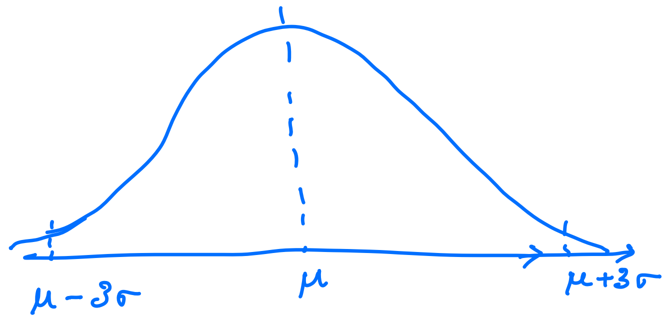
Last class — Discrete r.v
— Continous r.v

Gaussion /Normal random variable

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}; \ -\infty < x < +\infty$$

$$X \sim N(\mu, \sigma^2)$$



Property:

If $X \sim N(\mu, \sigma^2)$

$$Y = aX + b$$

$$Y \sim N(a\mu + b, \ a^2\sigma^2)$$

Proof:

$$F_Y(\alpha) = P(Y \leq \alpha)$$

$$= P(aX + b \leq \alpha)$$

$$= P\left(X \leq \frac{\alpha - b}{a}\right)$$

$$= F_X\left(\frac{\alpha-b}{a}\right)$$

$$= \int_{-\infty}^{\frac{\alpha-b}{a}} f_X(x)\, dx$$

$$= \int_{-\infty}^{\frac{\alpha-b}{a}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}\, dx$$

$$v = ax + b$$

$$= \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{v-b}{a}-\mu\right)^2/2\sigma^2} \frac{dv}{a}$$

$$= \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi(a\sigma)^2}} e^{-(v-b-a\mu)^2/2(a\sigma)^2} dv$$

$$= \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi(a\sigma)^2}} e^{-(v-(a\mu+b))/2(a\sigma)^2} dv$$

$$Y \sim N(a\mu + b, \ a^2\sigma^2)$$

Expectation of random variable.

Discrete random variable

$$\mathbb{E}(X) = \sum x \ p(x)$$

"Weighting the possible values of the r.v by the probability of taking them"

Example: Expectation of Bernoulli r.v.

$$P_X(1) = p \qquad P_X(0) = 1-p.$$

$$\mathbb{E}(X) = 1 \cdot p + 0 \cdot (1-p)$$

$$= p$$

Exercise: (i) $X \sim$ Binomial (n, p)

$$\mathbb{E}[X] = np.$$

Chap 5 of PP

(ii) $X \sim$ Geometric (p)

$$\mathbb{E}(X) = \frac{1}{p}$$

(iii) $X \sim$ Poisson (λ)

$$\mathbb{E}(X) = \frac{1}{\lambda}$$

Example: Contenstant in quiz

2 questions V_1 & V_2

$V_1 \to 200\$$ $\quad V_2 \to 100\$$

Contestant is 60% certain of the answer of V_1

80% " " " " of V_2

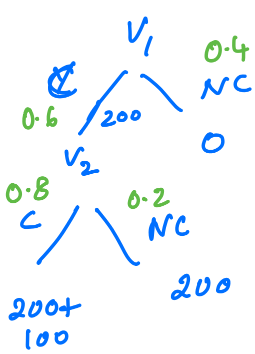
Attempt the qns in any order

Allowed to move to the 2^{nd} qns only if the first qn is correct.

What order should he attempt to maximize the expected reward

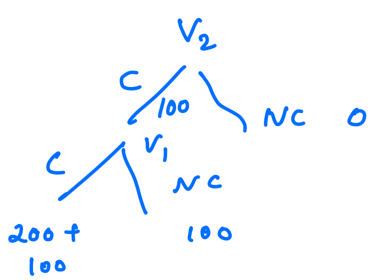
$(V_1 \; V_2)$

$$P = \begin{cases} 0 & 0.4 \\ 200 & 0.6 \times 0.2 \\ 200+100 & 0.6 \times 0.8 \end{cases}$$



$$E((V_1, V_2)) = 0 \times 0.4 + 200 \times 0.12 + 300 \times 0.48 = 168$$

$$(V_2, V_1) = \begin{cases} 0 & 0.2 \\ 100 & 0.32 \\ 300 & 0.48 \end{cases}$$



$$E((V_2, V_1)) = 0 \times 0.2 + 100 \times 0.32 + 300 \times 0.48$$

$= 156$ // (176)

