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ITERATIVE SVD METHOD FOR NOISE REDUCTION OF LOW-DIMENSIONAL CHAOTIC TIME SERIES

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A new simple method using singular value decomposition (SVD) is presented for reducing noise from a sampled signal where the deterministic signal is from a low-dimensional chaotic dynamical system. The technique is concerned particularly with improving the reconstruction of the phase portrait. This method is based on time delay embedding theory to form a trajectory matrix. SVD is then used iteratively to distinguish the deterministic signal from the noise. Under certain conditions, the method can be used almost blindly, even in the case of a very noisy signal (e.g. a signal to noise ratio of 6 dB). The algorithm is evaluated for a chaotic signal generated by the Duffing system, to which white noise is added.

1. INTRODUCTION

Over the last couple of decades, chaotic systems have been extensively studied and many remarkable results have been achieved in understanding very complex phenomena produced by simple non-linear dynamical systems, such as reconstruction of the phase portrait, and the estimation of Lyapunov exponents and fractal dimensions from a time series. The majority of work has been based on computer simulations. However, noise is always problematic in a practical situation. Noise reduction for a time series may be considered as the filtering of a noisy signal to extract a relatively clean signal. There are many methods of filter-based noise reduction, including optimal filtering (Wiener filter). However, the filtered chaotic time series may be altered fundamentally, so the inherent dynamical properties (dimensions, Lyapunov exponents, etc.) of the original noise-free chaotic signal may not be obtained successfully from the filtered time series. For example, Badii *et al*. [1] showed that filtering processes may introduce additional spurious Lyapunov exponents and may cause an increase of the fractal dimension. Broomhead *et al*. [2], however, proved that finite impulse response (FIR) filters (finite-order and non-recursive filters) do not have this effect. Thus, filters for chaotic time series must be FIR filters and applicable to non-stationary time series because chaotic time series are generally strongly non-stationary. Recently many different noise-reduction methods for chaotic time series have been developed [3–13], many of them based on the considerations of geometrical properties by using the embedding methods such as the 'method of delays' [14] and SVD (singular value decomposition) [15]. It is claimed that some produce very good noise reduction. However, these methods are usually very complicated and difficult to implement, and also require many aspects to be considered carefully. In this paper, a very simple and effective noise reduction method is presented. The term 'simple' means that there are only two parameters required to apply this method, namely the sampling rate

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and the embedding dimension. It is based on the algorithm of reconstruction of phase portrait in [15] which is very useful for the reconstruction of the phase portrait from a noisy signal.

This paper assumes that the noise is additive and white. Other than this, no prior knowledge of the noise-free signal is assumed. The signal used in this paper is from the Duffing equation

$$\ddot{x} + c\dot{x} - \varepsilon x(1 - x^2) = A \cos(\omega t) \tag{1}$$

with parameters for $\varepsilon = 1$, $c = 0.4$, $A = 0.4$, and $\omega = 1$. A sampled displacement signal $x(k)$ is obtained using one fourth-order Runge–Kutta method with a fixed integration step size of 0.1 s, giving a sampling frequency f_s of 10 Hz. This signal is assumed to be clean. The Gaussian white noise $n(k)$ is added to the clean signal, so the noise contaminated signal $s(k)$ is given by

$$s(k) = x(k) + n(k). \tag{2}$$

Two cases of the standard deviation of noise are considered: 10% and 50% of standard deviation of the clean signal (Fig. 1). The corresponding signal to noise ratios [SNR $= 10 \log(\sigma_x^2/\sigma_n^2)$] are 20 and 6 dB, respectively, where σ_x and σ_n are the standard deviation of the clean signal and the noise signal, respectively.

2. RECONSTRUCTION OF PHASE PORTRAIT AND SVD

A useful method of reconstruction of phase portraits based on SVD was introduced by Broomhead *et al*. [15]. This method provides phase portrait reconstruction with little noise reduction. From the measured discrete time series $\{v_i \mid i = 1, \ldots, N_T\}$ where N_T is the

