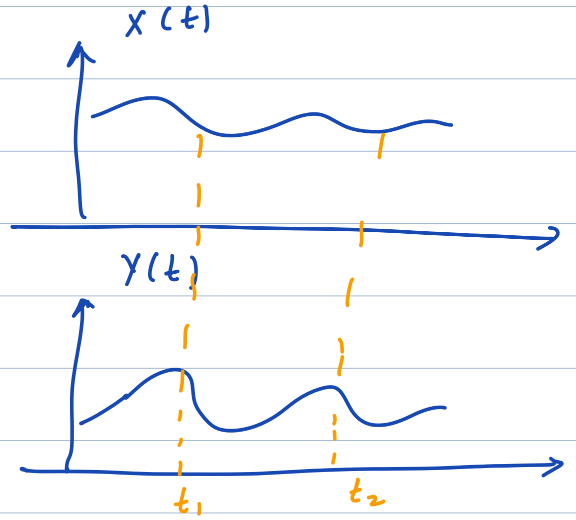
Announcements : (1) No lecture on Jan 3 (Friday Timetable)

(2) Vivas are scheduled on Jan 3.

Recap:



Cross- correlation

$$R_{XY}(t_1, t_2) = \mathbb{E}[X(t_1) \, Y(t_2)]$$

$$R_{YX}(t_1, t_2) = \mathbb{E}[Y(t_1) \, X(t_2)]$$

$$R_{XY}(t_1, t_2) \neq R_{YX}(t_1, t_2)$$

Properties (i) $R_{XY}(t, t) = R_{YX}(t, t)$

(ii) If the Random process $X(t)$ & $Y(t)$ are uncorrelated.

$$R_{XY}(t_1, t_2) = \mathbb{E}[X(t_1) \, Y(t_2)]$$

$$= \mathbb{E}[X(t_1)] \, \mathbb{E}[Y(t_2)]$$

$$= m_X(t_1) \, m_Y(t_2)$$

(iii) (Definition) Random Process X and Y are said to be orthogonal if

$$R_{xy}(t_1, t_2) = 0 \quad \forall\, t_1, t_2$$

X and Y are said to be "jointly" WSS
(i) X and Y are each WSS
(ii) $R_{xy}(t_1, t_2) = R_{xy}(t_1 - t_2)$
$[\,R_{xy}(\tau)\,]$

Example: Two random proces X and Y are defined as

$$X(t) = 2\cos(5t + \theta)$$
$$Y(t) = 10\sin(5t + \theta)$$

where $\theta \sim \text{Unif}(0, 2\pi)$

Check if X and Y are jointly WSS.

Exercise: Show that X and Y are individually WSS

$$R_{xy}(t+\tau, t)$$
$$= E[\,2\cos(5(t+\tau) + \theta)\;10\sin(5t + \theta)\,]$$

$$\boxed{\begin{aligned} &2\sin A \cos B \\ &= \sin(A+B) \\ &\quad + \sin(A-B) \end{aligned}}$$

$$= 10\, E[\,\underline{\sin(5\tau)} + \sin(10t + 5\tau + 2\theta)]$$

$$= 10\sin(5\tau) + 10\, E[\,\sin(10t + 5\tau + 2\theta)]$$

$$\int_0^{2\pi} \sin(2\theta + 10t + 5\tau)\, d\theta = 0$$

$$= 10\sin(5\tau)$$

$\uparrow$ Only on the time difference

X and Y are jointly WSS.

Properties (Jointly WSS)

* (1) $R_{XY}(\tau) = R_{YX}(-\tau)$ [Exercise]

(2) $R_{XY}(\tau) \leq [R_X(0)\ R_Y(0)]^{\frac{1}{2}}$

[Exercise: Use Cauchy-Schwartz inequality]

Cross-spectral density.

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau$$

FT (Cross-correlation)

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{j\omega\tau}\, d\omega$$

Example: The cross-correlation function of a jointly WSS process is given by.

$$R_{XY}(\tau) = 2e^{-2\tau} \quad \tau \geq 0$$
$$= 0 \quad \text{else.}$$

a) Find $S_{XY}(\omega)$

b) Find $S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-j\omega\tau}\, d\tau$

a) $S_{XY}(\omega) = FT[R_{XY}(\tau)]$

$$= \int_0^\infty 2\, e^{-2\tau}\, e^{-j\omega\tau}\, d\tau$$

$$= 2 \int_0^\infty e^{-(2+j\omega)\tau}\, d\tau$$

$$= 2/(j\omega + 2)$$

b) $$S_{YX}(\omega) = FT[\,R_{YX}(\tau)\,]$$

$$= FT[\,R_{XY}(-\tau)\,]$$

Exercise.

$$= 2/(j\omega + 2)$$

Property

$$S_{YX}(\omega) = FT[\,R_{YX}(\tau)\,]$$

$$= FT[\,R_{XY}(-\tau)\,]$$

$$= \int_{-\infty}^{\infty} R_{XY}(-\tau)\, e^{-j\omega\tau}\, d\tau$$

Change of variable $-\tau \to \tau$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{j\omega\tau}\, d\tau$$

$$= S^*_{XY}(\omega) \quad //$$

$$X \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h(\cdot)\end{matrix}} \xrightarrow{\;Y\;}$$

Theorem: X and Y are jointly WSS, if X is WSS

$$R_{YX} = h \circledast R_X$$

$$S_{YX}(\omega) = H(\omega) \cdot S_X(\omega)$$

Proof: $R_{YX}(\tau) = \mathbb{E}[\,Y(t)\;X(t-\tau)\,]$

$$Y(t) = \int h(\alpha)\; X(t-\alpha)\; d\alpha$$

$$= \mathbb{E}\left[\int h(\alpha)\; X(t-\alpha)\,d\alpha \quad X(t-\tau)\right]$$

$$= \int h(\alpha)\; \underbrace{\mathbb{E}[\,X(t-\alpha)\,X(t-\tau)\,]}_{R_X(\tau-\alpha)}\; d\alpha$$

$$= \int h(\alpha)\quad R_X(\tau-\alpha)\; d\alpha$$

$$R_{YX} = h \circledast R_X$$

$$S_{YX}(\omega) = H(\omega)\; S_X(\omega)$$

Exercise: $S_{XY}(\omega) = H^*(\omega)\; S_X(\omega)$

Exercise!: $S_Y(\omega) = H^*(\omega)\, S_{YX}(\omega)$

$$= H^*(\omega)\; H(\omega)\; S_X(\omega)$$

$$= |H(\omega)|^2\; S_X(\omega)$$