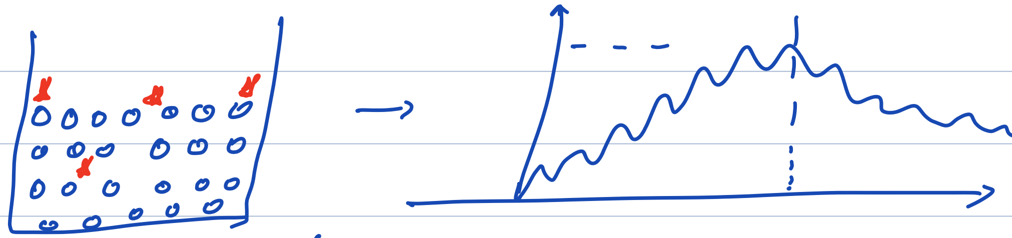
Announcements

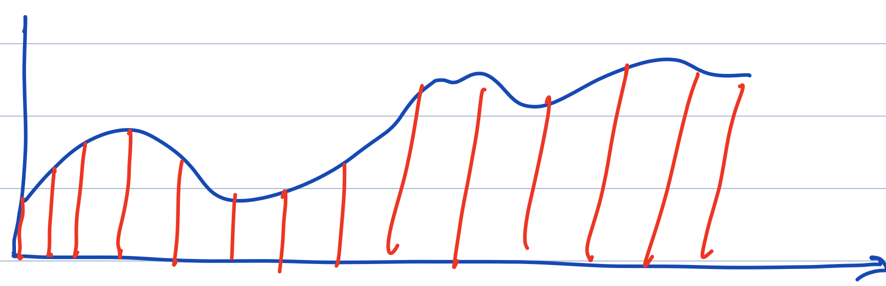
1. Report of Lab Experiment 2 due on Dec 26
   Ref : Ross



B_t $(B(t))$

- $B_0 = 0$
- Independent increment
- Gaussian increment

How would you simulate Gaussian Process?



$X = (X_0, \dots, X_n)$

↳ Gaussian random vector

mean function: 0

Covariance function: $C_X(t_1, t_2)$

$$\mathcal{N}(0, \Sigma)$$

$$\Sigma_{ij} = C_X(t_i, t_j)$$

Simulating sample paths of Gaussian Process

$\Rightarrow$ Take samples from $\mathcal{N}(0, \Sigma)$

randn(2)

$\swarrow$

independent r.vs

$\mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \underbrace{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}_{I}\right) \xrightarrow{H} \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \underbrace{\begin{bmatrix}1 & 0.4\\0.4 & 1\end{bmatrix}}_{\Sigma}\right)$

HH^T

$X \sim \mathcal{N}(0, \Sigma)$

$Y = HX$

$Y \sim \mathcal{N}(0, H\Sigma H^T)$

Find H such that

$HH^T = \Sigma$

"Cholesky decomposition"

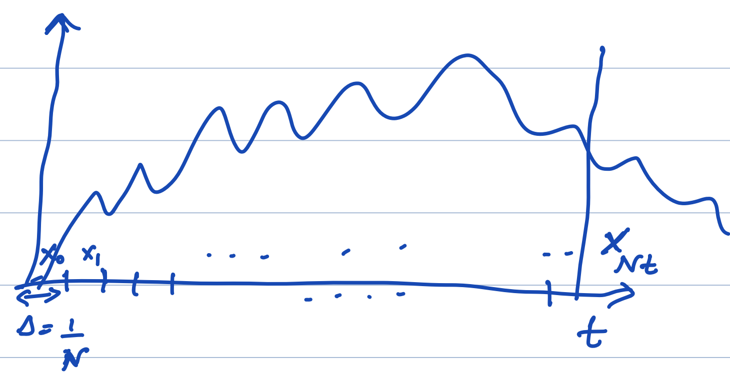
Simulating Brownian Motion?

$\hookrightarrow$ Gaussian process

$C_X(t, s) = \min(s, t)$

Property:

Brownian Motion as a limit of random walk.



X_i is iid

$P(X_i = +1) = P(X_i = -1) = \frac{1}{2}$

$$B_t^N = \underbrace{\varepsilon X_0 + \varepsilon X_1 + \cdots + \varepsilon X_{tN}}_{\text{small quantity}}$$

random walk

$$B_t^N \xrightarrow{N \to \infty} B_t \text{ (standard Brownian motion)}$$

(i) $B_0^N = 0$

(ii) Random walk → Independent increment

(iii) Gaussian property.

