Quantum Computing Laboratory Report

Detailed Derivations, Truth Tables, and Bloch Sphere Analysis

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Contents

Introduction & Reference Sheet

The Computational Basis

Information is stored in qubits, which can exist in superposition. The standard basis states are:

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \quad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

Bloch Sphere Representation: $|\psi\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle$.

Complete Quantum Gate Reference

1. Identity & Pauli Matrices (Single Qubit)

$$I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \quad X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \quad Y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}, \quad Z = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

2. Superposition & Phase Gates

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}, \quad S = \begin{bmatrix}1 & 0\\0 & i\end{bmatrix}, \quad S^\dagger = \begin{bmatrix}1 & 0\\0 & -i\end{bmatrix}$$

$$T = \begin{bmatrix}1 & 0\\0 & e^{i\pi/4}\end{bmatrix}, \quad T^\dagger = \begin{bmatrix}1 & 0\\0 & e^{-i\pi/4}\end{bmatrix}$$

3. Rotation Operators

$$R_x(\theta) = \begin{bmatrix}\cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}\\-i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix}e^{-i\theta/2} & 0\\0 & e^{i\theta/2}\end{bmatrix}$$

4. Multi-Qubit Gates

$$\text{CNOT} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix}, \quad \text{SWAP} = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

$$\text{Toffoli} = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}$$

5. Fundamental Identities

$$H^2 = I, \quad H^\dagger = H, \quad S^2 = Z, \quad T^2 = S, \quad T^4 = Z$$

$$R_X(\theta) = HR_Z(\theta)H, \quad H = R_Z(\pi)R_X(\pi/2)R_Z(\pi)$$

6. Bell States (Maximally Entangled)

$$|\Phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \quad |\Psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$$

1 Lab-1: Pauli X, Y, Z Gates

1.1 Pauli-X Gate (Quantum NOT)

Matrix:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Action:

$$X|0\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$$
$$X|1\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$

Bloch Sphere: Rotation of π (180°) around the X-axis.

1.2 Pauli-Y Gate

Matrix:

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Action:

$$Y|0\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = i|1\rangle$$
$$Y|1\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -i|0\rangle$$

Bloch Sphere: Rotation of π (180°) around the Y-axis.

1.3 Pauli-Z Gate (Phase-Flip)

Matrix:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Action:

$$Z|0\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle$$
$$Z|1\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -|1\rangle$$

Bloch Sphere: Rotation of π (180°) around the Z-axis.

1.4 Viva / Conceptual Points

- The Pauli matrices are Hermitian ($A^\dagger = A$) and Unitary ($A^\dagger A = I$).
- $X^2 = Y^2 = Z^2 = I$.
- They form a basis for the space of 2×2 Hermitian matrices.

2 Lab-2: Hadamard Gate

2.1 Construction

Matrix:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

2.2 Derivation of State Transformations

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$
$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$$

2.3 Truth Table

Input	Output State	Description
$\|0\rangle$	$\|+\rangle = \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	Superposition (Phase 0)
$\|1\rangle$	$\|-\rangle = \frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	Superposition (Phase π)

Bloch Sphere: Rotates π radians around the diagonal axis $\frac{X+Z}{\sqrt{2}}$.

2.4 Viva / Conceptual Points

- The Hadamard gate creates superposition from standard basis states.
- Measuring $|+\rangle$ or $|-\rangle$ in the standard basis yields 0 or 1 with 50% probability.
- It transforms the Z-basis to the X-basis.

3 Lab-3: Inverse of Hadamard Gate

Problem: Show that $H^{-1} = H$.

3.1 Algebraic Proof

Applying H twice returns the original state:

$$H(H|0\rangle) = H|+\rangle = H\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(H|0\rangle + H|1\rangle)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} + \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \frac{1}{2}(2|0\rangle) = |0\rangle$$

Similarly, $H(H|1\rangle) = |1\rangle$. Since $H^2 = I$, H is its own inverse.

3.2 Viva / Conceptual Points

- In quantum computing, all gates (except measurement) must be reversible.
- Since H is unitary and Hermitian, $H^\dagger = H^{-1} = H$.

4 Lab-4: Proof $H^2 = I$

Problem: Prove mathematically that $H^2 = I$.

4.1 Matrix Multiplication Derivation

$$H^2 = H \cdot H = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)$$

$$= \frac{1}{2} \begin{bmatrix} (1)(1) + (1)(1) & (1)(1) + (1)(-1) \\ (1)(1) + (-1)(1) & (1)(1) + (-1)(-1) \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Q.E.D.

