Last class $<$ Moments of R.V
M.G.F

Reminder : Quiz 1 on Sunday at 3 pm - 4.30pm

* Functions of random variables

$X \quad f_X(x)$

$Y = g(X) \quad F_Y(y) - f_Y(y)$

Idea: Try and express $F_Y(y)$ in terms of cdf of F_X

Ex $Y = aX + b$

$$F_Y(\alpha) = \Pr(Y \le \alpha)$$
$$= \Pr(aX + b \le \alpha)$$
$$= \Pr\left(X \le \frac{\alpha - b}{a}\right)$$

$a > 0 \quad F_Y(\alpha) = \Pr\left(X \le \frac{\alpha - b}{a}\right)$

$$= F_X\left(\frac{\alpha - b}{a}\right)$$

$a < 0 \quad F_Y(\alpha) = \Pr\left(X \le \frac{\alpha - b}{-|a|}\right)$

$$= \Pr\left(X \le \frac{b - \alpha}{|a|}\right)$$
$$= F_X\left(\frac{b - \alpha}{|a|}\right)$$

Ex $Y = X^2 \quad F_Y(\cdot)$ in term of $F_X(\cdot)$

$$F_Y(\alpha) = \Pr(Y \le \alpha)$$

$$= Pr(X^2 \le \alpha) \qquad \alpha > 0$$
$$= Pr(-\sqrt{\alpha} \le X \le +\sqrt{\alpha})$$
$$= F_X(\sqrt{\alpha}) - F_X(-\sqrt{\alpha})$$

$$F_Y(\alpha) = \begin{cases} F_X(\sqrt{\alpha}) - F_X(-\sqrt{\alpha}) & \alpha \ge 0 \\ 0 & \alpha < 0 \end{cases}$$

$$f_Y(\alpha) = \begin{cases} \frac{1}{2\sqrt{\alpha}}\left[f_X(\sqrt{\alpha}) + f_X(-\sqrt{\alpha})\right] & \alpha \ge 0 \\ 0 & \alpha < 0 \end{cases}$$

Exercise Ex 5-4 in PP (Hard - limiter)

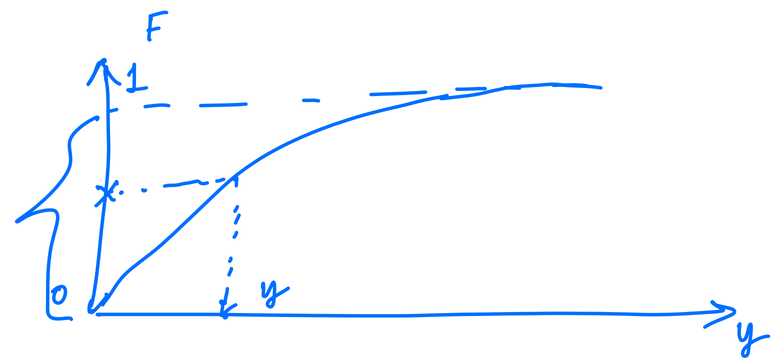
* Inverse transform method

$X \sim Unif(0,1) \qquad F_X(x) = x$

$$Y = g(X) = \underbrace{F^{-1}}_{\text{a function that satisfies all properties of a cdf}}(X)$$

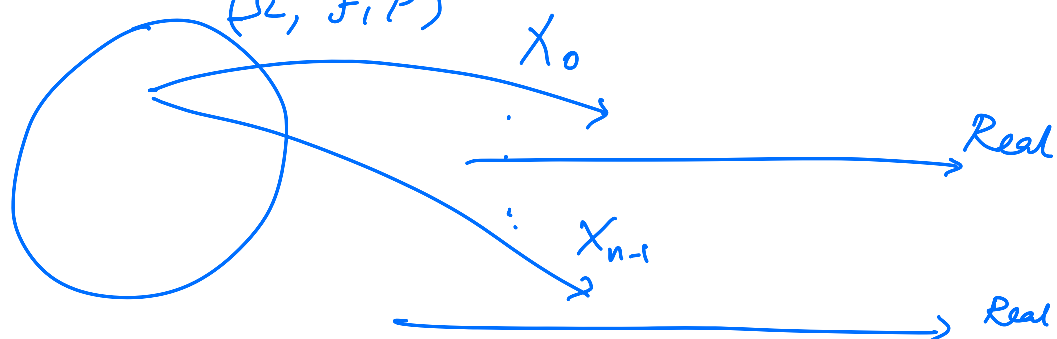
What is the c.d.f of Y

$$F_Y(y) = Pr(Y \le y)$$
$$= Pr(F^{-1}(X) \le y)$$
$$= Pr(X \underbrace{\le F(y)}_{\text{c.d.f of } X})$$
$$= F(y)$$



