Gaussian Process & Brownian Motion

GP = Stochastic Process

Finite dim dist (SP) = Gaussian random vector

Standard Brownian motion

(i) $B_0 = 0$ $\quad\quad$ | $B(t)$ or B_t

(ii) $B_t - B_s \perp\!\!\!\perp B_r$ $\quad\quad r \leq s$

Indp. increment $\quad\quad t \geq s$

(iii) Gaussian increment

$$B_t - B_s \sim \mathcal{N}(0, t-s)$$

Brownian motion with drift (i) & (ii)

(iii) $B_t - B_s \sim \mathcal{N}(\mu(t-s), \sigma^2(t-s))$

Drift parameter: μ.

Variance parameter: σ

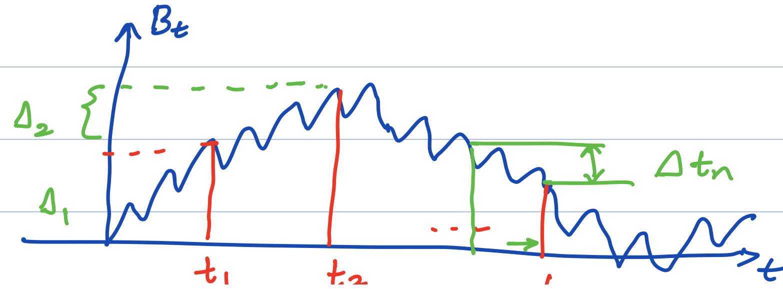
B_t : standard Brownian motion ($\mu = 0$, $\sigma^2 = 1$)

W_t : Brownian motion with drift (μ, σ)

$W_t = \mu t + \sigma \cdot B_t$ [Proof as exercise]

Properties (Standard Brownian Motion)

1. Brownian Motion is a Gaussian Process



$B = (B(t_1), B(t_2), \ldots, B(t_n))$

$\uparrow$ Gaussian random vector

Proof

$$\Delta_i = B(t_i) - B(t_{i-1})$$

$\uparrow$ Dist ?

Gaussian distributed $\mathcal{N}(0, t_i - t_{i-1})$

$$B(t_1) = \Delta_1$$

$$B(t_2) = \underbrace{B(t_2) - B(t_1)}_{\Delta_2} + \underbrace{B(t_1)}_{\Delta_1}$$

$$= \Delta_2 + \Delta_1$$

$$B(t_3) = \underbrace{B(t_3) - B(t_2)}_{\Delta_3} + \underbrace{B(t_2)}_{\Delta_2 + \Delta_1}$$

$$= \Delta_3 + \Delta_2 + \Delta_1$$

$\vdots$

$$B(t_n) = \Delta_n + \Delta_{n-1} + \cdots + \Delta_1$$

$$B = \begin{pmatrix} B(t_1) \\ B(t_2) \\ \vdots \\ B(t_n) \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 1 & 1 & \cdots & \cdots & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \Delta_1 \\ \vdots \\ \Delta_n \end{bmatrix}}_{\text{Gaussian random vector}}$$

$\to$ Gaussian random vector

B_t is a Gaussian process

Theorem: A stochastic process B_t is a std. Brownian motion iff

(i) B_t is a Gaussian process

(ii) $\mathbb{E}[B_t] = 0$

(iii) $\mathrm{Cov}(B_t, B_s) = \min(s,t)$

(iv) B has continous sample paths

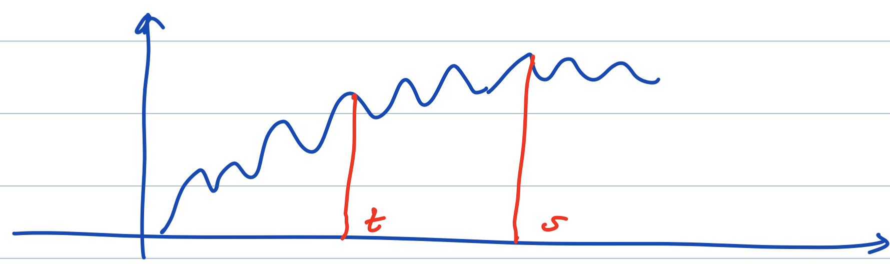
(ii) $\mathbb{E}[B_t] = 0$

$\cancel{B_t \sim N(0, 1)\ ?}$

$$B_t = B_t - \underbrace{B_0}_{0} \sim N(0, t)$$

$$\mathbb{E}[B_t] = 0$$

(iii) $\mathrm{Cov}(B_t, B_s) = \min(s,t)$



$t \le s$

$$\mathrm{Cov}(B_t, B_s) = \mathbb{E}[B_t B_s] - \cancel{m_s m_t}^{\,0}$$

$$= \mathbb{E}[B_t B_s]$$

$$B_s = \underbrace{B_s - B_t}_{\sim N(0,\, s-t)} + \underbrace{B_t - B_0}_{N(0,t)}$$

independent

$$\mathbb{E}[B_t \; B_s] = \mathbb{E}[\,B_t\,[\,B_s - B_t + B_t]]$$

$$= \mathbb{E}[\,\underbrace{B_t\;[B_s - B_t]}_{\text{Independent}}] + \mathbb{E}[B_t^2]$$

$$= \mathbb{E}[\cancel{B_t}]^{0}\;\mathbb{E}[B_s - \cancel{B_t}]^{0} + \underbrace{\mathbb{E}[B_t^2]}_{t}$$

$$= t$$

$Cov(B_t, B_s) = t \qquad t \leq s$

$Cov(B_t, B_s) = s \qquad t \geq s$ [Exercise!]

