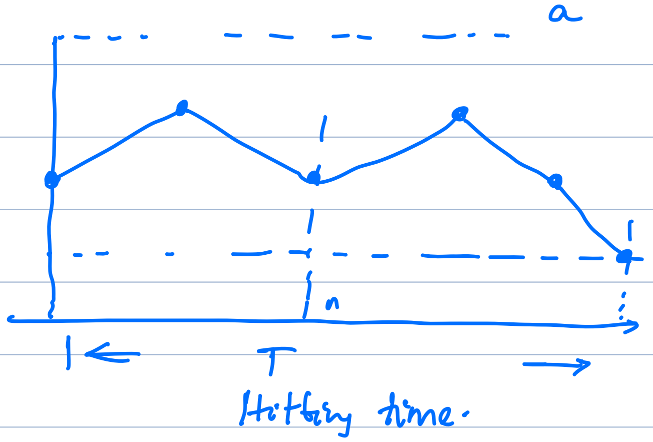
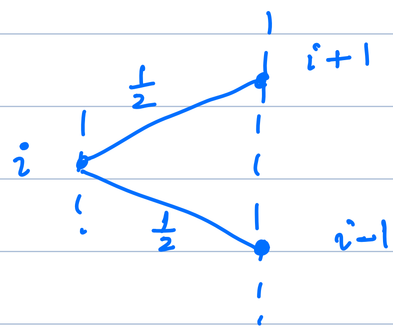
Last class : Random Walks



1. $f(i) = P[T = \infty \mid S_0 = i]$
   $= 0$
2. $r(i) = P[S_T = b \mid S_0 = i]$
   $= \dfrac{i-a}{b-a}$
3. $E[T \mid S_0 = i]$
   $\mu(i) =$

Expected hitting time

$\mu(i) = E[T \mid S_0 = i]$



$\mu(i)$
$= \frac{1}{2}[\underline{1} + \mu(i+1)] +$
$\frac{1}{2}[\underline{1} + \mu(i-1)]$

$\mu(i) = 1 + \frac{1}{2}\mu(i-1) + \frac{1}{2}\mu(i+1)$ — ①

$\mu(a) = 0, \quad \mu(b) = 0$ — ②

$2\mu(i) = 2 + \mu(i-1) + \mu(i+1)$

$\mu(i+1) - \mu(i) = \mu(i) - \mu(i-1) - 2$ — (1a)

$\cancel{\mu(a)} = 0$

$\cancel{\mu(a+1)} - \cancel{\mu(a)} = z$

$i = a+1 \quad \underbrace{\cancel{\mu(a+2)} - \cancel{\mu(a+1)}} = \underbrace{\mu(a+1) - \mu(a)}_{z} - 2$

$i = a+2 \quad \cancel{\mu(a+3)} - \cancel{\mu(a+2)} = \mu(a+2) - \mu(a+1) - 2$

$$= z - 2 - 2$$

$$= z - 4$$

$\vdots$

$$\mu(a+i) - \cancel{\mu(a+i-1)} = z - 2(i-1)$$

$$\mu(a+i) = i \cdot z - 2[0 + 1 + \cdots + i-1]$$

$$= i \cdot z - \cancel{2} \cdot \frac{(i-1) \cdot i}{\cancel{2}}$$

$$= i[z - i + 1]$$

$$\mu(b) = 0$$

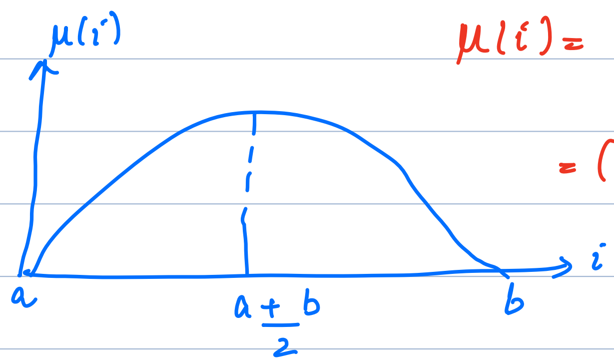
$$\mu(b) = \mu(\underline{a} + \underbrace{b-a}_{i})$$

