Announcements : Lab on Friday

: Quiz 1 on Sunday. (Oct 17) ~~11am~~ Afternoon (3pm) (4:30pm)

Last class : (i) Distribution of random variable

(ii) Probability mass function of r.v

(iii) Cumulative distribution function

Binomial distribution 0, 1

Bernoulli distribution : $p_X(i) = \binom{n}{i} p^i (1-p)^{n-i}$

Example:

Suppose an airplane engine will fail, in flight, with probablity $1-p$ independent from engine to engine. The airplane will make a successful flight if at least 50 % of engine remain operative.

Qn: For what value of p is a 4 engine plane preferable to a a 2 engine plane ?

4 engine plane flight successful - 2, 3, 4

Probability that 2 out of 4 engines are operational

$$= \binom{4}{2} p^2 (1-p)^2$$

$\Downarrow$

$Bin(4, p)$

Probablity that 3 out of 4 engines are operational

$$= \binom{4}{3} p^3 (1-p)$$

" " 4 out of 4 " " "

$$= \binom{4}{4} p^4 (1-p)^0$$

Probablity (4 engine flight successful)

$$= 6p^2(1-p)^2 + 4p^3(1-p) + p^4$$

Probability (2 engine flight successful)

$$= \binom{2}{1} p (1-p) + \binom{2}{2} p^2$$

$$= 2p(1-p) + p^2$$

$$= 2p - p^2$$

$$6p^2(1-p)^2 + 4p^3(1-p) + p^4 \geq 2p - p^2$$

$$p \geq 2/3 \ //$$

* Geometric random variable ($X \sim Geo(p)$)

1, 2, 3, ...

$$P_X(n) = (1-p)^{n-1} p$$

p Waiting time = 1

$(1-p)p$ Waiting time = 2

$(1-p)^{n-1} p$

Intution: # of Trials required for first success.

* Poisson random variable $X \sim Poisson(\lambda)$

Range space : 0, 1, 2, ...

$$P_X(n) = e^{-\lambda} \frac{\lambda^n}{n!}$$

$$\sum_{n=0}^{\infty} P_X(n) = 1$$

$$\sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}$$

$$= e^{-\lambda} \underbrace{\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \right]}_{e^{\lambda}}$$

$$= e^{-\lambda} \cdot e^{\lambda}$$

$$= 1 //$$

Property of cdf $\Rightarrow F_X(\infty) = 1$

$$F_X(\alpha) = \sum_{k=-\infty}^{\alpha} P_X(k)$$

$$F_X(\infty) = 1 \Rightarrow \sum_{k=-\infty}^{\infty} P_X(k) = 1$$

$$\sum_{k=0}^{\infty} P_X(k) = 1$$

Relation between Binomial r.v and Poisson r.v.

n is large $\quad p$ is small $\quad \lambda = np$

$$P_X(i) = \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \frac{n!}{(n-i)!\, i!} p^i \frac{(1-p)^n}{(1-p)^i} \quad [\lambda = np]$$

$$= \frac{n!}{(n-i)!\, i!} \left(\frac{\lambda}{n}\right)^i \frac{(1-\lambda/n)^n}{(1-\lambda/n)^i}$$

$$= \frac{\lambda^i}{i!} \underbrace{\frac{(n-i+1)\cdots n}{n^i}}_{1} \frac{(1-\lambda/n)^n}{(1-\lambda/n)^i}$$

i term

$e^{-\lambda}$

$(1-p)^i \approx 1$

$i = 10$

$n = 1000$

Wikipedia $\quad \lim_{n \to \infty} (1 - \lambda/n)^n = e^{-\lambda}$

$$\approx \frac{\lambda^i}{i!} e^{-\lambda} \Rightarrow \text{Poisson random variable.}$$

Continous random variables

$$P_X(F) = \Pr\{\omega : X(\omega) \in F\}$$

If there exists a function $f_X(x)$ such that $P_X(F) = \int_F f_X(x)\,dx$

Then f_X is called the probability density function of r.v. X
(pdf)

$$F_X(\alpha) = P_X((-\infty, \alpha])$$
$$= \int_{-\infty}^{\alpha} f_X(x)\,dx$$

$$f_X(\alpha) = \frac{d}{d\alpha} F_X(\alpha)$$

Prop1: $P_X([a,b]) = F_X(b) - F_X(a)$
$$= \int_{-\infty}^{b} f_X(x)\,dx - \int_{-\infty}^{a} f_X(x)\,dx$$
$$= \int_{a}^{b} f_X(x)\,dx$$

Prop 2: $F_X(\infty) = 1$
$$\int_{-\infty}^{\infty} f_X(x) = 1$$

"Important" continous random variables:

1. Uniform random variable $X \sim Unif([0,1])$

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{else.} \end{cases}$$

$$\int_{-\infty}^{\infty} f_X(x) = 1$$

$$P_X([a,b]) = \int_a^b 1 \cdot dx = b - a \quad //$$

$0 < a < b < 1$

Generalization $Unif([\alpha, \beta])$

$$f_X(x) = \begin{cases} \frac{1}{\beta - \alpha} & \alpha \le x \le \beta \\ 0 & \text{else.} \end{cases}$$

2. Exponential random variable $Exp(\lambda)$ $\lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

$$\underset{\text{c.df}}{F_X(x)} = \int_0^x \lambda e^{-\lambda x}$$

(Exercise !)

$$= 1 - e^{-\lambda x}$$

3. Gaussian / Normal random variable.