- Lab on Fri
- OH on Fri
- Quiz on Sun, 2pm

Markov Chains

- P (one-step transition prob.)
- P^n (n-step transition prob)
- First step analysis (General case: Textbook)
- What happens if we run our M.C for a long time?

Recurrent: $\sum_{n=1}^{\infty} P_{ii}(n) = \infty$

Transient: $\sum_{n=1}^{\infty} P_{ii}(n) < \infty$

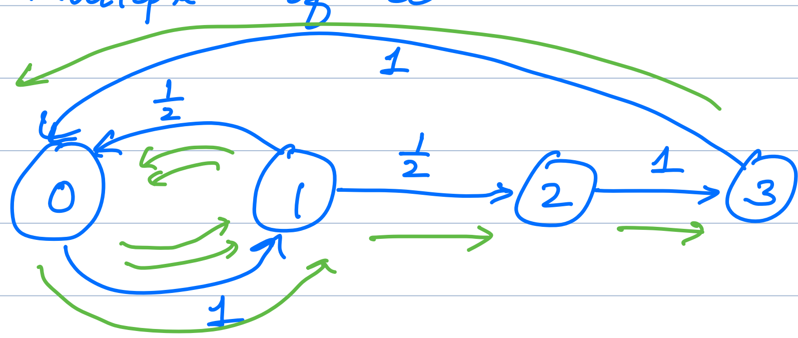
Irreducible: M.C with a single class

: A finite state irreducible M.C. is recurrent

Periodicity of Markov Chain

Defn: State i of M.C has a period d if $P_{ii}(n) = 0$ unless n is a multiple of d.

Example:



Consider state 0

$P_{00}(1) = 0$

$P_{00}(2) = \frac{1}{2}$

$P_{00}(3) = 0$

$P_{00}(4) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} = 3/4$

$\vdots$

$P_{00}(2n) > 0$

$P_{00}(2n+1) = 0$

State 0 has a period 2.

Exercise : Period for 1, 2, 3.

(Period 2)

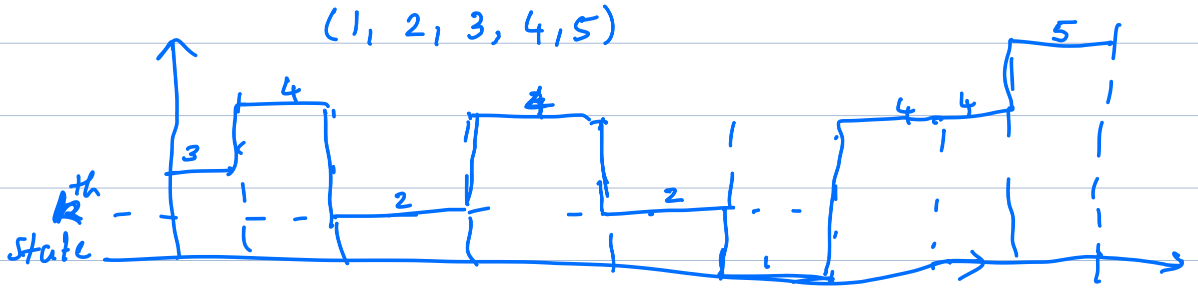
Property: All states in a class have the same period.

Regular M.C : $\boxed{\text{Finite state, irreducible}}$, aperiodic Markov Chain

recurrent

Long-run behaviour of Regular M.C

(1, 2, 3, 4, 5)



$N_k(n)$ # of times k^{th} state occurs if we run the M.C for n time steps

$$\pi_k = \lim_{n\to\infty}\left(\frac{N_k(n)}{n}\right)$$ [fraction of times k occurs in n steps]

" Limiting distribution "

Prop : 1. $\pi_k \geq 0$

2. $\sum_{k=1}^{n} \pi_k = 1$

$\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ — is a p.m.f

Theorem (Ross) (Proof in Ross : positive recurrent + aperiodic)

