Announcements: 1) Grades for Test 1 released
2) Problem set released.
3) Experiment 2

Markov Chain

↳ RP + Markov Prop + Discrete time + Discrete set

$$P_{ij} = P[X_{n+1} = j \mid X_n = i]$$
$$= P[X_1 = j \mid X_0 = i] \quad \text{"Time homogeneous"}$$

$$\underset{\text{transition prob. matrix}}{P} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \rightarrow \text{"State diagram"}$$

Example:
$$P = \begin{array}{c|ccc} & \boxed{0} & 1 & 2 \\ \hline 0 & .3 & .2 & .5 \\ \boxed{1} & .5 & .1 & .4 \\ 2 & .5 & .2 & .3 \end{array}$$

$$P[X_0 = 0] = P[X_0 = 1] = P[X_0 = 2] = \frac{1}{3}$$

$$P[X_0 = 1, X_1 = 1, X_2 = 0]$$
$$\overset{\text{Cond. prob}}{=} P[X_0 = 1]\, P[X_1 = 1, X_2 = 0 \mid X_0 = 1]$$
$$\overset{\text{Cond. prob}}{=} P[X_0 = 1]\, P[X_1 = 1 \mid X_0 = 1]\, P[X_2 = 0 \mid X_1 = 1, X_0 = 1]$$

$$= \underline{P[X_0 = 1]}\ P[\underline{X_1 = 1} \mid X_0 = 1]\ \underbrace{P[X_2 = 0 \mid X_1 = 1]}_{}$$ Markov

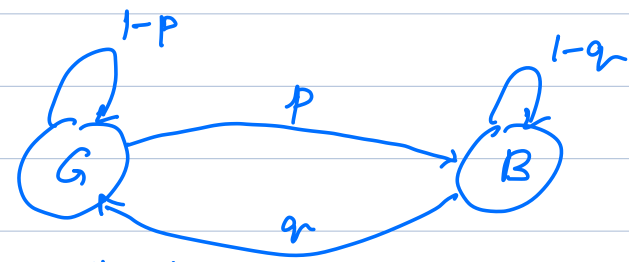
$$= \left(\frac{1}{3}\right)(0.1)(0.5) \ //$$

$P[X_1 = 0 \mid X_0 = 1]$

3. Property: The joint distribution of the Markov chain is fully characterized through $\pi_0 = P(X_0)$ and the transition prob. matrix P.

initial state dist

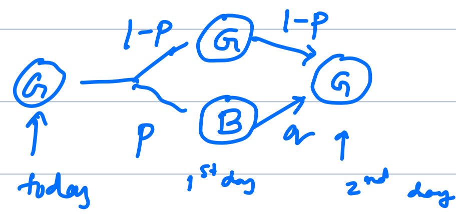
Example (from last class)



"n-step" transition prob

$n: 2$: $G \to G$

today 2 days



n step transition prob

$$P^{(2)}_{GG} = (1-p)(1-p) + p \cdot q$$

n: step transition prob : Chapman-Kolmogrov equation.

$$P^{(n)}_{ij} = P[X_{k+n} = j \mid X_k = i]$$

"Time homogenity"

$$= P[X_n = j \mid X_0 = i]$$

$$P^{(n+m)}_{ij} = P[X_{n+m} = j \mid X_0 = i]$$

Marginilizing or Total Law of prob.

$$= \sum_{k \in D} P[X_{n+m} = j,\ X_n = k \mid X_0 = i]$$

cond prob

$$= \sum_{k \in D} P[X_n = k \mid X_0 = i]$$

$$P[X_{n+m} = j \mid \underbrace{X_n = k,\ X_0 = i]}_{\text{Markov}}$$

$$\parallel \ \text{Markov}$$

$$P[X_{n+m} = j \mid X_n = k]$$

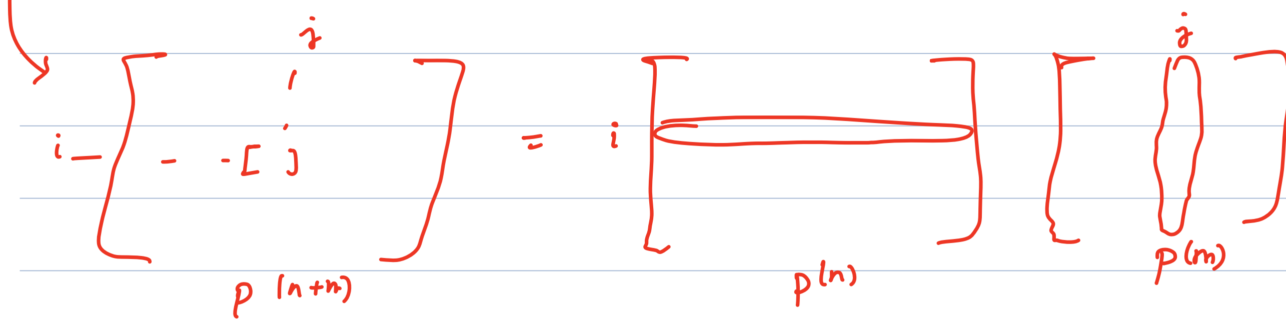
$$\parallel \ \text{Time Homogeneity}$$

$$P[X_m = j \mid X_0 = k]$$

$$= \sum_{k \in D} \underbrace{P[X_n = k \mid X_0 = i]}_{P_{ik}^{(n)}} \ \underbrace{P[X_m = j \mid X_0 = k]}_{P_{kj}^{(m)}}$$

