Last class : Correlation : $E[XY]$

Covariance $\text{Cov}(X,Y) = E[(X - E(X))(Y - E(Y))]$

Independence $\Rightarrow$ Uncorrelated
$\xleftarrow{\times}$

Covariance $= 0 \Longleftrightarrow$ Uncorrelated

Example: $X = \cos\theta$
$Y = \sin\theta$

$$E[XY] = \int_0^{2\pi} \cos\theta \, \sin\theta \left(\frac{1}{2\pi}\right) d\theta \quad \Bigg| \quad \theta \sim \text{Unif}(0, 2\pi)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sin\frac{2\theta}{2} \, d\theta$$

$$= \frac{1}{2\pi} \frac{1}{2} \left[ -\cos\frac{2\theta}{2} \right]_0^{2\pi}$$

$$= 0 //$$

$\text{Cov}(X,Y) = E[(X - E(X))(Y - E(Y))]$

Covariance matrix

$X = (X_0, X_1, \dots, X_{n-1})^T$

$\mu = E(X) = (\underbrace{E(X_0)}_{\mu_0}, \underbrace{E(X_1)}_{\mu_1} \cdots, \underbrace{E(X_{n-1})}_{\mu_{n-1}})^T$

Covariance matrix $\quad K = E[\underbrace{(X - \mu)}_{n \times 1} \underbrace{(X - \mu)^T}_{1 \times n}]$

$$\xleftarrow{\quad n \times n \quad} \rightarrow$$

$$K(i, j) = E[(X_i - \mu_i)(X_j - \mu_j)]$$

$$= Cov(X_i, X_j)$$

Properties:

1. $K$ is symmetric and positive semi definite.

$$K(i,j) = K(j,i)$$

$$Cov(X_i, X_j) = Cov(X_j, X_i)$$

Positive semi definite: $x^T Q x \geq 0 \quad \forall x$

$\uparrow (Q) \quad 1 \times n$

$n \times n$

Proof: $x^T K x \to \geq 0$

$1 \times n \quad n \times 1$

$$x^T \; \mathbb{E}[(X-\mu)(X-\mu)^T] \; x$$

$$\mathbb{E}[\boxed{x^T(X-\mu)}\boxed{(X-\mu)^T x}]$$

$1 \times n \quad n \times 1 \quad 1 \times n \quad n \times 1$

$$\mathbb{E}[[x^T(X-\mu)]^2] \geq 0$$

2. random vector is mutually uncorrelated.

$$K(i,j) = 0 \quad \forall i \neq j$$

$$Cov(X_i, X_j)$$

Expectation of random vectors (Properties)

Moment Generating Function (MGF)

$X = (X_0, X_1, \ldots X_{n-1})$

$$M_x(t) = \mathbb{E}[e^{tx}]$$

$$M_x(t) = \mathbb{E}[e^{t^T x}]$$

$\leftarrow$ random variable $\rightarrow$

(MGF) $\uparrow$

$(t_0, t_1, \ldots, t_{n-1})$

Properties of MGF

1. $\mathbb{E}[X_0 X_1 \cdots X_{n-1}] = \dfrac{\partial^n}{\partial t_0 \, \partial t_1 \cdots \, \partial t_{n-1}} M_X(t) \Big|_{t=0}$

$\nearrow$ Zero vector

Generalization:

$\mathbb{E}[X_0^{k_0} X_1^{k_1} \cdots X_{n-1}^{k_{n-1}}] \longrightarrow k = k_0 + k_1 + \cdots + k_{n-1}$

$= \dfrac{\partial^k}{\partial t_0^{k_0} \, \partial t_1^{k_1} \cdots \, \partial t_{n-1}^{k_{n-1}}} M_X(t) \Big|_{t=0}$

2. MGF and mutually independent random variables.

$X_0, X_1, \ldots, X_{n-1}$ are mutually independent.

$Y = X_0 + X_1 \cdots + X_{n-1}$

$M_Y(t) = \mathbb{E}[e^{tY}]$

$= \mathbb{E}[e^{t(X_0 + X_1 + \cdots + X_{n-1})}]$

$= \mathbb{E}[e^{tX_0} \cdot e^{tX_1} \cdots e^{tX_{n-1}}]$

$= \mathbb{E}[e^{tX_0}] \, \mathbb{E}[e^{tX_1}] \cdots \mathbb{E}[e^{tX_{n-1}}]$

$= \displaystyle\prod_{i=0}^{n-1} M_{X_i}(t) \quad \underset{M_{X_0}(t)}{\longrightarrow} \quad \underset{M_{X_1}(t)}{\longrightarrow} \quad \underset{M_{X_{n-1}}(t)}{\searrow}$

independent

Example: $X$ and $Y$ are $\overset{\wedge}{\text{Poisson}}$ random variables with parameters $\lambda_1$ and $\lambda_2$.

