Announcements 1. Lab tmrw : Finish Qns 1, 2, 4

2. ~~OH Proposal : Move to Monday~~

Test 2 : 9

## Poisson Process

**Thinning**

$$\begin{tikzcd} & & N_1(t) \\ {} \arrow[r, "N(t)"'] & {} \arrow[ur, "p"] \arrow[dr] & \\ & & N_2(t) \end{tikzcd}$$

$N(t) \sim PP(\lambda)$

$N_1(t) \sim PP(\lambda p)$

$N_2(t) \sim PP(\lambda(1-p))$

**Superposition**

$$\begin{tikzcd} N_1(t) \arrow[dr] & & \\ & {} \arrow[r] & N(t) \\ N_2(t) \arrow[ur] & & \end{tikzcd}$$

$N_1(t) \sim PP(\lambda_1)$

$N_2(t) \sim PP(\lambda_2)$

$N(t) = N_1(t) + N_2(t)$

$\sim PP(\lambda_1 + \lambda_2)$

**Superposition (Proof)**

$N(t) = N_1(t) + N_2(t)$

$N_1(t) \sim PP(\lambda_1)$

$N_2(t) \sim PP(\lambda_2)$

$N_1$ & $N_2$ are indp

Independent Increments

$N(t) - N(s) = N_1(t) + N_2(t) - (N_1(s) + N_2(s))$

$= (N_1(t) - N_1(s)) + (N_2(t) - N_2(s))$

$\perp\!\!\!\perp \; N_1(r) \quad r \le s \qquad \perp\!\!\!\perp \; N_2(r) \quad r \le s$

$= \perp\!\!\!\perp \; N(r) \quad r \le s$

Stationary Increment

$N(t) - N(s) = \underbrace{N_1(t) - N_1(s)}_{\text{Poisson}(\lambda_1(t-s))} + \underbrace{N_2(t) - N_2(s)}_{\text{Poisson}(\lambda_2(t-s))}$

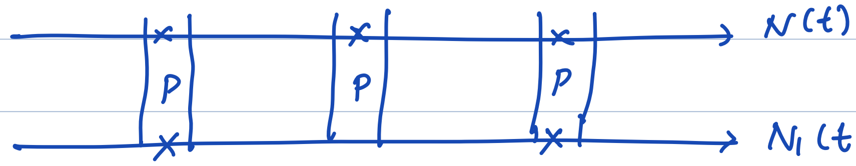
Review : $X \sim \text{Poisson}(\lambda_x) \quad Y \sim \text{Poisson}(\lambda_y)$

$X+Y \sim \text{Poisson}(\lambda_x + \lambda_y)$

$\sim \text{Poisson}((\lambda_1+\lambda_2)(t-s))$

$N(t) \sim PP(\lambda_1 + \lambda_2)$

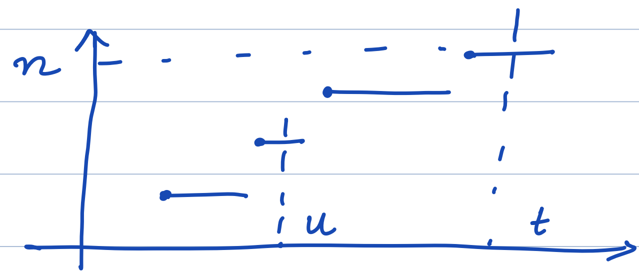
* Thinning : Exercise !




$\sim PP(\lambda p)$

Properties on Conditional Distributions of PP.

Prop1:




$$P[X(u)=k / X(t)=n] = \binom{n}{k}\left(\frac{u}{t}\right)^k\left(1-\frac{u}{t}\right)^{n-k}$$

$0 \le u \le t$

$0 \le k \le n$

Proof

$$P[X(u)=k \mid X(t)=n]$$

$$= \frac{P[X(u)=k,\ X(t)=n]}{P[X(t)=n]}$$

$$= \frac{P[\,X(u) = k,\; X(t) - X(u) = n-k\,]}{P[\,X(t) = n\,]}$$

Poisson$(\lambda(t-u))$

$P[\,X(t-u) = n-k\,]$

$$\overset{\text{independent increment}}{=} \frac{\underbrace{P[\,X(u) = k\,]}_{\text{Poisson}(\lambda u)}\; P[\,X(t) - X(u) = n-k\,]}{\underbrace{P[\,X(t) = n\,]}_{\text{Poisson}(\lambda t)}}$$