4.2 Viva / Conceptual Points

- I is the Identity matrix, which leaves a state unchanged.
- This property implies that applying the Hadamard gate twice is equivalent to doing nothing.

5 Lab-5: Two S Gates are Equivalent to Z

Problem: Show $S^2 = Z$.

5.1 Derivation

Matrix: $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$.

$$S^2 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z$$

Physical Interpretation: S rotates by 90° around Z. Two S gates rotate by 180°, equivalent to Z.

5.2 Viva / Conceptual Points

- The S gate is sometimes called the $\sqrt{Z}$ gate.
- It adds a relative phase of i ($e^{i\pi/2}$) to the $|1\rangle$ state.

6 Lab-6: Rotation Gates (R_x, R_y, R_z)

6.1 Rx Gate

$$R_x(\theta) = \begin{bmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

Bloch Sphere: Rotates the state vector by angle θ around the X-axis.

6.2 Ry Gate

$$R_y(\theta) = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$$

Bloch Sphere: Rotates the state vector by angle θ around the Y-axis.

6.3 Rz Gate

$$R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

Bloch Sphere: Rotates the state vector by angle θ around the Z-axis.

6.4 Viva / Conceptual Points

- Any single-qubit unitary gate can be decomposed into rotations around these axes.
- R_z is a phase shift gate; it only changes the longitude on the Bloch sphere, not the latitude.

7 Lab-7: Swap Gate Implementation

Problem: Implement SWAP using three CNOTs.

7.1 Swap Matrix

$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

7.2 Trace (Input $|10\rangle$)

Circuit: $CNOT_{1\to2} \to CNOT_{2\to1} \to CNOT_{1\to2}$. Input: $|q_1 = 1, q_2 = 0\rangle$.

1. $CNOT_{12}$: Control 1 is True. Target 2 flips ($0 \to 1$). State: $|11\rangle$.
2. $CNOT_{21}$: Control 2 is True. Target 1 flips ($1 \to 0$). State: $|01\rangle$.
3. $CNOT_{12}$: Control 1 is False. Target 2 unchanged. State: $|01\rangle$.

Result: $|10\rangle \to |01\rangle$.

7.3 Truth Table

Input	After $CNOT_{12}$	After $CNOT_{21}$	After $CNOT_{12}$	Final
$\|00\rangle$	$\|00\rangle$	$\|00\rangle$	$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$	$\|11\rangle$	$\|10\rangle$	$\|10\rangle$
$\|10\rangle$	$\|11\rangle$	$\|01\rangle$	$\|01\rangle$	$\|01\rangle$
$\|11\rangle$	$\|10\rangle$	$\|10\rangle$	$\|11\rangle$	$\|11\rangle$

7.4 Viva / Conceptual Points

- The SWAP gate exchanges the states of two qubits.
- It is essential for moving quantum information between non-adjacent qubits in physical architectures.

8 Lab-8: Classical Logic with Toffoli Gates

The Toffoli (CCNOT) gate maps $|a, b, c\rangle \rightarrow |a, b, c \oplus (a \cdot b)\rangle$.

8.1 Toffoli Matrix

$$CCNOT = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

8.2 Logic Gates Construction

- **AND Gate:** $CCNOT(A, B, 0) \implies Output = A \cdot B$.
- **NAND Gate:** $CCNOT(A, B, 1) \implies Output = \neg(A \cdot B)$.
- **NOT Gate:** $CCNOT(1, 1, A) \implies Output = \neg A$.

8.3 Viva / Conceptual Points

- Toffoli is a universal reversible gate for classical computation.
- Unlike classical AND, which loses information (irreversible), Toffoli keeps inputs A and B, making it reversible.

9 Lab-9: Half Adder Construction

Goal: Compute Sum ($S = A \oplus B$) and Carry ($C = A \cdot B$).

9.1 Circuit Design

Inputs: $|A\rangle, |B\rangle, |0\rangle$.

1. **Carry:** $CCNOT(A, B, 0) \rightarrow |A, B, AB\rangle$.
2. **Sum:** $CNOT(A, B) \rightarrow |A, A \oplus B, AB\rangle$.