Expectation of continous r.v

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

Example: Expectation of an exponential r.v

$X \sim Exp(\lambda)$

$$f_X(x) = \lambda e^{-\lambda x} \quad x > 0$$

$$\mathbb{E}(X) = \int_0^{\infty} x \, \lambda e^{-\lambda x} \, dx$$

$$= \lambda \int_0^{\infty} x \, e^{-\lambda x} \, dx$$

$$= \lambda \left[\cancel{x \frac{e^{-\lambda x}}{-\lambda}}^{\,0} \Big|_0^{\infty} - \int_0^{\infty} \frac{e^{-\lambda x}}{-\lambda} \, dx \right]$$

$$= \int_0^{\infty} e^{-\lambda x} \, dx$$

$$= \frac{e^{-\lambda x}}{-\lambda} \Big|_0^{\infty} = \frac{1}{\lambda}$$

Exercise:

1. $X \sim Unif([\alpha, \beta])$

   $\mathbb{E}(x)$

2. $X \sim \mathcal{N}(\mu, \sigma^2)$

   $\mathbb{E}(x) = \mu$.

   "Mean"

Fundamental theorem of expectation

$$\mathbb{E}[g(x)] = \sum g(x) \, P_X(x) \, dx.$$

$$\int g(x) \, f_X(x) \, dx$$

Example: Let X be Unif $[0,1]$ Calculate $\mathbb{E}[X^3]$

$$g(x) = x^3$$

$$\mathbb{E}[X^3] = \int_0^1 x^3 \cdot 1 \cdot dx$$

$$= \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$$

Example: If you are s minutes early to an appointment, then you incur a cost $c \cdot s$

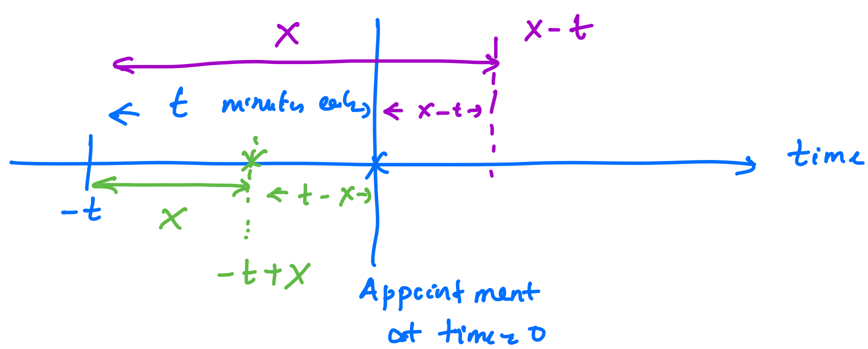
If you are s minutes late to an appointment, then you incur a cost of $k \cdot s$

Travel time to this appointment is a random variable with p.d.f f

$$f(t) = 0 \quad t < 0$$

$$f(t) = 0 \quad t > T$$

Determine the time at which you should depart to minimize expected cost?



$$\text{Cost} = \begin{cases} c\,(t-x) & x \le t \\ k\,(x-t) & x > t \end{cases}$$

$$\mathbb{E}[\text{Cost}] = \int_0^t c\,\underbrace{(t-x)}_{I_1} f(x)\,dx + \int_t^T k\,\underbrace{(x-t)}_{I_2} f(x)\,dx$$

Fundamental Thm of Expectation.

$$I_1 = \int_0^t c\,(t-x)\,f(x)\,dx$$

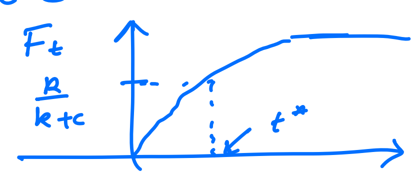
$$= c\left[t \int_0^t f(x)\,dx - \int_0^t x\,f(x)\,dx \right]$$

$$= c\left[t\,\cancel{F(t)} - x\,\cancel{F(x)}\Big|_0^t + \int_0^t F(x)\,dx \right]$$

$$= c \int_0^t F(x)\,dx$$

$$I_2 = \int_t^T k\,(x-t)\,f(x)\,dx$$

[Exercise]



$$\frac{d}{dt}\mathbb{E}[\text{Cost}] = 0$$

$$= c\,F(t) - k + k\,F(t)$$

At optimal $t = t^*$

$$c\,F(t^*) - k + k\,F(t^*) = 0$$

$$F(t^*) = \frac{k}{k+c}$$

$$= k\left[\int_t^T x\, f(x)\, dx - t\int_t^T f(x)\, dx\right]$$

$$= k\left[\, x\, F(x)\Big|_t^T - \int_t^T F(x)\, dx - t\,[F(T) - F(t)]\right]$$

$$= k\left[\, T\,F(T) - t\,\cancel{F(t)} - \int_t^T F(x)\,dx - t\,F(T) + \cancel{t\,F(t)}\right]$$

$$= k\left[T - t - \int_t^T F(x)\, dx\right]$$

$$\frac{d}{dt} I_1 = c\, F(t)$$

$$\frac{d}{dt} I_2 = k\,[-1 + F(t)]$$