

METHOD OF LEAST SQUARES FOR A LINE

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Problem Statement

Given a set of N ordered pairs (x_i, y_i) , find a line through these points such that the square of the error is minimized.

Derivation

For every given x_i , let the fitted y_i be obtained from the general equation of the line

$$y_{fit} = ax_i + b \quad (1)$$

where a is the slope of the line and b is the y-intercept.

Now, the error is given by,

$$E = \sum_{i=1}^N (y_{fit} - y_i)^2 \quad (2)$$

$$= \sum_{i=1}^N (ax_i + b - y_i)^2 \quad (3)$$

For a given set of x_i and y_i , we see that the error is a function of a and b , i.e., $E \equiv E(a, b)$.

The extremum (maximum or minimum) of a function occurs when its slope is zero, or mathematically put, when the first derivative of the function is zero. Therefore,

$$dE = \frac{\partial E}{\partial a} da + \frac{\partial E}{\partial b} db = 0 \quad (4)$$

$$\therefore \frac{\partial E}{\partial a} = 0, \quad \text{and} \quad \frac{\partial E}{\partial b} = 0 \quad (5)$$

Now,

$$\frac{\partial E}{\partial a} = \sum_{i=1}^N 2(ax_i + b - y_i) \cdot x_i = 0 \quad (6)$$

$$\Rightarrow \left(\sum_{i=1}^N x_i^2 \right) a + \left(\sum_{i=1}^N x_i \right) b = \sum_{i=1}^N x_i y_i \quad (7)$$

Similarly,

$$\frac{\partial E}{\partial b} = \sum_{i=1}^N 2(ax_i + b - y_i) \quad (8)$$

$$\Rightarrow \left(\sum_{i=1}^N x_i \right) a + (N) b = \sum_{i=1}^N y_i \quad (9)$$

This is a system of linear equations which can be solved for a and b

$$a = \frac{N \sum_{i=1}^N x_i y_i - \sum_{i=1}^N x_i \sum_{i=1}^N y_i}{N \sum_{i=1}^N x_i^2 - \left(\sum_{i=1}^N x_i \right)^2} \quad (10)$$

$$b = \frac{\sum_{i=1}^N y_i - \left(\sum_{i=1}^N x_i \right) a}{N} \quad (11)$$

There is an accompanying piece of code written in matlab which calculates and plots the fitted line and the given N ordered pairs.