Last class : Conditional distribution

Consider $n+m$ trials having a common prob of sucess.

Prob of sucess chosen from $U \sim [0,1]$

What is the cond. dist of sucess prob. given $n+m$ trials resulted in n success ?

X: Probability of success in a trial

$$f_X(x) = 1 \quad 0 < x < 1$$

$$X \sim \text{Unif } [0,1]$$

$$X = x$$

N: # of success in $(n+m)$ trials

$$N \mid X = x \sim \text{Binomial } (n+m, x)$$

$$f_{X|N}(x|n) = \frac{f_{X,N}(x,n)}{f_N(n)}$$

$$= \frac{f_{X,N}(x,n)}{\int f_{X,N}(x,n)\, dx}$$

$$= \frac{\overbrace{f_X(x)}^{1} \; \overbrace{f_{N|X}(n|x)}^{\text{Binomial }(n+m,\, x)}}{\int f_X(x)\, f_{N|X}(n|x)\, dx}$$

$$= \frac{1 \cdot \cancel{\binom{n+m}{n}} x^n (1-x)^m}{\int 1 \cdot \cancel{\binom{n+m}{n}} x^n (1-x)^m\, dx}$$

$$= \frac{x^n (1-x)^m}{\boxed{\int_0^1 x^n (1-x)^m \, dx}} \Rightarrow \text{Beta}(n+1, m+1)$$

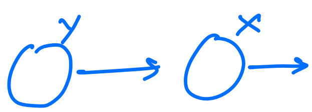
Beta function

$$\text{Beta}(n, m) = \int_0^1 x^{n-1} (1-x)^{m-1} \, dx$$

$$\text{Beta distribution } (\alpha, \beta) = \frac{x^{\alpha-1} (1-x)^{\beta-1}}{\text{Beta}(\alpha, \beta)}$$

$$X/N \sim \text{Beta}(n+1, m+1) \quad //$$

Exercise: Suppose a random variable Y has an exponential distribution with parameter θ. Given $Y = y$, the random variable X has a Poisson distribution with mean y. Find the conditional distribution of Y/X? [Hint: Gamma distribution]



Conditional Expectation

$$E[Y \mid X = x] = \begin{cases} \text{Discrete case} \\ \sum_y y \, P_{Y/X}(y|x) \\ \text{Continous case.} \\ \int y \, f_{Y/X}(y|x) \, dy \end{cases}$$

Example:

$$f_{X,Y}(x, y) = \frac{e^{-x/y} e^{-y}}{y} \qquad \begin{array}{l} 0 < x < \infty \\ 0 < y < \infty \end{array}$$

$$f_{X/Y}(x/y) = \frac{1}{y} e^{-x/y} \qquad [\text{Exercise}]$$

$$\mathbb{E}[X \mid y] = \int_0^\infty x \, f_{X/Y}(x/y) \, dx$$

Exercise!

$$= y$$

$$\mathbb{E}[X \mid Y=y] = y.$$

$$g(y)$$

$g(y)$ $\qquad$ $g(Y)$

↳ function over a random variable

Another random variable

$\mathbb{E}[X \mid Y=y]$ $\qquad$ $\mathbb{E}[X/Y]$

↓

Conditional Expectation of X given Y

Conditional Expectation.

$$\mathbb{E}[X/Y] = \sum_x x \cdot f_{X/Y}(x/Y)$$

Properties

(Discrete case)

1. $\mathbb{E}[\mathbb{E}[X/Y]]$

$$= \sum_y \mathbb{E}[X/Y] \, P_Y(y)$$

$$= \sum_y \sum_x x \cdot \underbrace{P_{X|Y}(x/y) \, P_Y(y)}$$

$$= \sum_y \sum_x x \cdot P_{X,Y}(x,y)$$

$$= \sum x \sum P_{X,Y}(x,y)$$

$x \qquad y$ ← Marginalization

$$= \sum_x x \cdot P_X(x) = \mathbb{E}[X]$$

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X/Y]]$$

Iterated expectation / Nested expectation.

