- Lab report due on Dec 29.

Recap:

WSS

$$X \longrightarrow \boxed{h(t)} \xrightarrow{\;Y\;} \text{WSS}$$

$$h[n]$$

$$m_Y = m_X \sum h[k]$$

$$C_Y(k,j) = \sum_n \sum_m h[n]\, h[m]\, C_X\big((k-j)-(n-m)\big)$$

$$\mathcal{H}(f) = \sum h[n]\, e^{-j2\pi f n}$$

$$m_Y = H(0)\, m_X$$

$$\tau = k-j$$

$$C_Y(\tau) = \sum_n \sum_m h[n]\, h[m]\, C_X(\tau-(n-m))$$

$$\mathcal{F}(C_Y(\tau)) = \sum_\tau C_Y(\tau)\, e^{-j2\pi f\tau}$$

$$= \sum_\tau \sum_n \sum_m h[n]\, h[m]\, C_X(\tau-(n-m))\, e^{-j2\pi f\tau}$$

$$= \sum_n \sum_m h[n]\, h[m]\, e^{-j2\pi f(n-m)} \sum_\tau C_X(\tau-(n-m))\, e^{-j2\pi f\tau}\, e^{j2\pi f(n-m)}$$

$$= \sum_n \sum_m h[n]\, h[m]\, e^{-j2\pi f(n-m)} \underbrace{\sum_\tau C_X(\tau-(n-m))\, e^{-j2\pi f(\tau-(n-m))}}_{\mathcal{F}(C_X)}$$

$$= \underbrace{\sum_n h[n]\, e^{-j2\pi f n}}_{\mathcal{H}(f)} \; \underbrace{\sum_m h[m]\, e^{j2\pi f m}}_{\mathcal{H}^*(f)} \; \mathcal{F}(C_x)$$

[LTI system has real coefficients]

$$= \mathcal{H}(f)\, \mathcal{H}^*(f)\; \mathcal{F}(C_x)$$

$$= |H(f)|^2 \; \mathcal{F}(C_x)$$

$$\mathcal{F}(C_y) = |H(f)|^2 \; \mathcal{F}(C_x) \quad - (*)$$

Express Eqn $(*)$ in terms of auto-correlation R_x & R_y

$$\mathcal{F}(R_y) = |H(f)|^2 \; \mathcal{F}(R_x)$$

[Exercise: show this!]

Power Spectral Density (PSD)

$S(f) = \mathcal{F}($Auto-correlation function$)$

– WSS process

$$S_x(f) = \mathcal{F}(R_x(\tau))$$

$$= \begin{cases} \int R_x(\tau)\, e^{-j2\pi f\tau}\, d\tau \\ \sum_k R_x(k)\, e^{-j2\pi k f} \end{cases}$$

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(f)\, e^{j2\pi f\tau}\, df$$

$$= \int_{-1/2}^{1/2} S_x(f)\, e^{j2\pi k f}\, df$$

0 to 2π : radians

($-\pi$ to π)

0 to 1 : f

($-\frac{1}{2}$ to $\frac{1}{2}$)

ω: notation ($\omega = 2\pi f$)

$$S_x(\omega) = \int R_x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_x(\tau) = \frac{1}{2\pi} \int S_x(\omega)\, e^{j\omega\tau}\, d\omega$$

$$\underset{\tau = 0}{R_x(\tau)} = \int_{-\infty}^{\infty} S_x(f)\, e^{j2\pi f\tau}\, df$$

$R_x(0) = E[x^2(t)]$ = Power of the process

$$R_x(0) = \int_{-\infty}^{\infty} S_x(f)\, df$$

Obtain power by integrating $S_x(f)$ over entire freq range.

[Power per unit freq / bandwidth]

