

Question 39.2023

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Let X_1, X_2, \dots, X_n be a random sample of size n from a population having probability density function

$$f(x; \mu) = \begin{cases} e^{-(x-\mu)}, & \text{if } \mu \leq x < \infty \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

where $\mu \in \mathbb{R}$ is an unknown parameter. If \hat{M} is the maximum likelihood estimator of the median of X_1 , then which one of the following statements is true?

A) $P(\hat{M} \leq 2) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

B) $P(\hat{M} \leq 1) = 1 - e^{-n \log_e 2}$ if $\mu = 1$

C) $P(\hat{M} \leq 3) = 1 - e^{-n(1-\log_e 2)}$ if $\mu = 1$

D) $P(\hat{M} \leq 4) = 1 - e^{-n(2 \log_e 2 - 1)}$ if $\mu = 1$

Solution: For continuous random variable X , median M is such that,

$$P(X \leq M) = 0.5 \quad (2)$$

The pdf of X is given by,

$$p_X(x) = \begin{cases} e^{-(x-\mu)}, & \text{if } \mu \leq x < \infty \\ 0, & \text{otherwise,} \end{cases} \quad (3)$$

Hence, cdf is given by

$$F(x; \mu) = \int_{\mu}^x e^{-(t-\mu)} dt \quad (4)$$

$$= e^{\mu} [-e^{-x} + e^{-\mu}] \quad (5)$$

$$= 1 - e^{-(x-\mu)} \quad (6)$$

Now,

$$F(x; \mu) = 0.5 \quad (7)$$

$$\implies 1 - e^{-(M-\mu)} = 0.5 \quad (8)$$

$$\implies \hat{M} = \mu + \ln(2) \quad (9)$$

Definition 1. L , the Maximum Likelihood Estimator of the distribution is given by,

$$L = \prod e^{-(x-\mu)} \quad (10)$$

$$= e^{n\mu - \sum x_i} \quad (11)$$

$$\log L = n\mu - \sum x_i \quad (12)$$

$$\frac{\partial(\log L)}{\partial \mu} = 0 \quad (13)$$

$$\implies n = 0 \text{ (Not possible)} \quad (14)$$

Hence L is maximum when μ is maximum.

$$\mu = \min(X_1, X_2, \dots) \quad (15)$$

$$= X_1 \quad (16)$$

Thus,

$$\hat{M} = X_1 + \ln(2) \quad (17)$$

$$P(\hat{M} \leq y) = P(X_1 \leq y - \ln(2)) \quad (18)$$

$$= F(y - \ln(2)) \quad (19)$$

Using exponential distribution,

$$f(X_1) = ne^{-n(x-\mu)} \quad (20)$$

$$F(X_1) = 1 - e^{-n(x-\mu)} \quad (21)$$

$$\implies F(y - \ln(2)) = 1 - e^{-n(y - \ln(2) - \mu)} \quad (22)$$

When $y = 2$ and $\mu = 1$,

$$P(\hat{M} \leq 2) = 1 - e^{-n(1 - \log_e 2)} \quad (23)$$

Therefore, option (A) is correct.