Example: " Random Sum "

$X_0, X_1, X_2 \ldots$ are iid random variables

$$X_i \sim F$$

$$Y = \sum_{i=0}^{N-1} X_i \quad \Big| \quad \mathbb{E}[X_i] = \mu.$$

N: another random variable taking values in the integer

$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=0}^{N-1} X_i\right]$$

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y/N]]$$

$$Y/N=n = \sum_{i=0}^{n-1} X_i$$

$$\mathbb{E}[Y/N=n] = \mathbb{E}\left[\sum_{i=0}^{n-1} X_i\right]$$

$$= \sum_{i=0}^{n-1} \mathbb{E}[X_i]$$

$$= n \cdot \mu$$

$$\mathbb{E}[Y/N] = \mu \cdot N$$

$$\mathbb{E}[\ \mathbb{E}[Y/N]\ ] = \mathbb{E}[Y] =$$

$$= \mathbb{E}[\mu \cdot N]$$

$$= \mu \cdot \mathbb{E}[N] \ //$$

2. $\mathrm{Var}(\mathbb{E}[X/Y])$

$$\left[\begin{aligned} &\mathrm{Var}(x) \\ &= \mathbb{E}[x^2] \\ &\quad - (\mathbb{E}[x])^2 \end{aligned} \right.$$

$$= \mathbb{E}[(\mathbb{E}[X/Y])^2]$$

$$- (\mathbb{E}[\ \underbrace{\mathbb{E}[X/Y]}\ ])^2$$

$$= \mathbb{E}[(\mathbb{E}[X/Y])^2] - (\mathbb{E}[X])^2$$

Generalization of Property 1

Generalized iterated expectation

$$\mathbb{E}[g(X)\ h(X,Y)]$$

$$= \underset{x}{\mathbb{E}}[g(x)\ \mathbb{E}[h(x,Y) | x]]$$

[Proof]

Exercise: $U_1, U_2, \dots$ are iid $U(0,1)$ random

(Ross textbook) variables. Find $\mathbb{E}[N]$

$$N = \min \left\{ n : \sum_{i=1}^{n} U_i > 1 \right\}$$

Conditional Variance

$$\mathrm{Var}(X|Y) = \mathbb{E}[(X - \mathbb{E}[X|Y])^2 | Y]$$

Example: Random Sum (Conditional Variance)

$$Y = \sum_{i=0}^{N-1} X_i \qquad X_i : \text{iid}$$

N : r.v (integer)

$$E[Y/N] = \mu \cdot N.$$

$Var(Y/N=n)$

$$= E[(Y - \underbrace{E[Y/N=n]}_{})^2 / N=n]$$

$$= E[(\sum_{i=0}^{n-1} X_i - \overset{n\cdot\mu}{n\mu})^2 / N=n]$$

$$= E[[\sum_{i=0}^{n-1}(X_i-\mu)]^2 / N=n]$$

$$= E[\sum_{i=0}^{n-1}(X_i-\mu)^2 + \sum_{i=0}^{n-1}\sum_{\substack{j=0 \\ i\neq j}}^{n-1}(X_i-\mu)(X_j-\mu) / N=n]$$

$$= \underbrace{\sum_{i=0}^{n-1} E[\underbrace{(X_i-\mu)^2}_{}]}_{} + \sum_i \sum_{\substack{j \\ i\neq j}} E[\underbrace{(X_i-\mu)(X_j-\mu)}_{}] / N=n$$

$Var(X_i) = \sigma^2$

$n \cdot \sigma^2$

$E[(X_i-\mu)]$ 0

$E[(X_j-\mu)]$ 0

$E X_i - \mu$

$\underbrace{\mu - \mu = 0}_{0}$

$$Var(Y/N=n) = n \cdot \sigma^2$$

$$Var(Y/N) = \sigma^2 \cdot N$$

Properties

1. $$\mathrm{Var}(X|Y) = \mathbb{E}[(X - \mathbb{E}[X|Y])^2 | Y]$$

$$= \mathbb{E}[X^2 - 2X\,\mathbb{E}[X|Y] + (\mathbb{E}[X|Y])^2 | Y]$$

$$= \mathbb{E}[X^2 | Y] - \mathbb{E}[(\mathbb{E}[X|Y])^2 | Y]$$

2. Law of Total Variance

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}\underbrace{(\mathbb{E}[X|Y])}_{\geq 0}$$

$$\Rightarrow \mathrm{Var}(X) \geq \mathbb{E}[\mathrm{Var}(X|Y)]$$

Var $\Longleftrightarrow$ "Uncertainity" $\rightsquigarrow$ Prediction.