Random vectors

$(\Omega, \mathcal{F}, P)$



$$X = \{ X_i \; ; \; i = 0, 1, \dots, n-1 \}$$

"finite collection of random variable"

Distribution of random vector:

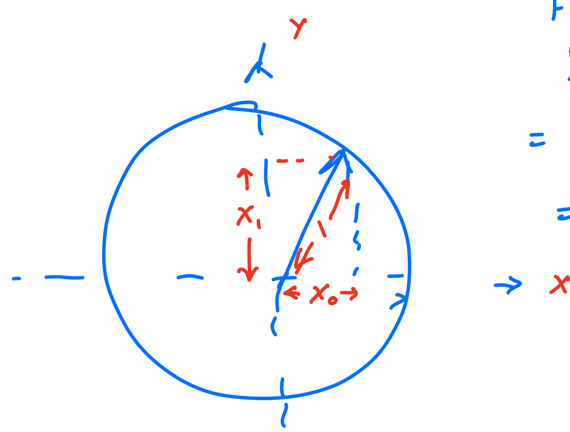
$$P_X(F) = P(X^{-1}(F))$$
$$= P(\{\omega : X(\omega) \in F\})$$

"Joint distribution"

Probability that random vector X is in some set F

$(X_0 \cdots X_{n-1})$

Example:



$F = \{ 0 \le X_0, X_1 \le 1 \}$

$$P_X(F) = P\left(0 < \theta < \frac{\pi}{2}\right) = \frac{\pi/2}{2\pi} = \frac{1}{4}$$

Special Case: Discrete random vectors

Multidimensional p.m.f (Joint pmf)

$$P_X(x) = P_X(\{x\})$$

$X \to (X_0 \dots X_{n-1})$, $\quad x = (x_0 \dots x_{n-1})$

Cumulative distribution function. (Joint cdf)

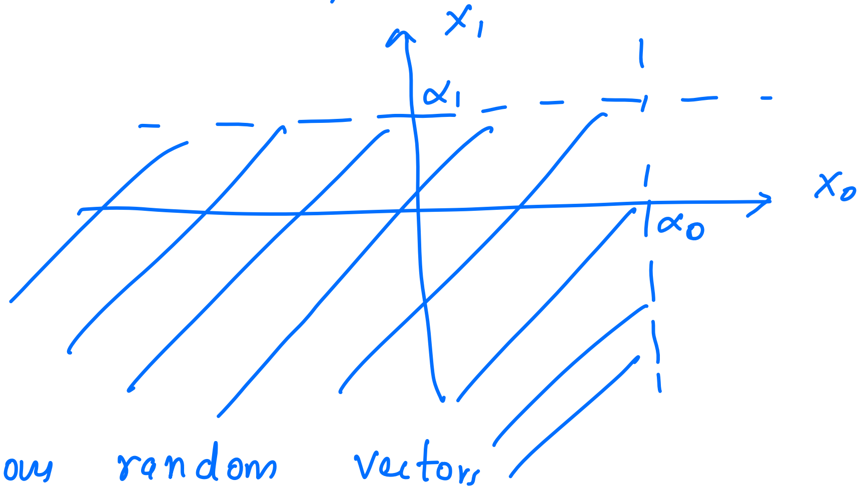
$$F_X(\alpha) = Pr(X \le \alpha)$$

Short hand notation for below expression

$X \to (X_0 \dots X_{n-1})$, $\quad \alpha = (\alpha_0 \dots \alpha_{n-1})$

$$= Pr(\{X_i \le \alpha_i \, ; \, i = 0, \dots n-1\})$$

Eg: $n = 2$ $\quad (X_0, X_1)$

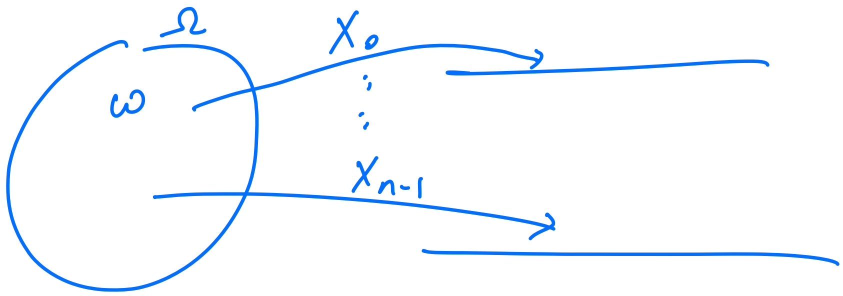


Continous random vectors

$$f_X(\alpha) = \frac{\partial^n}{\partial x_0 \, \partial x_1 \dots \partial x_{n-1}} F_X(\alpha)$$

Multidimensional pdf (Joint pdf)

Consistency between marginal & joint distribution.



$$P_{X_0}(G) = P(\{\omega : X_0(\omega) \in G\})$$

$a < X_0 < b$

Joint distribution $P_X(\cdot)$