For a regular M.C

1. $\pi_j = \lim_{n\to\infty} P_{ij}(n)$

2. π_j is the unique non-negative solution to

$$\pi_j = \sum_i \pi_i P_{ij}$$

1. π_j = n: step transition probability ($n \to \infty$)

doesn't depend on the starting state i

P

n: step transition prob: $\underline{P^n}$

$$\pi_j = \lim_{n\to\infty} \underline{P_{ij}(n)}$$

$$P_{ij}(n) = [P^n]_{i,j}$$

3- state regular M.C

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \end{array}$$

$$\pi_j = \lim_{n \to \infty} P_{ij}[n]$$

$$\lim_{n \to \infty} P^n = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{bmatrix}$$

rows of P^n converge to the limiting distribution.

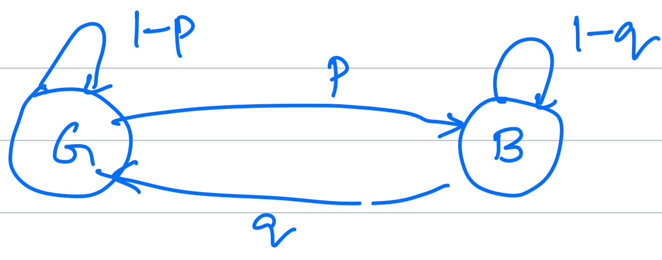
2. $$\pi_j = \sum_i \pi_i P_{ij}$$

Matrix Notation: $\pi = \pi \cdot P$

row vector: $(\pi_1, \dots \pi_n)$

Example:

Fraction of time the person is in a Good mood?



$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

$$\pi = \pi \cdot P$$

$$(\pi_G \quad \pi_B) = (\pi_G \quad \pi_B)\begin{bmatrix}1-p & p \\ q & 1-q\end{bmatrix}$$

$$\cancel{\pi_G} = (\cancel{1}-p)\,\pi_G + \pi_B\cdot q$$

$$p\cdot\pi_G = \pi_B\cdot q$$

$$\pi_G + \pi_B = 1$$

$$\pi_G = \frac{p}{p+q}$$

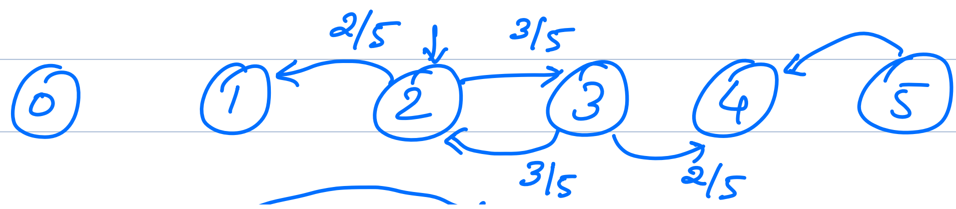
$$\pi_B = \frac{q}{p+q} \quad //$$

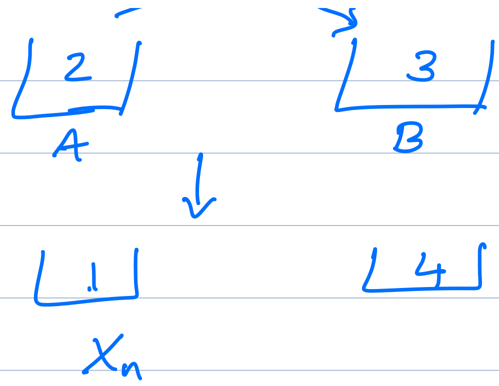
Example: (Ehrenfest Urn Model) Five balls are distributed b/w 2 urns.. A & B.
In each discrete time, one ball is chosen at random.
The ball is moved to the other urn.
What fraction of time is the urn A empty?

X_n = # of balls at any time in urn A

# of balls in B = $5 - X_n$

$X_n \in \{0, 1, 2, 3, 4, 5\}$





(Exercise!)

Use Matlab to figure out π_0 ?

Syllabus: RP + Limit Thm + Prob Ineqs + M.C