Last class: Independent random variables

$P[X \in A, Y \in B] = P(X \in A)\, P(Y \in B)$ $\quad [P((X,Y) \in A \times B)]$

Discrete: $p_{XY}(x,y) = p_X(x)\, p_Y(y)$

Continous: $f_{XY}(x,y) = f_X(x)\, f_Y(y)$

Mutual independence of n r.vs $X = (X_0, \dots X_{n-1})$

$$P_{X_0 X_1 \dots X_{n-1}}\left(\prod_{i=0}^{n-1} F_i\right) = \prod_{i=0}^{n-1} P_{X_i}(F_i)$$

Special Case: $\underbrace{\text{IID}}$ $\quad X = (X_0 \dots X_{n-1})$

Independent & Identically distributed

1. Independence

$$P_{X_0 \dots X_{n-1}}\left(\prod_{i=0}^{n-1} F_i\right) = \prod_{i=0}^{n-1} P_{X_i}(F_i)$$

$\swarrow$

2. Identically distributed

$$P_{X_i}(F_i) = P(F)$$

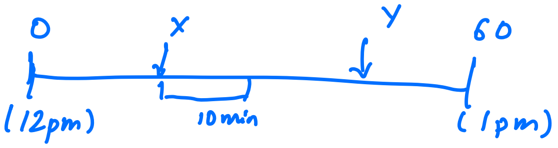
Discrete case: (IID)

$$P_{X_0 \dots X_{n-1}}(x_0 \dots x_{n-1}) = \prod_{i=0}^{n-1} p(x_i)$$

Continous case: (IID)

$$f_X(x) = \prod_{i=0}^{n-1} f_{X_i}(x_i) = \prod_{i=0}^{n-1} f(x_i)$$

Example: A man & a woman decide to meet at a certain location. If each of them independently arrive at a time uniformly distributed b/w 12pm & 1pm. Find the probability that the first person to arrive has to wait more than 10 mins ?



X : time at which man arrives

Y : " " " woman "

$f_X(x) = \frac{1}{60}$ $f_Y(y) = \frac{1}{60}$

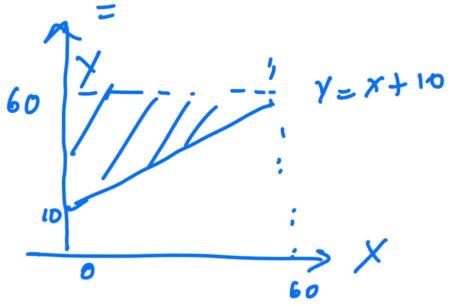
$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \left(\frac{1}{60}\right)^2$

2 cases

X arrives first

$P(X+10 < Y)$

=



Y arrives first

$P(Y+10 < X)$

$= 25/72$

$$\iint_{x+10<y} \underbrace{f_{XY}(x,y)}_{\left(\frac{1}{60}\right)^2} \, dx\, dy$$

$$= \left(\frac{1}{60}\right)^2 \int_{10}^{60} \int_0^{y-10} dx \quad dy .$$

$$= \left(\frac{1}{60}\right)^2 \int_{10}^{60} (y-10)\, dy .$$

$$= \frac{25}{72} //$$

P(first to arrive waits more than 10 min)

$= 2 \cdot 25/72 = 25/36 //$

Expectations : $\mathbb{E}[X] = \int x \cdot f_X(x)\, dx$ [Random variable]

(random vector)

Random vector

vector

$$\mathbb{E}[X] = \underbrace{\int \cdots \int}_{n \text{ integrals}} (x_0, x_1 \cdots x_{n-1}) \, f_{X_0 X_1 \cdots X_{n-1}}(x_0 \cdots x_{n-1}) \, dx_0 \, dx_1 \cdots dx_{n-1}$$

Discrete case

$$= \sum_{x_0 \cdots x_{n-1}} \underbrace{(x_0 \cdots x_{n-1})}_{\text{vector}} \; \underbrace{P_{X_0 \cdots X_{n-1}}}_{\text{number}} (x_0 \cdots x_{n-1})$$