9.2 Detailed Trace for $A = 1, B = 1$

1. Initial: $|1\rangle_A |1\rangle_B |0\rangle_{Aux}$.
2. CCNOT: Controls $A = 1, B = 1$. Aux flips $0 \rightarrow 1$. State: $|111\rangle$.
3. CNOT: Control $A = 1$. Target B flips $1 \rightarrow 0$. State: $|101\rangle$.
4. Result: Sum (Q_B) = 0, Carry (Q_{Aux}) = 1. ($1 + 1 = 10_2$).

9.3 Truth Table

A	B	Carry (Aux)	Sum (Qubit B)
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

9.4 Viva / Conceptual Points

- A Half Adder performs binary addition of two bits.
- The Quantum Half Adder must be reversible; hence we keep the input A.
- We need an ancilla qubit (initialized to $|0\rangle$) to store the Carry bit.

10 Lab-10: The Four Bell States

10.1 Circuit Structure

1. Hadamard on Qubit 1. 2. CNOT (Control 1, Target 2).

10.2 Derivations

1. **$|\Phi^+\rangle$ (from $|00\rangle$):**
$$|00\rangle \xrightarrow{H_1} \frac{|00\rangle + |10\rangle}{\sqrt{2}} \xrightarrow{CNOT} \frac{|00\rangle + |11\rangle}{\sqrt{2}} = |\Phi^+\rangle$$

2. **$|\Phi^-\rangle$ (from $|10\rangle$):**
$$|10\rangle \xrightarrow{H_1} \frac{|00\rangle - |10\rangle}{\sqrt{2}} \xrightarrow{CNOT} \frac{|00\rangle - |11\rangle}{\sqrt{2}} = |\Phi^-\rangle$$

3. **$|\Psi^+\rangle$ (from $|01\rangle$):**
$$|01\rangle \xrightarrow{H_1} \frac{|01\rangle + |11\rangle}{\sqrt{2}} \xrightarrow{CNOT} \frac{|01\rangle + |10\rangle}{\sqrt{2}} = |\Psi^+\rangle$$

4. **$|\Psi^-\rangle$ (from $|11\rangle$):**
$$|11\rangle \xrightarrow{H_1} \frac{|01\rangle - |11\rangle}{\sqrt{2}} \xrightarrow{CNOT} \frac{|01\rangle - |10\rangle}{\sqrt{2}} = |\Psi^-\rangle$$

10.3 Viva / Conceptual Points

- Bell states are maximally entangled quantum states.
- Measurement of one qubit instantaneously determines the state of the other, regardless of distance.
- They form an orthonormal basis for the two-qubit Hilbert space (Bell Basis).

11 Lab-11a: Identity $R_X(\theta) = HR_Z(\theta)H$

11.1 Derivation

We conjugate the Z-rotation with Hadamard gates.

$$HR_z(\theta)H = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\right)\begin{bmatrix}e^{-i\theta/2} & 0\\0 & e^{i\theta/2}\end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix}e^{-i\theta/2} & e^{i\theta/2}\\e^{-i\theta/2} & -e^{i\theta/2}\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}e^{-i\theta/2}+e^{i\theta/2} & e^{-i\theta/2}-e^{i\theta/2}\\e^{-i\theta/2}-e^{i\theta/2} & e^{-i\theta/2}+e^{i\theta/2}\end{bmatrix}$$

Using Euler's laws ($e^{ix} + e^{-ix} = 2\cos(x)$ and $e^{-ix} - e^{ix} = -2i\sin(x)$):

$$= \frac{1}{2}\begin{bmatrix}2\cos(\theta/2) & -2i\sin(\theta/2)\\-2i\sin(\theta/2) & 2\cos(\theta/2)\end{bmatrix} = \begin{bmatrix}\cos(\theta/2) & -i\sin(\theta/2)\\-i\sin(\theta/2) & \cos(\theta/2)\end{bmatrix} = R_x(\theta)$$

11.2 Viva / Conceptual Points

- This identity shows that an X-rotation is equivalent to a Z-rotation in the Hadamard basis.
- It demonstrates the relationship between X and Z bases via the Hadamard transformation.

