Last class : Conditional expectation

$\mathbb{E}[X \mid \underline{Y=y}] \rightarrow \mathbb{E}[\underline{X \mid Y}]$

Syllabus: Lecture 1 - Lecture 13

[One upload for all answers]

Properties

(1) $\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X]$

(2) $\mathrm{Var}(\mathbb{E}[X \mid Y]) = \mathbb{E}[(\mathbb{E}[X \mid Y])^2] - (\mathbb{E}[X])^2$

Conditional Variance

$$\mathrm{Var}(X \mid Y) = \mathbb{E}[(X - \mathbb{E}[X \mid Y])^2 \mid Y]$$
$$= \mathbb{E}[X^2 \mid Y] - (\mathbb{E}[X \mid Y])^2$$

Law of Total Variance

$$\mathrm{Var}(X) = \mathrm{Var}(\mathbb{E}[X \mid Y]) + \mathbb{E}[\mathrm{Var}(X \mid Y)]$$

Proof

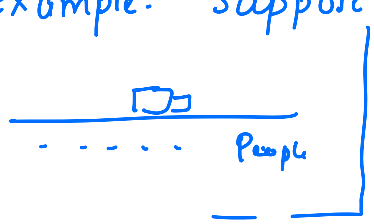
$$\mathrm{Var}(\mathbb{E}[X \mid Y]) = \underbrace{\mathbb{E}[(\cancel{\mathbb{E}}[X \mid Y])^2]}_{(A)} - (\mathbb{E}[X])^2$$

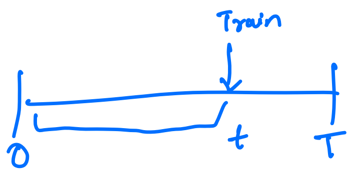
$$\mathbb{E}[\mathrm{Var}(X \mid Y)] = \mathbb{E}[\mathbb{E}[X^2 \mid Y] - (\mathbb{E}[X \mid Y])^2]$$
$$= \mathbb{E}[X^2] - \underbrace{\mathbb{E}[(\mathbb{E}[X \cancel{\mid Y}])^2]}_{(B)}$$

(A) + (B)

$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$
$$= \mathrm{Var}(X)$$

Example: Suppose at any time t, the number of people arriving in the train depot is a Poisson random variable with rate λt.





If the first train comes at the depot at a time uniformly distributed in $(0,T)$.
What is the mean and variance of the # of passangers in the train?

Y: Time of arrival of this train
N: # of passengers in the train

$\mathbb{E}[N]$? $Var[N]$?

$Y = t$

$N \mid Y=t \sim \text{Poisson}(\lambda t)$

$\mathbb{E}[N \mid Y=t] = \lambda t$

$\mathbb{E}[N \mid Y] = \lambda \cdot Y$

$$\mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N \mid Y]] = \mathbb{E}[\lambda \cdot Y]$$
$$= \lambda\, \mathbb{E}[Y]$$
$Y \sim \text{Unif}(0,T)$
$$= \lambda \cdot T/2$$

$$Var(N) = Var(\underbrace{\mathbb{E}[N \mid Y]}_{\lambda \cdot Y}) + \mathbb{E}[\underline{Var(N \mid Y)}]$$

$Y \sim \text{Unif}(0,T)$ $\quad Var(\lambda Y) = \lambda^2 \, Var(Y) = \lambda^2 \cdot T^2/12$

$N \mid Y=t \sim \text{Poisson}(\lambda t)$

$$Var(N \mid Y=t) = Var(\text{Poisson}(\lambda t))$$
$$= \lambda t$$

$$Var(N \mid Y) = \lambda \cdot Y$$

$$\mathbb{E}[\mathrm{Var}(N|Y)] = \lambda \cdot \mathbb{E}[Y] = \lambda \cdot T/2$$

$$\mathrm{Var}(N) = \frac{\lambda^2 T^2}{12} + \lambda \cdot \frac{T}{2}$$ //

Exercise: (Ross) A miner is trapped in a mine containing 3 doors

1^{st} door leads him to safety after 3 hrs of travel

2^{nd} door leads back to mine after 5 hrs of travel

3^{rd} door also leads backs to mine after 7 hrs of travel.

Find expected time & variance for the miner to reach safety?

Conditional expectation as Prediction

X, Y Joint pdf

$\hat{Y} = g(X)$: Prediction

Mean squared error $= \mathbb{E}[(Y - \hat{Y})^2]$

g should minimize $\mathbb{E}[(Y-\hat{Y})^2]$

$= \mathbb{E}[(Y - g(X))^2]$

What is the best g?

$g = \mathbb{E}[Y|X]$

Proof: Formal proof (text book)

Step1: One random variable Y

Problem: Predict this random variable Y using a constant c

[Mean of random variable]

$$\mathbb{E}[(Y-a)^2] = \mathbb{E}[(Y - \mathbb{E}(Y) + \mathbb{E}[Y] - a)^2]$$

$$= \mathbb{E}[(Y - \mathbb{E}(Y))^2] + \mathbb{E}[(\mathbb{E}[Y] - a)^2] + 2\,\mathbb{E}[(Y - \mathbb{E}(Y))(\mathbb{E}[Y]-a)]$$

$$2(\mathbb{E}[Y]-a)\,\mathbb{E}[Y - \mathbb{E}[Y]]$$

$$\mathbb{E}(Y) - \mathbb{E}[Y]$$

$$= 0$$

$$\mathbb{E}[(Y-a)^2] = \underbrace{\mathbb{E}[(Y - \mathbb{E}[Y])^2]}_{\substack{Var(Y) \\ \geq 0}} + \underbrace{\mathbb{E}[(\mathbb{E}[Y]-a)^2]}_{\geq 0}$$

$$a^* = \mathbb{E}[Y] //$$

Step2:

$$\mathbb{E}[(Y - g(X))^2] = \mathbb{E}[\mathbb{E}[(Y - g(X))^2 \mid X]]$$

(Discrete Case)

$$= \sum_x P_X(x)\, \underbrace{\mathbb{E}[(Y - \overbrace{g(x)}^{a})^2 \mid X = x]}$$

Optimal

$$g^*(x) = \mathbb{E}[Y \mid X = x]$$

$$g^*(X) = \mathbb{E}[Y \mid X]$$

g^* : minimizes the error (mean squared)

Example: Suppose the son of a man of height x (in inches) attain a height that is normally distributed with mean $x+1$ and variance 4. What is the best prediction of the height of the son of a man who is 6 feet (72 inches)

X: Height of father

Y: Height of son

$$Y \mid X = x \sim N(x+1, 4)$$

$$\mathbb{E}[Y \mid X] = X + 1$$

$$\hat{X} = 72 \text{ inches}$$

$$\mathbb{E}[Y \mid X = 72 \text{ inches}] = 72 + 1 = 73$$

Next class: Jacobians (Calculus)