Finding probabilities by conditional expectation

$P(E)$

$\downarrow$

Event E: $\quad Z > 0$

$\uparrow$ r.v

$$X = \begin{cases} 1 & E \text{ happens} \\ 0 & \text{else} \end{cases}$$

$\uparrow$

indicator r.v

$$\mathbb{E}[X] = 1 \cdot P_X(E) = P(E)$$

$$P(E) = \mathbb{E}[X]$$

$$= \mathbb{E}[\,\mathbb{E}[X / Y]\,]$$

$\uparrow$ some. r.v. Y

Functions of multiple random variable

$X, Y \qquad Z = g(X, Y) \quad , \quad F_Z(z)$

$$F_Z(z) = P[Z \le z]$$

$$= P[g(X,Y) \le z]$$

$$= P[(X, Y) \in D_z]$$

D_z is the region in the (x, y) space s.t

$g(x, y) \le z$

$$= \iint_{D_z} f_{X,Y}(x, y)\, dx\, dy$$

Example: Let X and Y be independent exponential random variable with parameters λ_x and λ_y.

$Z = \dfrac{X}{Y}$. Find distribution of Z?

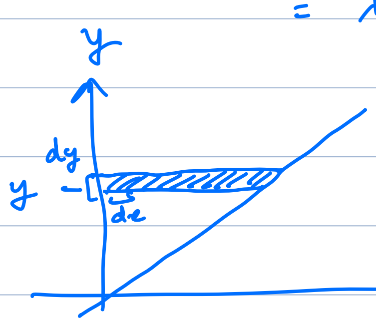
$$F_Z(z) = P(Z \le z)$$

$$= P\left(\frac{X}{Y} \le z\right)$$

$$= \iint_{(x,y)\,:\,\frac{x}{y} \le z} \underbrace{f_{XY}(x,y)}_{\substack{\text{Independence} \\ f_X(x)\, f_Y(y)}} \, dx\, dy$$

$$= \iint_{(x,y)\,:\, x/y \le z} \lambda_x e^{-\lambda_x x} \lambda_y e^{-\lambda_y y} \, dx\, dy.$$

$$= \lambda_x \lambda_y \iint_{(x,y)\,:\,\frac{x}{y} \le z} e^{-\lambda_x x} e^{-\lambda_y y} \, dx\, dy$$



$x = yz$

$$= \int_{y=0}^{\infty} \int_{x=0}^{yz} e^{-\lambda_x x} e^{-\lambda_y y} \, dx\, dy$$

$$\overset{\text{Exercise}}{=} \frac{\lambda_x \cdot z}{\lambda_x \cdot z + \lambda_y}$$

Examples in Sec 6.2 (PP)

Multiple functions of multiple random variables.

Requirements:

(i)

$(U, V) = (g(X,Y), h(X,Y))$

is a one-one mapping.

Notation X, Y, $f_{XY}(x,y)$

$U = g(X, Y)$

$V = h(X, Y)$

$f_{U,V}(u,v)$

Review (Calculus)

Change of variables

1 variable $\int_a^b f(x)\, dx$

$x(y)$

Change of variable from x to y.

$$\int_a^b f(x(y)) \left(\frac{dy}{dx}\right) \cdot dx$$

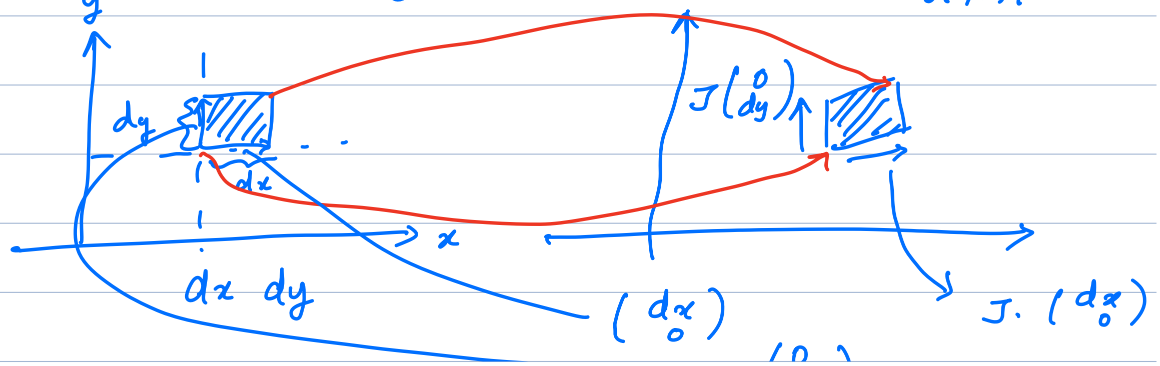
2 variables $\int\int f(x, y)\, dx\, dy$

Change of variables

$x = g(u, v)$

$y = h(u, v)$

$$\int\int f(g(u,v), h(u,v)) \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\text{Jacobian}} du\, dv$$



$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \longrightarrow (\breve{dy})$$

$$|J| \cdot dx \cdot dy.$$

$$f_{U,V}(u,v) = f_{X,Y}(x,y)\ |J|$$

Example: Suppose R is an exponential random variable with parameter $\frac{1}{2}$. Θ is a uniform random variable in $[0, 2\pi)$

$(R \perp\!\!\!\perp \Theta)$
R & Θ are independent

$$X = \sqrt{R}\ \cos\Theta$$

$$Y = \sqrt{R}\ \sin\Theta$$

Find $f_{X,Y}(x,y)$?

$$f_{R,\Theta}(r,\theta) = f_R(r)\ f_\Theta(\theta)$$

$$= \frac{1}{2} e^{-r/2} \cdot \frac{1}{2\pi}$$

$$= \frac{1}{2} \cdot \frac{1}{2\pi} e^{-r/2}$$

$$J = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix}$$

$$x^2 = r\cos^2\theta \quad , \quad y^2 = r\sin^2\theta$$

$$x^2 + y^2 = r \qquad \frac{y}{x} = \tan\theta \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial r}{\partial x} = 2x \qquad \frac{\partial r}{\partial y} = 2y$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + y^2/x^2} \left(-y/x^2\right) = \frac{-y}{x^2+y^2} = \frac{-y}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + y^2/x^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{x}{r}$$

$$|J| = \begin{bmatrix} 2x & 2y \\ -\frac{y}{r} & \frac{x}{r} \end{bmatrix} = \frac{2(x^2+y^2)}{r} = 2$$

$$f_{X,Y}(x,y) = f_{R,\theta}(r,\theta) \cdot 2$$

$$= \frac{1}{2} \cdot \frac{1}{2\pi} e^{-r/2} \cdot \cancel{2}$$

$$= \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

X and Y are independent standard Normal distribution.