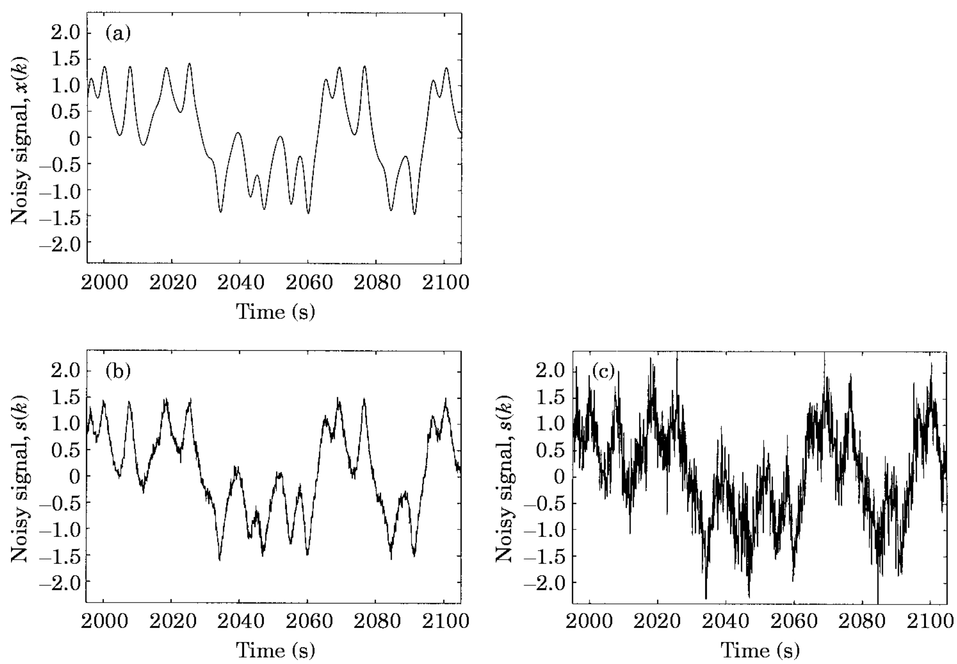

Figure 1. (a) Clean signal; (b) noisy signal, 10% white noise added (SNR = 20 dB); (c) noisy signal, 50% white noise added (SNR = 6 dB).

number of data points, a sequence of vectors $\{\mathbf{x}_i \in \Re^n \mid i = 1, \ldots, N\}$ can be generated and the trajectory matrix $\mathbf{X}$ can be constructed in n-embedding dimension

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \\ \vdots \\ \mathbf{x}_N^T \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ v_2 & v_3 & \cdots & v_{3n+1} \\ \vdots & & \ddots & \vdots \\ v_N & v_{N+1} & \cdots & v_{N+n-1} \end{bmatrix} \tag{3}$$

where $N = N_T - (n-1)$. Note that the matrix $\mathbf{X}$ is the pseudo-phase portrait (by the 'method of delays') in n-dimensional pseudo-phase space with a delay time of 'one unit'. The SVD of the trajectory matrix gives

$$\mathbf{X} = \mathbf{S}\Sigma\mathbf{C}^T \tag{4}$$

where, $\mathbf{S}$ is the $N \times n$ matrix of eigenvectors of $\mathbf{X}\mathbf{X}^T$ and $N \gg n$, $\mathbf{C}$ is the $n \times n$ matrix of eigenvectors of $\mathbf{X}^T\mathbf{X}$ and Σ is the $n \times n$ diagonal matrix consisting of singular values, i.e. diag $(\sigma_1, \sigma_2, \ldots, \sigma_n)$. Rearranging equation (4),

$$\mathbf{X}\mathbf{C} = \mathbf{S}\Sigma \tag{5}$$

The matrix $\mathbf{XC}$ is the trajectory matrix projected onto basis $\{\mathbf{c}_i\}$, where $\mathbf{c}_i$ is the ith column of $\mathbf{C}$. One can think of the trajectory as exploring, on average, an n-dimensional ellipsoid, where $\{\mathbf{c}_i\}$ represent directions and $\{\boldsymbol{\sigma}_i\}$ represent the lengths of the principal axes of the ellipsoid [15]. The main concept of this method is to extract the dimensionality n' (minimum embedding dimension) of the subspace containing the embedded manifold, where, $n' \leqslant n$. The dimensionality n' is the rank of the eigenvector matrices [rank($\mathbf{S}$) = rank($\mathbf{C}$)], where the rank is the number of non-zero singular values. At this point, one can intuitively think of the physical meaning of the dimensionality n' as an effective embedding dimension. In other words, the matrix $\mathbf{XC}$ with embedding dimension n' has no less information than the matrix with embedding dimension n. Also, note that the SVD ensures that each column of the matrix $\mathbf{XC}$ is linearly independent. In the presence of noise, the noise causes all the singular values of the trajectory matrix to be non-zero. However, assuming the noise is white, the noise will cause all the singular values of $\mathbf{X}$ to be shifted uniformly, i.e., they can be written as

$$\begin{aligned} &\sigma_i^2 = \bar{\sigma}_i^2 + \sigma_{noise}^2 \qquad\qquad i = 1, 2, \ldots, k \\ &\sigma_{k+1}^2 = , \ldots, = \sigma_n^2 = \sigma_{noise}^2 \end{aligned} \tag{6}$$

