Announcements : 1. Lab this Friday [Optional]

: 2. Office Hours : Suggested Fri 5.15 - 6.15

: 3: Quiz 2 : Nov 28 @ 2pm [15 Marks]

Syllabus : RP, Random Walk, Markov Chains (upto Nov 25)

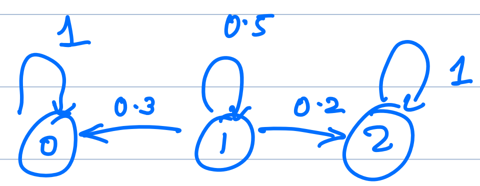
Review: Markov Chains

P - one - step transition prob. matrix

P^n n - step " " "

Example

$$P = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ .3 & .5 & .2 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$



" Absorbed in state 0 or State 2 "

- What is the prob of absorbtion happening in state 0 instead of state 2?
- Mean time for absorbtion?

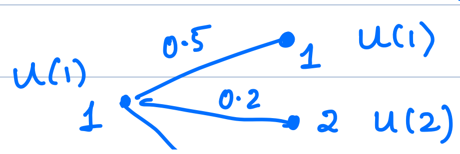
First step analysis

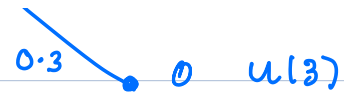
$$T = \min\{ n \geq 0 , X_n = 0 \text{ or } X_n = 2 \}$$

$$u(i) = P[X_T = 0 \mid X_0 = i]$$

$$u(0) = 1 \qquad u(2) = 0$$

$$u(1)$$





$$u(1) = 0.5\, u(1) + 0.2\, \overset{0}{u(2)} + 0.3\, \overset{1}{u(0)}$$

$$u(1) = 0.5\, u(1) + 0.3$$

$$0.5\, u(1) = 0.3$$

$$u(1) = 3/5$$

Proof: $u(i) = P(X_T = 0 \mid X_0 = i)$

$$= \sum_{s \in (0,1,2)} P(X_T = 0, X_1 = s \mid X_0 = i) \quad \text{" Total Law of Prob"}$$

$$= \sum_{s \in (0,1,2)} P(X_1 = s \mid X_0 = i) \underbrace{P(X_T = 0 \mid X_1 = s, X_0 = i)}_{\text{Markov}}$$

$$\underbrace{P(X_T = 0 \mid X_1 = s)}$$

$$\boxed{= \sum_{s \in \{0,1,2\}} P_{is}\, u(s)}$$

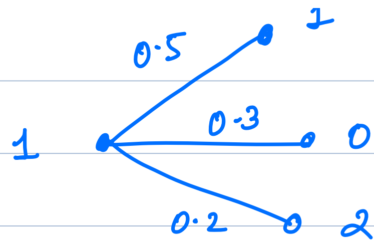
Mean time for absorption.

$$\vartheta(i) = \mathbb{E}[T \mid X_0 = i]$$

$$\vartheta(0) = \quad , \quad \vartheta(2) =$$

$$T = \min\{n \geq 0,\ X_n = 0 \text{ or } X_n = 2\}$$

$$T = 0 \text{ when } X_0 = 0,\ X_0 = 2$$



$$v(1) = 1 + 0.5\, v(1) + 0.3\, \cancel{v(2)}^{\,0} + 0.2\, \cancel{v(0)}^{\,0}$$

$$v(1) = 1 + 0.5\, v(1)$$

$$v(1) = 2$$

Proof: Exercise!