Power of the process from f_1 to f_2

$$R_x(f_1, f_2) = \int_{f_1}^{f_2} S_x(f)\, df$$

Properties : $S_x(f) \geq 0$ [real, symmetric] positive

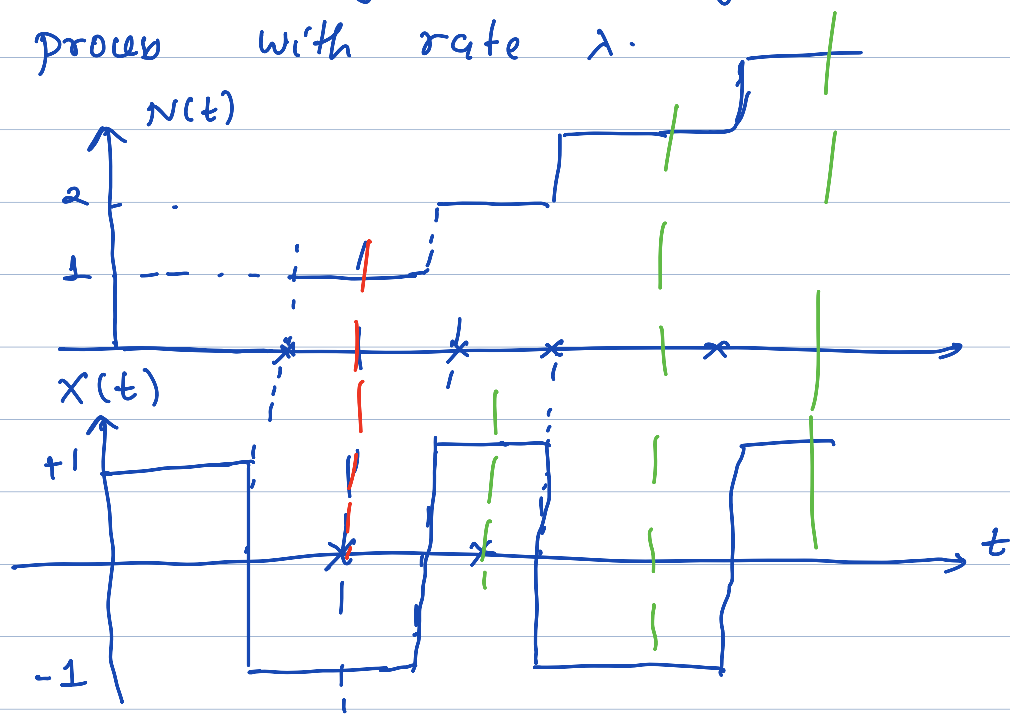
Proof: [$R_X(\tau)$ is real / even function]

Example: Random Telegraph Signal

Random process $X(t)$ takes ± 1

$$X(0) = \begin{cases} +1 & \text{w.p } \frac{1}{2} \\ -1 & \text{w.p } \frac{1}{2} \end{cases}$$

$X(t)$ changes according to a Poisson process with rate λ.



Mean, Covariance, PSD ?

Dist of $X(t)$

$$\underbrace{P[X(t) = 1 \mid X(0) = 1]}_{= P[X(t) = -1 \mid X(0) = -1]} = P[\underset{\substack{\uparrow \\ \#\text{ of events}}}{N_t} = 0, 2, 4 \ldots \text{(even)}]$$

$$P[\,N_t = 0, 2, 4, \dots\,]$$

$\downarrow$

Poisson (λt)

$$= \sum_{i=0,2,4\dots} (\lambda t)^i \frac{e^{-\lambda t}}{i!}$$

$$= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \dots\right]$$

$$e^{+\lambda t} = 1 + (\lambda t) + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \dots$$

$+$

$$e^{-\lambda t} = 1 + (-\lambda t) + \frac{(-\lambda t)^2}{2!} + \frac{(-\lambda t)^3}{3!} + \dots$$

$$e^{\lambda t} + e^{-\lambda t} = 2\left[1 + \frac{(\lambda t)^2}{2!} + \dots\right]$$

$$= e^{-\lambda t}\left[\frac{e^{\lambda t} + e^{-\lambda t}}{2}\right]$$

$$= \frac{1}{2}\left[1 + e^{-2\lambda t}\right] \quad //$$

$P[\,X(t)=1 \mid X(0)=+1]$

$P[\,X(t)=-1 \mid X(0)=-1]$

$$P[\,X(t)=-1 \mid X(0)=+1]$$

$P[\,X(t)=1 \mid X(0)=-1]$

$$= P[\,N_t = \text{odd}]$$

$\downarrow$

Poisson (λt)

$$= \sum_{i=1,3,5,\dots} e^{-\lambda t} \frac{(\lambda t)^i}{i!}$$

$$= e^{-\lambda t}\, \frac{1}{2}\left[e^{\lambda t} - e^{-\lambda t}\right]$$

$$= \frac{1}{2}\left[1 - e^{-2\lambda t}\right]$$

$$P[X(t) = i \mid X(0) = j] \quad ; \quad \begin{array}{l} i = \pm 1 \\ j = \pm 1 \end{array}$$

$$P[X(t) = 1]$$

$$\overset{\text{Law of Total Prob}}{=} \sum_{i \in \{-1, +1\}} P[X(t) = 1, X(0) = i]$$

$$= \sum_{i \in \{-1, +1\}} P[X(0) = i]\, P[X(t) = 1 \mid X(0) = i]$$

$$= P(X(0) = 1)\, P[X(t) = 1 \mid X(0) = 1] + P[X(0) = -1]\, P[X(t) = 1 \mid X(0) = -1]$$

$$= \frac{1}{2} \cdot \frac{1}{2}\left(1 + e^{-2\lambda t}\right) + \frac{1}{2} \cdot \frac{1}{2}\left(1 - e^{-2\lambda t}\right) = \frac{1}{2} //$$

$$P(X(t) = -1) = 1 - P(X(t) = 1) = \frac{1}{2}$$

$$m_X(t) = 1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0 //$$

$$C_X(t_1, t_2) = R_X(t_1, t_2)$$

$$= \mathbb{E}[X(t_1)\, X(t_2)]$$

$$X(t_1)\, X(t_2) = \begin{cases} +1 & \text{w.p} \quad \frac{1}{2}\left(1 + e^{-2\lambda |t_1 - t_2|}\right) \\ -1 & \text{w.p} \quad \frac{1}{2}\left(1 - e^{-2\lambda |t_1 - t_2|}\right) \end{cases}$$