where σ_{noise} are the singular values of the noise floor, and the trajectory matrix can also be written as

$$\mathbf{X} = \bar{\mathbf{X}} + \mathbf{N} = [\mathbf{S}_1 \quad \mathbf{S}_2]\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\begin{bmatrix} \mathbf{C}_1^T \\ \mathbf{C}_2^T \end{bmatrix} \tag{7}$$

where $\bar{\mathbf{X}}$ is the deterministic part of the trajectory matrix, $\mathbf{N}$ is the noise-dominated part, $\mathbf{S}_1 \in \Re^{N \times k}$, $\Sigma_1 \in \Re^{k \times k}$, and $\mathbf{C}_1 \in \Re^{n \times k}$. In order to separate the noise-dominated part from the trajectory matrix, one can estimate the deterministic part $\bar{\mathbf{X}}$ by either least squares or minimum variance estimate. The least squares estimate of $\bar{\mathbf{X}}$ is given by [16–18]

$$\bar{\mathbf{X}}_e = \mathbf{S}_1 \Sigma_1 \mathbf{C}_1^T \tag{8}$$

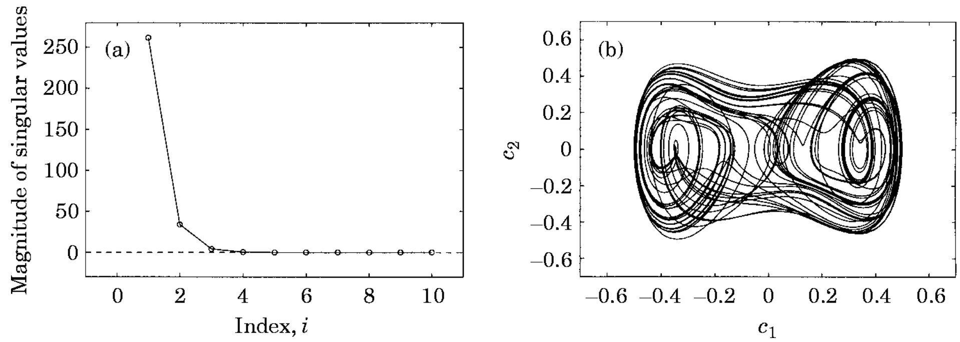

Figure 2. (a) Singular values of the trajectory matrix constructed by the clean signal $x(k)$. (b) Pseudo-phase portraits reconstructed by SVD (normalised version of the matrix $\bar{\mathbf{X}}_e\mathbf{C}_1$).

and the minimum variance estimate is given by [16, 17]

$$\bar{\mathbf{X}}_e = \mathbf{S}_1\Sigma_1^{-1}(\Sigma_1^2 - \sigma_{noise}^2\mathbf{I}_k)\mathbf{C}_1^T \tag{9}$$

where $\mathbf{I}_k$ is the $k \times k$ identity matrix. From equation (8) or (9), one can see that the deterministic part of the trajectory matrix can be estimated by using the SVD of $\mathbf{X}$. The matrix $\mathbf{XC}$ now becomes $\bar{\mathbf{X}}_e\mathbf{C}_1$ which is less noisy. However, if the smallest singular value of the deterministic part is not significantly greater than the noise level, the above method is a little problematic, especially for reconstruction of phase portraits.

Equation (6) has a special meaning in that the ratio of singular values $\sigma_i > \sigma_{noise}$ (above the noise floor) and σ_{noise} represents the signal-to-noise ratios which are associated with each singular vector $\mathbf{s}_i$ (each column of the matrix $\mathbf{S}$). This means that the SNR of each column is different, and so the signal-to-noise ratio of the i-th left singular vector can be written as

$$\mathrm{SNR}_i = 10 \log \frac{\sigma_i^2 - \sigma_{noise}^2}{\sigma_{noise}^2}. \tag{10}$$

Thus, '$\sigma_1^2/\sigma_{noise}^2$' represents the SNR of the first column of the matrix $\bar{\mathbf{X}}_e\mathbf{C}_1$, and '$\sigma_2^2/\sigma_{noise}^2$' represents the SNR of the second column of the matrix $\bar{\mathbf{X}}_e\mathbf{C}_1$, and so on. Thus, the reconstructed phase portraits using the above method may be degraded by the part associated with the singular values which are not significantly greater than σ_{noise}. The singular values of the trajectory matrix constructed by the clean signal $x(k)$ and the pseudo-phase portrait by SVD are shown in Fig. 2. The above problem is shown in Fig. 3 for 10% white noise and in Fig. 4 for 50% white noise. From these figures, it can be shown that the reconstructed phase portraits are degraded by the second column of the matrix $\bar{\mathbf{X}}_e\mathbf{C}_1$ which has far lower SNR compared to the first column of the matrix $\bar{\mathbf{X}}_e\mathbf{C}_1$. Also the dimensionality (number of non-zero singular values) n' is estimated at 2 for noisy signals [Figs. 3(a) and 4(a)] rather than 3 as for the clean signal [Fig. 2(a)].

