Last Class

- Expectation $\mathbb{E}[X] = \begin{cases} \sum x \cdot p_X(x) \\ \int x \cdot f_X(x)\,dx \end{cases}$

- Fundamental theorem of expectation

$$\mathbb{E}[g(X)] = \begin{cases} \sum g(x)\, p_X(x) \\ \int g(x)\, f_X(x)\,dx \end{cases}$$

Properties

1. X r.v such that $P(X \geq 0) = 1$

$\mathbb{E}[X] \geq 0$

$$\mathbb{E}[X] = \sum \underset{+ve}{\underset{\uparrow}{x}}\ \underset{+ve}{\underset{\uparrow}{p_X(x)}}$$

2. X is a r.v $P(X = c) = 1$

$\mathbb{E}[X] = c$

3. "Linearity Property"

$\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b$

$$\int (ax+b)\, f_X(x)\,dx$$

$$a \underbrace{\int x\, f_X(x)\,dx}_{\mathbb{E}[X]} + b \underbrace{\int f_X(x)\,dx}_{1}$$

$$a\,\mathbb{E}[X] + b$$

Moments of a random variable

n^{th} moment of a r.v

$$m_n = \mathbb{E}[X^n] = \begin{cases} \sum x^n p_X(x) \\ \int x^n f_X(x)\,dx \end{cases}$$

$n=1$ $\quad \mathbb{E}[X] = \mu = \begin{cases} \sum x \, p_X(x) \\ \int x \, f_X(x)\,dx \end{cases}$

"Mean"

$n=2$ $\quad \mathbb{E}[X^2] = \int x^2 f_X(x)\,dx$

$\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2]$

"Variance of random variable"

$$= \int (x-\mu)^2 f_X(x)\,dx \quad *$$

$$= \int x^2 f_X(x)\,dx - 2\mu \int x f_X(x)\,dx + \mu^2 \int f_X(x)\,dx$$

$$= \mathbb{E}[X^2] - 2\mu\,\mu + \mu^2 \cdot 1$$

$$= \mathbb{E}[X^2] - 2\mu^2 + \mu^2$$

$$= \mathbb{E}[X^2] - \mu^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Property of Variance:

$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$

$\mathbb{E}[aX+b] = a\mu + b$

$$\mathrm{Var}(aX+b) = \mathbb{E}[[(aX+\cancel{b}) - (a\mu+\cancel{b})]^2]$$

$$= \mathbb{E}[(a(X-\mu))^2]$$

$$= \int a^2 (x-\mu)^2 f_X(x)\,dx$$

Linearity

Homogenity Superposition

$$= a^2 \int \underbrace{(x-\mu)^2 f_X(x)\,dx}$$

$$= a^2 \, Var(X)$$

Exercise : 1. $Var(\text{Bin}(n,p)) = np(1-p)$

2. $Var(\text{Poisson}(\lambda)) = \lambda$

3. $Var(\text{Unif}(\alpha,\beta)) = \frac{1}{12}(\beta-\alpha)^2$

4. $Var(\text{Normal}(\mu,\sigma^2)) = \sigma^2$

$\mathbb{E}[\text{Normal}(\mu,\sigma^2)] = \mu$.

Moment Generating Function (MGF)

$$M(t) = \mathbb{E}[e^{tX}] = \begin{cases} \sum_x e^{tx} p_X(x) \\ \int e^{tx} f_X(x)\,dx \end{cases}$$

$$M'(t) = \frac{d}{dt} M(t)$$

$$= \frac{d}{dt} \mathbb{E}[e^{tX}]$$

$$= \mathbb{E}\left[\frac{d}{dt} e^{tX}\right]$$

$$= \mathbb{E}[X e^{tX}]$$

$$M'(t) = \mathbb{E}[X e^{tX}]$$

$$M'(t)\Big|_{t=0} = \mathbb{E}[X] = \mu$$

$$M^{n}(t)\Big|_{t=0} = \frac{d^n}{dt^n} M(t)\Big|_{t=0} = \mathbb{E}[X^n] = m_n$$