12 Lab-11b: Identity $H = R_Z(\pi)R_X(\pi/2)R_Z(\pi)$

12.1 Step-by-Step Matrix Multiplication

1. Calculate $R_z(\pi)$:

$$R_z(\pi) = \begin{bmatrix} e^{-i\pi/2} & 0 \\ 0 & e^{i\pi/2} \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

2. Calculate $R_x(\pi/2)$:

$$R_x(\pi/2) = \begin{bmatrix} \cos(\pi/4) & -i\sin(\pi/4) \\ -i\sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

3. Compute Product $R_z(\pi)R_x(\pi/2)$:

$$\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -i(1) & -i(-i) \\ i(-i) & i(1) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix}$$

4. Compute Final Product with $R_z(\pi)$:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} (-i)(-i) & (-1)(i) \\ (1)(-i) & (i)(i) \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -i \\ -i & -1 \end{bmatrix} = -1 \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

(Holds up to global phase)

12.2 Viva / Conceptual Points

- This is an Euler angle decomposition of the Hadamard gate.
- It allows implementing H on hardware that only supports Z and X rotations natively.

13 Lab-12: T Gate Equivalence

Problem: Show that applying the T gate 4 times is equivalent to a Z gate.

13.1 Derivation

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

Applying T twice (T^2) gives the S gate:

$$T^2 = \begin{bmatrix} 1 & 0 \\ 0 & (e^{i\pi/4})^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = S$$

Applying T four times (T^4) is equivalent to S^2:

$$T^4 = (T^2)^2 = S^2 = \begin{bmatrix} 1 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z$$

13.2 Bloch Sphere Visualization

- **T Gate:** Rotates $\pi/4$ (45°) around Z.
- **4 Applications:** $4 \times 45° = 180°$ (π radians).
- A 180° rotation around Z is the definition of the Z gate.

13.3 Viva / Conceptual Points

- The T gate is the $\sqrt{S}$ gate or the $\sqrt[4]{Z}$ gate.
- It is crucial for fault-tolerant quantum computing (Magic State distillation).

14 Lab-13: CNOT and XOR

14.1 Matrix and Truth Table

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Control (A)	Target (B)	Output Control	Output Target	XOR ($A \oplus B$)
0	0	0	0	0
0	1	0	1	1
1	0	1	1	1
1	1	1	0	0

Comparing the last two columns, we see that the Output Target is exactly $A \oplus B$. Thus, CNOT implements reversible XOR logic.

14.2 Viva / Conceptual Points

- CNOT (Controlled-NOT) flips the target qubit if and only if the control qubit is 1.
- It is an entangling gate; combined with single-qubit gates, it forms a universal set.

15 Lab-14: T-Dagger Gate

Problem: Prove $T^\dagger = R_z(-\pi/4)$.

15.1 Derivation

Given $T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$. The conjugate transpose $T^\dagger$ is found by taking the complex conjugate of the diagonal:

$$T^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix}$$

Now consider $R_z(-\pi/4)$:

$$R_z(-\pi/4) = \begin{bmatrix} e^{-i(-\pi/4)/2} & 0 \\ 0 & e^{i(-\pi/4)/2} \end{bmatrix} = \begin{bmatrix} e^{i\pi/8} & 0 \\ 0 & e^{-i\pi/8} \end{bmatrix}$$

Factor out the global phase $e^{i\pi/8}$ from R_z:

$$e^{i\pi/8} \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix} = e^{i\pi/8} T^\dagger$$

Since global phase factors do not affect measurement statistics, $T^\dagger$ is physically equivalent to $R_z(-\pi/4)$.

15.2 Viva / Conceptual Points

- $T^\dagger$ is the inverse of the T gate.
- It rotates by $-45°$ around the Z-axis.

16 Lab-15: $T^\dagger$ Phase Analysis

16.1 Action on Superposition

Let the arbitrary quantum state be $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Applying $T^\dagger$:

$$T^\dagger|\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta e^{-i\pi/4} \end{bmatrix}$$

$$= \alpha|0\rangle + \beta e^{-i\pi/4}|1\rangle$$

16.2 Measurement Probabilities

Probability of measuring $|0\rangle$:

$$P(0) = |\alpha|^2$$

Probability of measuring $|1\rangle$:

$$P(1) = |\beta e^{-i\pi/4}|^2 = |\beta|^2|e^{-i\pi/4}|^2 = |\beta|^2(1) = |\beta|^2$$

16.3 Conclusion

The gate rotates the relative phase of the $|1\rangle$ component by $-45°$ ($-\pi/4$ radians). It does **not** change the measurement probabilities (magnitudes), only the interference properties (relative phase).

16.4 Viva / Conceptual Points

- Phase gates like $T^\dagger$ change the relative phase between superposition components.
- This phase change is invisible to standard measurement but can be detected via interference (e.g., using Hadamard gates).