STOCKHOLM DOCTORAL PROGRAM IN ECONOMICS

Handelshögskolan i Stockholm | Stockholms universitet

Paul Klein
Email: paul.klein@su.se
http://paulklein.se/newsite/teaching/38030.php

EC38030: Mathematics for macroeconomics
Fall 2022

Lecture 2: Pontryagin's maximum principle

1 A heuristic approach

Problem:

$$\max \int_0^T f(t, x(t), u(t))dt$$

subject to

$$\dot{x}(t) = g(t, x(t), u(t))$$

and where $x(0)$, $x(T)$ might or might not be fixed. T is nearly always fixed in economics, since we rarely endogenize lifespans.

Objective: Find conditions corresponding to

$$\frac{\partial}{\partial x}\left[f(x) + \lambda \cdot g(x)\right]$$

where $\cdot$ represents the Euclidean inner product. To do this, form the "Lagrangian"

$$\mathcal{L}(t, \mathbf{x}, \mathbf{u}) := \int_0^T f(t, x(t), u(t))dt + \int_0^T \lambda(t)\left[g(t, x(t), u(t)) - \dot{x}(t)\right]dt.$$

Now integrate by parts!

$$\mathcal{L} = \int_0^T \left[f + \lambda \cdot g + \dot{\lambda} \cdot x \right] dt + \lambda(T) \left[x(T) - x(0) \right].$$

Conjecture: Optimum conditions are

$$\text{“}\frac{\partial \mathcal{L}}{\partial x(t)} = 0\text{” for each } t \in [0, T]$$

and

$$\text{“}\frac{\partial \mathcal{L}}{\partial u(t)} = 0\text{” for each } t \in [0, T].$$

This makes no sense; changing the value at one point makes no difference at all, these conditions are vacuous. Never mind! Let's take an engineer's pragmatic approach to math and say that

$$\frac{\partial}{\partial f(s)} \int_0^T g(f(t))dt = g'(f(s))ds$$

as would be the case with a discrete sum, viz.

$$\frac{d}{dx_i} \sum_{k=1}^{n} g(x_k)\Delta x_k = g'(x_i)\Delta x_i.$$

If so, then our optimality conditions become

$$\left\{ \begin{array}{rcl} \dfrac{d}{du}\left[f + \lambda \cdot g \right] & = & 0 \\ \dfrac{d}{dx}\left[f + \lambda \cdot g \right] + \dot{\lambda} & = & 0. \end{array} \right.$$

This, roughly speaking, is Pontryagin's maximum principle, or PMP. The usual notation is the following.

$$\mathcal{H}(t,x,u,\lambda) := f(t,x,u) + \lambda \cdot g(t,x,u)$$

where $\mathcal{H}$ is called the "Hamiltonian". The optimality then become

$$\mathcal{H}_x(t,x(t),u(t),\lambda(t)) + \dot{\lambda}(t) = 0$$

and

$$u(t) \in \operatorname*{argmax}_u \mathcal{H}(t,x(t),u).$$

Incidentally, continuing in the heuristic vein, we may notice the following. Suppose neither f nor g depend explicitly on t. Then, assuming more differentiability than we really have a legal entitlement to,

$$\dot{\mathcal{H}} = f_x\dot{x} + f_u\dot{u} + \dot{\lambda}g + \lambda[g_x\dot{x} + g_u\dot{u}].$$

At an optimum, we have $f_u + \lambda g_u = 0$ so this reduces to

$$\dot{\mathcal{H}} = f_x\dot{x} + \dot{\lambda}g + \lambda g_x\dot{x}.$$

We also have $f_x + \lambda g_x = -\dot{\lambda}$, so this in turn reduces to

$$\dot{\mathcal{H}} = -\dot{\lambda}\dot{x} + \dot{\lambda}g.$$

But since $\dot{x} = g$, this reduces to

$$\dot{\mathcal{H}} = 0$$

so that the Hamiltonian is constant along an optimal trajectory.

Incidentally, suppose we have no constraint on $x(T)$ but that we do have endpoint evaluation so that the objective function is something like

$$\int_0^T f(t, x(t), u(t))dt + \Phi(x(T)).$$

Then the endpoint optimality condition is $\lambda(T) = \Phi'(x(T))$. This fact doesn't perhaps fit in here, but where else does it belong?

2 Mangasarian's sufficient conditions

The following result is based on Mangasarian (1966) via Lang (1993).

Consider

$$\max_u \int_0^T f\left(t, x\left(t\right), u\left(t\right)\right) dt$$

$$\text{s.t.} \begin{cases} \dot{x}\left(t\right) = g\left(t, x\left(t\right), u\left(t\right)\right) \\ x\left(0\right) = x_0 \\ \text{Some suitable NPG condition} \\ u\left(t\right) \in U \subset \mathbb{R}^k \ \text{ for all } t \in \left[0, T\right]. \end{cases} \tag{1}$$

Note that the endpoint condition $x\left(T\right) = b$ has been replaced by the somewhat vague phrase "some suitable NPG condition". We will make this precise later. But, broadly speaking, an NPG condition means that, at the endpoint T, the agent should leave behind a portfolio of assets with a nonnegative value. Notice that for the value of a portfolio to be defined, we need asset prices. But these are determined in equilibrium, so the NPG condition cannot be stated when looking merely at one agent's problem and ignoring the market interaction. It can only be stated if we know the prices $\boldsymbol{\lambda}$. Given these prices, the NPG condition is

$$\lambda(T) \cdot x(T) \geq 0. \tag{2}$$

This must hold for any feasible ("admissible") x, not just the optimal one. If $T = +\infty$ then the appropriate requirement is that $\lambda(t) \cdot x(t)$ eventually stop ever dipping below zero. The mathematical statement of this is

$$\liminf_{t \to \infty} \lambda(t) \cdot x(t) \geq 0. \tag{3}$$

Now let every agent in the economy solve (1), and, to motivate why a competitive equilibrium is the appropriate solution concept, suppose each agent is so small that her behavior cannot influence prices. We now consider sufficient conditions for the profile $(\mathbf{x}^*, \mathbf{u}^*)$ to be a competitive equilibrium allocation enforced by the equilibrium prices $\boldsymbol{\lambda}$. The sense in which $\boldsymbol{\lambda}$ is a set of prices is that $\lambda(t)$ will turn out to be the marginal value from the point of view of an individual of increasing $x(t)$. This idea will be stated more precisely in Section 2.1. We are now ready to state (an economic version of) Mangasarian's theorem formally.