Choose $\varepsilon = \frac{1}{\sqrt{N}}$

$$B_t^N = \frac{\sqrt{t}}{\sqrt{Nt}}(\underbrace{X_0 + X_1 + \cdots + X_{tN}}_{tN \text{ r.v.s}}) \to N(0,1)$$

[Central Limit Theorem]

X_i are i.i.d $\xrightarrow{N \to \infty}$

$$\mathbb{E}[X_i] = +1 \cdot \frac{1}{2} + (-1)\frac{1}{2} = 0$$

$$\text{Var}[X_i] = 1 \cdot \left(\frac{1}{2}\right) + 1 \cdot \frac{1}{2} = 1$$

CLT : $\frac{1}{\sqrt{N}}(X_1 + X_2 + \cdots + X_n) \xrightarrow[\text{i.i.d}]{N \to \infty} N(0,1)$

$\mathbb{E}[X_i] = 0$

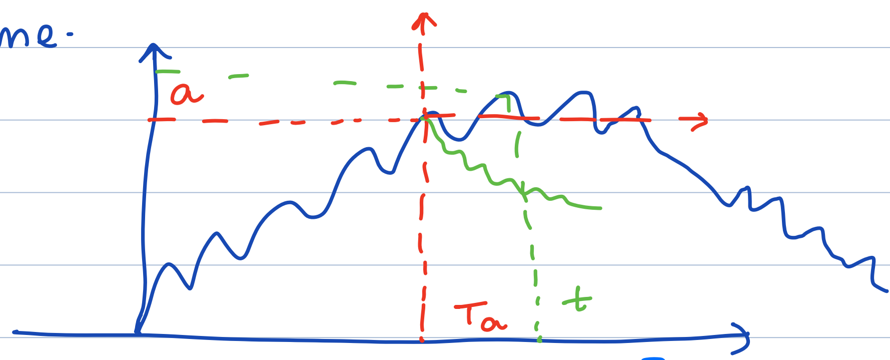
$\text{Var}[X_i] = 1$

$$\frac{1}{\sqrt{tN}}(X_0 + X_1 + \cdots + X_{tN}) \xrightarrow{N \to \infty} N(0,1)$$

$$\sqrt{t}\, N(0,1)$$

$$B_t^N \to \overset{\mathcal{N}(0,t)}{\mathcal{N}(0,t)} \qquad B_t \sim \mathcal{N}(0,t)$$

Hitting Time.



$$T_a = \min\{t : B_t \geq a\}$$

Distribution of T_a ?

Relate distribution of T_a to dist of B_t (Normal)

Claim : $P(B_t > a \mid T_a \leq t) = \frac{1}{2}$

Proof : Renewal Prop + Random walk.

$$P(B_t > a \mid T_a \leq t)\, P(T_a \leq t)$$
$$= \frac{1}{2} \cdot P(T_a \leq t)$$

$$P(B_t > a,\ T_a \leq t) = \frac{1}{2} P(T_a \leq t)$$

$$B_t > a \Rightarrow T_a \leq t$$

$$P(B_t > a) = \frac{1}{2} P(T_a \leq t)$$

$\downarrow$

Normal

$$B_t \sim N(0, t)$$

$$P(T_a \leq t) = 2 \, P(N(0,t) > a)$$

$$= 2 \, P\left(\frac{1}{\sqrt{t}} N(0,t) > \frac{a}{\sqrt{t}}\right)$$

$$= 2 \, P\left(\underline{\underline{N(0,1)}} > \frac{a}{\sqrt{t}}\right)$$

$$= 2 \int_{a/\sqrt{t}}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx$$

$$F_{T_a}(t) = \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{\infty} e^{-x^2/2} \, dx$$

$$f_{T_a}(t) = \frac{d}{dt} F_{T_a}(t)$$

skipped

$$= \frac{a \, e^{-a^2/2t}}{\sqrt{2\pi t^3}} \qquad t \geq 0$$

———— x ————

$$\frac{1}{\sqrt{n}} (X_1 + X_2 + \dots + X_n) \longrightarrow N(0,1)$$

$E[X_i] = 0$

$Var[X_i] = 1$

$$X_1 + X_2 + \dots + X_n \not\longrightarrow \sqrt{n} \, N(0,1)$$

$$N(0, n)$$