Theorem: $\{X_n\}$ is a sequence of uncorrelated random variables; such that $\mathbb{E}[X_n] = X$ (not a r.v) and $Var(X_n) = \sigma^2 < \infty$

"Finite variance"

Sample average: $S_n = \frac{1}{n} \sum_{i=0}^{n-1} X_i$

$$S_n \xrightarrow{m.s} X$$

Proof: $\lim_{n\to\infty} \mathbb{E}[\,|S_n - X|^2] = 0$

$$S_n = \frac{1}{n} \sum_{i=0}^{n-1} X_i$$

$$\mathbb{E}[S_n] = \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[X_i] = \frac{1}{n} \sum_{i=0}^{n-1} X = X$$

$$= \mathbb{E}\left[\left(S_n - \mathbb{E}[S_n]\right)^2\right]$$

$$= \mathbb{E}\left[\left(\frac{1}{n} \sum_{i=0}^{n-1} X_i - \frac{1}{n} \sum_{i=0}^{n-1} X\right)^2\right]$$

$$= \frac{1}{n^2} \mathbb{E}\left[\left(\sum_{i=0}^{n-1} (X_i - X)\right)^2\right]$$

$$= \frac{1}{n^2} \mathbb{E}\left[\sum_i (X_i - X)^2 + \sum_i \sum_{\substack{j \\ i \neq j}} (X_i - X)(X_j - X)\right]$$

$$= \frac{1}{n^2} \sum_i \underbrace{\mathbb{E}[(X_i - X)^2]}_{\substack{Var(X_i) \\ = \sigma^2}} + \sum_i \sum_{\substack{j \\ i \neq j}} \underbrace{\mathbb{E}[(X_i - X)(X_j - X)]}_{0}$$

$$= \frac{1}{n^2} \sum_i \sigma^2 = \frac{1}{n^2} \cdot n \cdot \sigma^2 = \frac{\sigma^2}{n}$$

$$\lim_{n\to\infty} \mathbb{E}[(S_n - X)^2] = 0$$

Weak law of large numbers

Suppose $\{X_n\}$ is a sequnu of uncorrelated r.vs such that $\mathbb{E}[X_n] = X$ and $\mathrm{Var}(X_n) = \sigma^2 < \infty$

$$S_n = \frac{1}{n}\sum_{i=0}^{n-1} X_i \xrightarrow{P} X$$

Strong law of large numbers

Suppose $\{X_n\}$ is an iid sequence of random variables with finite mean and variance.

$$S_n = \frac{1}{n}\sum_{i=0}^{n-1} X_i \to X \text{ with probability } 1$$

Central Limit Theorem

Let $\{X_n\}$ be a sequnu of iid random variables with finite mean $(m)$, and variance $(\sigma^2)$

$$R_n = \frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\left(\frac{X_i - m}{\sigma}\right) \quad (\ne \sqrt{n}\, S_n)$$

"Normalization"

$$\text{then } R_n \xrightarrow{d} \mathcal{N}(0,1)$$

$$\hat{X}_i = \frac{X_i - m}{\sigma}$$

$$\mathbb{E}[\hat{X}_i] = 0, \quad \mathrm{Var}(\hat{X}_i) = 1 \quad [\text{Exercise}]$$

$$R_n = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \hat{X}_i$$

Need to show $F_{R_n} \to$ Standard Normal cdf

$M_{R_n} \to$ MGF of standard normal

$$M_{R_n}(t) = \mathbb{E}\left[e^{t R_n}\right]$$

$$= \mathbb{E}\left[e^{\frac{t}{\sqrt{n}} \sum_{i=0}^{n-1} \hat{X}_i}\right]$$

$$= \mathbb{E}\left[e^{t\hat{X}_0/\sqrt{n}} \, e^{t\hat{X}_1/\sqrt{n}} \cdots e^{t\hat{X}_{n-1}/\sqrt{n}}\right]$$

$$\overset{\text{indpendence}}{=} \mathbb{E}\left[e^{t\hat{X}_0/\sqrt{n}}\right] \mathbb{E}\left[e^{t\hat{X}_1/\sqrt{n}}\right] \cdots \mathbb{E}\left[e^{t\hat{X}_{n-1}/\sqrt{n}}\right]$$

$$\boxed{\begin{aligned} &M_{X_i}(t/\sqrt{n}) \\ &= M(t/\sqrt{n}) \end{aligned}} \quad = \prod_{i=0}^{n-1} \underbrace{\mathbb{E}\left[e^{t\hat{X}_i/\sqrt{n}}\right]}_{M_{X_i}(t/\sqrt{n})}$$

$$\overset{\text{identical}}{=} \left(M(t/\sqrt{n})\right)^n$$

$$\lim_{n\to\infty} \left(\underbrace{M(t/\sqrt{n})}\right)^n \to \underbrace{\text{MGF of } N(0,1)}_{e^{t^2/2}}$$

$$M(s) \approx M(0) + s \cdot M'(0) + \frac{s^2}{2} M''(0)$$

↑ Taylor series expansion around $s=0$

$M$: MGF of $\hat{X}_i$

$$M(0) = \mathbb{E}[e^{0\cdot \hat{X}_i}] = \mathbb{E}[1] = 1$$

$$M'(0) = \frac{d}{dt}\mathbb{E}[e^{t\hat{X}_i}]\Big|_{t=0} = \mathbb{E}[\hat{X}_i] = 0$$

$$M''(0) = \frac{d^2}{dt^2}\mathbb{E}[e^{t\hat{X}_i}]\Big|_{t=0} = \mathbb{E}[\hat{X}_i^{\,2}] = 1$$

$$M(s) \approx 1 + s^2/2$$

$$\lim_{n\to\infty}\left(M(t/\sqrt{n})\right)^n = \lim_{n\to\infty}\left(1 + \frac{t^2}{2n}\right)^n$$

