Quiz 1 Graded: - Qn 4 Part involving X ignored

- Grades mapped to /10

$x/15 \rightarrow x/10$

- Test 1 for 20 points

Lab Exp 1: Nov 10

Conditional distribution

Recap: $P(E|F) = \frac{P(E \cap F)}{P(F)}$

X, Y : random variables

Discrete: Conditional pmf $P_{Y|X}(y|x)$

$$P_{Y|X}(y|x) = P(Y=y \mid X=x)$$

$$= \frac{P(Y=y, X=x)}{P(X=x)}$$

$$= \frac{P(\{\omega: Y(\omega)=y\} \cap \{\omega: X(\omega)=x\})}{P(\{\omega: X(\omega)=x\})}$$

$$= \frac{P_{XY}(x,y)}{P_X(x)}$$

$$P_{Y|X}(y|x) = \frac{P_{XY}(x,y)}{P_X(x)}$$

Properties:

1. $P_{Y|X}(y|x)$ is a pmf

$$\sum_y P_{Y|X}(y|x) = \sum_y \frac{P_{X,Y}(x,y)}{P_X(x)}$$

$$= \frac{1}{P_X(x)} \sum_y \underbrace{P_{XY}(x,y)}_{\rightarrow \text{ Marginilzation}} = \frac{1}{P_X(x)} P_X(x) = 1$$

2. Bayes' theorem

$$P_{XY}(x,y) = P_X(x)\, P_{Y/X}(y|x)$$

$$P_{X/Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)}$$

$$= \frac{P_X(x)\, P_{Y/X}(y|x)}{P_Y(y)}$$

3. Chain Rule

$$P_{X_0X_1}(x_0, x_1) = P_{X_0}(x_0)\, P_{X_1|X_0}(x_1|x_0)$$

$$P_{X_0\underbrace{X_1X_2}_{Y}}(x_0, x_1, x_2) = P_{X_0}(x_0)\, P_{Y/X_0}(y|x_0)$$

$$= P_{X_0}(x_0)\, \underbrace{P_{X_1X_2|X_0}(x_1x_2|x_0)}$$

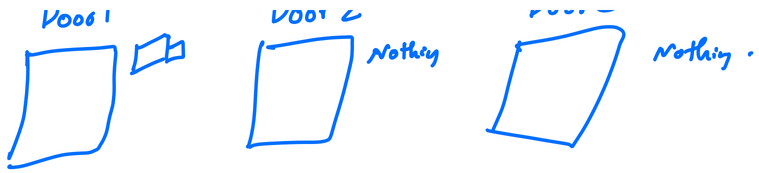
$$= P_{X_0}(x_0)\, P_{X_1|X_0}(x_1|x_0)\, P_{X_2|X_1X_0}(x_2|x_0x_1)$$

$X_0\ X_1 \dots X_{n-1}$

$$P_{X_0X_1\dots X_{n-1}}(x_0, x_1, \dots, x_{n-1}) = P_{X_0}(x_0) \prod_{k=1}^{n-1} P_{X_k|X_0\dots X_{k-1}}(x_k|x_0\dots x_{k-1})$$

Example: Monty Hall Problem

D 1 Door 2 Door 3



"Host knowns the location of car"

Step 1: Contestont comes & chooses one of the door. Host doesn't open the door the contestant chooses

Step 2: Host will open one of the 2 doors contestant has not chosen

Step 3: Contestont given an option

(i) either to stick with previous choice

(ii) switch to the other closed door.