Definition 2.1 *An* admissible allocation $(\mathbf{x}, \mathbf{u})$ *at market prices* $\boldsymbol{\lambda}$ *for the problem (1) is a pair of functions* $\mathbf{x} : [0, T] \to \mathbb{R}^n$ *and* $\mathbf{u} : [0, T] \to \mathbb{R}^m$ *such that*

1. $u(t) \in U$ *for each* $t \in [0, T]$,
2. $x(t) = x_0 + \int_0^t g(s, x(s), u(s))\, ds$ *for each* $t \in [0, T]$, *and*
3. $\lambda(T) \cdot x(T) \geq 0$ *if* $T < \infty$ *and* $\liminf_{t \to \infty} \lambda(t) \cdot x(t) \geq 0$ *if* $T = \infty$.

Definition 2.2 *The profile* $(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda})$ *is said to be a* competitive equilibrium *of the economy where all agents' preferences and constraints are given by (1) if* $\mathbf{u}^*$ *solves (1) and the prices* $\boldsymbol{\lambda}$ *clear the market for the 'assets'* $\mathbf{x}$ *clears in every period, i.e. if agents could trade assets freely at prices* $\boldsymbol{\lambda}$, *then each agent would demand the quantities* $\mathbf{x}^*$. *The profile* $(\mathbf{x}^*, \mathbf{u}^*)$ *is then called a competitive equilibrium* allocation *enforced by the equilibrium* prices $\boldsymbol{\lambda}$.

Remark 2.1 *This definition perhaps sounds a bit loose and non-mathematical. For example, we have not defined what it means to 'trade freely'. But we know that this can be done rigorously; see Debreu (1959).*

Theorem 2.1 (Mangasarian) *Let* $(\mathbf{x}^*, \mathbf{u}^*)$ *be an admissible allocation of (1) at market prices* $\boldsymbol{\lambda}$. *Let* $\boldsymbol{\lambda} : [0, T] \to \mathbb{R}$ *be a continuous and (except possibly at countably many points) differentiable function. Moreover, suppose the set* $U \in \mathbb{R}^m$ *is convex. Now define the function* $H : [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \to \mathbb{R}$ *via*

$$\mathcal{H}(t, x, u, \lambda) = f(t, x, u) + \lambda \cdot g(t, x, u)\,. \tag{4}$$

Suppose now that $\mathcal{H}$ *is continuously differentiable with respect to* x *on its entire domain. Suppose also that* $H(t, x, u, \lambda(t))$ *is concave in* (x, u) *for each* $t \in [0, T]$. *Finally, suppose that*

1. $\dfrac{\partial \mathcal{H}(t, x^*(t), u^*(t), \lambda(t))}{\partial x} + \dot{\lambda}(t) = 0$ *for all* $t \in [0, T]$ *except possibly at finitely many points*,
2. $u^*(t) \in \underset{u \in U}{\operatorname{argmax}}\, \mathcal{H}(t, x^*(t), u, \lambda(t))\,,$ *for all* $t \in [0, T]$ *and*
3. $\lambda(T) \cdot x^*(T) = 0$ *if* $T < \infty$ *and* $\lim_{t \to \infty} \lambda(t) \cdot x^*(t) = 0$ *if* $T = \infty$.

Then $(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda})$ *is a competitive equilibrium.*

Remark 2.2 *If all the prices* $\boldsymbol{\lambda}$ *are strictly positive,* $\lambda(T) \cdot x(T) = 0$ *implies* $x(T) = 0$.

Remark 2.3 *The NPG condition* $\lambda(T) \cdot x(T) \geq 0$ *or* $\liminf_{t\to\infty} \lambda(t) \cdot x(t) \geq 0$ *is a* constraint. *The condition* $\lambda(T) \cdot x^*(T) = 0$ *or* $\lim_{t\to\infty} \lambda(t) \cdot x^*(t) = 0$ *is called a* transversality *condition, and is not a constraint but an optimality condition. The intuitive meaning of the NPG condition is 'You mustn't leave any debt behind' and the intuitive meaning of the transversality condition is 'Since you can't leave any debt behind, and there is no point in leaving assets behind, set the net worth of your bequest to zero'.*

Before the proof starts, we need some preliminaries. Eventually the strategy will be to show that $\mathbf{u}^*$ solves (1) by showing that it delivers a higher value of the objective than any admissible alternative. Then we will invoke an envelope theorem to show that $\boldsymbol{\lambda}$ clears the asset market. But for now, we introduce some definitions and results that we'll need in the proof.

Definition 2.3 *Let* $A \subset \mathbb{R}^n$ *be a convex set. A function* $f : A \to BbbR$ *is said to be* concave *if for all* $x^0, x^1 \in A$ *and all* $\lambda \in [0,1]$ *we have*

$$f\left(\lambda x^0 + (1-\lambda)x^1\right) \geq \lambda f(x_0) + (1-\lambda) f\left(x^1\right). \tag{5}$$

Example 2.1 *Let* $x, x^1 \in A$. *Define* $\Delta x = x^1 - x$ *and set* $\lambda = 1 - \frac{1}{M}$ *where* $M > 0$. *Then* $\lambda x + (1-\lambda)x^1 = x + \frac{\Delta x}{M}$ *and if* f *is concave, then*

$$f\left(x + \frac{\Delta x}{M}\right) \geq \frac{1}{M}\left[f(x + \Delta x) - f(x)\right]. \tag{6}$$

