Gaussian Process & Brownian Motion

Review:

Gaussian random vector / Jointly Gaussian

$X = (X_1, X_2, \ldots, X_n)$

$$f_X(x) = f_{X_1 \cdots X_n}(x_1, x_2, \ldots x_n)$$

$$= \frac{1}{\sqrt{(2\pi)^n \det(\Sigma)}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}$$

$X \sim \mathcal{N}(\mu, \Sigma)$

Σ: positive semi definite.

$\mu_i = \mathbb{E}[X_i]$

$\Sigma_{ij} = \text{Cov}(X_i, X_j)$

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} \xrightarrow{\quad X_i's \quad}$$

Each component is a Gaussian r.v

$$f_{X_i}(x_i) \sim \mathcal{N}(\mu_i, \Sigma_{ii})$$

$\uparrow$

i^{th} diagonal entry of Σ

$y_1, y_2, \ldots, y_n$ are individually Gaussian

$Y_i \sim N(\mu_i, \sigma_i^2)$

$Y = (y_1, y_2, \ldots, y_n)$

Is Y a Gaussian random vector?

In General No.

$X: N(0, 1)$

$Z = \begin{cases} 1 & w.p \ \frac{1}{2} \\ -1 & w.p \ \frac{1}{2} \end{cases}$

$Y = ZX$ is $N(0,1)$

X and Y are not jointly Gaussian

$X_1, X_2, \ldots X_n$ are individually Gaussian
and they are mutually independent

$X = (X_1, X_2, \ldots, X_n)$

X is a Gaussian r.v / $X_1, X_2, \ldots X_n$ are
jointly Gaussian.

$X_i \sim N(\mu_i, \sigma_i^2)$

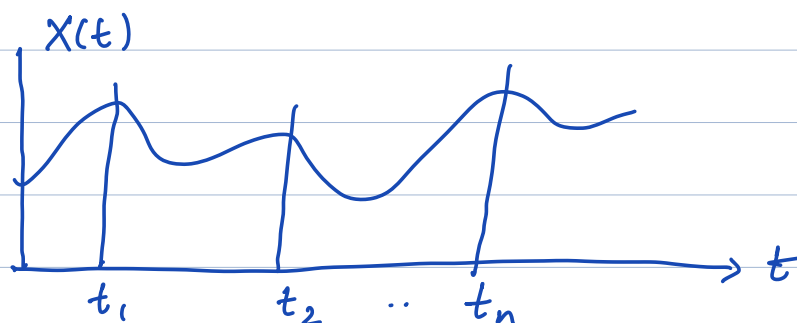$X \sim N(\mu, \text{diag}(\sigma_1^2, \sigma_2^2, \ldots \sigma_n^2))$

Most important property: (Linear transformation)

$X \sim N(\mu, \Sigma)$

$Y = AX + b$

$Y \sim N(A\mu + b, A\Sigma A^T)$

Gaussian Process

$$X = (X(t_1), X(t_2) \cdots, X(t_n))$$
$$X = \text{Gaussian random vector.}$$

Defn: $\{X_t\}$ is a Gaussian process if
for any time instants $t_1, t_2, \ldots, t_n$
(any n)
$$X = (X(t_1), X(t_2) \cdots X(t_n))$$
$$\text{and} \quad X \sim N(\mu, \Sigma)$$

Example: Discrete-time i.i.d Gaussian process
$\{W_n\}$ $W_n \overset{iid}{\sim} N(0, \sigma^2)$

$$W = (W_{n_1}, W_{n_2}, \cdots W_{n_k}) \quad \rightarrow \text{Yes! Gaussian}$$
$$W \sim N(0, \sigma^2 I)$$

Example: Discrete-time Gaussian sum process
$W_n \overset{iid}{\sim} N(0, \sigma^2)$
$$S_n = W_1 + W_2 + \cdots + W_n$$
Is S_n a Gaussian process?
$\rightarrow$ At each n, S_n is a Gaussian random

variable

Finite dimensional dist (S_n)

$$S = (S_{n_1}, S_{n_2} \dots S_{n_k})$$

$\downarrow$

Gaussian random vector ?

Example
k=2 $\boxed{S} = (S_1, S_4)$?

$S_1 = W_1$

$S_4 = W_1 + W_2 + W_3 + W_4$

$$S = \begin{pmatrix} S_1 \\ S_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \boxed{\begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix}}$$

Gaussian   Gaussian
random vector   r. vector

$$S = (S_{n_1}, S_{n_2}, \dots, S_{n_k}) \quad : S \text{ is a Gaussian}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad r. \text{ vector}$$

$\mathbb{E}(S) = 0$

$Cov(S_{n_i}, S_{n_j})$

Example: $Cov(S_1, S_4)$

$\quad = \mathbb{E}[S_1 S_4]$

$\quad = \mathbb{E}[W_1 (W_1 + W_2 + W_3 + W_4)]$
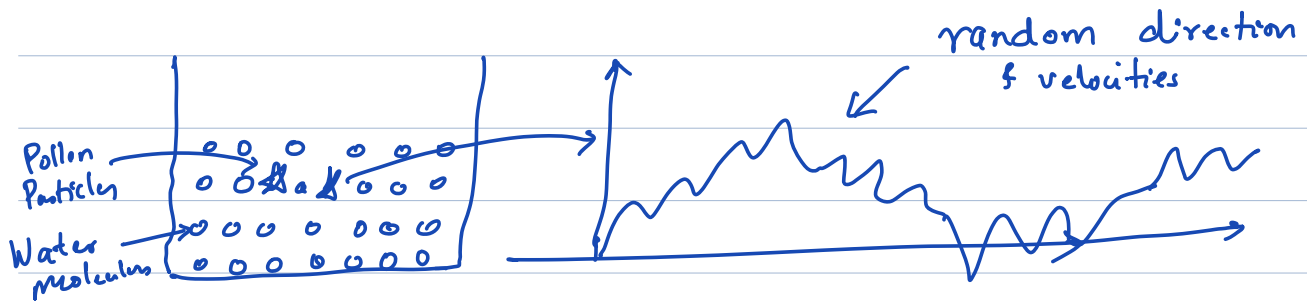
$\quad = \mathbb{E}[W_1^2 + W_1 W_2 + W_1 W_3 + W_1 W_4]$

$\quad = \mathbb{E}[W_1^2] + \mathbb{E}[W_1 W_2] + \mathbb{E}[W_1 W_3]$
$\qquad\qquad \downarrow \qquad\quad + \mathbb{E}[W_1 W_4]$
$\quad = \sigma^2 \qquad + \mathbb{E}[W_1]^{\;0} \mathbb{E}[W_2]^{\;0} +$

$$\mathbb{E}[\vec{w_i}] \ \mathbb{E}[\vec{w_j}]^0 + \ \mathbb{E}[\vec{w_i}] \ \mathbb{E}[\vec{w_4}]$$

$$= \sigma^2$$

$$Cov \ (S_{n_i}, S_{n_j}) \ = \ Min(n_i, n_j) \ \sigma^2 \quad [\ Exercise \ !]$$

Brownian Motion (Weiner Process)

random direction
& velocities

Standard
Defn: Brownian Motion

A continous time random process $\{X_t\}$ is called a standard Brownian motion if

(i) $X_0 = 0$

(ii) Independent increment
$$X_t - X_s \perp\!\!\!\perp X_r \qquad \begin{array}{l} r \le s \\ t \ge s \end{array}$$

(iii) Gaussian Stationary Increment
$$X_t - X_s \ \sim \ N(0, t-s) \quad t \ge s$$