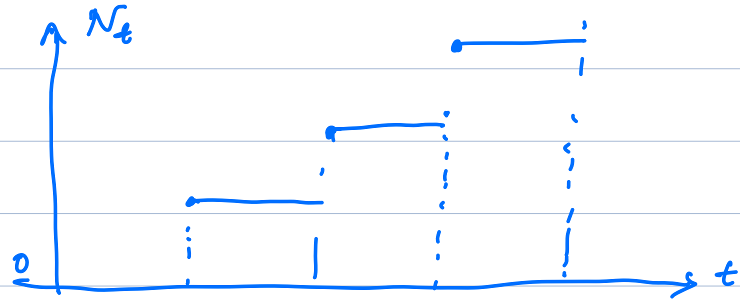
Announcements

1. Lab viva schedule posted
2. Lab this week. Complete Qns 1, 2, 4

Counting Process



A random process N_t is a counting process if

(i) $N_0 = 0$

(ii) N_t is an increasing function of t

(iii) N_t increases by only one whenever it changes

Example: Bernoulli Counting Process
(Discrete time)

$S_0 = 0$

$S_n = X_1 + X_2 + \cdots + X_n$

X_n's are iid Bernoulli random variables

$$X_i = \begin{cases} 1 & \text{w.p } p \\ 0 & \text{w.p } 1-p \end{cases}$$

Is this a counting process?

"3 properties of counting process"

Properties (Binomial Counting process)

1. $(S_t - S_s) \perp\!\!\!\perp S_s \quad t \geq s$

Generalization: $(S_t - S_s) \perp\!\!\!\perp S_r \quad \begin{matrix} t \geq s \\ r \leq s \end{matrix}$

$$S_t = X_1 + X_2 + \cdots + X_s + \underbrace{X_{s+1} + \cdots + X_t}$$
$$S_s = X_1 + X_2 + \cdots + X_s$$
$$S_t - S_s = X_{s+1} + \cdots + X_t$$
$$(S_t - S_s) \perp\!\!\!\perp S_s$$

" Independent increment property "

2. $S_0 = 0$

$$S_1 = X_1$$
$$S_2 = X_1 + X_2$$
$$S_3 = X_1 + X_2 + X_3$$
$$S_3 - S_1 = X_2 + X_3$$

Distribution of $S_3 - S_1 = \text{Dist}(X_2 + X_3)$

$$= \text{Binomial}(\underset{2}{n}, p)$$

$$= \text{Binomial}(\underset{(3-1)}{\underset{\sim}{2}}, p)$$

Distribution of $S_2 = \text{Dist}(X + X_2)$

$$= \text{Binomial}(2, p)$$

$\text{Dist}(S_t - S_s) = \text{Dist}(S_{t-s})$

" Stationary increment property "

Poisson Process

A counting process N_t is called a Poisson process with parameter λ ('rate') if it satisfies the following 2 properties

(ii) Independent increment

$$N_t - N_s \perp\!\!\!\perp N_r \qquad \begin{matrix} r \le s \\ t \ge s \end{matrix}$$

(iii) Stationary : $N_t - N_s \sim \text{Poisson}(\lambda(t-s))$

$$N_{t-s}$$

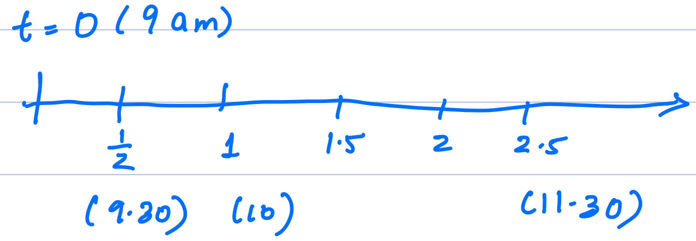
Review:

Poisson distribution (Discrete) $X \sim \text{Poisson}(\lambda)$

p.m.f : $P_X(k) = P(X = k)$

$$= e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

Example: Customers arrive in a certain store according to a Poisson process of rate $\lambda = 4/\text{hr}$. Given that the store opens at 9 am. What is the probability that we have exactly 1 customer at 9.30 am and a total of 5 customers at 11.30 am ?