3. THE ITERATIVE SVD METHOD AND APPLICATIONS

The method used to overcome the problem described in the previous section is named the 'iterative SVD method'. For the purpose of noise reduction (not reconstruction of phase portrait), if one can use only the first singular value in equation (8) or (9), then the signal-to-noise ratio can be maximised. In order to do this, the first singular value must contain most of the energy of the deterministic signal. This will happen when dealing with low-dimensional systems (Lorenz equation, Duffing equation, etc.). First, an example

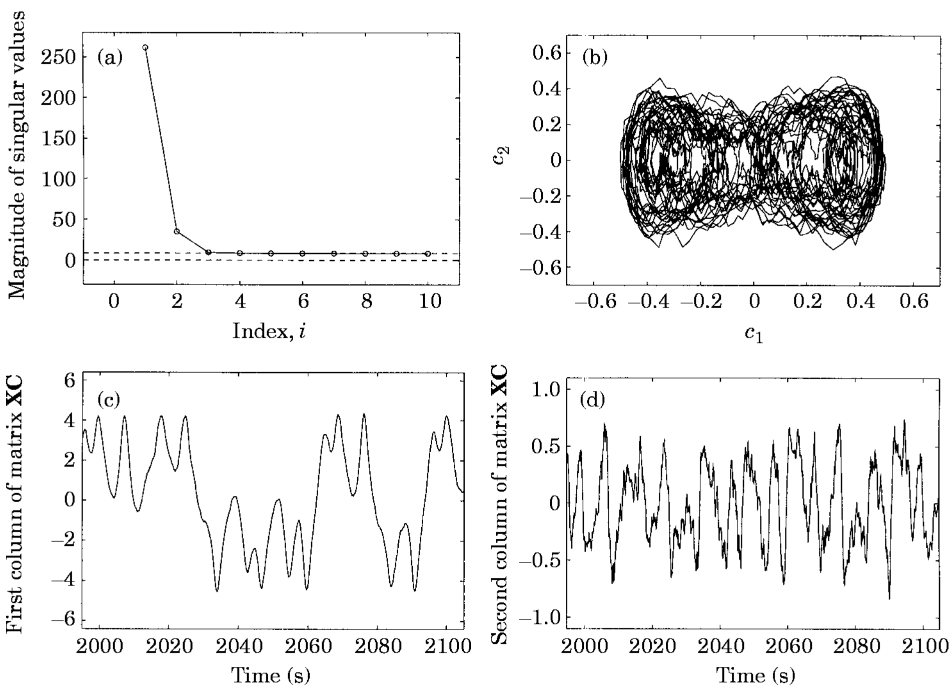

Figure 3. Reconstructed phase portraits by using the SVD for noisy signal (10% white noise): (a) singular values; (b) pseudo-phase portraits reconstructed by SVD (normalised version of the matrix $\bar{\mathbf{X}}_e\mathbf{C}_1$); (c) first column of the matrix $\bar{\mathbf{X}}_e\mathbf{C}_1$; (d) second column of the matrix $\bar{\mathbf{X}}_e\mathbf{C}_1$.

