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Let X_1, X_2, ... , X_n be a random sample of size n from a population having probability density function

$$f(x;\mu) = \begin{cases} e^{-(x-\mu)}, & \text{if } \mu \le x < \infty \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $\mu \in \mathbb{R}$ is an unknown parameter. If $\hat{M}$ is the maximum likelihood estimator of the median of X_1, then which one of the following statements is true?

A) $\mathrm{P}(\hat{M} \le 2) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

B) $\mathrm{P}(\hat{M} \le 1) = 1 - e^{-n\log_e 2}$ if $\mu = 1$

C) $\mathrm{P}(\hat{M} \le 3) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

D) $\mathrm{P}(\hat{M} \le 4) = 1 - e^{-n(2\log_e 2-1)}$ if $\mu = 1$

Solution: For continuous random variable X, median M is such that,

$$P(X \le M) = 0.5 \tag{2}$$

The pdf of X is given by,

$$f(x;\mu) = \begin{cases} e^{-(x-\mu)}, & \text{if } \mu \le x < \infty \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

Hence, cdf is given by

$$F(x;\mu) = \int_{\mu}^{x} e^{-(t-\mu)}dt \tag{4}$$

$$= e^{\mu}[-e^{-x} + e^{-\mu}] \tag{5}$$

$$= 1 - e^{-(x-\mu)} \tag{6}$$

Now,

$$F(x;\mu) = 0.5 \tag{7}$$

$$\implies 1 - e^{-(M-\mu)} = 0.5 \tag{8}$$

$$\implies M = \mu + \ln(2) \tag{9}$$

The MLE $\hat{M}$ of the median in a random sample is the middle value of the ordered sample. Since $M = \mu + \ln(2)$, the MLE of the median is $\hat{M} = \hat{\mu} + \ln(2)$, where $\hat{\mu}$ is the MLE of μ. Now, to calculate $P(\hat{M} \leq y)$,

$$P(\hat{\mu} + \ln(2) \leq y) = P(\hat{\mu} \leq y - \ln(2)) \tag{10}$$

CDF of $\hat{\mu}$ is same as CDF of μ, as MLE is an unbiased estimator, so,

$$F(\hat{\mu}) = F(\mu) \tag{11}$$

$$\text{As there are 'n' different samples} \tag{12}$$

$$\implies P(\hat{M} \leq y) = 1 - e^{-n(y-M)} \tag{13}$$

When $y = 2$ and $\mu = 1$,

$$P(\hat{M} \leq 2) = 1 - e^{-n(1-\log_e 2)} \tag{14}$$

Therefore, option (A) is correct.