Question 9.3.7

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There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?
Solution:

Parameter	Values	Description
n	10	Number of items
p	0.05	Probability of being defective
q	0.95	Probability of not being defective
X	1 if defective 0 if not defective	Bernoulli Random Variable
Y	$\sum_{i=1}^{n} X_i$	Binomial Random Variable
$\mu = np$	0.5	Mean
$\sigma^2 = npq$	0.475	Variance

Table 1: Definition of parameters.

The cdf using binomial is given by

$$F_Y(n) = \Pr(Y \le n) \tag{1}$$

$$= \sum_{k=0}^{n} {}^{10}C_k p^k (1-p)^{10-k} \tag{2}$$

We require $\Pr(Y \le 1)$. Since $n = 1$,

$$F_Y(1) = \Pr(Y \le 1) \tag{3}$$

$$= \sum_{k=0}^{1} {}^{10}C_k (0.05)^k (0.95)^{10-k} \tag{4}$$

$$= 0.9138 \tag{5}$$

The gaussian distribution function is defined as:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x \in Y) \tag{6}$$

So,

$$p_Y\left(0\right) = \frac{1}{\sqrt{2\pi\left(0.475\right)}} e^{-\frac{(0-0.5)^2}{2(0.475)}} \tag{7}$$

$$= \frac{1}{\sqrt{2\pi\left(0.475\right)}} e^{-\frac{5}{19}} \tag{8}$$

$$= 0.4450 \tag{9}$$

$$p_Y\left(1\right) = \frac{1}{\sqrt{2\pi\left(0.475\right)}} e^{-\frac{(1-0.5)^2}{2(0.475)}} \tag{10}$$

$$= \frac{1}{\sqrt{2\pi\left(0.475\right)}} e^{-\frac{5}{19}} \tag{11}$$

$$= 0.4450 \tag{12}$$

Hence required probability,

$$p_Y = p_Y\left(0\right) + p_Y\left(1\right) \tag{13}$$

$$= 0.8901 \tag{14}$$

From (5) and (14)

$$F_Y\left(1\right) \approx p_Y \tag{15}$$

Solving using Q function
Q function is defined

$$Q\left(x\right) = \int_x^{\infty} f\left(x\right) dx \tag{16}$$

then CDF of Y is:

$$\Pr\left(Y < x\right) = \int_{-\infty}^{x} f\left(x\right) dx \tag{17}$$

$$= 1 - \int_x^{\infty} f\left(x\right) dx \tag{18}$$

$$= 1 - Q\left(x\right) \tag{19}$$

and for finding $\Pr\left(\frac{X-\mu}{\sigma}\right)$ Using approximation,

$$\Pr\left(\frac{Y-\mu}{\sigma}\right) \approx \Pr\left(\frac{Y+0.5-\mu}{\sigma} < \frac{Y-\mu}{\sigma} < \frac{Y-0.5-\mu}{\sigma}\right) \tag{20}$$

$$\approx \Pr\left(\frac{Y-\mu}{\sigma} < \frac{Y+0.5-\mu}{\sigma}\right) - \Pr\left(\frac{Y-\mu}{\sigma} < \frac{Y-0.5-\mu}{\sigma}\right) \tag{21}$$

$$\approx Q\left(\frac{Y-0.5-\mu}{\sigma}\right) - Q\left(\frac{Y+0.5-\mu}{\sigma}\right) \tag{22}$$

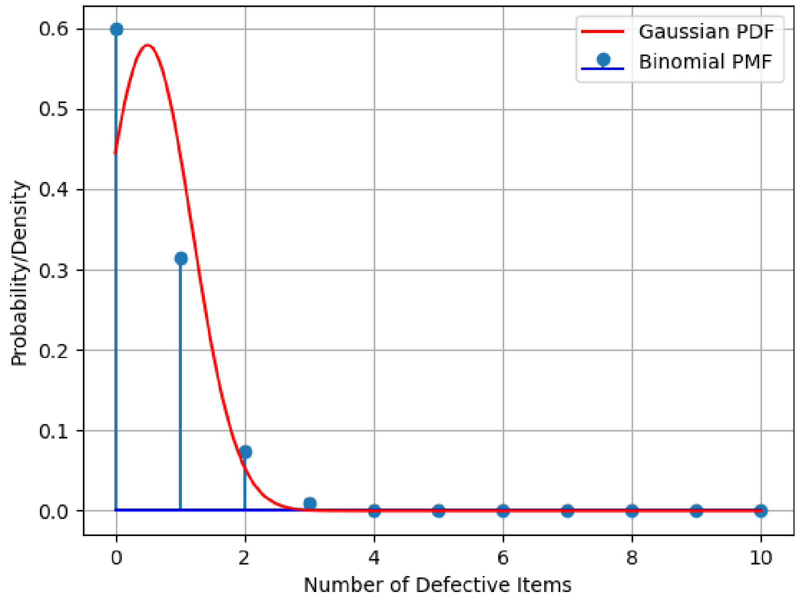

Figure 1: Binomial pmf vs Gaussian pdf

$$Y = 1 \tag{23}$$

$$\Pr\left(Z = 1.053\right) \approx Q\left(0\right) - Q\left(1.45\right) \tag{24}$$

$$\approx 0.4265 \tag{25}$$