HOMEWORK 1

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Instructions: This is a background self-test on the type of math we will encounter in class. If you find many questions intimidating, we suggest you drop 760 and take it again in the future when you are more prepared.
Use this latex file as a template to develop your homework. Submit your homework on time as a single pdf file to Canvas. There is no need to submit the latex source or any code. Please check Piazza for updates about the homework.

1 Logistics [5 pts]

Read the course webpage at http://pages.cs.wisc.edu/~jerryzhu/cs760.html and answer the following questions:

1. Where do we post announcements and clarifications?
   Announcements and clarifications are posted on CS760 Piazza page.
2. What time of day (hour:minute) are all homeworks due?
   10:59 AM (a minute before class starts on the due date)
3. Will late homework be accepted?
   Late homework submissions will not be accepted.
4. Tom received the following scores on his 7 homeworks: 0, 59, 92, 95, 98, 100, 100. According to the homework policies, what is Tom's final average homework score?
   Final average score is calculated by dropping one lowest homework score. So, for Tom :
   Final Average Score = (59, 92, 95, 98, 100, 100) / 6 = 90.67
5. How can you discuss homework questions with fellow students while avoiding the impression of cheating?
   We can form study groups and make broad discussions with our peers, TAs and instructors but all assignments must be written-up individually. For deep peer discussions, we must declare in the homework assignment.

2 Vectors and Matrices [2 pts]

Consider the matrix X and the vectors $\mathbf{y}$ and $\mathbf{z}$ below:

$$X = \begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \mathbf{z} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

1. Compute $\mathbf{y}^\top X\mathbf{z}$

   $\mathbf{y}^\top = \begin{pmatrix} 2 & 3 \end{pmatrix} \qquad X\mathbf{z} = \begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix} \begin{pmatrix} 7 \\ 6 \end{pmatrix} = \begin{pmatrix} 84 \\ 110 \end{pmatrix}$

   $\mathbf{y}^\top X\mathbf{z} = \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 84 \\ 110 \end{pmatrix} = \boxed{\begin{pmatrix} 498 \end{pmatrix}}$

2. Is X invertible? If so, give the inverse, and if no, explain why not.
   Yes X is invertible.

   $|X| = -2 \qquad \mathbf{Adj}X = \begin{pmatrix} 9 & -7 \\ -8 & 6 \end{pmatrix}$

$$X^{-1} = -\frac{1}{2}\begin{pmatrix} 9 & -7 \\ -8 & 6 \end{pmatrix} = \boxed{\begin{pmatrix} -4.5 & 3.5 \\ 4 & -3 \end{pmatrix}}$$

3 Calculus [1 pts]

1. If $y = e^x + \tan(z)x^{6z} - \ln(\frac{7x+z}{x^4})$, what is the partial derivative of y with respect to x?

   $\frac{\partial y}{\partial x} = \frac{\partial(e^x)}{\partial x} + \frac{\partial(\tan(z)x^{6z})}{\partial x} - \frac{\partial(\ln(\frac{7x+z}{x^4}))}{\partial x}$

   $= e^x + 6z\tan(z)x^{6z-1} - (\frac{x^4}{7x+z})\frac{\partial(\frac{7x+z}{x^4})}{\partial x}$

   $= e^x + 6z\tan(z)x^{6z-1} - (\frac{x^4}{7x+z})(\frac{7x^4-4x^3(7x+z)}{x^8})$

   $= e^x + 6z\tan(z)x^{6z-1} - (\frac{1}{7x+z})(\frac{-(21x+4z)}{x})$

   $= \boxed{e^x + 6z\tan(z)x^{6z-1} + \frac{21x+4z}{x(7x+z)}}$

4 Probability and Statistics [4 pts]

Consider a sequence of data $S = (0, 1, 1, 0, 1, 1, 1)$ created by flipping a coin x seven times, where 0 denotes that the coin turned up heads and 1 denotes that it turned up tails.

1. What is the probability of observing this data, assuming it was generated by flipping a biased coin with $p(x = 1) = 0.7$?
   $p(x = 1) = 0.7$. So, $p(x = 0) = 0.3$
   Therefore, probability of observing data $S = (0.7)^5(0.3)^2 = \boxed{0.0151263}$

2. Note that the probability of this data sample could be greater if the value of $p(x = 1)$ was not 0.7, but instead some other value. What is the value that maximizes the probability of S? Please justify your answer.
   Let $p(x = 1)$ be θ. So, $p(x = 0) = (1 - \theta)$.
   Probability of observing data $S = f(\theta) = \theta^5(1 - \theta)^2$.

