lab report by Nov 10]

Convergence of random variables & limit Theorems

Probability inequalities

1. Boole's Inequality / Union Bound)

$(\Omega, \mathcal{F}, P)$

$E_1, E_2, \ldots$

$$P\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} P(E_i)$$

Proof: E_1 and E_2 $\boxed{n=2}$

$$P(E_1 \cup E_2) = \underbrace{\underset{\geq 0}{P(E_1)} + \underset{\geq 0}{P(E_2)} - \underset{\geq 0}{P(E_1 \cap E_2)}}$$

$$\leq P(E_1) + P(E_2)$$

Proceed by induction to prove the general case.

Let us assume that this theorem holds for $n = n-1$

$$P\left(\bigcup_{i=1}^{n-1} E_i\right) \leq \sum_{i=1}^{n-1} P(E_i)$$

$$P\left(\bigcup_{i=1}^{n} E_i\right) = P\left(\left(\bigcup_{i=1}^{n-1} E_i\right) \cup (E_n)\right)$$

$$\leq P\left(\bigcup_{i=1}^{n-1} E_i\right) + P(E_n)$$

$$\leq \sum_{i=1}^{n-1} P(E_i) + P(E_n)$$

$$= \sum_{i=1}^{n} P(E_i)$$

2. Markov Inequality: Suppose X is a positive random

variable

$$P(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

Proof:

$$\mathbb{E}[X] = \int_0^\infty x \cdot f_X(x)\, dx$$

$$= \underbrace{\int_0^a \underset{\geq 0}{x} \cdot \underset{\geq 0}{f_X(x)}\, dx}_{\geq 0} + \int_a^\infty x \cdot f_X(x)\, dx$$

$$\geq \int_a^\infty \underset{\geq a}{x} \cdot f_X(x)\, dx$$

$$\geq \int_a^\infty a \cdot f_X(x)\, dx$$

$$= a \underbrace{\boxed{\int_a^\infty f_X(x)\, dx}}_{P(X \geq a)}$$

$$P(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

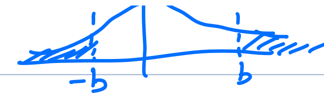
Example: $X \sim Bin(n, p)$

$$P(X \geq q \cdot n) \leq \frac{\mathbb{E}[X]}{q \cdot n} = \frac{np}{q \cdot n} = \frac{p}{q}$$

$p = \frac{1}{3}$, $q = 2/3$

$$P(X \geq \tfrac{2}{3} n) \leq \tfrac{1}{2}$$

3. Chebychev Inequality.



Suppose X is a random variable, then for any $b > 0$

$$P(|X - \mathbb{E}[X]| \geq b) \leq \frac{Var(X)}{b^2}$$

Proof:

$$Y = (X - \mathbb{E}[X])^2$$

Markov Ineq

$$P(Y \geq b^2) \leq \frac{\mathbb{E}[Y]}{b^2}$$

$$\mathbb{E}[Y] = \mathbb{E}[(X - \mathbb{E}[X])^2] = Var(X)$$

$$= \frac{Var(X)}{b^2} \quad //$$

4. Chernoff Bound

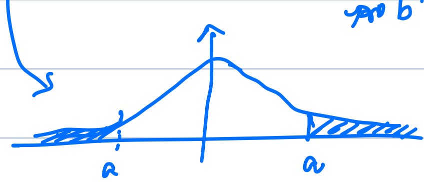
Suppose X is a random variable, for any a

$$P(X \geq a) \leq \min_{t>0} e^{-ta} M_X(t)$$

"upper tail prob"

$$P(X \leq a) \leq \min_{t<0} e^{-ta} M_X(t)$$

" lower tail prob"



Proof: $P(X \geq a) \overset{t>0}{=} P(tX \geq ta)$

$$= P(\underbrace{e^{tX}}_{\text{"a new random variable"}} \geq e^{ta})$$

positive random variable

$$\overset{\text{Markov}}{\leq} \frac{\boxed{\mathbb{E}[e^{tX}]}}{e^{ta}} = e^{-ta} M_X(t)$$

"MGF of X"

$$\underbrace{P(X \geq a)}_{h} \leq \underbrace{e^{-ta} M_X(t)}_{f(t)}$$

$$\leq \min_{t>0} e^{-ta} M_X(t)$$

$h \leq f(t)$

t	$f(t)$
$t=1$	10
$t=2$	5
$t=3$	2

$h \leq 10$

$h \leq 5$

$h \leq 2$

$\Rightarrow h \leq 2$

$2 = $ min of all upper bounds.

$$h \leq \min_t f(t)$$

Example: $X \sim \mathrm{Bin}(n, p)$

$$P(X \geq qn) \leq \min_t e^{-tqn} M_X(t)$$

$$M_X(t) = (pe^t + (1-p))^n$$

$$\frac{d}{dt}\left[e^{-tqn}(pe^t + (1-p))^n\right]\Big|_{t=t^*} = 0$$

$$e^{t^*} = \frac{q(1-p)}{(1-q)p}$$

$$P(X \geq qn) \leq e^{-t^*qn}(e^{t^*}p + (1-p))^n$$

$$= \left(\frac{p}{q}\right)^{qn} \left(\frac{1-p}{1-q}\right)^{(1-q)n}$$

$q = \frac{2}{3}, \quad p = \frac{1}{3}$

$$P(X \geq 2/3\, n) \leq \frac{1}{2}$$

$$P(X \geq 2/3\, n) \leq 2^{-n/3}$$

"tail bounds in term of "n""