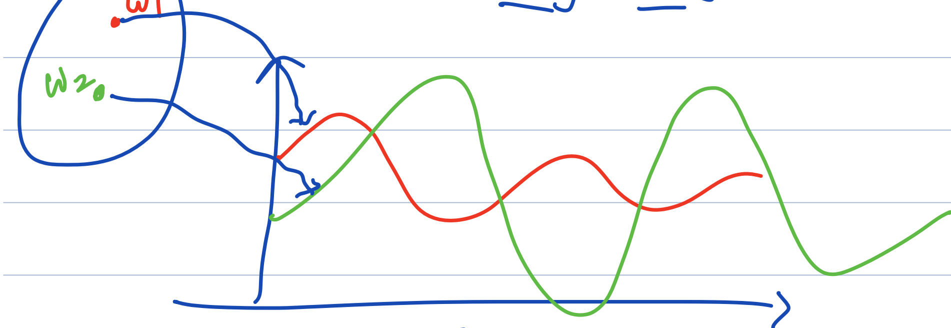
Announcements

1. Problem Set on M3 posted
2. Syllabus for final: M2 & M3
   Marks: 20 + 10 (Bonus)
3. Please fill out course feedback form
4. No OH on Friday.
5. Problem Solving : Thu or $\boxed{\text{Monday}}$
   (Optional)

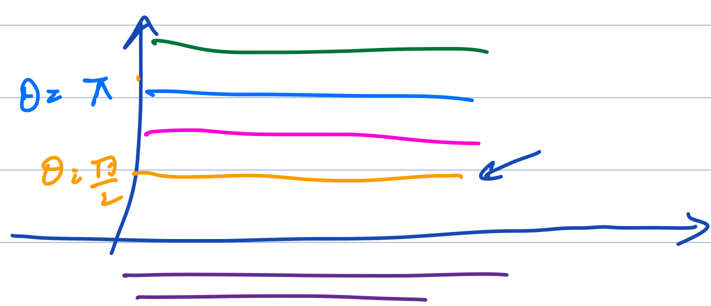
Ergodicity



$$\mu_X \overset{?}{=} \lim_{T \to \infty} \frac{1}{T} \int_0^T X(t)\, dt$$

⟵ Time average ⟶

Example: $X(t) = \theta \quad t \geq 0 \quad \theta \sim \text{Unif}(-\pi, \pi)$



$$\mathbb{E}[X(t)] = 0$$

$$\frac{1}{T}\int_0^T X(t)\, dt = \theta \neq 0$$

$t \geq 0$

Example: $X(t) = A \cos(\omega t + \theta)$; $\theta \sim \text{Unif}(0, 2\pi)$

$\mathbb{E}[X(t)] = 0$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\, dt$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T A\cos(\omega t + \theta)\, dt$$

$$\lim_{T\to\infty} A\,\frac{\sin(\omega t + \theta)}{\omega T}\Big|_0^T$$

$$\lim_{T\to\infty} -\frac{A}{\omega T} < A\,\frac{\sin(\omega T + \theta)}{\omega T} < \frac{A}{\omega T}$$

$$= 0 \,//$$

WSS + Ergodicity

Definition: A WSS process is "ergodic in mean" if

$$\hat{m}(T) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\, dt \xrightarrow{m.s} \mu_x$$

"$\langle x(t) \rangle$"

$$\mathbb{E}[\hat{M}(T)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \underbrace{\mathbb{E}(X(t))}_{\mu_x}\, dt$$

$$= \mu_X$$

$$\mathbb{E}[\hat{M}^2(T)] = \mathbb{E}\left[\lim_{T\to\infty} \frac{1}{(2T)^2}\int_{-T}^{T} x(t_1)\, dt_1 \int_{-}^{T} x(t_2)\, dt_2\right]$$

$$= \lim_{T\to\infty} \frac{1}{(2T)^2} \int_{-T}^{T}\int_{-T}^{T} \underbrace{E[X(t_1)X(t_2)]}_{R_X(t_1-t_2)}\, dt_1\, dt_2$$

$$Var[\hat{M}(t)] = E[\hat{M}^2(T)] - E[\hat{M}(t)]^2$$

$$= \underbrace{\lim_{T\to\infty} \frac{1}{(2T)^2} \int_{-T}^{T}\int_{-T}^{T} C_X(t_1-t_2)\, dt_1\, dt_2}$$

Mean square convergence

$$E[(\hat{M}(T) - \mu_X)^2] = Var(\hat{M}(T)) \to 0$$

$$\to \quad \to 0$$

Proof skipped (Textbook)

$$\lim_{T\to\infty} \frac{1}{(2T)} \int_{-T}^{T} \left(1 - \frac{|\tau|}{2T}\right) C_X(\tau)\, d\tau \to 0$$

"Condition for ergodicity in mean"

Sufficient condition

$$\int_{-T}^{T} |C_X(\tau)|\, d\tau < \infty$$

Finite value of integral of covariance

Exercise: Show that $X(t) = A\cos(\omega t + \Theta)$ satisfies the sufficient condition.

Ergodicity in auto-correlation.

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$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t+\tau)\, x(\tau)\, d\tau \xrightarrow{m.s} R_x(\tau)$$

Ergodicity in mean square

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X^2(t)\, dt \xrightarrow{m.s} R_x(0) \quad E[X^2(t)]$$

Conditions on auto-covariance in textbook.

Pointwise Ergodic Theorem (Strong Law of Large Numbers)

Given a discrete time stationary (WSS) and ergodic process $\{X_n\}$ with $E[X_n] = \mu_X$ then.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} X_k \to \mu_X \text{ with probability one}$$

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