Question 39.2023

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Let X_1, X_2, ... , X_n be a random sample of size n from a population having probability density function

$$f(x;\mu) = \begin{cases} e^{-(x-\mu)}, & \text{if } \mu \leq x < \infty \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $\mu \in \mathbb{R}$ is an unknown parameter. If $\hat{M}$ is the maximum likelihood estimator of the median of X_1, then which one of the following statements is true?

A) $\mathrm{P}(\hat{M} \leq 2) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

B) $\mathrm{P}(\hat{M} \leq 1) = 1 - e^{-n\log_e 2}$ if $\mu = 1$

C) $\mathrm{P}(\hat{M} \leq 3) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

D) $\mathrm{P}(\hat{M} \leq 4) = 1 - e^{-n(2\log_e 2-1)}$ if $\mu = 1$

Solution: For continuous random variable X, median M is such that,

$$P(X \leq M) = 0.5 \tag{2}$$

The pdf of X is given by,

$$p_X(x) = \begin{cases} e^{-(x-\mu)}, & \text{if } \mu \leq x < \infty \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

Hence, cdf is given by

$$F(x;\mu) = \int_{\mu}^{x} e^{-(t-\mu)}dt \tag{4}$$

$$= e^{\mu}[-e^{-x} + e^{-\mu}] \tag{5}$$

$$= 1 - e^{-(x-\mu)} \tag{6}$$

Now,

$$F(x;\mu) = 0.5 \tag{7}$$

$$\implies 1 - e^{-(M-\mu)} = 0.5 \tag{8}$$

$$\implies \hat{M} = \mu + \ln(2) \tag{9}$$

Definition 1. *L, the Maximum Likelihood Estimator of the distribution is given by,*

$$L = \prod e^{-(x-\mu)} \tag{10}$$
$$= e^{-(\sum x_i - n\mu)} \tag{11}$$
$$\tag{12}$$

For the Likelihood function to be maximum, $\sum x_i - n\mu$ should be minimum Hence,

$$X_i > \mu \tag{13}$$
$$\implies \sum x_i > n\mu \tag{14}$$
$$\sum x_i - n\mu > 0 \tag{15}$$
$$\implies \mu = \frac{\sum x_i}{n} \tag{16}$$

Given,

$$p_X(x) = e^{-(x-\mu)} \tag{17}$$
$$\tag{18}$$

X_i follows an exponential distribution.

Definition 2. *We know that if,*

$$p_X(x) = \lambda_i e^{-\lambda_i x} \tag{19}$$
$$S = X_1 + X_2 + ... + X_n \tag{20}$$
$$p_S(n) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} \tag{21}$$

which is gamma distribution with parameters n and λ

Hence, pdf of

$$Y = \sum_{i=1}^{n} x_i \tag{22}$$
$$\text{where,} \tag{23}$$
$$p_X(x) = e^{-(x-\mu)} \tag{24}$$

will follow gamma distribution with parameter n and $\lambda = 1$, given by,

$$p_Y(x; n, 1) = \frac{x^{n-1} e^{-x}}{(n-1)!} \tag{25}$$

Hence, cdf is given by,

$$f_Y(x;n) = \int_1^x \frac{t^{n-1}e^{-t}}{(n-1)!}\,dt \tag{26}$$
$$= 1 - \Gamma(n,x) \tag{27}$$

where $\Gamma(n,x)$ is incomplete gamma function. Thus,

$$P(\hat{M} \le k) = P(\frac{\sum x_i}{n} + \ln(2) \le k) \tag{28}$$
$$= P(Y/n + \ln(2) \le k) \tag{29}$$
$$= P(Y \le n(k - \ln(2))) \tag{30}$$
$$= f_Y(n(k - \ln(2))) \tag{31}$$
$$= 1 - \Gamma(n, n(k - ln2)) \tag{32}$$

It needs a value of n to be computed.