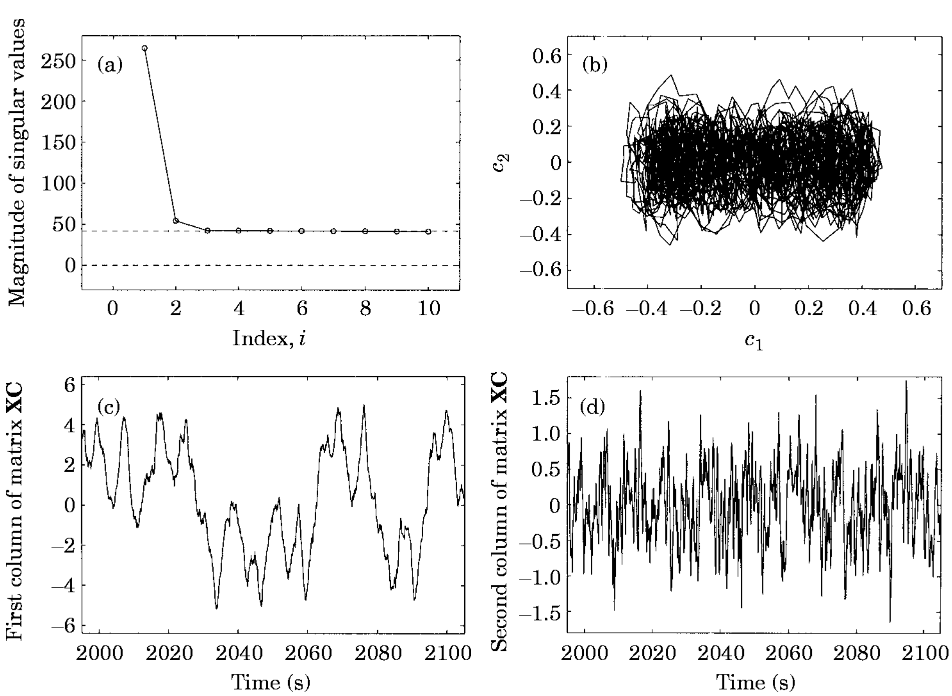

Figure 4. Reconstructed phase portraits by using the SVD for noisy signal (50% white noise): (a) singular values; (b) pseudo-phase portraits reconstructed by SVD (normalised version of the matrix $\bar{\mathbf{X}}_e\mathbf{C}_1$); (c) first column of the matrix $\bar{\mathbf{X}}_e\mathbf{C}_1$; (d) second column of the matrix $\bar{\mathbf{X}}_e\mathbf{C}_1$.

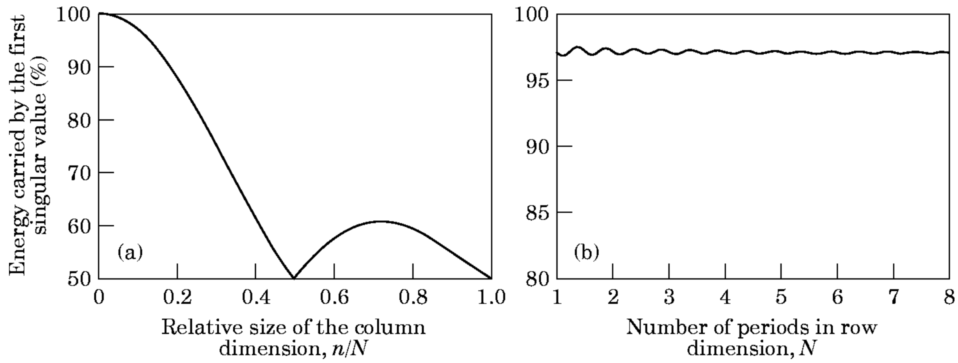

Figure 5. Energy carried by the first singular value (one sinusoid) (a) with different embedding dimension n and row dimension and sampling rate fixed, and (b) with different row dimension N and embedding dimension and sampling rate fixed.

using a sinusoid is considered since in this case the first singular value carries most of the energy of the signal. If the trajectory matrix is constructed from a sinusoidal signal, then

$$\mathbf{X} = \begin{bmatrix} \sin(\omega t) & \sin(\omega t + \phi) & \cdots & \sin(\omega t + (n-1)\phi) \\ \sin(\omega t + \phi) & \sin(\omega t + 2\phi) & \cdots & \sin(\omega t + n\phi) \\ \vdots & \vdots & \ddots & \vdots \\ \sin(\omega t + (N-1)\phi) & \sin(\omega t + N\phi) & & \sin(\omega t + (N+n-2)\phi) \end{bmatrix} \tag{11}$$

where, the phase delay $n\phi$ corresponds to the time delay $\omega n T_s$, and T_s is the sampling time. Assuming each column of the matrix contains exact periods of the signal, the autocovariance matrix $\mathbf{X}^T\mathbf{X}$ becomes

$$\mathbf{X}^T\mathbf{X} = \sigma_s^2 \begin{bmatrix} 1 & \cos(\phi) & \cos(2\phi) & \cdots & \cos((n-1)\phi) \\ \cos(\phi) & 1 & \cos(\phi) & \cdots & \vdots \\ \cos(2\phi) & \cos(\phi) & & \cdots & \cos(2\phi) \\ \vdots & \vdots & \vdots & \vdots & \cos(\phi) \\ \cos((n-1)\phi) & \cos((n-2)\phi) & \cdots & \cos(\phi) & 1 \end{bmatrix} \tag{12}$$

where, σ_s^2 is the variance of the signal. It is shown that the rank of the matrices (11) and (12) is 2, and the two non-zero eigenvalues of matrix (12) are given by [19]

$$\lambda_1, \lambda_2 = \frac{\sigma_s^2}{2}\left[n \pm \frac{\sin(n\phi)}{\sin(\phi)}\right]. \tag{13}$$

