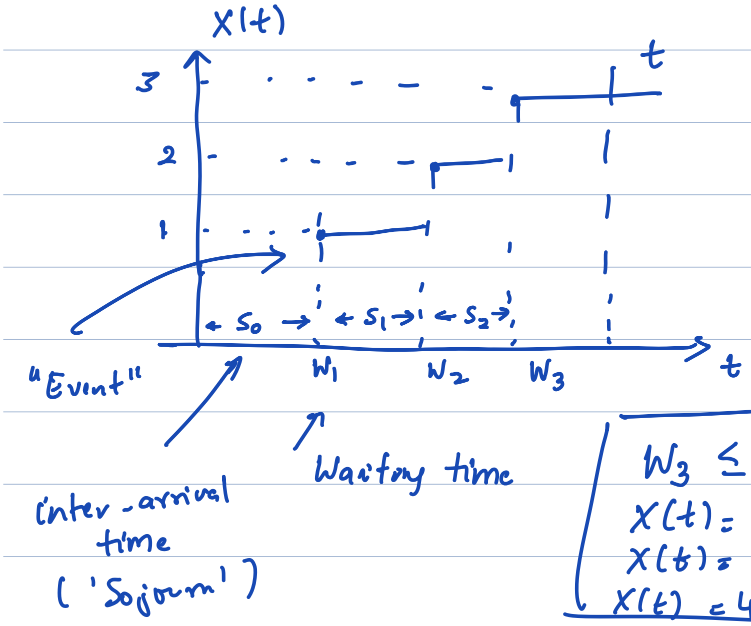
Poisson Process



$$\boxed{\begin{array}{c} W_3 \le t \\ X(t) = 0\,? \\ X(t) = 1 \\ X(t) = 4\,? \end{array}}$$

$X(t)$: Poisson Process (λ)

$$P[\,X(t+h) - X(t) = 0\,]$$
$$= e^{-\lambda h} \approx 1 - \lambda h + o(h)$$
(Small h)

$$P[\,X(t+h) - X(t) = 1\,]$$
$$\approx \lambda h + o(h)$$
(Small h)

Theorem: The waiting time W_n has the following distribution:

$$f_{W_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \qquad n = 1, 2, \dots \quad t \geq 0$$

"Gamma distribution" / "Erlang distribution"

Proof:

$$F_{W_n}(t) = P(W_n \le t)$$
↳ Change to $X(t)$

$$= P(X(t) \geq n)$$

$$= 1 - P_r(\underbrace{X(t)}_{\text{Poisson}(\lambda t)} < n)$$

$$= 1 - \sum_{k=0}^{n-1} (\lambda t)^k \frac{e^{-\lambda t}}{k!}$$

$$f_{W_n}(t) = \frac{d}{dt} F_{W_n}(t)$$

$$= - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t} (-\lambda) +$$

$$- \sum_{k=1}^{n-1} \frac{e^{-\lambda t}}{\cancel{k!}} \cancel{k} \cdot (\lambda t)^{k-1} \lambda$$

$$= \lambda e^{-\lambda t} \underbrace{\sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}}_{\cancel{1} + \cancel{(\lambda t)} + \cdots \cancel{+} \boxed{\frac{(\lambda t)^{n-1}}{(n-1)!}}} - \lambda e^{-\lambda t} \underbrace{\sum_{k=1}^{n-1} \frac{(\lambda t)^{k-1}}{(k-1)!}}_{\cancel{1} + \cancel{(\lambda t)} + \cdots + \cancel{\frac{(\lambda t)^{n-2}}{(n-2)!}}}$$

$$= \lambda \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!}$$

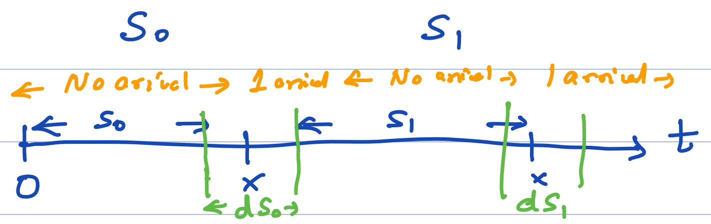
$$= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$$

Theorem: The inter-arrival times $S_0, S_1, \dots$ are independent and identical with density

$$f_{S_n}(s) = \lambda e^{-\lambda s} \quad ; \quad s \geq 0$$

"Exponential dist"

Proof : "$n=2$"



$$\Pr(s_0 < S_0 < s_0 + ds_0,\ s_1 < S_1 < s_1 + ds_1)$$

$$= f_{S_0 S_1}(s_0, s_1)\, ds_0\, ds_1$$

$$\int_{s_1}^{s_1+\Delta s_1} \int_{s_0}^{s_0+\Delta s_0} f_{S_0 S_1}(s_0, s_1)\, ds_0\, ds_1 = \Pr(s_0 < S_0 < s_0 + \Delta s_0,\ s_1 < S_1 < s_1 + \Delta s_1)$$

Convert LHS to $X(t)$ instead of S

$$\Pr[X(s_0) = 0,\ X(s_0 + ds_0) - X(s_0) = 1,\ X(s_0 + s_1 + ds_0) - X(s_0 + ds_0) = 0,\ X(s_0 + s_1 + ds_0 + ds_1) - X(s_0 + s_1 + ds_0 +) = 1]$$