Lemma 2.1 (Mean value theorem) *Let* $x, y \in \mathbb{R}^n$ *and let*

$$A(x,y) := \{z \in \mathbb{R}^n : z = \lambda x + (1-\lambda) y \text{ for some } \lambda \in (0,1)\}. \quad (7)$$

Then $A(x,y)$ *is the (open) line segment between* x *and* y*. Let* $f : A(x,y) \to \mathbb{R}$ *be continuously differentiable. Then there is a* $w \in A(x,y)$ *such that*

$$f(x) - f(y) = \nabla f(w) \cdot (x - y). \quad (8)$$

Proof. Omitted. ■

Lemma 2.2 *Let* $\mathcal{H}$ *and* U *be as in Theorem 2.1. Let* x^* *and* λ *be arbitrary fixed vectors in* $\mathbb{R}^n$ *(and hence* not *functions), let* $t \in [0,T]$ *be fixed and let* u^* *be such that*

$$u^* \in \underset{u \in U}{\operatorname{argmax}} \, \mathcal{H}(t, x^*, u, \lambda). \quad (9)$$

Then for any $u \in U$ *and* $x \in \mathbb{R}^n$ *we have*

$$\mathcal{H}(t, x^*, u^*, \lambda) - \mathcal{H}(t, x, u, \lambda) \geq \frac{\partial \mathcal{H}(t, x^*, u^*, \lambda)}{\partial x} \cdot [x^* - x]. \quad (10)$$

Proof.

Let $x \in \mathbb{R}^n$ and $u \in U$ be fixed vectors and define $\Delta x = x - x^*$, $\Delta u = u - u^*$. For each

$M > 0$ we have

$$\begin{aligned}
&\mathcal{H}\left(t, x^* + \frac{\Delta x}{M}, u^* + \frac{\Delta u}{M}, \lambda\right) \geq \{\text{concavity!}\} \geq \\
&\geq \mathcal{H}\left(t, x^*, u^*, \lambda^*\right) + \frac{1}{M}\left[\mathcal{H}\left(t, x, u, \lambda\right) - \mathcal{H}\left(t, x^*, u^*, \lambda\right)\right] \geq \\
&\geq \left\{u^* \in \underset{u \in U}{\operatorname{argmax}}\, \mathcal{H}\left(t, x^*, u, \lambda\right) \text{ and } \left(u^* + \frac{\Delta u}{M}\right) \in U \text{ since } U \text{ is convex}\right\} \geq \\
&\geq \mathcal{H}\left(t, x^*, u^* + \frac{\Delta u}{M}, \lambda\right) + \frac{1}{M}\left[\mathcal{H}\left(t, x, u, \lambda\right) - \mathcal{H}\left(t, x^*, u^*, \lambda\right)\right].
\end{aligned} \tag{11}$$

Hence

$$\begin{aligned}
&\mathcal{H}\left(t, x^*, u^*, \lambda\right) - \mathcal{H}\left(t, x, u, \lambda\right) \geq \\
&\geq M\left[\mathcal{H}\left(t, x^*, u^* + \frac{\Delta u}{M}, \lambda\right) - \mathcal{H}\left(t, x^* + \frac{\Delta x}{M}, u^* + \frac{\Delta u}{M}, \lambda\right)\right] = \\
&= \{\text{Mean value theorem!}\} \\
&= -\frac{\partial}{\partial x}\mathcal{H}\left(t, \widehat{x}_M, u^* + \frac{\Delta u}{M}, \lambda\right) \cdot \Delta x
\end{aligned} \tag{12}$$

for some $\widehat{x}_M \in A\left(x^*, x^* + \dfrac{\Delta x}{M}\right)$ where $A\left(x^*, x^* + \dfrac{\Delta x}{M}\right)$ is the open line segment between x^* and $x^* + \dfrac{\Delta x}{M}$. Now let $M \to \infty$ and invoke the continuity of $\mathcal{H}_x$. ■

The proof can now begin in earnest. Note first that

$$\int_0^T f\left(t, x\left(t\right), u\left(t\right)\right) dt = \int_0^T \left[\mathcal{H}\left(t, x\left(t\right), u\left(t\right)\right) - \lambda\left(t\right) \dot{x}\left(t\right)\right] dt \tag{13}$$

Now consider the candidate optimal allocation $(\mathbf{x}^*, \mathbf{u}^*)$ which by assumption satisfies $u^*(t) \in \underset{u \in U}{\operatorname{argmax}}\, \mathcal{H}(t, x^*(t), u, \lambda(t))$ for each $t \in [0, T]$ and let (x, u) be an admissible allocation. We will now show that $(\mathbf{x}^*, \mathbf{u}^*)$ delivers a value of the objective function no smaller than does $(\mathbf{x}, \mathbf{u})$.

$$\int_0^T f\left(t, x^*(t), u^*(t)\right) dt - \int_0^T f\left(t, x(t), u(t)\right) dt =$$

$$= \int_0^T \left[\mathcal{H}\left(t, x^*(t), u^*(t), \lambda(t)\right) - \mathcal{H}\left(t, x(t), u(t), \lambda(t)\right)\right] dt - \int_0^T \lambda(t) \cdot \left[\dot{x}^*(t) - \dot{x}(t)\right] dt \geq$$

$$\geq \{\textbf{Lemma 2.2!}\} \geq$$

$$\geq \int_0^T \left\{\mathcal{H}_x\left(t, x^*(t), u^*(t), \lambda(t)\right) \cdot \left[x^*(t) - x(t)\right]\right\} dt - \int_0^T \lambda(t) \cdot \left[\dot{x}^*(t) - \dot{x}(t)\right] dt =$$

$$= \{\textbf{Integration by parts!}\} =$$

$$= \int_0^T \left[\mathcal{H}_x\left(t, x^*(t), u^*(t), \lambda(t)\right) + \dot{\lambda}(t)\right] \cdot \left[x^*(t) - x(t)\right] dt -$$

$$-\lambda(T)\left[x^*(T) - x(T)\right] + \lambda(0)\left[x^*(0) - x(0)\right]. \tag{14}$$

