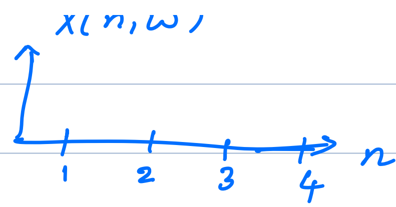
Random Process



Discrete - time R.P

I.I.D Process

X_n is a discrete time i.i.d process

" $X(\omega,n)$ " X_1 is sample of RP at time n_1

X_2 " " " n_2

$$F_{X_1 X_2 \dots X_k}(x_1, x_2, \dots, x_k)$$

$$= P[X_1 \le x_1, X_2 \le x_2, \dots, X_k \le x_k]$$

$$\overset{\text{"indp"}}{=} P[X_1 \le x_1]\, P[X_2 \le x_2] \dots P[X_k \le x_k]$$

$$= F_{X_1}(x_1)\, F_{X_2}(x_2) \dots F_{X_k}(x_k)$$

$F_{X_i}(\cdot) = F_X$: identical

$$= F_X(x_1)\, F_X(x_2) \cdots F_X(x_k)$$

Example : Bernoulli Random Process [Exercise]

Property 1 : Mean Function

$$m_X(n) = \mathbb{E}[X_n] = m$$

"Mean of dist F_X"

Property 2: Auto-covariance function

$$C_X(n_1, n_2) = \mathbb{E}[(X_{n_1} - m)(X_{n_2} - m)]$$

$$= \mathbb{E}[X_{n_1} X_{n_2}] - m \overset{m}{\boxed{\mathbb{E}[X_{n_1}]}} - m \overset{m}{\boxed{\mathbb{E}[X_{n_2}]}} + m^2$$

$$= \mathbb{E}[X_{n_1} X_{n_2}] - m^2$$

$n_1 \ne n_2$ $$= \underbrace{\mathbb{E}[X_{n_1}]}_{m}\, \underbrace{\mathbb{E}[X_{n_2}]}_{m} - m^2 = 0$$

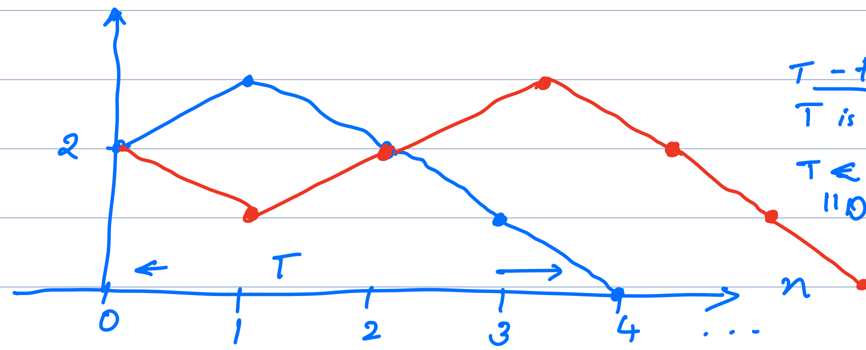
$n_1 = n_2$ $$= \mathbb{E}[X_{n_1}^2] - m^2 = \mathrm{Var}(F_X(\cdot)) = \sigma^2$$

$$C_X(n_1, n_2) = \sigma^2\, \delta(n_1, n_2)$$ $\delta(m,n) = 1$ if $m = n$

0 else-

Exercise: Mean, Auto-correlation, Auto-covariance of Bernoulli R.P.

Random Walks



T - time for gambler to go bankrupt

T is a r.v

$T < \infty$

"Does gambler go bankrupt"

$E(T)$

S_n: # Money that the Gambler has at any time n

$$S_n = S_0 + X_0 + X_1 + \cdots + X_n$$

S_0: Initial money

X_i's are independent and identical r.vs

$$P(X_i = +1) = P(X_i = -1) = \frac{1}{2}$$

Def: Random Walk.