   For local maxima, $\frac{\mathrm{d}f}{\mathrm{d}\theta} = 0$

   $\frac{\mathrm{d}f}{\mathrm{d}\theta} = -2\theta^5(1 - \theta) + 5\theta^4(1 - \theta)^2 = 0$

   $= 7\theta^6 - 12\theta^5 + 5\theta^4 = 0$

   $\theta^4(\theta - 1)(7\theta - 5) = 0$

   Since, $\theta = 0$ or $\theta = 1$ is not possible, $\theta = \frac{5}{7} = \boxed{0.7142857}$

3. Consider the following joint probability table where both A and B are binary random variables:

A	B	$P(A, B)$
0	0	0.4
0	1	0.3
1	0	0.2
1	1	0.1

   (a) What is $P(A = 0|B = 1)$?
   $P(A = 0|B = 1) = \frac{P(A=0 \wedge B=1)}{P(B=1)} = \frac{0.3}{0.3+0.1} = \boxed{0.75}$

   (b) What is $P(A = 0 \vee B = 0)$?
   $P(A = 0 \vee B = 0) = 0.4 + 0.3 + 0.2 = \boxed{0.9}$

5 Big-O Notation [3 pts]

For each pair (f, g) of functions below, list which of the following are true: $f(n) = O(g(n))$, $g(n) = O(f(n))$, both, or neither. Briefly justify your answers.

1. $f(n) = \frac{n}{2}$, $g(n) = \log_2(n)$.
   $g(n) = O(f(n))$ is true because $\log_2(n) \leq \frac{n}{2}$ for all $n \geq 4$.
   $f(n) = O(g(n))$ is false because $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

2. $f(n) = \ln(n)$, $g(n) = \log_2(n)$.
   Here, both $g(n) = O(f(n))$ and $f(n) = O(g(n))$ are true.
   Since $\log_2 n = \frac{\ln(n)}{\ln(2)}$, therefore for n tending to ∞, $\frac{f(n)}{g(n)} = \frac{g(n)}{f(n)} = O(1)$. Hence, both expressions are true.

3. $f(n) = n^{100}$, $g(n) = 100^n$.
   $f(n) = O(g(n))$ is true because $\lim_{n\to\infty} \frac{f(n)}{g(n)} = O(1)$.
   $g(n) = O(f(n))$ is false because $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$.

6 Probability and Random Variables

6.1 Probability [5 pts]

State true or false. Here Ω denotes the sample space and A^c denotes the complement of the event A.

1. For any $A, B \subseteq \Omega$, $P(A|B)P(B) = P(B|A)P(A)$.
   True

2. For any $A, B \subseteq \Omega$, $P(A \cup B) = P(A) + P(B) - P(A|B)$.
   False

3. For any $A, B, C \subseteq \Omega$ such that $P(B \cup C) > 0$, $\frac{P(A \cup B \cup C)}{P(B \cup C)} \geq P(A|B \cup C)P(B \cup C)$.
   True

4. For any $A, B \subseteq \Omega$ such that $P(B) > 0, P(A^c) > 0$, $P(B|A^C) + P(B|A) = 1$.
   False

5. For any n events $\{A_i\}_{i=1}^n$, if $P(\bigcap_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$, then $\{A_i\}_{i=1}^n$ are mutually independent.
   False

6.2 Discrete and Continuous Distributions [5 pts]

Match the distribution name to its probability density / mass function. Below, $|\boldsymbol{x}| = k$.