Since the square roots of the eigenvalues of equation (12) are the singular values of equation (11), the two non-zero singular values can be written as:

$$\sigma_1, \sigma_2 = \sqrt{\frac{\sigma_s^2}{2}\left[n \pm \left|\frac{\sin(n(\phi)}{\sin(\phi)}\right|\right]}. \tag{14}$$

Equation (14) can be used to determine when the first singular value carries the most energy of the signal. Note that it is related to both embedding dimension (n) and sampling time ($\phi = \omega T_s$). For a given $N \times n$ trajectory matrix, the energy carried by the first singular value can be expressed by

$$E_1 = \frac{\sigma_1^2}{\sum_{i=1}^{n} \sigma_i^2} \tag{15}$$

and when the signal contains white noise this becomes

$$E_1 = \frac{\sigma_1^2 - \sigma_{noise}^2}{\sum_{i=1}^{k} (\sigma_i^2 - \sigma_{noise}^2)}. \tag{16}$$

Given the sampling rate, assuming that each column of the matrix has exactly one period, the ratio of the energy carried by the first singular value of equation (11) is shown in Fig. 5(a). From this figure, it is shown that if the dimension of the trajectory matrix is $(n/N) < 0.1$, then the first singular value carries more than 97% of the energy. In other words, each row of the matrix contains 1/10 of the period. This also indicates the necessary sampling rate. For example, if the embedding dimension (or column dimension) n is estimated as 5, then more than 50 samples per period is required. The row dimension N of the trajectory matrix does not significantly affect the nature of the signal compression as long as the sampling rate and the column dimension are fixed [Fig. 5(b)]. Similar results are obtained for the multiple sinusoid case [20]. The above results require very high sampling rates especially when the embedding dimension is estimated to be large, i.e. the sampling rate becomes more than $10n$ times the highest-frequency component. However, for low-dimensional dynamical systems, such as the Lorenz and Duffing equations, the sampling rate of approximately more than 10 times the cut-off frequency is shown to be satisfactory. The case of the Duffing equation can also be found in [20]. For the Duffing equation, if the sampling rate is roughly 10 times the cut-off frequency and the embedding dimension is set to $n = T_s/T_c$ where T_s is the sampling time, $T_c = 1/f_c$ and f_c is the cut-off frequency, then the size of row dimension N is not very important.

Once it is ensured that the energy of the signal is compressed toward the first singular value, then the first singular value only can be used to estimate $\bar{\mathbf{X}}_e$ in equation (8) or (9), and this will maximise the signal-to-noise ratio of the recovered signal, i.e. the equations become

$$\bar{\mathbf{X}}_{e1} = \sigma_1 \mathbf{s}_1 \mathbf{c}_1^T \tag{17}$$

$$\bar{\mathbf{X}}_{e1} = \left(\frac{\sigma_1^2 - \sigma_{noise}^2}{\sigma_1} \right) \mathbf{s}_1 \mathbf{c}_1^T \tag{18}$$

where, s_1 and $\mathbf{c}_1$ are the first columns of the corresponding singular vectors in equation (8) or (9). This procedure can be considered as optimal filtering. Consider a linear FIR filter which can be expressed as

$$y_i = \mathbf{w}^T \mathbf{x}_i \tag{19}$$

where $\mathbf{x}_i$ is the measured sequence with length n ($\mathbf{x}_i = [x_i x_{i-1} \cdots x_{i-n+1}]^T$, and $\mathbf{w}$ is the filter with length n ($\mathbf{w} = [w_1 \; w_2 \; \cdots \; w_n]^T$). One can find the FIR filter which maximises the output variance subject to the constraint

$$\sum_{i=1}^{n} w_i^2 = \mathbf{w}^T\mathbf{w} = 1, \text{ i.e.}$$

$$\text{maximise } E[y_i^2] = E[\mathbf{w}^T\mathbf{x}\mathbf{x}^T\mathbf{w}] = \mathbf{w}^T\mathbf{R}\mathbf{w} \quad \text{subject to} \quad \mathbf{w}^T\mathbf{w} = 1. \tag{20}$$

To solve this optimisation problem, the method of Lagrange multiplier is used.

$$\frac{\partial}{\partial \mathbf{w}}\{\mathbf{w}^T\mathbf{R}\mathbf{w} + \lambda(\mathbf{w}^T\mathbf{w} - 1)\} = \mathbf{R}\mathbf{w} + \lambda\mathbf{w} = 0 \tag{21}$$

$$\mathbf{R}\mathbf{w} = -\lambda\mathbf{w} \tag{22}$$

where $\mathbf{R}$ is the autocovariance matrix of $\mathbf{x}$. Thus, $\mathbf{w}$ are the eigenvectors of the autocovariance matrix $\mathbf{R}$, and $-\lambda$ are the eigenvalues of the matrix $\mathbf{R}$. Since $\mathbf{w}^T\mathbf{R}\mathbf{w} = E[y_i^2] = -\lambda$, the filter $\mathbf{w}$ (eigenvector) associated with the largest eigenvalue gives maximum output variance. Also the matrix $\mathbf{X}^T\mathbf{X}$ from the trajectory matrix is the autocovariance matrix. Hence the singular vector associated with the largest singular value is also the FIR filter which maximise the output power. Thus, equation (17) or (18) can be considered as the optimal FIR filter.