What we want to show, of course, is that the last expression is ≥ 0 under the assumption of our Theorem. The first term vanishes by the $\mathcal{H}_x + \dot{\lambda} = 0$ condition. The final term vanishes since $x^*(0) - x(0) = x_0$ (recall that x and $\mathbf{x}^*$ are admissible!). This leaves us with the middle term

$$\lambda(T)\,x(T) - \lambda(T)\,x^*(T)\,.$$

Consider the case $T < \infty$ first. Then the admissibility of x guarantees that $\lambda(T)\,x(T) \geq 0$. Meanwhile, the transversality condition says that $\lambda(T)\,x^*(T) = 0$. This finishes the proof that u^* is optimal for a finite T. On the other hand, if $T = \infty$, we consider the difference

$$\lambda(t)\,x(t) - \lambda(t)\,x^*(t) \tag{15}$$

and let $t \to \infty$. By the NPG condition, $\lambda(t)\,x(t)$ eventually stops ever dipping below zero as $t \to \infty$, and by the transversality condition, $\lambda(t)\,x^*(t) \to 0$ as $t \to \infty$. The final part of the proof shows that the prices $\boldsymbol{\lambda}$ really clear the market. This will be true if each agent's marginal values (in terms of her objective function) of the assets x are equal to the market prices $\boldsymbol{\lambda}$. This is the Envelope Theorem; see Section 2.1. ∎

2.1 The Envelope Theorem

Theorem 2.2 *Let* $(\mathbf{x}^*, \mathbf{u}^*, \boldsymbol{\lambda})$ *satisfy the sufficient conditions for being a competitive equilibrium of an economy described by (1). Suppose the sufficient conditions define* u^* *uniquely. (There is a further technicality that we sweep under the rug here. See*

Lang (1993).) Define the value function (indirect utility function) via

$$V(t,x) = \max_{\mathbf{u}} \int_t^T f(s, x(s), u(s))\, ds$$

$$\textit{s.t.} \quad \begin{cases} \dot{x}(s) = g(s, x(s), u(s)) \\ x(t) = x \\ \textit{NPG (relative to market clearing prices)} \\ u(s) \in U \subset \mathbb{R}^k \;\; \textit{for all } s \in [t, T]. \end{cases} \tag{16}$$

Then, for all $t = [0, T)$, *we have*

$$\frac{\partial V(t, x^*(t))}{\partial x} = \lambda(t). \tag{17}$$

and if we define $\dfrac{\partial V(T, x^*(T))}{\partial x} = \lim_{t \to T} \dfrac{\partial V(t, x^*(t))}{\partial x}$ *then the statement folds for* $t = T$ *as well.*

Proof. See Lang (1993). ■

Remark 2.4 *The equation (17) is only valid along the optimal path* $\mathbf{x}^*$.

Example 2.2 *Consider the optimal consumption problem*

$$\max_{c} \int_{0}^{1} \ln c(t)\, dt$$

$$\text{s.t. } \begin{cases} \dot{k}(t) = -c(t) \\ k(0) = 1 \\ \textit{NPG} \end{cases} \tag{18}$$

The Hamiltonian is

$$\mathcal{H}(t, x(t), \lambda(t)) = \ln c(t) - \lambda(t) c(t) \tag{19}$$

Our optimality conditions become

$$\dot{\lambda}(t) = 0 \tag{20}$$

and

$$\frac{1}{c(t)} = \lambda(t) \tag{21}$$

Hence $\lambda(t) \equiv \lambda$ *and* $c(t) \equiv c = \frac{1}{\lambda}$*. According to the law of motion for the capital stock,*

$$k(t) = 1 - \int_{0}^{t} c ds = 1 - tc. \tag{22}$$

The transversality condition now becomes

$$\frac{1-c}{c} = 0. \tag{23}$$

This implies that $c = 1$ *and we have solved the problem. Note that* $\lambda = 1$ *and*

$k^*(t) = 1 - t$. *To find the value function, we now solve*

$$\max_c \int_t^1 \ln c(s)\, ds$$

$$\textit{s.t.} \begin{cases} \dot{k}(s) = -c(s) \\ k(t) = k \\ \textit{NPG} \end{cases} \tag{24}$$

In this case, the solution is $c(t) \equiv c = \dfrac{k}{1-t}$. *Hence the value function is*

$$V(t,k) = \int_t^1 \ln\left(\frac{k}{1-t}\right) ds = (1-t)\left[\ln k - \ln(1-t)\right] \tag{25}$$

and

$$\frac{\partial V(t,k)}{\partial k} = \frac{1-t}{k}. \tag{26}$$

Consequently

$$\frac{\partial V(t,k^*(t))}{\partial k} = \frac{1-t}{1-t} = 1. \tag{27}$$

and we have confirmed the envelope theorem in a special case.

3 Current value costate

Sometimes the time-dependence of a dynamic optimization problem takes the form of geometric discounting only, i.e. we have a problem of the form

$$\max_u \int_0^T e^{-\rho t} f\left(x\left(t\right), u\left(t\right)\right) dt$$

$$\text{s.t.} \begin{cases} \dot{x}\left(t\right) = g\left(x\left(t\right), u\left(t\right)\right) \\ x\left(0\right) = x_0 \\ \text{NPG} \\ u\left(t\right) \in U \subset \mathbb{R}^k \ \text{ for all } t \in \left[0, T\right]. \end{cases} \tag{28}$$

(By "NPG" we mean a no-Ponzi-scheme condition. As we've seen, it's problematic to state in general but we want it to imply that $\liminf_{t\to\infty} \lambda(t)x(t) \geq 0$.)

In this case it is often useful to redefine the costate $\lambda\left(t\right)$ as the current rather than the present value of $x\left(t\right)$. The definition is

$$\lambda^c\left(t\right) = e^{\rho t}\lambda^p\left(t\right) \tag{29}$$

where the "present value" costate $\lambda^p\left(t\right)$ is just the costate that we have been working with so far.

