

Question 39.2023

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Let X_1, X_2, \dots, X_n be a random sample of size n from a population having probability density function

$$f(x; \mu) = \begin{cases} e^{-(x-\mu)}, & \text{if } \mu \leq x < \infty \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

where $\mu \in \mathbb{R}$ is an unknown parameter. If \hat{M} is the maximum likelihood estimator of the median of X_1 , then which one of the following statements is true?

A) $P(\hat{M} \leq 2) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

B) $P(\hat{M} \leq 1) = 1 - e^{-n \log_e 2}$ if $\mu = 1$

C) $P(\hat{M} \leq 3) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

D) $P(\hat{M} \leq 4) = 1 - e^{-n(2 \log_e 2 - 1)}$ if $\mu = 1$

Solution: Let \hat{M} = Maximum likelihood estimator of median of $X_1 = m$
By definition,

$$\int_{\mu}^m f(x; \mu) dx = \frac{1}{2} \quad (2)$$

$$\Rightarrow \int_{\mu}^m e^{-(x-\mu)} dx = \frac{1}{2} \quad (3)$$

$$\Rightarrow -e^{-(m-\mu)} + 1 = \frac{1}{2} \quad (4)$$

$$\Rightarrow e^{-(m-\mu)} = \frac{1}{2} \quad (5)$$

$$\Rightarrow m = \mu + \log_e 2 \quad (6)$$

Now,

$$P(\hat{M} \leq y) = P(\mu < y - \log_e 2) \quad (7)$$

$$= F(y - \log_e 2) \quad (8)$$

$$X_i \sim \exp(1)$$

$$X_1 \sim \exp(n)$$

$$\text{Hence, cdf of } F(X_1) = 1 - e^{-n(x-\mu)}$$

$$\implies P(\hat{M} \leq y) = 1 - e^{-n(y - \log_e 2 - \mu)} \quad (9)$$

When $y = 2$ and $\mu = 1$,

$$P(\hat{M} \leq 2) = 1 - e^{-n(1 - \log_e 2)} \quad (10)$$

Therefore, option (A) is correct.