From equation (17) or (18), it is easily noticed that $\bar{\mathbf{X}}_{e1}$ is an $N \times n$ matrix, where N is the length of each column vector of the trajectory matrix and n is the embedding dimension. Each column of the matrix $\bar{\mathbf{X}}_{e1}$ can be considered as a candidate for the

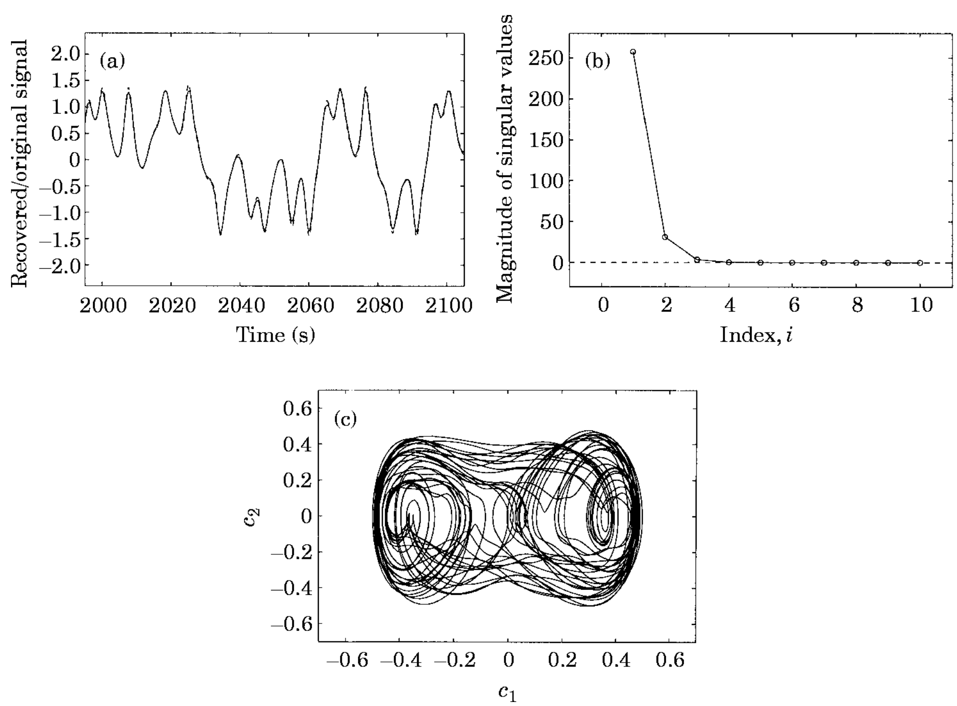

Figure 6. Reconstructed phase portraits and time series by using the iterative SVD method (10% white noise). (a) Recovered signal (——) and original clean signal (– – –) (the recovered signal is obtained by one iteration of the iterative SVD method). (b) Singular values of the trajectory matrix constructed from the recovered signal. (c) Pseudo-phase portraits reconstructed from the recovered signal (normalised version of the matrix $\bar{\mathbf{X}}_e\mathbf{C}_1$).