(a) Laplace (h)
(b) Multinomial (i)
(c) Poisson (l)
(d) Dirichlet (k)
(e) Gamma (j)

(f) $f(\boldsymbol{x}; \boldsymbol{\Sigma}, \boldsymbol{\mu}) = \frac{1}{\sqrt{(2\pi)^k \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu})\right)$

(g) $f(x; n, \alpha) = \binom{n}{x} \alpha^x (1 - \alpha)^{n-x}$ for $x \in \{0, \ldots, n\}$; 0 otherwise

(h) $f(x; b, \mu) = \frac{1}{2b} \exp\left(-\frac{|x-\mu|}{b}\right)$

(i) $f(\boldsymbol{x}; n, \boldsymbol{\alpha}) = \frac{n!}{\Pi_{i=1}^k x_i!} \Pi_{i=1}^k \alpha_i^{x_i}$ for $x_i \in \{0, \ldots, n\}$ and $\sum_{i=1}^k x_i = n$; 0 otherwise

(j) $f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ for $x \in (0, +\infty)$; 0 otherwise

(k) $f(\boldsymbol{x}; \boldsymbol{\alpha}) = \frac{\Gamma(\sum_{i=1}^k \alpha_i)}{\prod_{i=1}^k \Gamma(\alpha_i)} \prod_{i=1}^k x_i^{\alpha_i - 1}$ for $x_i \in (0, 1)$ and $\sum_{i=1}^k x_i = 1$; 0 otherwise

(l) $f(x; \lambda) = \lambda^x \frac{e^{-\lambda}}{x!}$ for all $x \in Z^+$; 0 otherwise

6.3 Mean and Variance [5 pts]

1. Consider a random variable which follows a Binomial distribution: $X \sim \text{Binomial}(n, p)$.

   (a) What is the mean of the random variable?
   np

   (b) What is the variance of the random variable?
   $np(1-p)$

2. Let X be a random variable and $\mathbb{E}[X] = 1, \text{Var}(X) = 1$. Compute the following values:

   (a) $\mathbb{E}[3X]$
   3

   (b) $\text{Var}(3X)$
   9

   (c) $\text{Var}(X + 3)$
   1

6.4 Mutual and Conditional Independence [4 pts]

1. If X and Y are independent random variables, show that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.
   $\mathbb{E}[XY]$ can be defined as $\int_x \int_y xyP(x,y)dxdy$ where $P(x,y)$ is the probability of event (x,y)

   Since X and Y are independent random variables, $P(x,y) = P(x)P(y)$.

   Hence, $\mathbb{E}[XY] = \int_x \int_y xyP(x)P(y)dxdy = \int_x xP(x)dx \int_y yP(y)dy = \mathbb{E}[X]\mathbb{E}[Y]$

2. If X and Y are independent random variables, show that $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.
   Hint: $\text{Var}(X + Y) = \text{Var}(X) + 2\text{Cov}(X, Y) + \text{Var}(Y)$
   $\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$

   $= \mathbb{E}[XY - X\mathbb{E}[Y] - Y\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y]] = \mathbb{E}[XY] - \mathbb{E}[Y]\mathbb{E}[X] - \mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y]$

   $= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

   Since, X and Y are independent random variables, $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$. Hence, $\text{Cov}(X, Y) = 0$.

   Therefore, $\text{Var}(X + Y) = \text{Var}(X) + 2\text{Cov}(X, Y) + \text{Var}(Y) = \text{Var}(X) + \text{Var}(Y)$

3. If we roll two dice that behave independently of each other, will the result of the first die tell us something about the result of the second die?
   No, result of the first dice will not tell us anything about the result of the second dice.

   If, however, the first die's result is a 1, and someone tells you about a third event — that the sum of the two results is even — then given this information is the result of the second die independent of the first die?

   The result of second dice is independent of the first dice. But, the results of the two die are not conditionally independent given the information that the sum of the two is even.

6.5 Central Limit Theorem [1 pts]

Provide one line explanation.

1. Let $X_i \sim \mathcal{N}(0, 1)$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then the distribution of $\bar{X}$ satisfies

$$\sqrt{n}\bar{X} \xrightarrow{n\to\infty} \mathcal{N}(0, 1)$$

   According to the Central Limit Theorem, for a sequence of independent and identically distributed random variables X_i with mean μ, variance σ^2, and sample average defined by $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then as n approaches infinity, the random variables $\sqrt{n}(\bar{X} - \mu)$ converge in distribution to a normal $\mathcal{N}(0, \sigma^2)$. Since, here, $\mu = 0$ and $\sigma^2 = 1$, $\sqrt{n}\bar{X} \xrightarrow{n\to\infty} \mathcal{N}(0, 1)$ is true.