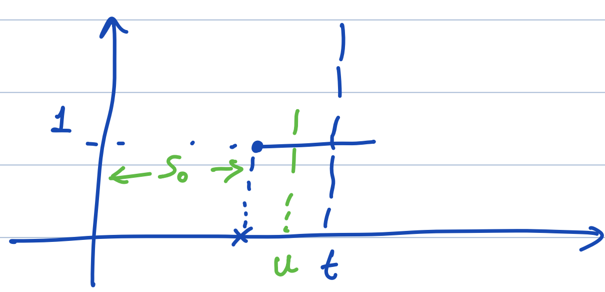
$$= \frac{\cancel{e^{-\lambda u}} \dfrac{(\cancel{\lambda} u)^k}{k!} \; \cancel{e^{-\lambda(t-u)}} \dfrac{(\cancel{\lambda}(t-u))^{n-k}}{(n-k)!}}{\cancel{e^{-\lambda t}} \dfrac{(\cancel{\lambda} t)^n}{n!}}$$

$$= \frac{n!}{k!\,(n-k)!} \; \frac{u^k \,(t-u)^{n-k}}{t^n}$$

$t^k \qquad t^{(n-k)}$

$$= \binom{n}{k} \left(\frac{u}{t}\right)^k \left(1 - \frac{u}{t}\right)^{n-k} \sim Bin\left(n, \frac{u}{t}\right)$$

Prop 2:




$$P[\,S_0 \le u \mid X(t) = 1\,] = \frac{u}{t}$$

" Uniform arrival in $[0, t]$ "

Proof:

$$\frac{P(\boxed{S_0 \le u},\; X(t)=1)}{P(X(t)=1)}$$

$X(u)=1$

$$= \frac{P[X(u)=1,\; X(t)=1)}{P(X(t)=1)}$$

$$= \frac{P(X(u)=1,\; X(t)-X(u)=0)}{P(X(t)=1)}$$

$$= \frac{P(X(u)=1)\, P(X(t)-X(u)=0)}{P(X(t)=1)}$$

$$= \frac{\cancel{(\lambda u)}\, \cancel{e^{-\lambda u}} \cdot \cancel{e^{-\lambda(t-u)}}}{(\cancel{\lambda} t)\, \cancel{e^{-\lambda t}}}$$

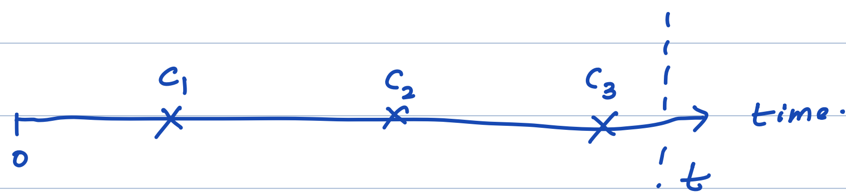
$$= \frac{u}{t} \quad //$$

Generalization

**Property*:** Given $X(t) = n$, the $n$ inter-arrival times are independent uniformly distributed in $(0,t)$
[ Order these independent & uniform distributed in $(0,t)$ ]

Proof : Text book.

**Example:** Insurance claims are made at times distributed according to a $PP(\lambda)$




Each claim value is an independent random variable with mean $\mu$. (independent of claim arrival)

$C_1, C_2, C_3 \sim G$ (mean $\mu$)

Insurance company wants to figure out the total expected discounted cost upto time $t$ ( Discount rate is $\alpha$ )

$$D(t) = \sum_{i=1}^{X(t)} C_i e^{-\alpha S_i}$$

↑ discounted cost

random sum

?

$$E[D(t)] = E\left[ \sum_{i=1}^{\boxed{X(t)}} C_i e^{-\alpha S_i} \right]$$

random variable

Random sum trick $= \mathbb{E}\left[\mathbb{E}\left[D(t) \mid X(t)=n\right]\right]$

$$= \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{n} C_i \, e^{-\alpha S_i}\right]\right]$$

$$= \sum_{n=0}^{\infty} \boxed{\mathbb{E}\left[\sum_{i=1}^{n} C_i \, e^{-\alpha S_i}\right]_{X(t)=n}} \; P[X(t)=n]$$

$X(t) \sim$ Poisson$(\lambda t)$

$$\mathbb{E}\left[\sum_{i=1}^{n} C_i \, e^{-\alpha S_i} \mid X(t)=n\right]$$

$$= \sum_{i=1}^{n} \mathbb{E}\left[C_i \, e^{-\alpha S_i} \mid X(t)=n\right]$$

independence

$$= \sum_{i=1}^{n} \underbrace{\mathbb{E}\left[C_i \mid X(t)\right]}_{\mu} \mathbb{E}\left[e^{-\alpha S_i} \mid X(t)=n\right]$$

$$= \mu \cdot \sum_{i=1}^{n} \mathbb{E}\left[e^{-\alpha S_i} \mid X(t)=n\right]$$

$$= n \cdot \mu \int_0^t e^{-\alpha \tau} \underbrace{f_{S_i}(\tau)}_{\frac{1}{t}} \, d\tau$$

$$= \frac{n\mu}{t}\left(1 - e^{-\alpha t}\right)$$

$$= \sum_{n=0}^{\infty} \frac{n\mu}{t}\left(1 - e^{-\alpha t}\right) e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

Exercise

$$= \frac{\mu\lambda}{\alpha}\left(1 - e^{-\alpha t}\right) \quad //$$