$N(t)$: # of customers in the store at time t

$$P\left(N\left(\tfrac{1}{2}\right) = 1,\ N(2.5) = 5\right)$$

$= P\left(N\left(\tfrac{1}{2}\right) = 1, \quad N(2.5) - N\left(\tfrac{1}{2}\right) = 4 \right)$

$\overset{\text{independent increment}}{=}$

$N(t) - N(s) \perp\!\!\!\perp N(s)$

$= \underbrace{P\left(N\left(\tfrac{1}{2}\right) = 1 \right)} \; \underbrace{P\left(N(2.5) - N\left(\tfrac{1}{2}\right) = 4 \right)}_{\text{stationarity increment}}$

$\mathrm{Dist}\left(N(t) - N(s) \right)$

$= \mathrm{Dist}\left(N(t-s) \right)$

$N\left(\tfrac{1}{2}\right) \sim \mathrm{Poisson}\left(\lambda \tfrac{1}{2}\right)$

$= \mathrm{Poisson}(2)$

$P\left(\underbrace{N(2)} = 4 \right)$

$\mathrm{Poisson}(2 \cdot 4)$

$= \mathrm{Poisson}(8)$

$= e^{-2} \dfrac{2}{1!}$

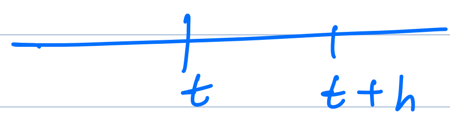
$e^{-8} \dfrac{8^4}{4!}$

$= e^{-10} \cdot 2 \cdot \dfrac{8^4}{4!}$

Is Poisson process a counting process?

- Does $N(t)$ change by atmost 1?

Probability that $N(t)$ changes by more than 1 $\overset{?}{=} 0$



$N(t+h) - N(t) \sim \mathrm{Poisson}(h \cdot \lambda)$

0 arrivals on the interval $(t, t+h)$

$$P(\underbrace{N(t+h) - N(t)}_{\text{Poisson}(h\cdot\lambda)} = 0) = e^{-h\lambda}\frac{(h\lambda)^0}{0!}$$

$$= e^{-h\lambda}$$

1 arrival in the interval $(t, t+h) \approx 1 - h\lambda$

$$P(\underbrace{N(t+h) - N(t)}_{\text{Poisson}(h\cdot\lambda)} = 1) = e^{-h\lambda}\cdot\frac{(h\lambda)^1}{1!}$$

$$= e^{-h\lambda}\cdot(h\cdot\lambda)$$

$$\approx (1-h\lambda)(h\lambda)$$

$$= h\lambda - h^2\lambda^2$$

$$P(N(t+h) - N(t) = 1 \text{ or } 0)$$

$$= \underbrace{1 - h\lambda}_{0 \text{ arrivals}} + \underbrace{h\lambda - h^2\lambda^2}_{1 \text{ arrival}}$$

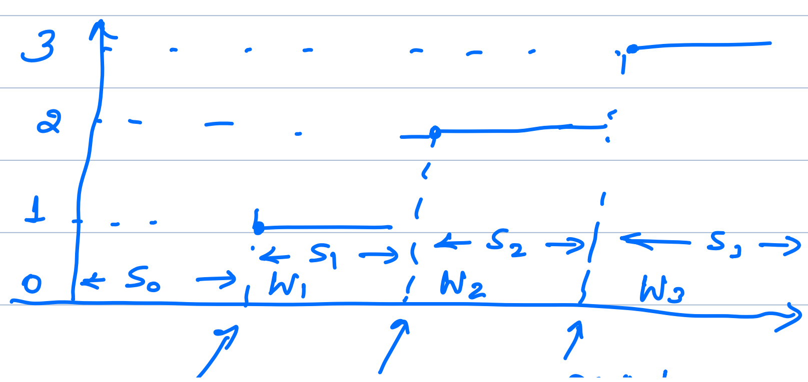
$$= 1 - h^2\lambda^2$$

$$\lim_{h\to 0} P(N(t+h) - N(t) = 1 \text{ or } 0)$$

$$= 1 \quad //$$

"$N(t)$ increases by at most 1."

$\Rightarrow$ Counting process.



arrival arrival arrival
(event)

W_n : Waiting time for the n^{th} arrival

$S_n = W_{n+1} - W_n$ inter-arrival time (Sojourn time)