Last class: $(\Omega, \mathcal{F}, P)$

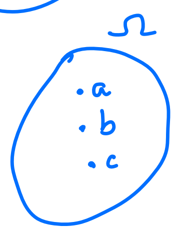
Ω: outcomes of random exp

$\mathcal{F}$: Event space

Discrete Probablity space : P.m.f p

$\mathcal{F}_2 = \{ \{H\}, \{T\}, \{H,T\}, \phi \}$: 4

$\{H,T\} \to \Omega$



$\mathcal{F}_3 = \{ \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}, \phi \}$: 8

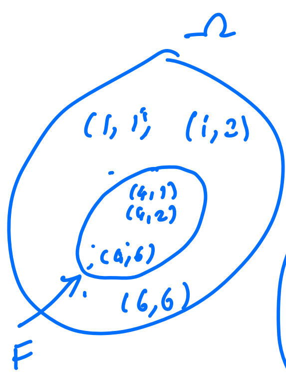
$|\Omega| = n$

$\mathcal{F}$ = Power set

Example: Experiment with 2 dice throw

Information: The first dice came up 4

Calculate the probablity of the sum of the dice being 6 ?



$E \cap F$

(4,1)	$\boxed{(4,2)}$	(4,3)	(4,4)	(4,5)	(4,6)
1/6	$\boxed{1/6}$	1/6	1/6	1/6	1/6

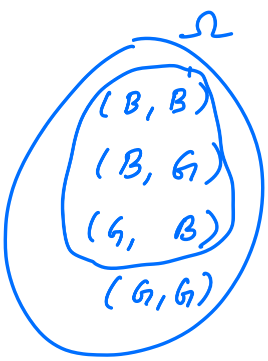
Probability = 1/6

Conditional Probablity : Event F has happened

Probabity of Event E

$$\boxed{\begin{aligned} P(E|F) &= \frac{P(E\cap F)}{P(F)} \\ \Rightarrow P(E\cap F) &= P(F)\; P(E|F) \end{aligned}} \;\; *$$

Example: A family has 2 children. What is the probability of both of them being boys given one of them is a boy? (Equally likely)



F: One of them is a boy

$$F = \{(B,B), (B,G), (G,B)\}$$

$$P(F) = p(B,B) + p(B,G) + p(G,B)$$
$$= 1/4 + 1/4 + 1/4$$
$$= 3/4.$$

E: Both of them boys

$$E = \{(B,B)\}$$

$$E \cap F = E$$

$$P(E\cap F) = 1/4$$

$$P(E|F) = 1/4 \,/\, 3/4 = 1/3 \,//$$

Total Law of Probability

Suppose we have $A_1, A_2, \dots A_n$: Events

"Partition of sample space" $\begin{cases} 1. & A_1 \cup A_2 \cdots \cup A_n = \Omega \\ 2. & A_i \cap A_j = \phi \quad i \neq j \end{cases}$

$$P(B) = P(A_1)\,P(B|A_1) + P(A_2)\,P(B|A_2) \cdots\cdots + P(A_n)\,P(B|A_n)$$

$$B = B \cap \Omega$$
$$= B \cap \left(\bigcup_{i=1}^{n} A_i \right)$$
$$= \bigcup_{i=1}^{n} (A_i \cap B)$$

$$P(B) = P\left(\bigcup_{i=1}^{n} (A_i \cap B) \right)$$
$$= \sum_{i=1}^{n} P(A_i \cap B)$$
$$= \sum_{i=1}^{n} P(A_i)\, P(B \mid A_i) \quad //$$

Bayes' Theorem

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)}$$

$$\boxed{P(A_i \mid B) = \frac{P(A_i)\, P(B \mid A_i)}{P(B)}}$$

$$P(A_i \mid B) = \frac{P(A_i)\, P(B \mid A_i)}{\sum_{i=1}^{n} P(A_i)\, P(B \mid A_i)}$$

Total Law of Probability to denominator

Example 2.15 (PP)

A certain test for cancer is known to be 95% accurate. A person submits to this test and result is positive. Suppose it is known that this person comes from a population of 100,000 where 2,000 suffers from this disease. What is the probablity that the person has cancer?

T: Outcome of the test

H: Healthy person (no cancer)

C: Person with cancer.