Properties

1. Fundamental theorem of expectation

$$\mathbb{E}[g(X)] = \begin{cases} \int g(x) \, f_X(x) \, dx \\ \sum g(x) \, P_X(x) \, dx \end{cases}$$

$X = (X_0, X_1)$

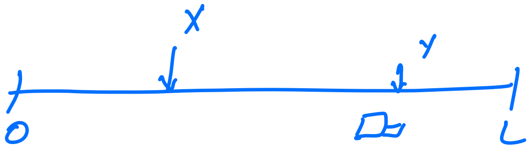
$g(X) = X_0 + X_1$

scalar

$g(X) = (X_0 + X_1, \; X_0 - X_1)$

vector

Example: An accident occurs at a point uniformly distributed on a road of length L. An ambulance is at a location Y that is also uniform distributed on the road of length L. Find the expected distance b/w point of accident & ambulance?



$$\mathbb{E}[|X-Y|] = \int \int |x-y| \, f_{XY}(x,y) \, dx \, dy$$

$(\mathbb{E}[g(X)])$

$$f_X(x) = \frac{1}{L} \quad f_Y(y) = \frac{1}{L} \quad f_{XY}(x,y) = \left(\frac{1}{L}\right)^2$$

$$= \int_0^L \int_0^L |x-y| \left(\frac{1}{L}\right)^2 dx \, dy$$

$$= \left(\frac{1}{L}\right)^2 \int_0^L \int_0^L |x-y| \, dx \, dy$$

Idea: $\int_0^L |x-y| \, dx$

$$= \int_0^y -(x-y)dx + \int_y^L (x-y) \, dx$$

[Exercise]

$$\mathbb{E}[\,|X-Y|\,] = \frac{L}{3} \; //$$

Exercise: (Coupon collecting problem)

N different coupons

Each time you are equally likely to get one of these N coupons.

Find the expected number of coupons one need to collect to have a complete set ?

2. Property (Linearity)

$$\mathbb{E}[aX + bY] = a\,\mathbb{E}[X] + b\,\mathbb{E}[Y]$$

3. For any two independent random variable X and Y

$$\mathbb{E}[g(X)\,h(Y)] = \mathbb{E}[g(X)]\,\mathbb{E}[h(Y)]$$

Proof: $\int\int g(x)\,h(y)\,\underbrace{f_{XY}(x,y)}\,dx\,dy$

[Extend

' to multiple rvs]

$f_X(x) \quad f_Y(y)$

$\int g(x)\, f_X(x) \qquad \int h(y)\, f_Y(y)$

Special Case*: $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$

Correlation

Correlation b/w random variables X and Y is defined as $\mathbb{E}[XY]$

General case (random vector) $\mathbb{E}[X_0\, X_1 \ldots X_{n-1}]$ (Correlation)

Two random variables are uncorrelated if $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$

$$\text{Independence} \Longrightarrow \text{Uncorrelated}$$

$$\overset{?}{\Longleftarrow}$$

In general "No"

Theorem: Two random variables X and Y are independent only if $g(X)$ and $h(Y)$ are uncorrelated for all functions g and h (technical conds skipped)

Covariance (2 random variables)

$$\mathrm{Cov}(X,Y) \triangleq \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

$$= \mathbb{E}[XY - \mathbb{E}[X]Y - \mathbb{E}[Y]X + \mathbb{E}[X]\,\mathbb{E}[Y]]$$

$$= \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] - \cancel{\mathbb{E}[Y]\,\mathbb{E}[X]} + \cancel{\mathbb{E}[X]\,\mathbb{E}[Y]}$$

$$= \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$$

← Correlation →

Property: Two random variables X and Y are uncorrelated $\boxed{\text{iff}}$ $\mathrm{Cov}(X, Y) = 0$

if & only if

Properties of $\mathrm{Cov}(X, Y)$

1. $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$
2. $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$
3. $\mathrm{Cov}(aX, Y) = a\,\mathrm{Cov}(X, Y)$
4. $\mathrm{Cov}\left(\sum_i X_i, \sum_j Y_j\right) = \sum_i \sum_j \mathrm{Cov}(X_i, Y_j)$

Next class - Covariance matrix