Given our new definition of the costate, we can define new optimality conditions in terms of a current value Hamiltonian in the following way. The old-fashioned (present-value) Hamiltonian as applied to our discounted case is of course

$$\mathcal{H}^p\left(t, x\left(t\right), u\left(t\right), \lambda\left(t\right)\right) = e^{-\rho t} f\left(x\left(t\right), u\left(t\right)\right) + \lambda^p\left(t\right) \cdot g\left(x\left(t\right), u\left(t\right)\right). \tag{30}$$

Rewriting in terms of the current value, we get

$$\mathcal{H}^p\left(t, x\left(t\right), u\left(t\right), \lambda\left(t\right)\right) = e^{-\rho t} f\left(x\left(t\right), u\left(t\right)\right) + e^{-\rho t} \lambda^c\left(t\right) \cdot g\left(x\left(t\right), u\left(t\right)\right) \tag{31}$$

and maximizing this with respect to $u\left(t\right)$ is of course the same as maximizing the current value Hamiltonian

$$\mathcal{H}^c\left(x\left(t\right), u\left(t\right), \lambda\left(t\right)\right) = f\left(x\left(t\right), u\left(t\right)\right) + \lambda^c\left(t\right) \cdot g\left(x\left(t\right), u\left(t\right)\right). \tag{32}$$

The condition $\mathcal{H}_x + \dot{\lambda} = 0$ is a little bit trickier. The correct condition is of course $\mathcal{H}_x^p + \dot{\lambda}^p = 0$. Since $\mathcal{H}_x^p = e^{-\rho t}\mathcal{H}_x^c$ and $\dot{\lambda}^p = -\rho e^{-\rho t}\lambda^c + e^{-\rho t}\dot{\lambda}^c$, this condition becomes $\mathcal{H}_x^c + \dot{\lambda}^c - \rho\lambda^c$ in terms of the current value Hamiltonian and costate. Notice that, conveniently, this does not explicitly involve time! In summary, our optimality conditions become

$$\begin{cases} \dfrac{\partial \mathcal{H}^c\left(x^*\left(t\right), u^*\left(t\right), \lambda\left(t\right)\right)}{\partial x} + \dot{\lambda}^c\left(t\right) - \rho\lambda^c\left(t\right) = 0 \\ u^*\left(t\right) \in \operatorname*{argmax}_{u \in U} \mathcal{H}^c\left(x^*\left(t\right), u, \lambda\left(t\right)\right) \\ \lambda^c\left(T\right) \cdot x\left(T\right) = 0 \text{ if } T < \infty \text{ and } \lim_{t \to \infty} e^{-\rho t}\lambda^c\left(t\right) \cdot x\left(t\right) = 0 \text{ if } T = \infty \end{cases} \tag{33}$$

where the current value Hamiltonian is defined via

$$\mathcal{H}^c\left(x\left(t\right), u\left(t\right), \lambda\left(t\right)\right) = f\left(x\left(t\right), u\left(t\right)\right) + \lambda^c\left(t\right) \cdot g\left(x\left(t\right), u\left(t\right)\right). \tag{34}$$

4 The neoclassical growth model in continuous time

4.1 Prequel: an economy with exogenous saving

The Solow (1956) model was set in continuous time. It is described by the following ordinary differential equation.

$$\dot{k}(t) = sk^{\alpha}(t) - \delta k(t)$$

where s is the fraction of output saved (as opposed to consumed). This differential equation is of the Bernoulli type, and can be solved explicitly. Let's do it! We begin by introducing the auxiliary variably y, defined via

$$y(t) := k^{1-\alpha}(t).$$

Given this definition, we have, by the chain rule,

$$\dot{y}(t) = (1-\alpha)k^{-\alpha}(t)\dot{k}(t).$$

Alternatively, we may write

$$\dot{k}(t) = \frac{1}{1-\alpha} \cdot k^{\alpha}(t) \cdot \dot{y}(t).$$

Substituting this into Solow's differential equation, we have

$$\frac{1}{1-\alpha} \cdot k^{\alpha}(t) \cdot \dot{y}(t) = sk^{\alpha}(t) - \delta k(t).$$

Multiplying by $1-\alpha$ and dividing by k^α (assuming, as we should, that $k > 0$), we get

$$\dot{y}(t) = (1-\alpha)s - \delta(1-\alpha)y(t).$$

This is a linear differential equation, and it can be solved in two steps. First we find a "particular" solution which in this case is the steady state. Setting $\dot{y} = 0$, we have

$$y^* = \frac{s}{\delta}.$$

Now notice that

$$\dot{y}(t) = -\delta(1-\alpha)[y(t) - y^*].$$

It follows that

$$y(t) - y^* = \exp\{-\delta(1-\alpha)t\} \cdot [y(0) - y^*].$$

Hence

$$k(t) = \left\{\exp\{-\delta(1-\alpha)t\} \cdot \left[k^{\frac{1}{1-\alpha}}(0) - \frac{s}{\delta}\right] + \frac{s}{\delta}\right\}^{\frac{1}{1-\alpha}}.$$

Notice that, as expected, the solution does converge to the steady state as long as $k(0) > 0$. Notice also that the rate of convergence to the steady state increases in the depreciation rate δ and decreases in the capital share α.

4.2 An economy with endogenous saving

Suppose a social planner maximizes

$$\int_0^\infty e^{-\rho t} \ln(c(t))dt$$

subject to $k(0) = k_0 > 0$ given, $k(t) \geq 0$, $c(t) \geq 0$ and

$$\dot{k}(t) = f(k(t)) - c(t). \tag{35}$$

The log specification of utility is of course a bit special, but in general we would like to capture the assumption that people like their consumption smooth.