$P(T=P \mid H) = 0.05$
$P(T=N \mid H) = 0.95$
$P(T=N \mid C) = 0.05$
$P(T=P \mid C) = 0.95$

$P(H) = 0.98$
$P(C) = 2000/100,000 = 0.02$

$P(C \mid T=P)$

" Probablity that person has cancer given test is positive)

$$= \frac{P(T=P \mid C)\ P(C)}{P(T=P)}$$

$$= \frac{P(T=P \mid C)\ P(C)}{P(H)\ P(T=P \mid H) + P(C)\ P(T=P \mid C)}$$

$$= \frac{0.95 \times 0.02}{0.98 \times 0.05 + 0.95 \times 0.02}$$

$$\approx 0.28$$

$\boxed{\text{Self-Study: (PP) Example 2-18, 2-20, 2-22}}$

Independent events

Two events E and F are said to be independent if $P(E|F) = P(E)$

$$= \frac{P(E \cap F)}{P(F)}$$

$$\Downarrow$$

$$\underline{P(E \cap F) = P(E)\,P(F)}$$

Example: Consider the following, we select x and y uniformly from 0 to 1.

$$A = \left\{ y > \frac{1}{2} \right\}$$

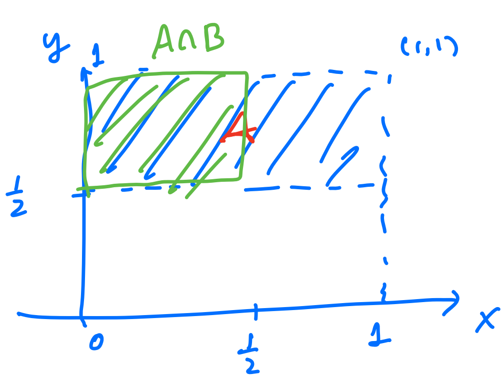
$$B = \left\{ x < \frac{1}{2} \right\}$$

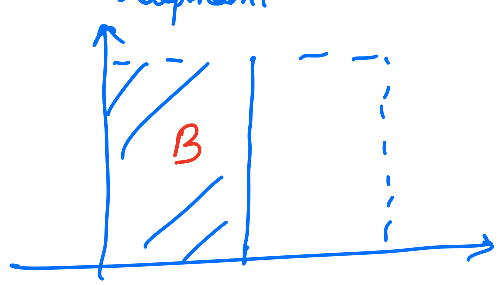
$$C = \left\{ x < \frac{1}{2},\ y < \frac{1}{2} \right\} \cup \left\{ x > \frac{1}{2},\ y > \frac{1}{2} \right\}$$

Show the following: A is independent of B ($A \perp\!\!\!\perp B$)

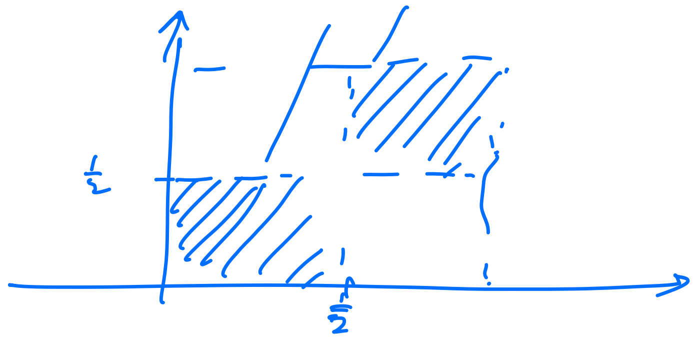
$B \perp\!\!\!\perp C$, $C \perp\!\!\!\perp A$

But A, B, C are not jointly independent.





C



$A \perp\!\!\!\perp B$?

$[\, B \perp\!\!\!\perp C$

Exercise $C \perp\!\!\!\perp A \,]$

$$P(A \mid B) \stackrel{?}{=} P(A)$$

$$P(A) = \frac{1}{2}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$P(A \mid B) = P(A)$$

A, B, C are not jointly independent

$$P(E \cap F) = P(E)\ P(F)$$

$$\underbrace{P(A \cap B \cap C)}_{=\,0} = \underbrace{P(A)}_{=\,\frac{1}{2}}\ \underbrace{P(B)}_{=\,\frac{1}{2}}\ \underbrace{P(C)}_{=\,\frac{1}{2}}$$