$Z = X + Y$

$M_X(t) = e^{\lambda_1(e^t - 1)}$

$M_Y(t) = e^{\lambda_2(e^t - 1)}$

$M_Z(t) = M_X(t) \, M_Y(t)$

$$z \quad e^{\lambda_1 (e^t - 1)} \quad e^{\lambda_2 (e^t - 1)}$$

$$= \quad e^{(\lambda_1 + \lambda_2)(e^t - 1)}$$

$$Z \sim \text{Poisson} (\lambda_1 + \lambda_2)$$

Exercise: $X$ & $Y$ are independent Gaussian random variables

$$X \sim N(\mu_1, \sigma_1^2)$$
$$Y \sim N(\mu_2, \sigma_2^2)$$

Find distribution of $Z = X + Y$ ?

[ Ans: $Z \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ ]

## Gaussian random vectors

[ Jointly Gaussian random variables]

$$X \sim N(m, \Lambda) \longrightarrow \text{Symmetric & positive semi definite matrix}$$
$$\downarrow$$
$$(X_0, X_1, \dots, X_{n-1}) \downarrow$$

vector $m = (m_0, m_1, \dots, m_{n-1})$

Density function.

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^n \det(\Lambda)}} \exp\left(- \frac{(x-m)^T \Lambda^{-1} (x-m)}{2}\right)$$

$\uparrow$ n dimensional

## Properties

1. $\mathbb{E}[X] = m$ (mean vector)
2. $\text{Cov}(X) = \Lambda$ (co-variance matrix)
3. $M_X(t) = e^{t^T m + \frac{1}{2} t^T \Lambda t}$

## Special Cases

$$X = (Z_0, Z_1, \ldots, Z_{n-1})$$
$$Z_i \sim \mathcal{N}(0,1) \qquad Z_i\text{'s are all independent}$$

↑ standard normal

$$X \sim \mathcal{N}(m, \Lambda)$$
$$m = (0, 0, \ldots, 0)$$
$$\text{Cov}(Z_i, Z_j) = 0$$
$$\text{Cov}(Z_i, Z_i) = \text{Var}(Z_i) = 1.$$
$$\Lambda = I$$

Property:  Suppose $X$ is a $k$-dimensional Gaussian random vector.

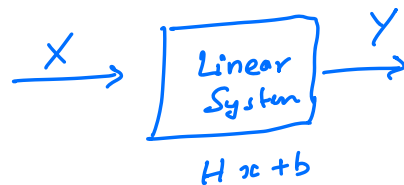$$X \sim \mathcal{N}(m, \Lambda)$$

Suppose $Y = HX + b$

$$\dim(H) = n \times k \qquad \Rightarrow \dim(Y) = n \times 1$$
$$\dim(b) = n \times 1$$

$Y$ is a Gaussian random vector.

$$Y \sim \mathcal{N}(Hm + b, \ H\Lambda H^T)$$



$$X \longrightarrow \boxed{\begin{array}{c}\text{Linear} \\ \text{System}\end{array}} \xrightarrow{\ Y\ }$$

$H x + b$

Proof:  $M_Y(t) = \mathbb{E}\left[ e^{t^T Y} \right]$

$$= \mathbb{E}\left[ e^{t^T (HX + b)} \right]$$

$$= \mathbb{E}\left[ e^{t^T H x} \; e^{t^T b} \right]$$
$$\leftarrow \text{(constant} \to)$$

$$= e^{t^T b} \quad \mathbb{E}\left[ e^{t^T H x} \right]$$

$$= e^{t^T b} \quad \mathbb{E}\left[ e^{(H^T t)^T x} \right]$$

$$= e^{t^T b} \qquad M_x(H^T t)$$

$$= e^{t^T b} \quad e^{(H^T t)^T m \; + \; \frac{1}{2} (H^T t)^T \wedge (H^T t)}$$

$$= e^{t^T b} \qquad e^{t^T H m \; + \; \frac{1}{2} t^T (H \wedge H^T) t}$$

$$= e^{t^T (H m + b) \; + \; \frac{1}{2} t^T (H \wedge H^T) t}$$

$$Y \sim N(H m + b, \; H \wedge H^T)$$

Next class : Conditional pdf / Expectation