$t_1 \geq t_2$

$X(t_2) = +1 \quad , \quad X(t_1) = +1$ $\qquad \frac{1}{2}\left(1 + e^{-2\lambda(t_1 - t_2)}\right)$

$X(t_2) = -1 \quad , \quad X(t_1) = -1$

$$P[\underbrace{X(t_1) = +1 \mid X(t_2) = +1}]$$

$$= \underbrace{N(t_1) - N(t_2)}_{\text{Poisson}(\lambda(t_1 - t_2))} = (\text{even})$$

$$P[X(t_2) = +1, \ X(t_1) = +1]$$

$$= P[X(t_2) = +1] \ P[X(t_1) = +1 \mid X(t_2) = +1]$$

$$= \frac{1}{2} \cdot \frac{1}{2}\left(1 + e^{-2\lambda(t_1 - t_2)}\right)$$

$$P[X(t_2) = -1, \ X(t_2) = -1]$$

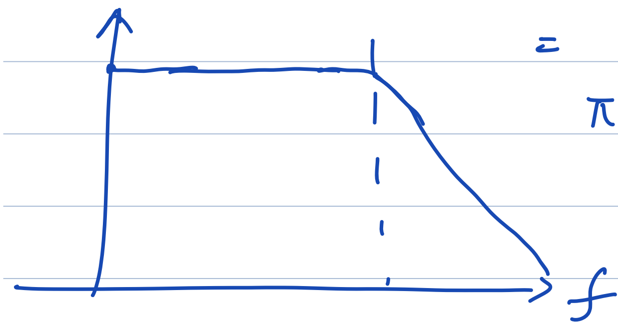
$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \left(1 + e^{-2\lambda(t_1 - t_2)}\right)$$

$$C_X(t_1, t_2) = R_X(t_1, t_2) = +1 \cdot \frac{1}{2}\left(1 + e^{-2\lambda|t_1 - t_2|}\right)$$
$$-1 \quad \frac{1}{2} \cdot \left(1 - e^{-2\lambda|t_1 - t_2|}\right)$$

$$= e^{-2\lambda|t_1 - t_2|}$$

PSD $\qquad S_X(f) = \int R_X(\tau) \, e^{-j2\pi f\tau} \, d\tau$

$$= \int e^{-2\lambda|\tau|} \, e^{-j2\pi f\tau} \, d\tau$$



$$= \frac{\lambda}{\pi^2 f^2 + \lambda^2} \quad (\text{Exercise}) \; //$$

LPF

White noise (Discrete case)

Theorem: If $\{X_n\}$ is a WSS process, uncorrelated with mean m and variance σ^2.

$$C_X[k] = \sigma^2 \delta[k]$$

$$R_X[k] = \sigma^2 \delta[k] + m^2$$

Delta function.

Proof: $\mathbb{E}[X_n] = m$

$$C_X[k] = \mathbb{E}[(X[n]-m)(X[n-k]-m)]$$

$k \neq 0$ $\quad \overset{\text{uncorrelated}}{=} \mathbb{E}[\cancel{(X[n]-m)}]^{0} \, \mathbb{E}[\cancel{(X[n-k]-m)}]^{0}$

$$= 0$$

$k = 0$ $\quad \mathbb{E}[(X(n)-m)(X(n)-m)]$

$$= \sigma^2$$

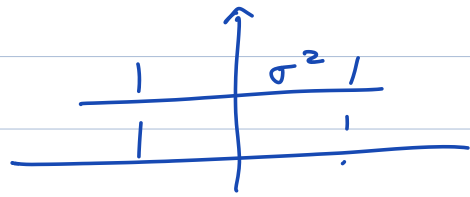
$$C_X[k] = \sigma^2 \delta[k]$$

$$R_X[k] = \sigma^2 \delta[k] + m^2$$

Zero-mean, uncorrelated, WSS

$$R_X[k] = \sigma^2 \delta[k]$$

$$S_X(f) = \sigma^2 \quad \forall k$$



$\Rightarrow$ white noise process

Theorem: A discrete time white noise process (flat p.s.d) iff it is zero mean, WSS and uncorrelated.

White noise process $\xrightarrow{X} \boxed{h[n]} \xrightarrow{Y}$

$$S_Y(f) = |H(f)|^2 \underbrace{S_X(f)}_{\sigma^2}$$

$$= \sigma^2 |H(f)|^2$$

$$R_Y(k) = IFT(S_Y(f))$$

$$= \sigma^2 \sum_{n=0}^{\infty} h[n]\, h[n-k]$$

(Causal filter)

$$= \sigma^2 \sum_{n=k}^{\infty} h[n]\, h[n-k]$$