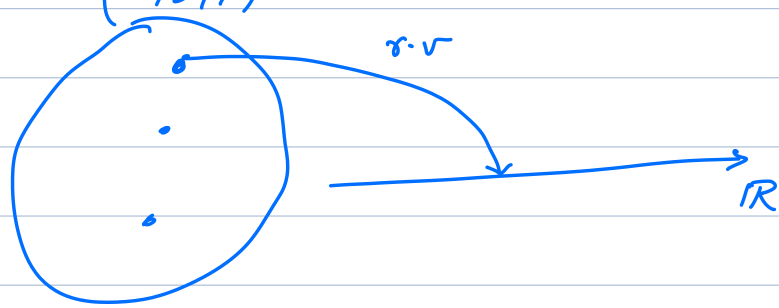
$$= e^{t^2/2} \quad //$$

"MGF of standard normal"

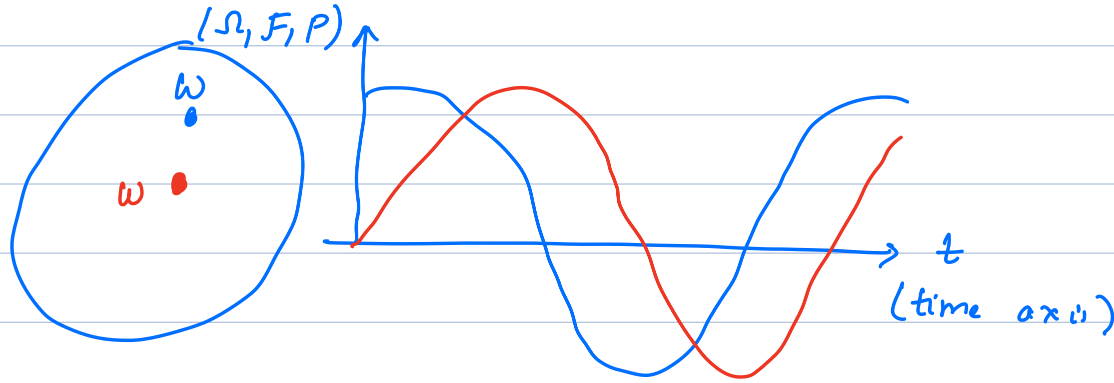
## Random Process

$(\Omega, \mathcal{F}, P)$

random vectors $= (X_0, X_1 \ldots, X_{n-1})$



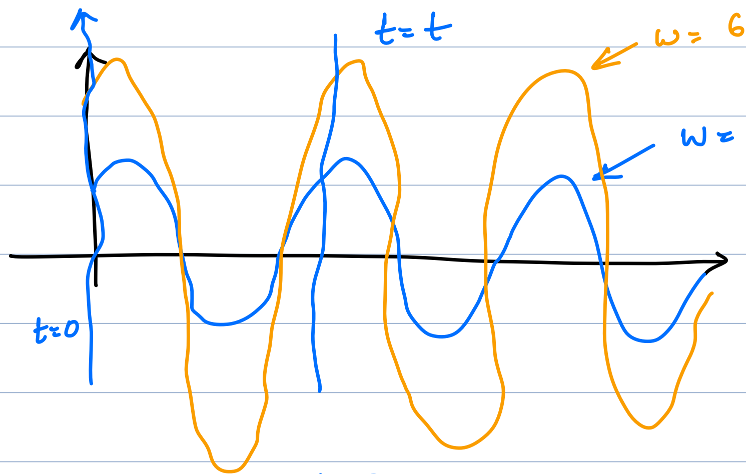

$(\Omega, \mathcal{F}, P)$



Example: Random Experiment: Throw a dice.

$\omega = \{1, 2, \dots, 6\}$

$X(\omega, t) = \omega \cdot \cos(t)$




" Sample paths"

" Realization"

$X_0$ is a random variable

$X_t$ is a random variable.

$$= \begin{cases} \cos t & \text{w.p } \frac{1}{6} \\ 2 \cdot \cos t & \text{w.p } \frac{1}{6} \\ \vdots & \\ 6 \cos t & \text{w.p } \frac{1}{6} \end{cases}$$

$X_{t_0} = X(\omega, t = t_0)$ is a random variable.

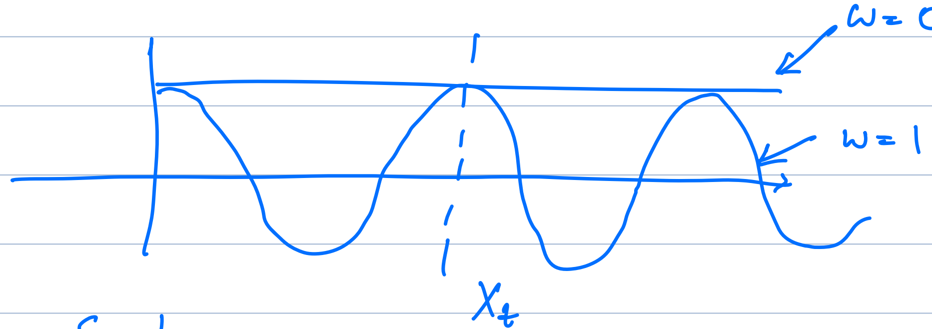
Example: $\Omega = \{0, 1\}$

$P(0) = \alpha$

$P(1) = 1-\alpha$

$X(t, \omega) = \cos(\omega t)$

What are the sample paths ?




$$X_t = \begin{cases} 1 & \text{w.p } \alpha \\ \cos t & \text{w.p } 1-\alpha \end{cases}$$