A discrete time random process $\{S_n\}$ is called a random walk if

$$S_n = S_0 + X_1 + X_2 + \cdots + X_n$$

where X_i's are iid random variables.

Simple random walk: $X_i \in \{+1, -1\}$

Symmetric random walk: Simple random walk

$$P(X_i = +1) = P(X_i = -1) = \frac{1}{2}$$

Property (RW) : Random walk has independent increment property

$n_1 < n_2 < n_3$

$S_{n_1} = S_0 + X_1 + \cdots + X_{n_1}$

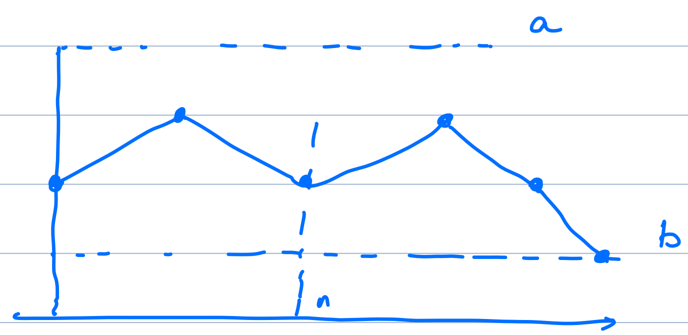
$S_{n_2} = S_0 + X_1 + \cdots + X_{n_1} + \cdots + X_{n_2}$

$= S_{n_1} + X_{n_1+1} + \cdots + X_{n_2}$

$S_{n_3} = S_{n_2} + X_{n_2+1} + \cdots + X_{n_3}$

$\underbrace{(S_{n_2} - S_{n_1})}_{X_{n_1+1} + \cdots + X_{n_2}} \perp\!\!\!\perp \underbrace{(S_{n_3} - S_{n_2})}_{X_{n_2+1} + \cdots + X_{n_3}}$

Symmetric Random Walk.



$T = \min \{ n \geq 0 ; S_n = a \text{ or } S_n = b \}$

"Hitting time"

Will the gambler ever stop playing Is $T < \infty$?

What is the probability that the gambler wins

$P[S_T = a]$

Expected time the gambler will play $\mathbb{E}[T]$

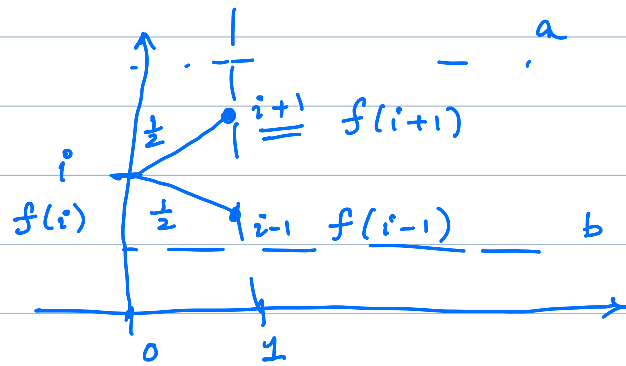
$P[T = \infty]$

$$P[T = \infty \mid S_0 = i] = f(i)$$

$$f(a) = P[T = \infty \mid S_0 = a] = 0$$

$$f(b) = P[T = \infty \mid S_0 = b] = 0$$

Technique: First step analysis
"one"



$$f(i) = \frac{1}{2} f(i+1) + \frac{1}{2} f(i-1)$$

Proof: $f(i) = P[T = \infty \mid S_0 = i]$

$$= \sum_{s_1} P[T = \infty, S_1 = s_1 \mid S_0 = i]$$

"Marginalization" "Total law of probability"

