Last class: Random vectors - Joint & Marginal distributions

Joint: $P_{X_0 \cdots X_{n-1}}(G)$

Marginal: $P_{X_i}(\cdot)$ $\qquad P_{X_0}(G_0)$

Consisteny b/w Joint & Marginal

$$P_{X_0}(G_0) = P\{\omega : X_0(\omega) \in G_0\}$$

$$P_X(G_0) = P\{\omega : X_0(\omega) \in G_0, \; -\infty < X_1 < \infty, \; \vdots \; -\infty < X_{n-1} < \infty\}$$

Discrete Random vector.

pmf $\quad P_{X_0}(\alpha) = P\{X_0 = \{\alpha\}, \; -\infty < X_1 < \infty, \; \vdots \; -\infty < X_{n-1} < \infty\}$

$$= \sum_{x_1 \cdots x_{n-1}} P_X((\alpha, x_1, \ldots x_{n-1}))$$

Eg: Throw 2 dice

X_1: Value of first dice

X_2: Value of 2nd dice.

$X = (X_1, X_2)$

$$P_{X_1}(\{4\}) = 1/6$$
$$= P_X(\{4,1\}) + P_X(\{4,2\}) + \cdots P_X(\{4,6\})$$

Continous random vector:

f_{X_i} : Marginal pdf

f_X : Joint pdf

$(X = X_0 \dots X_{n-1})$

"Marginilization"

$$f_{X_0}(\alpha) = \int f_X(\alpha, x_1 \dots x_{n-1})\, dx_1 \dots dx_{n-1}$$

Properties random vector (X, Y)

$$* \ 1. \quad P_r\{(X, Y) \in C\} = \iint_{x,y \in C} f_{XY}(x, y)\, dx\, dy$$

Example: (Ross)

$$f(x,y) = \begin{cases} 2e^{-x} e^{-2y} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{else} \end{cases}$$

Random variable: $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

$$F_X(\infty) = 1$$

Random vector: $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_0 X_1 \dots X_{n-1}}(x_0, x_1, \dots, x_{n-1})\, dx_0\, dx_1 \dots dx_{n-1}$

$$= 1$$

$$F_X(+\infty, +\infty, \dots, +\infty) = 1$$

$$P\{X > 1,\ Y < 1\} \quad \text{(Property } *)$$

$$= \int_{y=0}^{1} \int_{x=1}^{\infty} f_{XY}(x, y)\, dx\, dy$$

$$= \int_0^1 \int_1^\infty 2e^{-x} e^{-2y}\, dx\, dy$$

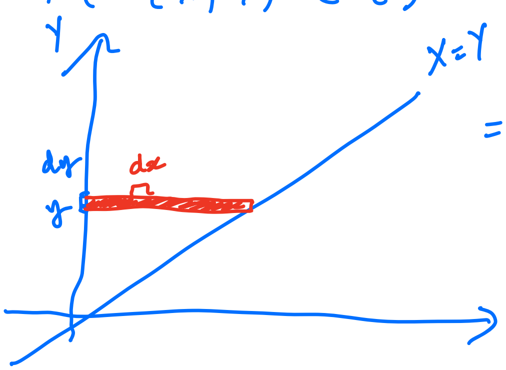
$$= \int_0^1 2e^{-2y} \left. \frac{e^{-x}}{-1} \right|_1^\infty$$

$$= 2e^{-1} \int_0^1 e^{-2y}$$

$$= \cancel{2}e^{-1} \left. \frac{e^{-2y}}{-\cancel{2}} \right|_0^1 = e^{-1}(1 - e^{-2})$$

$$P(X < Y) = \iint_{x,y\,:\,x<y} f_{XY}(x, y)\, dx\, dy$$

$$P((X, Y) \in C)$$



$$= \int_0^\infty \int_0^y 2e^{-x} e^{-2y}\, dx\, dy$$

$$= \frac{1}{3}$$

Exercise: $f_{X,Y}(x, y) = \begin{cases} e^{-(x+y)} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{else.} \end{cases}$

Find $P\left(\frac{X}{Y} \le a\right)$

$$
\begin{tikzcd}
f_{X_0} \arrow[dr] & \\
f_{X_1} \arrow[r] & \times f_{X_0 X_1 \cdots X_{n-1}} \arrow[dl, bend left] \\
\vdots \\
f_{X_{n-1}} \arrow[uur] &
\end{tikzcd}
$$

Independent random variable.

Remember: Independent events

A, B are said to be independent

if $P(A \cap B) = P(A)\,P(B)$

2 random variables X and Y are independent

$$P(\underbrace{X \in A,\ Y \in B}_{\text{Notation}}) = P(X \in A)\,P(Y \in B)$$

$$\Downarrow$$

$$P((X,Y) \in A \times B)$$

Discrete random variables X and Y

$$P_{XY}(\alpha, \beta) = P_X(\alpha)\,P_Y(\beta)$$

Continuous random variables X and Y

$$f_{XY}(\alpha, \beta) = f_X(\alpha)\,f_Y(\beta)$$

$X_0 \cdots X_{n-1}$ are independent if

$$P_X\left(X \in \prod_{i=0}^{n-1} F_i\right) = P_{X_0}(F_0)\,P_{X_1}(F_1) \cdots P_{X_{n-1}}(F_{n-1})$$