Example: MGF of Binomial distribution (n, p)

$$P_X(i) = \binom{n}{i} p^i (1-p)^{n-i}$$

$$M(t) = \mathbb{E}[e^{tX}] = \sum_{i=0}^{n} e^{ti} P_X(i)$$

$$= \sum_{i=0}^{n} e^{ti} \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{n} \binom{n}{i} (e^t p)^i (1-p)^{n-i}$$

$$= (e^t p + 1 - p)^n$$

$$E[X] = \frac{d}{dt} M(t)\Big|_{t=0}$$

$$= \frac{d}{dt} (e^t p + (1-p))^n \Big|_{t=0}$$

$$= n (e^t p + (1-p))^{n-1} e^t p \Big|_{t=0}$$

$$= np (p + (1-p))^{n-1} = np$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}(X))^2$$

$$\mathbb{E}[X^2] = \frac{d^2}{dt^2} M(t)\Big|_{t=0}$$

$$= \frac{d}{dt} \; np \; e^t (e^t p + 1 - p)^{n-1} \Big|_{t=0}$$

$$= np \left[e^t (e^t p + 1-p)^{n-1}) + e^t (n-1)(e^t p + 1-p)^{n-2} e^t p \right] \Big|_{t=0}$$

$$= np \left[1 \cdot (p + 1 - p)^{n-1} + 1 \cdot (n-1)(p + 1 - p)^{n-2} 1 \cdot p \right]$$

$$= np \left[1 + (n-1) p \right]$$

$$\text{Var}(X) = \mathbb{E}[X^2) - (\mathbb{E}(X))^2$$

$$= np + n(\cancel{n}-1) p^2 - \cancel{(np)^2}$$

$$= np - np^2$$

$$= np(1-p) \;//$$

Exercise 1. $X \sim \text{Poisson}(\lambda)$

$$M(t) = e^{\lambda(e^t - 1)}$$

2. $X \sim \text{Exp}(\lambda)$

$$M(t) = \frac{\lambda}{\lambda - t} \; ; \; \lambda < t$$

Example: Gaussian / Normal r.v $Z \sim N(0, 1)$

"Standard normal"

Mean, Variance

$$M_Z(t) = \mathbb{E}[e^{tZ}]$$

$$= \int e^{tz} \underbrace{f_Z(z)}_{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}} dz$$

$$\sigma^2 = 1, \quad \mu = 0$$

$$= \int e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \int \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{z^2}{2} - tz\right)} dz$$

$$= \int \frac{1}{\sqrt{2\pi}} e^{-\left(z^2/2 - tz + t^2/2 - t^2/2\right)} dz$$

$$= \int \frac{1}{\sqrt{2\pi}} e^{-\left(z^2/2 - tz + t^2/2\right)} e^{t^2/2} dz$$

$$= e^{t^2/2} \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-(z-t)^2/2} dz}_{\mathcal{N}(t,\,1) \; = 1}$$

$$= e^{t^2/2} \quad //$$

$X \sim \text{Normal}(\mu, \sigma^2)$

$Z \sim \text{Normal}(0, 1)$

$X = \sigma Z + \mu$

$$\mathbb{E}[X] = \sigma \cdot \underset{\overset{\shortparallel}{0}}{\mathbb{E}[Z]} + \mu = \mu$$

$$\text{Var}(X) = \text{Var}(\sigma Z + \mu) = \sigma^2 \underset{\overset{\shortparallel}{1}}{\text{Var}(Z)}$$

$$= \sigma^2$$

$X \sim \text{Normal}(\mu, \sigma^2)$

$$
\begin{aligned}
M_X(t) &= \mathbb{E}[e^{tX}] \\
&= \mathbb{E}[e^{t(\sigma Z + \mu)}] \\
&= \mathbb{E}[e^{t\sigma Z} e^{t\mu}] \\
&= e^{t\mu} \, \mathbb{E}[e^{t\sigma Z}] \\
&= e^{t\mu} \, e^{t^2\sigma^2/2} \\
&= e^{t\mu + t^2\sigma^2/2} \quad //
\end{aligned}
$$