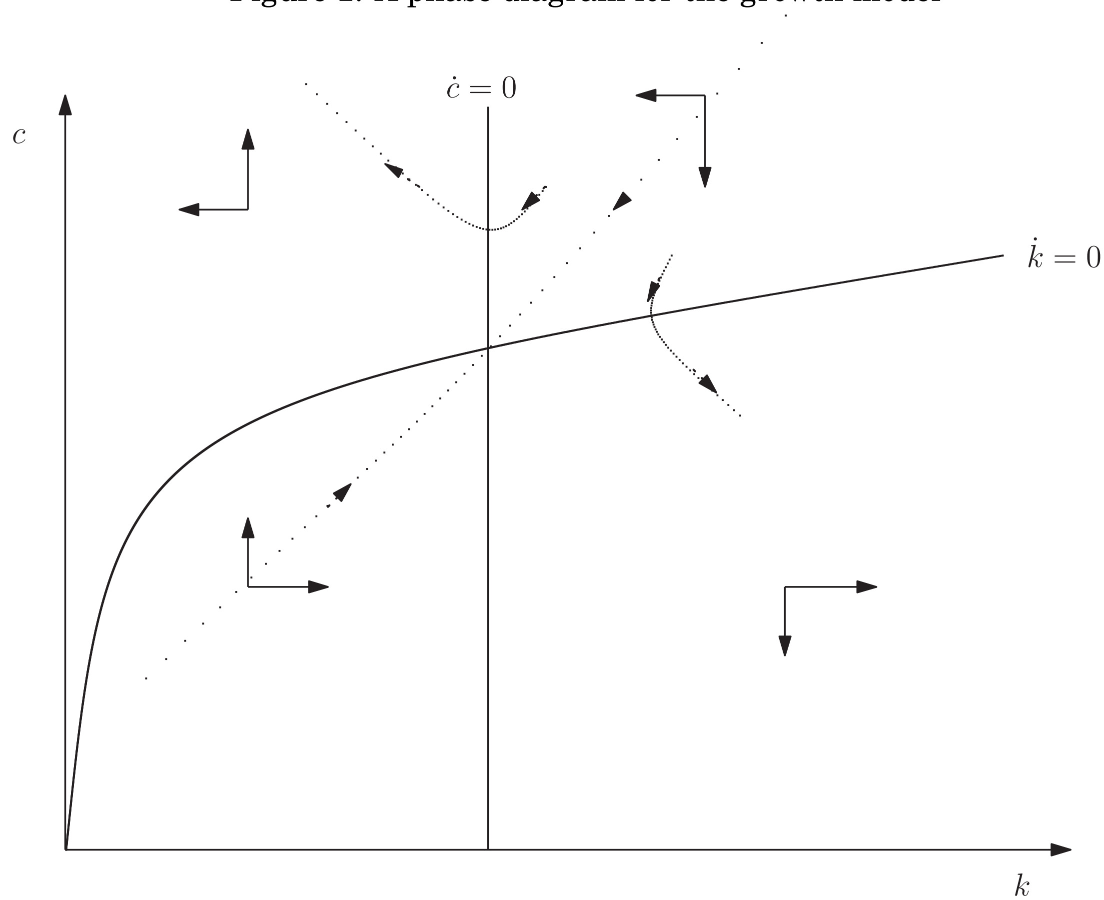
Using Pontryagin's maximum principle, we may characterize the solution via the following differential equation.

$$\dot{c}(t) = [f'(k(t)) - \rho]c(t).$$

What this means intuitively is that consumption should increase over time if it pays to postpone consumption. Notice that ρ is a measure of impatience, i.e. a measure of the cost of postponing consumption. The marginal product of capital $f'(k(t))$, on the other hand, is a measure of how much we gain by postponing consumption (by forgoing consumption today and instead investing more). If the cost of postponing consumption exceeds its benefits, consumption should gradually decline. And vice versa.

Combining this differential equation with Equation (35), we have a two-dimensional system of differential equations. We may characterize its solution with the help of a phase diagram.

Figure 1: A phase diagram for the growth model



What about transversality? Well, if we stay on the saddle path, then we know that $\lambda(t)$ will go to zero. Why? Because

$$\lambda(t) = \frac{e^{-\rho t}}{c(t)}$$

so that if $c(t)$ converges to a finite constant, then $\lambda(t)$ goes to zero. Because $\lambda(t)$ goes to zero and $k(t)$ goes to a finite constant, $\lambda(t)k(t)$ goes to zero as $t \to \infty$.

The paths outside of the saddle path are a bit trickier. Certainly some of them veer off into infeasible territory, eventually violating $k(t) \geq 0$. Some of them imply not necessarily that $c(t)$ goes negative but that $c(t) \to 0$. This in turn means that $\lambda(t)$ blows up, violating transversality. Of course, the transversality condition is not a necessary condition for optimality, so it does not follow that such paths are suboptimal (though they probably are). What we *can* say is that it *is* optimal to follow the saddle path.

5 Evolutionary biology: bang-bang in the beehive

Suppose we want to maximize the number of (fertile) queens and drones at the end of the season, $y(T)$, subject to

$$\dot{x}(t) = bu(t) - \gamma x(t),$$

$$\dot{y}(t) = c(1 - u(t))x(t),$$

$$x(0) = 1,$$

$$y(0) = 0$$

and

$$u(t) \in [0, 1] \text{ for all } t \in [0, T].$$

We assume $b > \gamma > 0$ and $c > 0$. Interpretation: x is the number of sterile workers, while u is the fraction of time spent rearing workers (as opposed to queens and drones). Also, $\dot{x}(t) = m(t)$ should be interpreted as

$$x(t) = x(0) + \int_0^t m(s)ds$$

to allow for the possibility that m is discontinuous and hence x not being differentiable everywhere.

