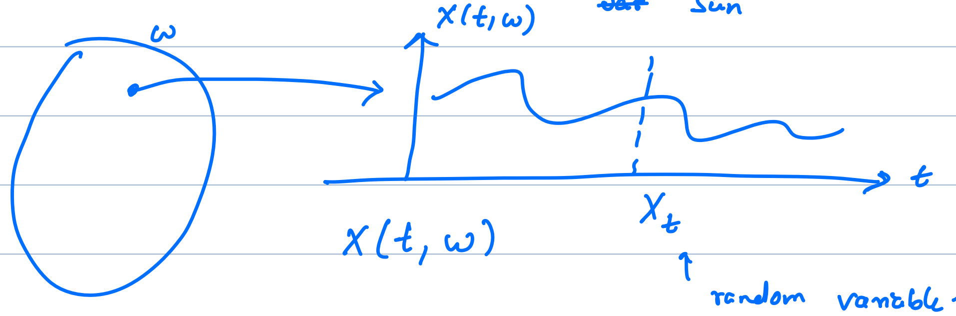
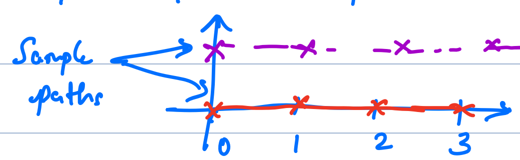
Announcements: Quiz 2 on Nov ~~27~~/28 2pm
~~Sat~~ Sun



Example 1: $\Omega = \{0, 1\}$ $P(0) = p$ $P(1) = 1-p$.

$x(n, \omega) = \omega$

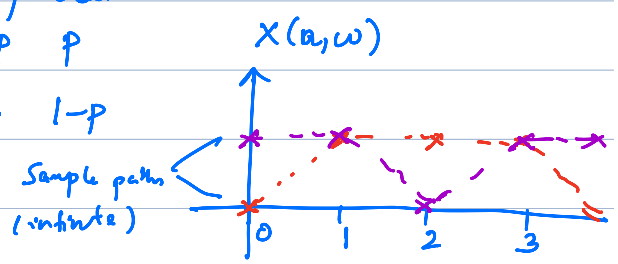
discrete time



Example 2: Bernoulli Random process

$$X_n = \begin{cases} 0 & \text{w.p } p \\ 1 & \text{w.p } 1-p \end{cases}$$

X_n's are iid



Example 1: $$X_n = \begin{cases} 0 & \text{w.p } p \\ 1 & \text{w.p } 1-p \end{cases}$$

Example 2: $$X_n = \begin{cases} 0 & \text{w.p } p \\ 1 & \text{w.p } 1-p \end{cases}$$

Finite dimensional distribution of samples.

Let $X_1, X_2, \ldots X_k$ be k random variables obtained by sampling the process $X(t,\omega)$ $[X(t)]$ at times $t_1, t_2, \ldots t_k$ $X_1 = x(t_1, \omega)$ $X_2 = x(t_2, \omega)$

- $P_{X_1 X_2 \ldots X_k}$

Discrete-valued

$$P_{X_1 X_2 \ldots X_k}(x_1, x_2, \ldots, x_k) = P[X_1 = x_1, \ldots, X_k = x_k]$$

"Joint pmf"

Continous-valued

$$f_{X_1 X_2 \ldots X_k}(x_1, x_2, \ldots x_k) = \frac{\partial^k}{\partial x_1 \ldots \partial x_k} F_{X_1 \ldots X_k}(x_1, x_2, \ldots x_k)$$

$$F_{X_1 X_2 \ldots X_k}(x_1, x_2, \ldots x_k) = P[X_1 \le x_1, \ldots, X_k \le x_k]$$

Example: Bernoulli Random Process

$X(n)$, $X_i = X(t_i)$

$\downarrow$ sampled $X(n)$ at $t = t_1, t_2, \ldots t_k$

$$P[X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k]$$
$$= P[X_1 = x_1]\, P[X_2 = x_2] \cdots P[X_k = x_k]$$
$$= (1-p)^{\#1s}\; p^{\#0s}$$

$$\#1s = \sum_i 1\{x_i = 1\}$$
$$\#0s = \sum_i 1\{x_i = 0\}$$

Kolmogrov Extension Theorem

Suppose we are given a "consistent" family of finite dimensional distribution $P_{X_1 X_2 \ldots X_k}$ for all positive integer k and for all posible times $t_1, t_2, \ldots t_k$.

Then there exists a random process "consistent" with this family.