P(Winning the car | Stick to original choice) $= \frac{1}{3}$

P(Win the car if you switch) $\begin{cases} 2/3 \\ 1/2 \end{cases}$

C: Choice of contestant

H: Choice of Host (to open the door)

$P[H = 1 \mid C] = 0$

" Host never opens the door with car"

$P[H = 3 \mid C = 2] = 1$

$P[H = 2 \mid C = 3] = 1$

$P[H = 2 \mid C = 1] = 1/2$

$$P[\ H = 3 \mid C = 1\ ] = 1/2$$

P[Winning the Car when you switch]

$$= \underbrace{P[\ C = 2\ ]}_{\frac{1}{3}}\ \underbrace{P[H = 3 \mid C=2]}_{1} + \overset{\nearrow \frac{1}{3}}{P[\ C = 3\ ]}\ \underbrace{P[\ H = 2 \mid C = 3\ ]}_{1}$$

$$= \frac{1}{3} + \frac{1}{3} = 2/3\ //$$

Example: If X and Y are independent Poisson random variable with rate λ_1 and λ_2.

Conditional probability of $P_{X|_{X+Y}}(x \mid X + Y = n)$

$$P(\ X = k \mid X + Y = n)$$

$$= \frac{P(\ X = k,\ X + Y = n)}{P(\ X + Y = n)}$$

$$= \frac{P(X = k\ ,\ Y = n-k)}{P(\ X + Y = n)}$$ independence

$$= \frac{P(\ X = k)\ P(\ Y = n-k)}{P(\ X + Y = n)}$$

$X \sim$ Poisson (λ_1)

$Y \sim$ Poisson (λ_2)

$X + Y \sim$ Poisson $(\lambda_1 + \lambda_2)$

$$= \cancel{e^{-\lambda_1}}\ \frac{\lambda_1^{k}}{k!}\qquad \cancel{e^{-\lambda_2}}\ \frac{\lambda_2^{n-k}}{(n-k)!}$$

$$e^{-\cancel{(\lambda_1 + \lambda_2)}} \frac{(\lambda_1 + \lambda_2)^n}{n!}$$

$$= \underbrace{\frac{n!}{k!\,(n-k)!}}_{\binom{n}{k}} \frac{\lambda_1^k \, \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^k \, (\lambda_1+\lambda_2)^{n-k}}$$

$$\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$$\text{Binomial}\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

Conditional distribution (Continuous)

Conditional pdf

$$f_{Y/X}(y|x) \triangleq \frac{f_{XY}(x,y)}{f_Y(y)}$$

Properties

1. $\int f_{Y/X}(y|x)\, dy = 1$

2. Chain Rule , Baye's rule.

3. $P(Y \in F \mid X = x) = \int_F f_{Y/X}(y|x)\, dy$

Example: $f(x,y) = \begin{cases} e^{-x/y} \, e^{-y} & 0 < x < \infty \\ & 0 < y < \infty \\ 0 & \text{else} \end{cases}$

$P[X > 1 \mid Y = y]$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_0^\infty f(x, y)\, dx$$

$$= \int_0^\infty e^{-x/y} e^{-y}\, dx$$

$$= e^{-y} \cdot \frac{e^{-x/y}}{-1/y} \Big|_0^\infty$$

$$= e^{-y} y$$

$$f_{X/Y}(x/y) = \frac{e^{-x/y} \cancel{e^{-y}}}{y \cdot \cancel{e^{-y}}}$$

$$= \frac{1}{y} e^{-x/y}$$

$$P(X > 1 / Y = y) = \int_1^\infty \frac{1}{y} e^{-x/y} \cdot dx$$

$$= 1/y \; \frac{e^{-x/y}}{-1/y} \Big|_1^\infty$$

$$= e^{-1/y} \; //$$

Exercise: $f(x, y) = \begin{cases} \frac{12}{5} x(2 - x - y) & 0 < x < 1 \\ & 0 < y < 1 \\ 0 & \text{else.} \end{cases}$

Find $f_{X/Y}(x/y)$

Exercise: Consider $\overset{10}{n} + \overset{10}{m}$ trials of an experiment having common prob of sucess.
The probability of success is unknown but is chosen from a $U(0,1)$ distributn.

What is the conditional probability of the sucess given n sucess (out of n+m trials).

Coin → Toss it 20 times

"p"

$p \sim U(0, 1)$ 10 times H (Success)

$p = 0.33$ 10 T (Fail)

$$f_{P \mid T = n}$$

T ↓ # of trials