Form the Hamiltonian

$$\mathcal{H} = \lambda(t)[bu(t)x(t) - \gamma x(t)] + \mu(t)c(1 - u(t))x(t).$$

This is linear, so concavity is guaranteed! In particular, it is linear in $u(t)$ with slope

$$\lambda(t)bx(t) - c\mu(t)x(t).$$

So maximizing the Hamiltonian with respect to $u(t)$ implies $u(t) = 1$ if this slope is positive and $u(t) = 0$ if it is negative. Since $x(t) \geq 0$, the sign of the slope is the same as that of the "switch" function

$$\sigma(t) := b\lambda(t) - c\mu(t).$$

Notice that Pontryagin guarantees that $\sigma(t)$ is a continuous function of t. This will be useful later when we establish that it can switch sign at most once. What about the other optimality conditions? Well, we have

$$\dot{\lambda}(t) + \lambda(t)[bu(t) - \gamma] + \mu(t)c(1 - u(t)) = 0,$$

$$-\dot{\mu}(t) = 0,$$

$$\lambda(T) = 0$$

because $x(T)$ is free and not evaluated and, finally,

$$\mu(T) = 1$$

because that is the slope of the endpoint evaluation function $\Phi(y(T)) = y(T)$. We conclude that $\mu(t) \equiv 1$ so that

$$\sigma(t) = b\lambda(t) - c$$

and

$$\begin{cases} \dot{\lambda}(t) &= [\gamma - bu(t)]\lambda(t) - c(1 - u(t)) \\ \lambda(T) &= 0. \end{cases}$$

We now ask how many switches there are between $t = 0$ and $t = T$. We will find out by examining the sign of $\dot{\sigma}$ at $\sigma = 0$. If $\sigma = 0$, we have $\lambda = c/b$. Meanwhile, $\dot{\sigma} = b\dot{\lambda}$ so that, when $\sigma = 0$ we have

$$\dot{\sigma} = b\left[(\gamma - bu) \cdot \frac{c}{b} - c(1 - u)\right] = c(\gamma - b) < 0.$$

So whenever $\sigma(t)$ switches signs, it does it from positive to negative, never the other way around. So there is (at most) one switch, from $u(t) = 1$ to $u(t) = 0$. From focussing entirely on rearing (sterile) workers to focussing entirely on rearing (fertile) queens and drones.

When is the switching time? Call it s. If we find that $s < 0$, then $u(t) = 0$ always. If $s > T$, then $u(t) = 1$ always, though this is hardly optimal because it implies $y(T) = 0$. Anyhow, consider the ODE for λ between $t = s$ and $t = T$ when $u(t) = 0$. We get

$$\begin{cases} \dot{\lambda}(t) &= \gamma\lambda(t) - c \\ \lambda(T) &= 0. \end{cases}$$

It follows that, for $s \leq t \leq T$, we have

$$\lambda(t) = \frac{c}{\gamma}[1 - e^{\gamma(t-T)}].$$

Meanwhile, s is defined via $\sigma(s) = 0$, so

$$b\lambda(s) = c$$

and hence

$$s = T + \frac{1}{\gamma} \underbrace{\ln\left(1 - \frac{\gamma}{b}\right)}_{<0}.$$

We conclude that $s < T$. But it could happen that $u(t) = 0$ always.

6 Optimal taxation à la Diamond-Saez

Pontryagin techniques are not only useful in the context of continuous *time*, but also in the context of mechanism design problems with a continuous state space. The excellent book Laffont and Tirole (1993) is full of interesting applications. More recent references include Diamond (1998) and Saez (2001). Fortunately, Heathcote and Tsujiyama (2021) have translated this into a language I can understand, and these notes are based on their work.

Let each worker/consumer be characterized by her idiosyncratic productivity $\theta \in \Theta$, which is private information. The distribution of θ is defined by the measure μ. Each person's preferences are represented by

$$\ln c - \frac{1}{1 + \frac{1}{\varepsilon}} h^{1+\frac{1}{\varepsilon}}$$

where c is consumption and h is effort. Output y is determined by

$$y = \theta h.$$

By the revelation principle, we can focus on mechanisms where truthful reporting happens in equilibrium. Let $c(\theta)$ be the consumption of a person of productivity θ reporting truthfully and, similarly let $y(\theta)$ be the output of a person truthfully revealing their θ. Evidently such a person will have to make an effort of $h = y(\theta)/\theta$. Also, to be incentive compatible, the mechanism must satisfy

$$\ln c(\theta) - \frac{1}{1+\frac{1}{\varepsilon}} \left(y(\theta)/\theta\right)^{1+\frac{1}{\varepsilon}} \geq \ln c\left(\hat{\theta}\right) - \frac{1}{1+\frac{1}{\varepsilon}} \left(y(\hat{\theta})/\theta\right)^{1+\frac{1}{\varepsilon}}$$

for all $\theta, \hat{\theta} \in \Theta$.

We begin by characterizing incentive compatibility via the following first-order condition.

$$\frac{1}{c(\theta)} c'(\theta) - \left(\frac{1}{\theta}\right)^{1+\frac{1}{\varepsilon}} \cdot y(\theta)^{1/\varepsilon} y'(\theta) = 0.$$

This in turn implies that

$$\frac{c(\theta)}{c'(\theta)} = \left(\frac{1}{\theta}\right)^{1+\frac{1}{\varepsilon}} \cdot y(\theta)^{1/\varepsilon} y'(\theta).$$

We then define the function

$$U(\theta) := \ln(c(\theta)) - \frac{1}{1+\frac{1}{\varepsilon}} \left(\frac{y(\theta)}{\theta}\right)^{1+\frac{1}{\varepsilon}}.$$

The derivative of this function (which we will use as the law of motion for the "state" in our "dynamic" optimization problem) is

$$U'(\theta) = \frac{c'(\theta)}{c(\theta)} - \left(\frac{y(\theta)}{\theta}\right)^{1/\varepsilon} \left[\frac{\theta y'(\theta) - y(\theta)}{\theta^2}\right].$$

Invoking the first-order condition ensuring that truth-telling is optimal, we have

$$U'(\theta) = \left(\frac{1}{\theta}\right)^{1+\frac{1}{\varepsilon}} \cdot y(\theta)^{1/\varepsilon} y'(\theta) - \left(\frac{y(\theta)}{\theta}\right)^{1/\varepsilon} \left[\frac{\theta y'(\theta) - y(\theta)}{\theta^2}\right] =$$

$$= \frac{y^{1+\frac{1}{\varepsilon}}(\theta)}{\theta^{2+\frac{1}{\varepsilon}}}$$