$$P_{ij}^{(n+m)} = \sum_{k \in D} \underline{P_{ik}^{(n)}} \ P_{kj}^{(m)} \quad \Updownarrow$$

$$P^{(n+m)} = P^{(n)} \ P^{(m)}$$



$$P^{(n+m)} = P^{(n)} \ P^{(m)}$$

$$P^{(1)} = P$$

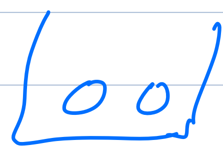
$$P^{(2)} = P^{(1+1)} = P^{(1)} \cdot P^{(1)} = P \cdot P = P^2$$

$$\vdots$$

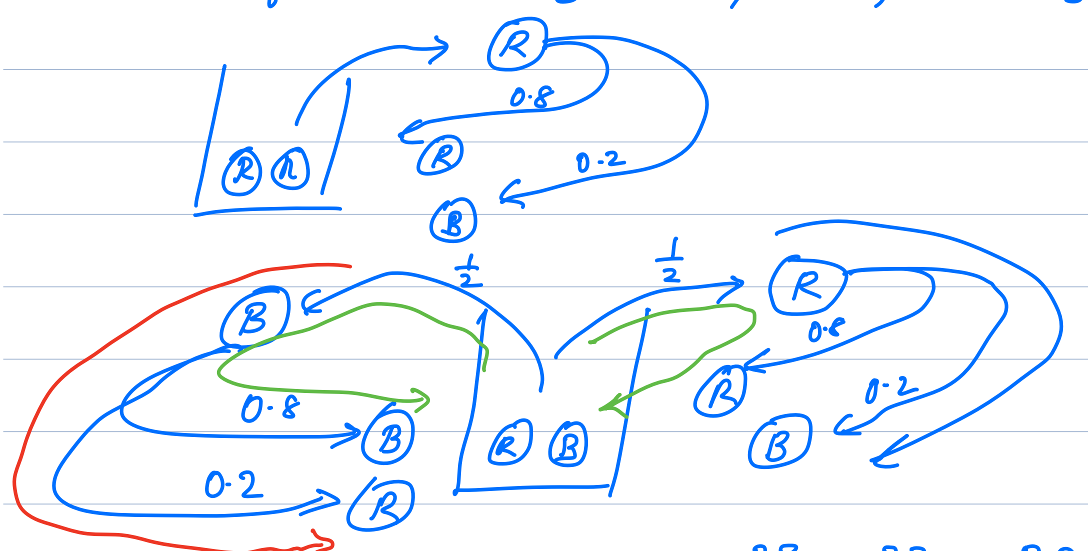
$$P^{(n)} = P^n \quad //$$

Example:

An urn contains 2 balls. Ball colors are red & blue.
At each time, a ball is randomly chosen and replaced by a new ball.
The new ball is of the same color (as the chosen color) with prob 0.8 and of the opposite color with prob 0.2. If initially both balls are red, find the prob that the fifth ball selected is red.



State of M.C = { RR, RB, BB }



$$P = \begin{array}{c|ccc} & RR & RB & BB \\ \hline RR & 0.8 & 0.2 & 0 \\ RB & 0.1 & 0.8 & 0.1 \\ BB & 0 & 0.2 & 0.8 \end{array}$$

$P(\text{fift selection is red} \mid X_0 = RR)$

0 1 2 3 4

RR

RR – 1

RB – $\frac{1}{2}$

~~BB~~

$$1 \cdot P^{(4)}_{RR,\,RR} + \frac{1}{2} \cdot P^{(4)}_{RR,\,RB}$$

$$= \sum_{s \in \{RR,\, RB,\, BB\}} P(\text{Fifth} = \text{Red},\ X_4 = s \mid X_0 = RR)$$

$$= \sum_{s} P(X_4 = s \mid X_0 = RR)\ P(\text{Fift} = \text{Red} \mid X_4 = s,\ X_0 = RR)$$

$$= \sum_{s} P(X_4 = s \mid X_0 = RR)\ P(\text{Fift} = \text{Red} \mid X_4 = s)$$

$$= P^{(4)}_{RR,\,RR} \cdot 1 + P^{(4)}_{RR,\,RB} \cdot \frac{1}{2} + P^{(4)}_{RR,\,BB} \cdot 0$$

$$= P^{(4)}_{RR,\,RR} + \frac{1}{2} P^{(4)}_{RR,\,RB}$$

$$= [P^4]_{RR,\,RR} + \frac{1}{2} \cdot [P^4]_{RR,\,RB}$$