P}(\hat{M} \leq 2) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

B) $\mathrm{P}(\hat{M} \leq 1) = 1 - e^{-n \log_e 2}$ if $\mu = 1$

C) $\mathrm{P}(\hat{M} \leq 3) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

D) $\mathrm{P}(\hat{M} \leq 4) = 1 - e^{-n(2\log_e 2-1)}$ if $\mu = 1$

Solution: For continuous random variable X, median M is such that,

$$P(X \leq M) = 0.5 \tag{2}$$

The pdf of X is given by,

$$p_X(x) = \begin{cases} e^{-(x-\mu)}, & \text{if } \mu \leq x < \infty \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

Hence, cdf is given by

$$F(x;\mu) = \int_{\mu}^{x} e^{-(t-\mu)} dt \tag{4}$$

$$= e^{\mu}[-e^{-x} + e^{-\mu}] \tag{5}$$

$$= 1 - e^{-(x-\mu)} \tag{6}$$

Now,

$$F(x;\mu) = 0.5 \tag{7}$$

$$\implies 1 - e^{-(M-\mu)} = 0.5 \tag{8}$$

$$\implies \hat{M} = \mu + \ln(2) \tag{9}$$

Definition 1. *L, the Maximum Likelihood Estimator of the distribution is given by,*

$$
\begin{aligned}
L &= \prod e^{-(x-\mu)} && (10)\\
&= e^{-(\sum x_i - n\mu)} && (11)\\
& && (12)
\end{aligned}
$$

For the Likelihood function to be maximum, $\sum x_i - n\mu$ should be minimum Hence,

$$
\begin{aligned}
X_i &> \mu && (13)\\
\implies \sum x_i &> n\mu && (14)\\
\sum x_i - n\mu &> 0 && (15)\\
\implies \mu &= \frac{\sum x_i}{n} && (16)
\end{aligned}
$$

Given,

$$
\begin{aligned}
p_X(x) &= e^{-(x-\mu)} && (17)\\
&= e^{-(x-\frac{\sum x_i}{n})} && (18)
\end{aligned}
$$

Let

$$
\begin{aligned}
Y = g(x) &= \frac{\sum x_i}{n} && (19)\\
g^{-1}(x) &= nx && (20)
\end{aligned}
$$

To find pdf of y,

$$
\begin{aligned}
p_Y(x) &= p_X(g^{-1}(x))|\frac{d}{dx}g^{-1}(x)| && (21)\\
&= e^{-(nx-\frac{n\sum x_i}{n})}|n| && (22)\\
&= ne^{-n(x-\frac{\sum x_i}{n})} && (23)\\
&= ne^{-n(x-\mu)} && (24)\\
\implies f_Y(x) &= 1 - e^{-n(x-\mu)} && (25)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
P(\hat{M} \leq k) &= P(\frac{\sum x_i}{n} + \ln(2) \leq k) && (26)\\
&= P(Y + \ln(2) \leq k) && (27)\\
&= P(Y \leq k - \ln(2)) && (28)\\
&= f_Y(k - \ln(2)) && (29)\\
&= 1 - e^{-n(y-\ln(2)-\mu)} && (30)
\end{aligned}
$$

When $y = 2$ and $\mu = 1$,

$$P(\hat{M} \leq 2) = 1 - e^{-n(1-\log_e 2)} \tag{31}$$

Therefore, option (A) is correct.