Announcements: No lab this week.

Convergence of random variables

"Convergence": $x_1, x_2, \ldots$

$$\lim_{n\to\infty} x_n = x$$

$\varepsilon > 0$ "Small positive value" $\quad |x_n - x| < \varepsilon \quad \forall n > N$

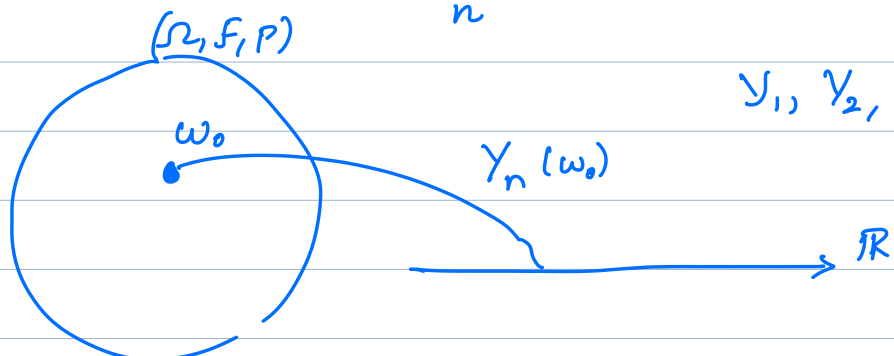
depends on ε

Ex: $x_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$\varepsilon > 0 \quad \left|\frac{1}{n} - 0\right| < \varepsilon$

$\frac{1}{n} < \varepsilon \quad n > N = \left\lceil \frac{1}{\varepsilon} \right\rceil$



$Y_1, Y_2, Y_3, \ldots$

Fix ω $\quad Y_1(\omega), Y_2(\omega), Y_3(\omega) \ldots$

$$\lim_{n\to\infty} Y_n(\omega) = Y(\omega) \quad \forall \omega$$

"Pointwise convergence"

Idea: Maybe pointwise convergence for a "big" subset of Ω, maybe "few" ω's where there is no convergence

define using prob measure.

$$P\left(\omega : \lim_{n\to\infty} Y_n(\omega) = Y(\omega)\right) = 1$$

"Convergence with probability 1"

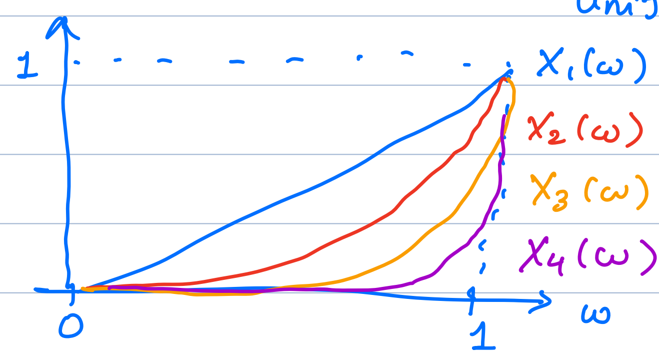
"almost sure" (a.s)

"almost everywhere"

Example:

$$X_n(\omega) = \omega^n \qquad \omega \in [0,1]$$

"Uniform"



$$\lim_{n\to\infty} X_n(\omega) = X(\omega)$$

$$X(\omega) = 0 \quad \forall \omega$$

Pointwise convergence? $\omega \in (0,1]$

$0 < \omega < 1$

$$|\omega^n - 0| < \varepsilon \qquad n > N$$

$$\omega^n < \varepsilon$$

$$n \log \omega < \log \varepsilon$$

$$n > \frac{\log \varepsilon}{\log \omega}$$

$\omega = 1$

$$|1 - 0| < \varepsilon$$

$$P\left(\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right) = 1 - \underbrace{P(\{1\})}_{0}$$

$$= 1 \;//$$

Convergence in mean square sense (m.s)

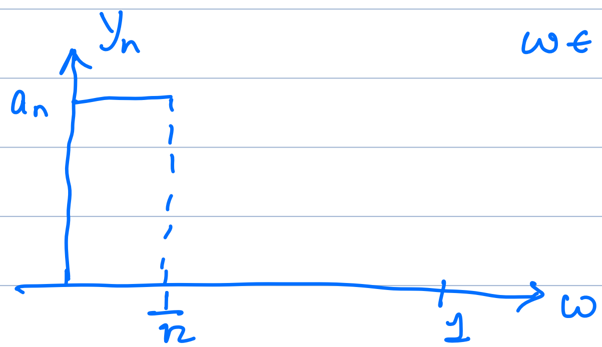
$Y_1, Y_2, \ldots\ldots$

$$Y_n \xrightarrow{m.s} Y$$

if $\lim_{n\to\infty} \mathbb{E}\left[\,|Y_n - Y|^2\right] = 0$

Example:

$\omega \in (0, 1]$



$$Y_n = \begin{cases} a_n & \text{with probablity } \frac{1}{n} \\ 0 & \quad \text{"} \quad 1 - \frac{1}{n} \end{cases}$$

$$Y = 0 \;//$$

$$\mathbb{E}\left[\,|Y_n - Y|^2\right] = \mathbb{E}\left[Y_n^2\right]$$

$$= \frac{a_n^2}{n}$$

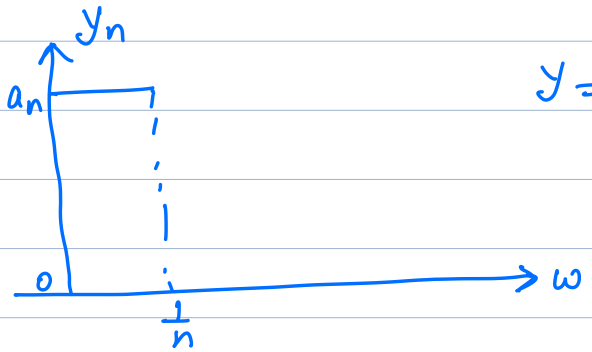
$$\lim_{n\to\infty} \mathbb{E}\left[\,|Y_n - Y|^2\right] = 0 \quad \text{if } \frac{a_n^3}{n} \longrightarrow 0$$

Exercise: There is no pointwise convergence?

Convergence in probability. (p) $\xrightarrow{P}$

A sequence of random variables Y_n is said to convergence in probability to Y

$$\lim_{n\to\infty} P(|Y_n - Y| > \varepsilon) = 0 \quad \forall \varepsilon > 0$$



$Y = 0$?

$\varepsilon > 0$

$$Y_n = \begin{cases} a_n & \text{w.p } \frac{1}{n} \\ 0 & \text{w.p } 0 \end{cases}$$

$P(|Y_n - 0| > \varepsilon)$

$P(Y_n > \varepsilon)$

$n > \frac{1}{\varepsilon}$ $\to 0$

$$Y_n \xrightarrow{P} Y = 0$$

Convergence in distribution ($\xrightarrow{d}$)

A sequence of random variables $\{Y_n\}$ Converges in distribution to random variable Y if

cdf

$$\lim_{n\to\infty} F_{Y_n}(y) = F_Y(y) \quad \forall y$$

* [Technical requirement: Continuity points]

Theorem (Relation b/w notions of convergence)

$$\begin{array}{l} a.s \Rightarrow \\ \qquad\qquad p \Rightarrow d \\ m.s \Rightarrow \end{array}$$

Proof: Bruce Hajek
(Skipped)