Classes of states in a Markov Chain

In a MC state j is accessible from state i $\quad i \to j$

if $P[X_n = j \mid X_0 = i] > 0$ for some n

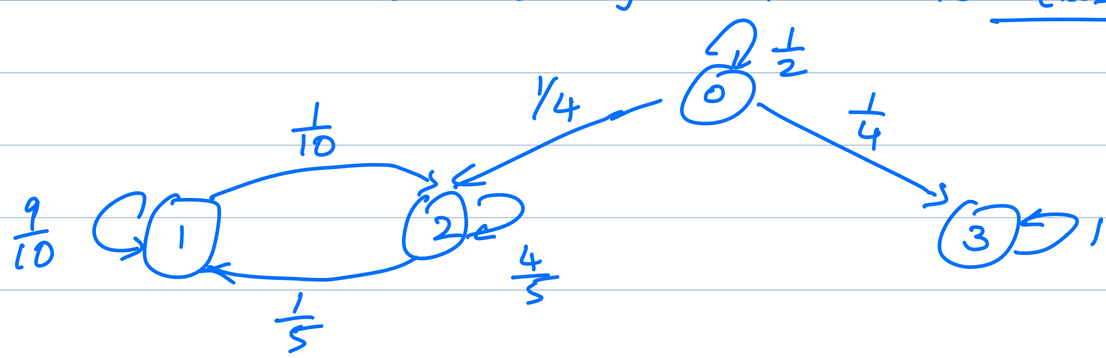
Notation: $P_{ij}(n) \qquad P_{ii}(0) = 1$

States i and j communicate $\quad i \leftrightarrow j$

if $i \to j$ & $j \to i$

States that communicate belong to the same class

Ex



$1 \to 2$ ✓ $\qquad 0 \to 2$ ✓ $\qquad 0 \to 3$ ✓

$2 \to 1$ ✓ $\qquad 2 \to 0$ ✗ $\qquad 3 \to 0$ ✗

$1 \leftrightarrow 2$

$\boxed{1,2}$ $\boxed{0}$ $\boxed{3}$

M.C has 3 classes

Prop: If $i \leftrightarrow j$, $j \leftrightarrow k$ then $i \leftrightarrow k$

<u>Irreducible</u>: If all states in a M.C belong to a single class

Recurrence:

Start M.C in state i

f_i = P[ever returing to the state i] $= 1$

State i is "recurrent"

$f_i < 1 \implies$ State i is "transient"

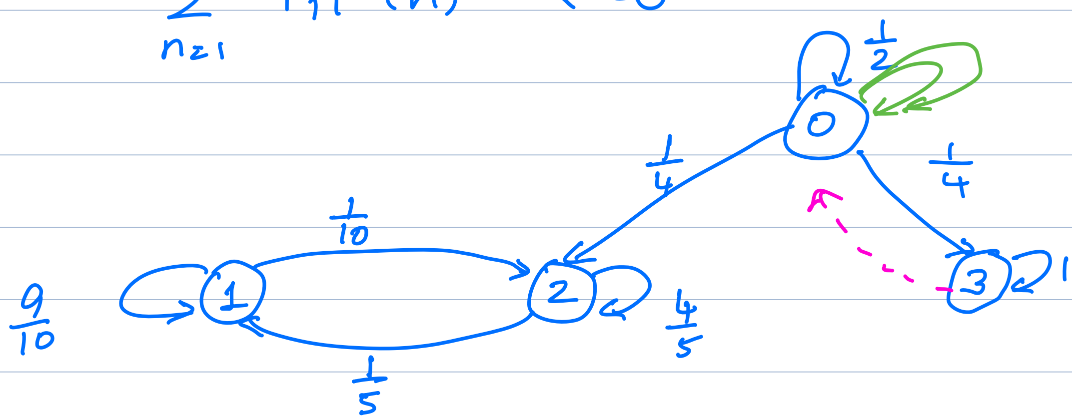
Prop: State i is recurrent iff

$$\sum_{n=1}^{\infty} P_{ii}(n) = \infty$$

State i is transient iff

$$\sum_{n=1}^{\infty} P_{ii}(n) < \infty$$

Example:



$$\sum^{\infty} P_{00}(n) = \sum^{\infty} \left(\tfrac{1}{2}\right)^n = 1 < \infty$$

$n=1$ $\qquad$ $n=1$ (2)

$$P_{00}(1) = \frac{1}{2}$$

$$P_{00}(2) = \left(\frac{1}{2}\right)^2$$

$$\vdots$$

$$P_{00}(n) = \left(\frac{1}{2}\right)^n$$

State 0 is transient

State 3 is recurrent

State 1 & 2 is also recurrent

Prop: Recurrence is a "class property"

All states in a class is either recurrent or transient

Prop: All states of a finite state irreducible Markov chain is recurrent.