Independent increment

$$= P[X(s_0) = 0]$$
$$P[X(s_0 + ds_0) - X(s_0) = 1]$$
$$P[X(s_0 + s_1 + ds_0) - X(s_0 + ds_0) = 0]$$
$$P[X(s_0 + s_1 + ds_0 + ds_1) - X(s_0 + s_1 + ds_0) = 1]$$

Stationary increment

$$P[\underline{X(ds_0) =} 1]$$
$$P[X(s_1) = 0]$$
$$P[X(ds_1) = 1]$$

small ds_1, ds_0

$$\approx \quad e^{-\lambda s_0} \quad \lambda \cdot ds_0 \quad e^{-\lambda s_1} \quad \lambda\, ds_1$$

$$f_{s_0 s_1}(s_0, s_1)\, ds_0\, ds_1 = \lambda e^{-\lambda s_0} \quad \lambda e^{-\lambda s_1}\, ds_0\, ds_1$$

$$f_{s_0 s_1}(s_0, s_1) = \underbrace{\lambda e^{-\lambda s_0}} \quad \underbrace{\lambda e^{-\lambda s_1}}$$

independent
identical exponential
dist

Example:
Suppose people immigrate into a territory following a poisson process at a rate of $\lambda = 1 / \text{day}$.

a) Expected time till the 10th immigrant arrives

b) Probability that the time b/w tenth and eleventh arrival exceeds 2 days?

a) $\mathbb{E}[W_{10}] = \mathbb{E}[S_0 + S_1 + \cdots + S_9]$

$$W_{10} = S_0 + S_1 + \cdots + S_9$$

$$= \mathbb{E}[S_0] + \mathbb{E}[S_1] + \cdots + \mathbb{E}[S_9]$$

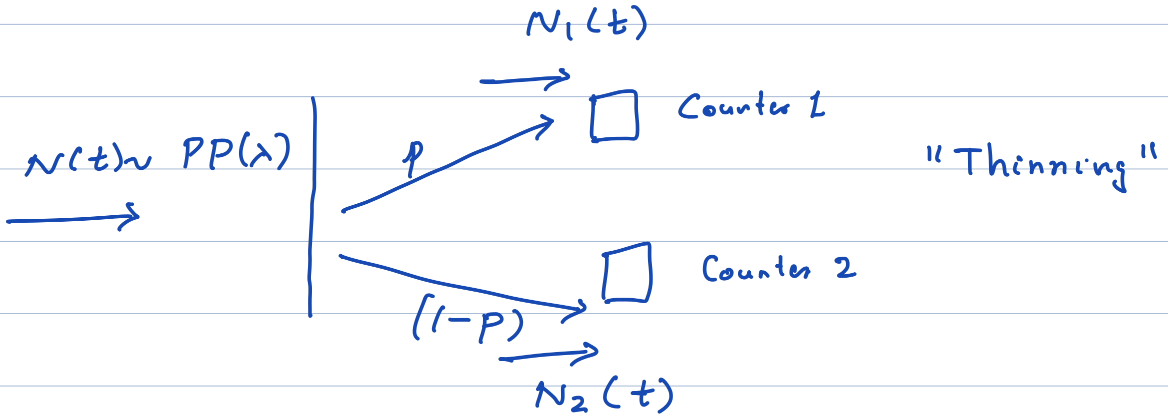
Exponential dist
with parameter $\lambda = 1$

$$= 10/\lambda = 10 \text{ days} //$$

b) $P[\, S_{10} \geq \overset{t}{2} \,] = e^{-\lambda t} = e^{-2}$

$= 0.133$ //

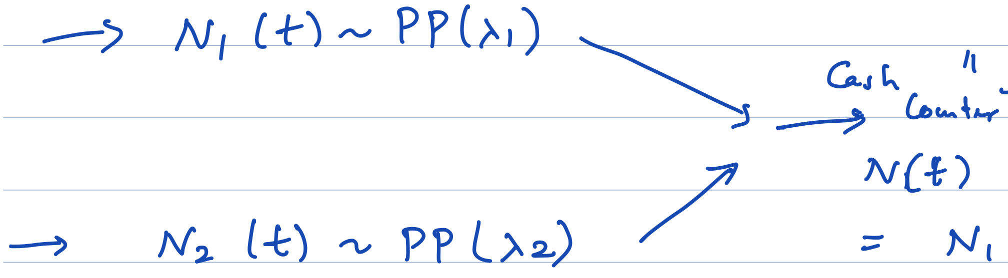
$\downarrow$ Exponential dist $\lambda = 1$



"Thinning"

$N_1(t)$ – Poisson Process $(\lambda \cdot p)$

$N_2(t)$ – Poisson Process $(\lambda(1-p))$



"Superposition"

$N(t) = N_1(t) + N_2(t)$

$N(t)$ = Poisson Process with rate $\lambda_1 + \lambda_2$

"Test 2 is scheduled next week"

Syllabus : Module 2 covered till next week

RP, Limit Thms, Prob ineqs, Markov Chain,

Poisson Process