$Cov(B_t, B_s) = \min(s, t)$

2. Sample paths of Brownian motion are continous but nowhere differentiable.

(Intitution)

$$\frac{d}{dt} B_t = \lim_{h \to 0} \frac{(B_{t+h} - B_t)}{h}$$

$\mathcal{N}(0, h)$

$\left(\frac{1}{h}\;\mathcal{N}(0, h)\right)$

$\sim \mathcal{N}\left(0, \frac{1}{h}\right)$

$= \lim \; \mathcal{N}(0, 1,$

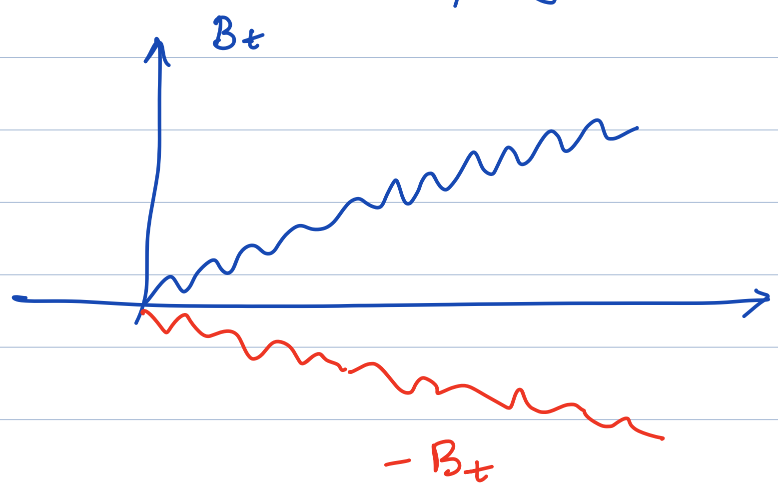
$h \to 0$ $\underbrace{\quad h)}$

infinite variance as $h \to 0$

$\Rightarrow$ Not differentiable.

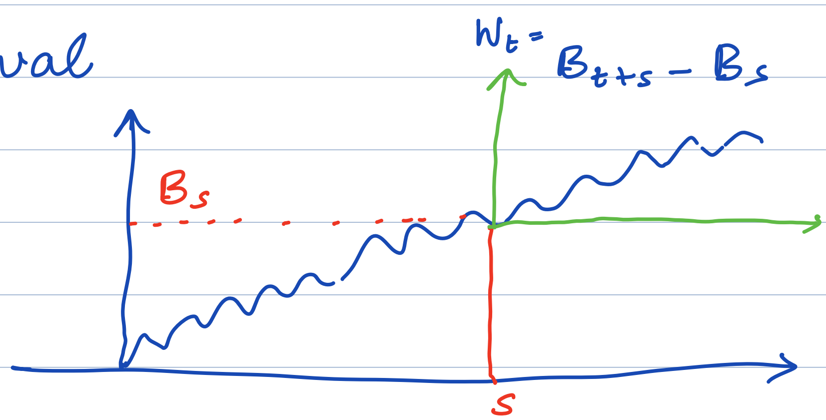
$B(t+h) - B(t) \sim \mathcal{N}(0, h)$

3. Reflection Property



B_t is a std Brownian motion so is $\{-B_t\}$
[Proof as exercise]

4. Renewal

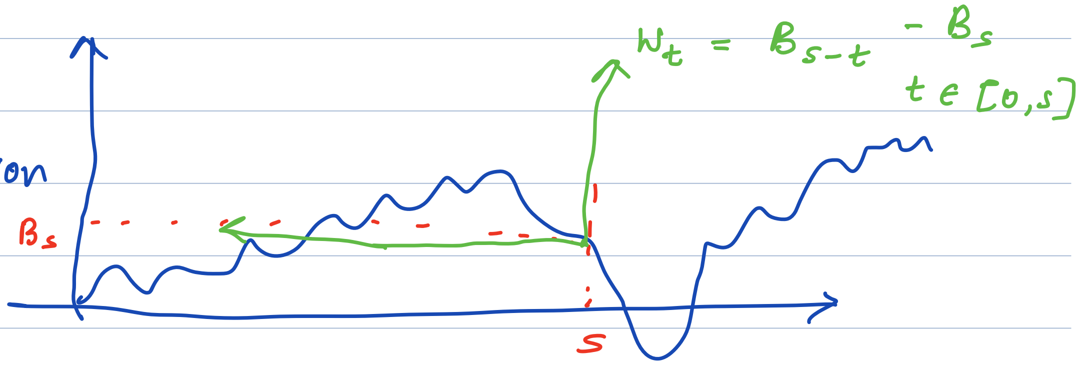


If B_t is a std Brownian motion

So is W_t

[Proof - Exercise!]

5. Time inversion



If B_t is a std Brownian motion so is

$$W_t = B_{s-t} - B_s \quad ; \quad t \in [0,s]$$

[Proof : Exercise !]

6. Brownian motion is the limit of a simple symmetric random walk.