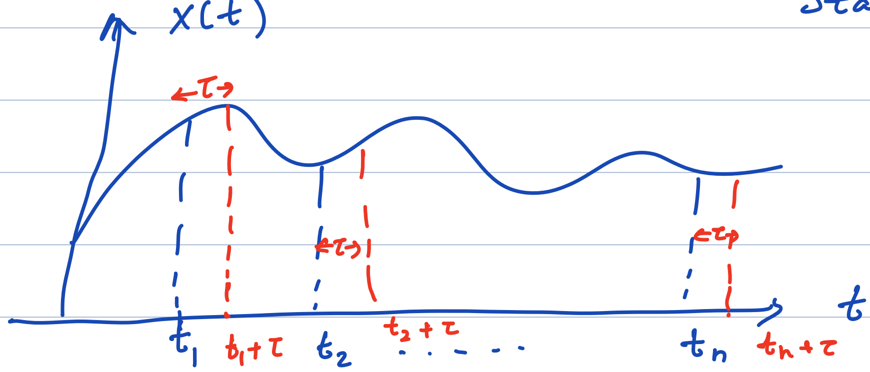
Module 3: Second-order theory

Stochastic Process

- Stationary random process [Strict Sense Stationary (SSS)]



Cdf of finite dimensional distribution

$$F_{X(t_1)\,X(t_2)\cdots X(t_n)}(x_1, x_2, \ldots x_n)$$

$$= F_{X(t_1+\tau)\,X(t_2+\tau)\cdots X(t_n+\tau)}(x_1, x_2, \ldots, x_n)$$

Stationary

Example: Discrete-time iid process

$$X_n \overset{iid}{\sim} F$$

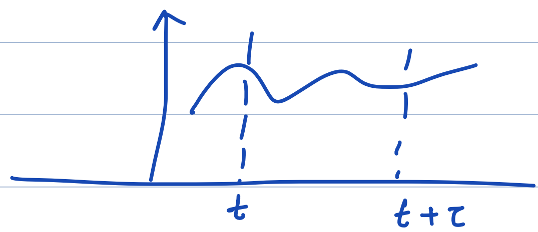
$$F_{X_1 X_2 \ldots X_K}(x_1, x_2, \ldots x_k) = F(x_1)\,F(x_2)\cdots F(x_k)$$

$$F_{X_{1+\tau}\,X_{2+\tau}\cdots X_{k+\tau}}(x_1, x_2 \ldots x_k) = F(x_1)\,F(x_2)\cdots F(x_k)$$

X_n — SSS process

Properties

1. $n = 1$ (First order statistics)



$$F_{X(t)} = F_{X(t+\tau)} = F$$

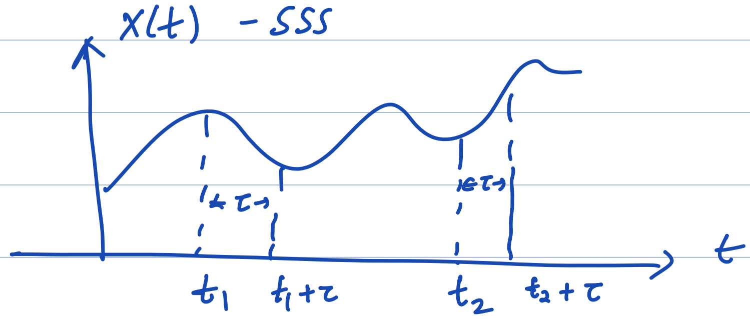
First order cdf is independent of t.

$$m_{X(t)} = E[X(t)] = \text{Mean}(F) = m \quad (\text{Constant})$$

$$\text{Var}(X(t)) = \text{Var}(F) = \sigma^2 \quad (\text{Constant})$$

2. Second - order statistics ($n = 2$)

X(t) - SSS



$$F_{X(t_1)\,X(t_2)}(x_1, x_2) = F_{X(t_1+\tau)\,X(t_2+\tau)}(x_1, x_2)$$

$\tau = -t_1$

$$F_{X(t_1)\,X(t_2)}(x_1, x_2) = F_{X(0)\,X(t_2-t_1)}(x_1, x_2)$$

Second - order cdf depends only on the time difference b/w 2 sampling instants.

Example: $X(t) = A \cos(\omega t + \theta)$ [Not a SSS]

$\theta \sim \text{Unif}(0, 2\pi)$

$m_{X(t)} = 0$

$C_X(t_1, t_2) = \mathbb{E}[X(t_1) X(t_2)]$

$= \mathbb{E}[A \cos(\omega t_1 + \theta) \; A \cos(\omega t_2 + \theta)]$

$= \frac{A^2}{2} \cos(\omega (t_2 - t_1))$

Second-order statisc depends only on time diff b/w t_1 and t_2.

Wide-sense stationary process (WSS)

A process is WSS if

(i) Mean function is independent of time

$$m_{X(t)} = m \quad \forall t$$

(ii) $C_X(t_1, t_2) = C_X(t_1 - t_2)$

Properties

(i) SSS $\Longrightarrow$ WSS

$\Longleftarrow$

In genral, 'No'

Expection: Gaussian process

(ii) If X is a Gaussian random process and is WSS $\Longrightarrow$ SSS

[Proof : Exercise !]

Mean function,
Covariance function

Auto-correlation function (WSS)

$$R_X(t_1, t_2) = C_X(t_1, t_2) - m_{X(t_1)}\, m_{X(t_2)}$$

$$\Downarrow \qquad = C_X(t_1 - t_2) - m^2$$

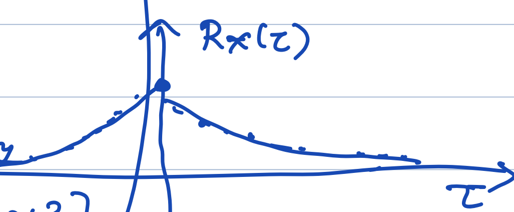
$$R_X(\underbrace{t_1 - t_2}_{\tau})$$

$$R_X(\tau)$$

Properties of $R_X(\tau)$

1. $R_X(0) = \mathbb{E}[X(t+\tau)\, X(t)]$
$\qquad = \mathbb{E}[X^2(t)]$
" Power / Energy of random process "

2. $|R_X(\tau)| \le R_X(0)$



Proof: Cauchy - Schwartz inequality

$$\mathbb{E}^2[XY] \le \mathbb{E}[X^2]\, \mathbb{E}[Y^2]$$

$$R_X^2(\tau) = \left(\mathbb{E}[X(t)\, X(t+\tau)]\right)^2$$

$$\le \mathbb{E}[X^2(t)]\, \mathbb{E}[X^2(t+\tau)]$$

$$= R_X(0) \qquad R_X(0)$$

$R_X^2(\tau) \leq R_X^2(0)$

$|R_X(\tau)| \leq R_X(0)$

3. $R_X(\tau)$ is an even function of τ

$R_X(\tau) = E[X(t)\ X(t+\tau)]$

$R_X(-\tau) = E[X(t+\tau)\ X(t)]$

$R_X(\tau) = R_X(-\tau)$

4. [Proof skipped]

If $R_X(T) = R_X(0)$ for some $T \neq 0$

then all the sample paths $X(t)$

$= X(t+T)$ (periodic) with probability 1.

Textbook : Garcia