Definitions

1- Independent increment property.

A RP is said to have the independent increment property if

$t_1 < t_2 < t_3 \cdots < t_k$

then if the random variables

$X(t_2)-X(t_1),\ X(t_3)-X(t_2) \cdots,\ X(t_k)-X(t_{k-1})$

are mutually independent.

2. Markov property.

A RP is said to have the Markov property if the "future of the process given the present is independent of the past"

$t_1 < t_2 < \cdots < t_k$

" Discrete - case "

$$P[\ X(t_k)=x_k \mid X(t_{k-1})=x_{k-1}, \ldots, X(t_1)=x_1]$$

$\leftarrow$ future $\rightarrow$ $\leftarrow$ Present $\rightarrow$ $\leftarrow$ Past $\rightarrow$

k $\quad k-1 \quad 0 \cdots k-2$

$$= P[\ X(t_k)=x_k \mid X(t_{k-1})=x_{k-1}]$$

" Continous - case "

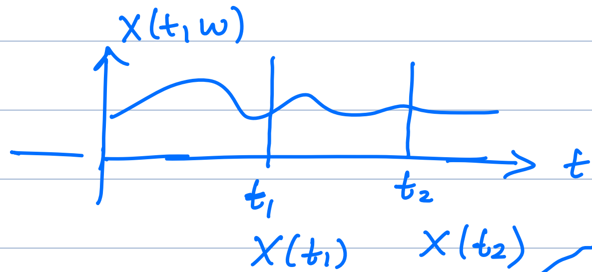
$$f_{X(t_k) \mid X(t_{k-1}) \cdots X(t_1)}(x_k \mid X(t_{k-1})=x_{k-1}, \cdots X(t_1)=x_1)$$

$$= f_{X(t_k) \mid X(t_{k-1})} \left(x_k \mid X(t_{k-1}) = x_{k-1} \right)$$

3. Mean function

$$m_X(t) = \mathbb{E}[\, X(t) \,]$$

$$= \int x \; \underbrace{f_{X(t)}(x)}_{\text{p.d.f of the r.v } X(t) \text{ sampled at } t=t} \, dx$$



4.)

Auto-correlation

$$R_X(t_1, t_2) = \mathbb{E}[\, X(t_1) \; X(t_2) \,]$$

$$\boxed{\text{Cross-correlation } X, Y \quad \mathbb{E}[\, X(t_1) \; Y(t_2) \,]}$$

Auto-covariance

$$C_X(t_1, t_2) = \mathbb{E}\left[\left(X(t_1) - m_X(t_1) \right) \left(X(t_2) - m_X(t_2) \right) \right]$$

Exercise!

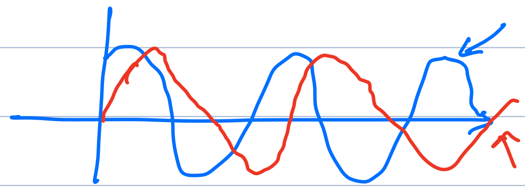
$$= R_X(t_1, t_2) - m_X(t_1)\, m_X(t_2)$$

Variance of the r.v $X(t)$

$$\operatorname{Var}[X(t)] = \mathbb{E}\left[\left(X(t) - m_X(t) \right)^2 \right]$$

$$= C_X(t, t)$$

Example: Let $X(t) = \cos(\omega t + \theta)$, where θ is uniformly distributed in the interval $(-\pi, \pi]$. Find mean, autocorrelation and autocovariance ?



$$m_x(t) = E[\cos(\omega t + \theta)]$$

$$= \int_{-\pi}^{\pi} \cos(\omega t + \theta) \cdot \frac{1}{2\pi} \, d\theta$$

$$= 0 \quad //$$

$$R_x(t_1, t_2) = E[\cos(\omega t_1 + \theta) \cos(\omega t_2 + \theta)]$$

$$= E\left[\frac{\cos(\omega t_1 - \omega t_2) - \cos(\omega t_1 + \omega t_2 + 2\theta)}{2} \right]$$

$$= \frac{\cos(\omega(t_1 - t_2))}{2} - \frac{1}{2} \underbrace{E[\cos(\omega t_1 + \omega t_2 + 2\theta)]}_{\int_{-\pi}^{\pi} \cos(\omega t_1 + \omega t_2 + 2\theta) \cdot \frac{1}{2\pi} \cdot d\theta}$$

$$= \frac{1}{2} \cos(\omega(t_1 - t_2)) \quad // \qquad = 0$$

Auto-covariance : Exercise !

Ref: "Garcia"