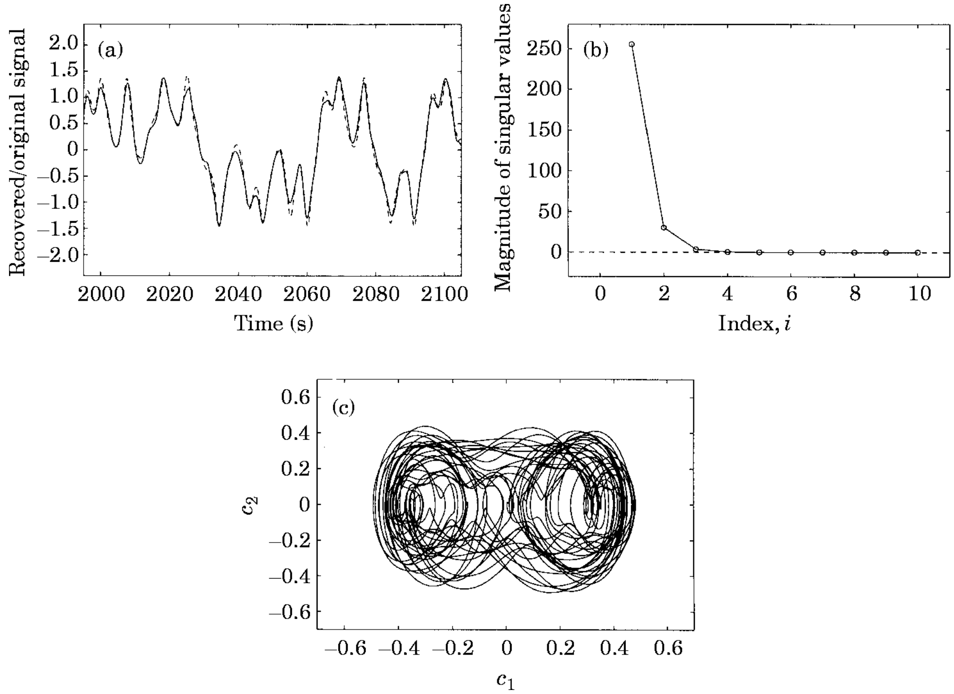

Figure 7. Reconstructed phase portraits and time series by using the iterative SVD method (50% white noise). (a) Recovered time series (——) and original clean signal (– – –) (the recovered signal is obtained by two iterations of the iterative SVD method). (b) Singular values of the trajectory matrix constructed from the recovered signal. (c) Pseudo-phase portraits reconstructed from the recovered signal (normalised version of the matrix $\bar{\mathbf{X}}_e\mathbf{C}_1$).

noise-reduced signal which is delayed by $(n-1)$ sampling unit. To maximise the SNR, average each column of the matrix $\bar{\mathbf{X}}_{e1}$ can be arranged by compensating for the delays. Then one obtains the new noise reduced signal $x_{e1}(k)$. Because $\bar{\mathbf{X}}_{e1}$ is only an estimate of the true deterministic part $\bar{\mathbf{X}}$, the recovered signal is not noise free and thus it may need several iterations. From the noise-reduced signal $x_{e1}(k)$, one can construct a new trajectory matrix, and then apply the SVD and construct $\bar{\mathbf{X}}_{e1}$ again to obtain the further noise-reduced signal. This procedure is iterated until $\sigma_{noise} \approx 0$ or $\|\sigma^m_{noise} - \sigma^{m-1}_{noise}\| < \varepsilon$, where σ^m_{noise} is the mth iterated noise floor, and ε is the tolerance which determines that the noise floor does not change significantly further. The procedure of this iteration method is summarised in below.

(1) Construct the trajectory matrix $\mathbf{X}$ from the noise signal $s(k)$.
(2) Apply the singular value decomposition, $\mathbf{X} = \mathbf{S}\Sigma\mathbf{C}^T$.
(3) Construct the matrix $\bar{\mathbf{X}}_{e1}$ using equation (17) or (18).
(4) Obtain the noise reduced signal $x_{e1}(k)$ by averaging each column of $\bar{\mathbf{X}}_{e1}$.
(5) Repeat the above steps until $\sigma_{noise} \approx 0$ or $\|\sigma^m_{noise} - \sigma^{m-1}_{noise}\| < \varepsilon$.

For white noise, as considered in this paper, very few iterations are required to recover the noise-reduced signal, just one iteration for 10% white noise and two iterations for 50% white noise. The noise-reduced signals, using equation (17), are shown in Fig. 6(a) for 10% white noise and in Fig. 7(a) for 50% white noise. It is observed that equations (17) and (18) do not differ much for this example. From this noise-reduced signal, the pseudo-phase portrait can be reconstructed by SVD described in Section 2. The singular values of the trajectory matrix constructed from the recovered signals are shown in Figs. 6(b) and 7(b), respectively. It can be seen that the dimensionality n' is well recovered and estimated as 3. The reconstructed pseudo-phase portraits are shown in Figs. 6(c) and 7(c), respectively, and show remarkable recovery compared to the noisy pseudo-phase portrait in Figs. 3 and 4.

4. CONCLUDING REMARKS

It has been shown that the iterative SVD method is a very useful and simple method to suppress white noise. Also, it is not only applicable to chaotic time series but to ordinary deterministic signals. Experimental results of both chaotic and non-chaotic signals can be found in references [20, 21]. This method has the additional advantage that information about the minimum embedding dimension of the system can be obtained. This is particularly important since choice of the embedding dimension has a strong influence on the reconstruction of phase portrait.

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