$$= (b-a)[z - (b-a) + 1] = 0$$

$$z = (b-a-1)$$

$$\mu(a+i) = i \cdot (b - a - 1 - i + 1) = i(b-a-i)$$

$$\mu(i) = (i-a)(b-i)$$



$$\mu(i) = \mu(\underline{a} + \underbrace{i-a}_{i})$$

$$= (i-a)(b-a-(i-a))$$

$$= (i-a)(b-i)$$

Markov Chain

Markov Process : A RP that satisfies Markov Property

Markov Chain : Markov Process + Discrete time + Process takes values in a finite set.

Defn:

A discrete time Random process $\{X_n\}$ is a Markov chain if X_n takes values in a finite set D and satisfies the following Markov property

$$P[\underbrace{X_{n+1} = x_{n+1}}_{\text{Future}} \mid \underbrace{X_n = x_n}_{\text{Present}}, \underbrace{\ldots, X_0 = x_0}_{\text{Past}}]$$

$$= \underbrace{P[\underbrace{X_{n+1} = x_{n+1}}_{\text{Future}} \mid \underbrace{X_n = x_n}_{\text{Present}}]}$$

$$P[X_{n+1} = j \mid X_n = i] = P_{ij}^{(n)}$$

$$P[X_{n+1} = j \mid X_n = i] = P_{ij}$$

" Time homogeneous Markov Chain"

$i, j \in D$

Example: Suppose a person (in any day) can either be in a good mood or bad mood.

Suppose he is in a bad mood today, he recovers tmrw with probability p.

Suppose he is in a good mood today, his mood could go bad tmrw with probability q.

X_n: Mood of person at each day

$D = \{G, B\}$

$P[X_{n+1} = G \mid X_n = B] = p$.

$P[X_{n+1} = B \mid X_n = B] = 1 - p$

$P[X_{n+1} = B \mid X_n = G] = q$

$P[X_{n+1} = G \mid X_n = G] = 1 - q$

Next state of RP

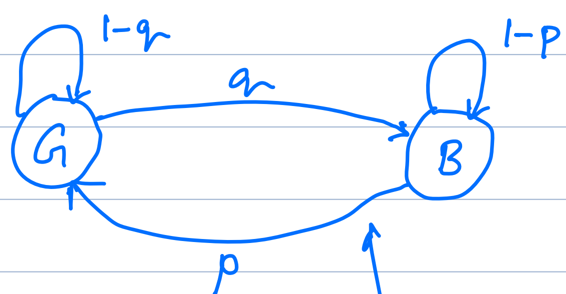
$$\mathbb{P} = \begin{array}{c|cc} & G & B \\ \hline G & 1-q & q \\ B & p & 1-p \end{array}$$

" Transition prob. matrix "

Current state of RP

$|D| \times |D|$

" State Transition Diagram "



Each "circle"

Arrows represent

represents the "state" the chain can be in

transition among state
Besides the arrow we write the prob of transition

Properties of P (transition prob. matrix)

1. $0 \leq P_{ij} \leq 1$

2. Row Sum(P) = 1

$$\sum_j P_{ij} = 1 \quad , \forall i$$

Example: A Markov chain has the following prob. transition matrix

$$P = \begin{bmatrix} .3 & .2 & .5 \\ .5 & .1 & .4 \\ .5 & .2 & .3 \end{bmatrix}$$

$D = \{0, 1, 2\}$

Markov chain "starts" in one of the three states uniformly at random.

$$P[X_0 = 0] = P[X_0 = 1] = P[X_0 = 2] = \frac{1}{3}$$

Find $Pr[X_0 = 1, X_1 = 1, X_2 = 0]$