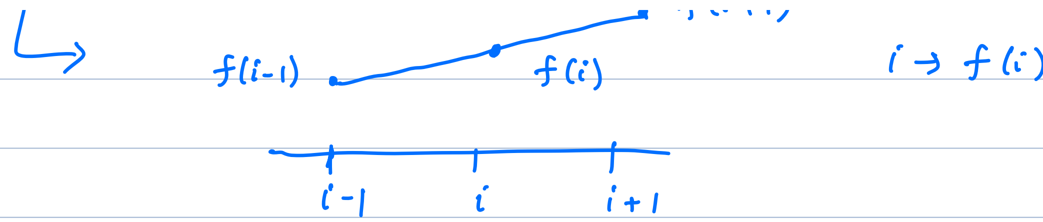
$$= P[T = \infty, S_1 = i-1 \mid S_0 = i] +$$
$$P[T = \infty, S_1 = i+1 \mid S_0 = i]$$

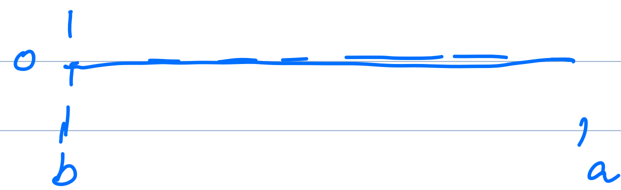
$$= \underbrace{P[S_1 = i-1 \mid S_0 = i]}_{\frac{1}{2}} \, \underbrace{P[T = \infty \mid S_0 = i, S_1 = i-1]}_{f(i-1)}$$

$$\underbrace{P[S_1 = i+1 \mid S_0 = i]}_{\frac{1}{2}} \, \underbrace{P[T = \infty \mid S_0 = i, S_1 = i+1]}_{f(i+1)}$$

$$\begin{cases} f(i) = \frac{1}{2} f(i-1) + \frac{1}{2} f(i+1) \\ f(a) = f(b) = 0 \end{cases}$$

$f(i+1)$





$$f(i) = 0 \quad \forall i$$

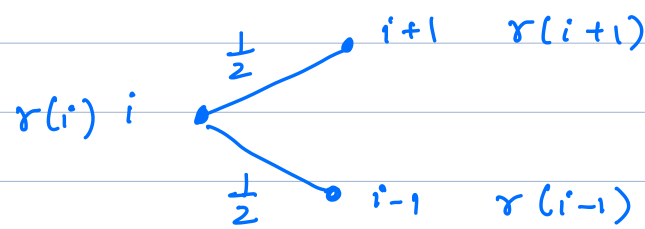
$$P[T = \infty \mid S_0 = i] = 0$$

"T is finite"

$$P[S_T = b \mid S_0 = i] = r(i)$$

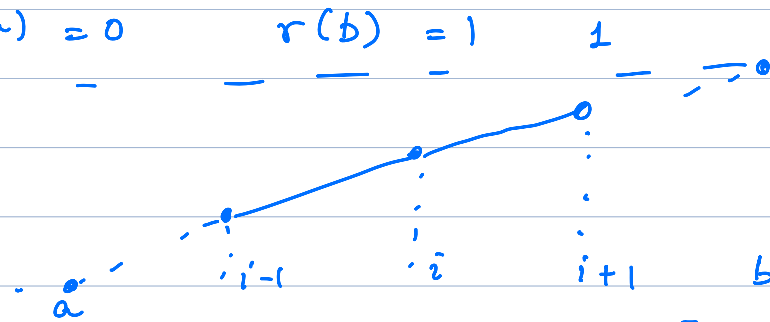
"Probability that the gambler goes bankrupt"

$r(b) = 1$ $\qquad$ $r(a) = 0$



$$r(i) = \frac{1}{2} r(i-1) + \frac{1}{2} r(i+1)$$ [Exercise: Proof]

$r(a) = 0$ $\qquad$ $r(b) = 1$



$$r(i) = \frac{i-a}{b-a} = P[S_T = b \mid S_0 = i]$$

Quiz 2 : 2pm on Nov 28

Syllabus : Everything till Nov 27