Announcements : (i) Lab Experiment 2 due on Dec 26

(ii) Please come prepared for lab viva

Recap : SSS vs WSS

WSS :
- $m_X(t) = m_X$
- $C_X(t_1, t_2) = C_X(t_1 - t_2)$

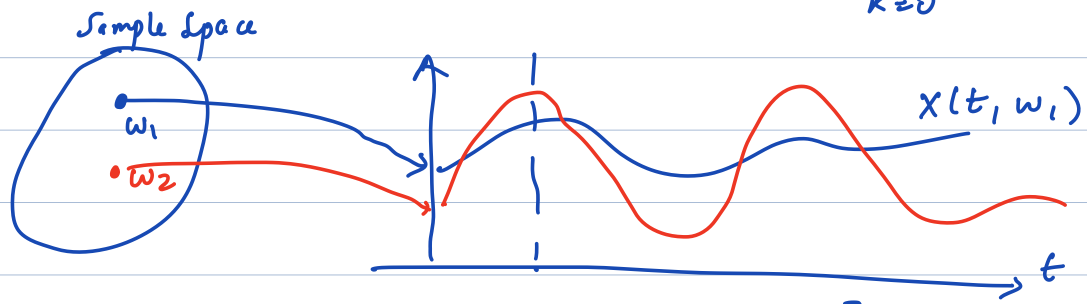
$[R_X(t_1, t_2) = R_X(t_1 - t_2)]$

"Causal" : $h(\tau) = 0 \ (\tau < 0)$

$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = \int_0^\infty h(\tau)\, x(t-\tau)\, d\tau$

LTI

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$



$X(t) \rightarrow \boxed{h(t)} \rightarrow Y(t) \overset{?}{=} \int_0^\infty h(\tau)\, X(t-\tau)\, d\tau$ ✗

Fix

$X(t, \omega) \rightarrow \boxed{h(t)} \rightarrow Y(t, \omega) \overset{?}{=} \int_0^\infty h(\tau)\, X(t-\tau, \omega)\, d\tau$

Convergence w.p. 1

$P\left(\left\{\omega : Y(t, \omega) = \int_0^\infty h(\tau)\, X(t-\tau, \omega)\right\}\right) \ \boxed{\forall \omega}$

$= 1 \quad - (*)$

Conditions where convergence w.p. 1 happen

↳ $h[n]$ is an FIR filter.

$h[n]$ is not FIR (IIR)

↳ mean square convergence if

$\sum_n |h[n]| < \infty$ (BIBO stability criteria)

$X(t)$: finite mean & variance.

From now on, $(*)$ is satisfied.

$$Y(t) = \int h(\tau)\, X(t-\tau)\, d\tau$$

Discrete time $\quad Y[n] = \sum_k h[k]\, X[n-k]$

Output mean:

$$\mathbb{E}[Y[n]] = \mathbb{E}\Big[\sum_k h[k]\, X[n-k]\Big]$$

$$= \sum_k h[k]\, \mathbb{E}[X[n-k]]$$

$$= \sum_k h[k]\, \underline{m_X[n-k]}$$

Special Case: X is WSS process

$$= \sum_k h[k]\, m_X$$

$$\underline{m_Y[n]} = m_X \sum_k h[k]$$

Mean of Y also doesn't depend on time!

(Aside)

$m_y[n]$ should exist

$m_y[n] < \infty \quad \forall n$

$\sum_k |h[k]| < \infty \Rightarrow m_y[n] < \infty$

For example $h[n] = 1 \quad \forall n \geq 0$

$\sum_k |h[k]| \to \infty$

Covariance of y

(Discrete time case)

$$C_y(k, j) \triangleq E\left[(y_k - E[y_k])(y_j - E[y_j]) \right]$$

$$= E\left[\left(\sum_n h[n]\, x[k-n] - \sum_n h[n]\, m_x[k-n] \right) \left(\sum_m h[m]\, x[j-m] - \sum_m h[m]\, m_x[j-m] \right) \right]$$

$$= E\left[\left(\sum_n h[n] \left(x[k-n] - m_x[k-n] \right) \right) \left(\sum_m h[m] \left(x[j-m] - m_x[j-m] \right) \right) \right]$$

$$= \sum_n \sum_m h[n]\, h[m]\, \underbrace{E\left[(x[k-n] - m_x[k-n])(x[j-m] - m_x[j-m]) \right]}_{C_x[k-n,\ j-m)}$$

$$C_Y(k,j) = \sum_n \sum_m h[n]\, h[m]\, C_X(k-n, j-m)$$

Special case: X is a WSS

$$C_X(k-n, j-m) = C_X((k-n)-(j-m))$$

$$= C_X((k-j)-(n-m))$$

$$C_Y(k,j) = \sum_n \sum_m h[n]\, h[m]\, C_X((k-j)-(n-m))$$

$C_Y(k,j)$ also depends only on $(k-j)$

If X is a WSS process, Y is also a WSS process

Fourier Transform I/O relations for WSS process.

$$h[n] \longleftrightarrow H(f) = \sum_{k=0}^{\infty} h[k]\, \underline{e^{-j2\pi fk}}$$

$$m_Y = m_X \sum h[k]$$

F.T $\quad m_Y = m_X\, H(0)$