We can now set up the "dynamic" optimization problem as follows, where $U(\theta)$ will be our "state" and $y(\theta)$ our "control" variable. Notice that $c(\theta)$ can quite easily be backed out given $U(\theta)$ and $y(\theta)$. Anyhow, we choose the functions $U(\theta)$ and $y(\theta)$ so as to maximize

$$\int_{\Theta} U(\theta) d\mu$$

subject to the incentive compatibility condition

$$U'(\theta) = \frac{y^{1+\frac{1}{\varepsilon}}(\theta)}{\theta^{2+\frac{1}{\varepsilon}}}$$

and the budget constraint

$$\int_{\Theta} [y(\theta) - c(\theta; U, y)] d\mu = G$$

where $c(\theta; U, y)$ is defined via

$$U(\theta) \equiv \ln c(\theta; U, y) - \frac{1}{1+\frac{1}{\varepsilon}} \left(\frac{y(\theta)}{\theta}\right)^{1+\frac{1}{\varepsilon}}.$$

Alas, we will need an explicit definition in order to derive the optimality conditions. We have

$$c(\theta; U, y) = \exp\left\{U(\theta) + \frac{1}{1+\frac{1}{\varepsilon}} \left(\frac{y(\theta)}{\theta}\right)^{1+\frac{1}{\varepsilon}}\right\}.$$

Suppose now that $\Theta = [\underline{\theta}, \overline{\theta}]$ is an interval and that the measure μ admits a density f so that, for any interval $[\theta_1, \theta_2] \subset \Theta$, we have

$$\mu([\theta_1, \theta_2]) = \int_{\theta_1}^{\theta_2} f(\theta) d\theta.$$

We can then set up the Hamiltonian as follows.

$$\mathcal{H} = \{U(\theta) + \lambda[y(\theta) - c(\theta; U, y) - G]\} f(\theta) + \mu(\theta) \cdot \frac{y^{1+\frac{1}{\varepsilon}}(\theta)}{\theta^{2+\frac{1}{\varepsilon}}}.$$

Applying PMP, we first maximize the Hamiltonian with respect to the control y. The first-order condition is

$$\lambda \left[1 - \frac{c(\theta) y^{1/\varepsilon}(\theta)}{\theta^{1+\frac{1}{\varepsilon}}}\right] f(\theta) + \mu(\theta) \frac{1 + \frac{1}{\varepsilon}}{\theta^{2+\frac{1}{\varepsilon}}} y^{1/\varepsilon}(\theta) = 0. \tag{36}$$

where we have suppressed the dependence of c on the functions U and y.

Next, the derivative of the Hamiltonian with respect to the state plus the derivative of the costate with respect to θ should be zero:

$$f(\theta)[1 - \lambda c(\theta)] + \mu'(\theta) = 0. \tag{37}$$

Finally, we notice that there are no participation constraints, so no constraints on either $U(\underline{\theta})$ or on $U(\overline{\theta})$, and no endpoint evaluation either, so we must have

$$\mu(\underline{\theta}) = \mu(\overline{\theta}) = 0.$$

Integrating Equation (37) from θ to $\overline{\theta}$ and invoking $\mu(\overline{\theta}) = 0$, we get

$$\mu(\theta) = \int_{\theta}^{\overline{\theta}} [1 - \lambda c(t)] f(t) dt.$$

Invoking $\mu(\underline{\theta}) = 0$, we get an expression for λ:

$$\lambda = \frac{1}{\int_{\underline{\theta}}^{\overline{\theta}} c(t) f(t) dt}.$$

Denoting average consumption by $\overline{c}$, we have

$$\lambda = \frac{1}{\overline{c}}.$$

It would be nice to describe the optimal policy in terms of an optimal income tax schedule $T(y)$. A consumer facing the tax schedule $T(y)$ will maximize, with respect to h,

$$\ln(\theta h - T(\theta h)) - \frac{1}{1 + \frac{1}{\varepsilon}} h^{1+\frac{1}{\varepsilon}}.$$

The first-order condition is

$$\frac{1}{\theta h - T(\theta h)} \cdot [\theta - T'(\theta h) \cdot \theta] = h^{1/\varepsilon}.$$

Now notice that $\theta h - T(\theta h) = y - T(y)$ is in fact consumption and that $h = y/\theta$. We then get

$$\frac{\theta}{c(\theta)} [1 - T'(y(\theta))] = [y(\theta)/\theta]^{1/\varepsilon}$$

or, equivalently,

$$T'(y(\theta)) = 1 - \frac{c(\theta)}{\theta} \left(\frac{y(\theta)}{\theta} \right)^{1/\varepsilon}.$$

Equation (36) can now be rewritten as

$$\lambda T'(y(\theta))f(\theta) + \left(1 + \frac{1}{\varepsilon}\right)\mu(\theta)\frac{[1 - T'(y(\theta))]}{\theta c(\theta)} = 0.$$

Using what we know about $\mu(\theta)$ and rearranging, we get

$$\frac{T'(y(\theta))}{1 - T'(y(\theta))} = \frac{1 + \frac{1}{\varepsilon}}{\theta f(\theta)c(\theta)}\int_{\theta}^{\overline{\theta}} [c(t) - \overline{c}]\, f(t)dt,$$

or, to be even more explicit,

$$\frac{T'(y(\theta))}{1 - T'(y(\theta))} = \frac{1 + \frac{1}{\varepsilon}}{\theta f(\theta)[y(\theta) - T(y(\theta))]}\int_{\theta}^{\overline{\theta}} [y(t) - T(y(t)) - \overline{c}]\, f(t)dt, \tag{38}$$

In particular,

$$T'\big(y\left(\overline{\theta}\right)\big) = 0,$$

the famous “no distortion at the top” result. Beyond that, finding the optimal tax function $T(\theta)$ for all values of θ is non-trivial. Equation (38) is a fully-fledged functional equation in that it expresses marginal taxes $T'(y(\theta))$ in terms of total taxes $T(s)$ for a range of values for s. Numerical methods are needed.

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