7 Linear algebra

7.1 Norms [3 pts]

Draw the regions corresponding to vectors $\mathbf{x} \in \mathbb{R}^2$ with the following norms:

1. $||\mathbf{x}||_1 \leq 1$ (Recall that $||\mathbf{x}||_1 = \sum_i |x_i|$)
2. $||\mathbf{x}||_2 \leq 1$ (Recall that $||\mathbf{x}||_2 = \sqrt{\sum_i x_i^2}$)
3. $||\mathbf{x}||_\infty \leq 1$ (Recall that $||\mathbf{x}||_\infty = \max_i |x_i|$)

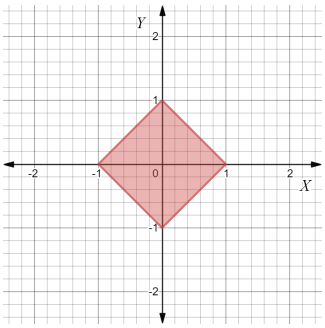


Fig.1: $||\mathbf{x}||_1 \leq 1$

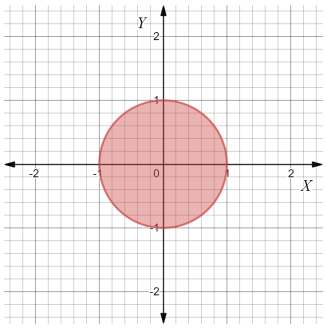


Fig.2: $||\mathbf{x}||_2 \leq 1$

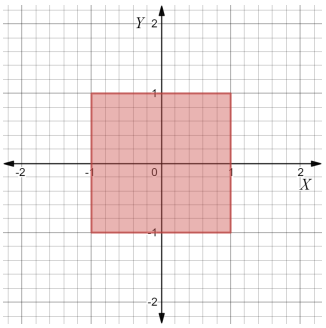


Fig.3: $||\mathbf{x}||_\infty \leq 1$

7.2 Geometry [2 pts]

Prove the following. Provide all steps.

1. The smallest Euclidean distance from the origin to some point $\mathbf{x}$ in the hyperplane $\mathbf{w}^\top \mathbf{x} + b = 0$ is $\frac{|b|}{||\mathbf{w}||_2}$. You may assume $\mathbf{w} \neq 0$.
   The normal vector of the hyperplane $\mathbf{w}^\top \mathbf{x} + b = 0$ is given by $\mathbf{w}$. If $\mathbf{x}$ is a point on the hyperplane, the smallest Euclidean distance from origin to $\mathbf{x}$ can be obtained by the projection of $\mathbf{x}$ on $\mathbf{w}$:
   $= \frac{|\mathbf{w}.\mathbf{x}|}{||\mathbf{w}||_2} = \frac{|\mathbf{w}^\top \mathbf{x}|}{||\mathbf{w}||_2} = \frac{|-b|}{||\mathbf{w}||_2} = \frac{|b|}{||\mathbf{w}||_2}$

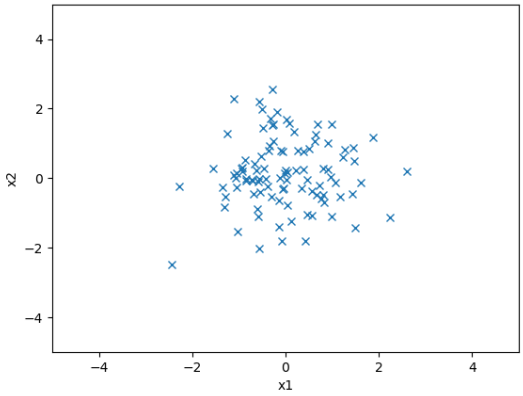
2. The Euclidean distance between two parallel hyperplane $\mathbf{w}^\top \mathbf{x} + b_1 = 0$ and $\mathbf{w}^\top \mathbf{x} + b_2 = 0$ is $\frac{|b_1 - b_2|}{||\mathbf{w}||_2}$ (Hint: you can use the result from the last question to help you prove this one).

   If we translate the X-axis by b_1 units such that one of the planes passes through the origin, the equations of the hyperplanes become $\mathbf{w}^\top \mathbf{x} = 0$ and $\mathbf{w}^\top \mathbf{x} + b_2 - b_1 = 0$. Since the hyperplanes are parallel the distance between them remains same in the new coordinate system. Since origin lies on the plane $\mathbf{w}^\top \mathbf{x} = 0$, the distance between the two planes is the distance from origin to plane $\mathbf{w}^\top \mathbf{x} + b_2 - b_1 = 0$ which is $\frac{|b_2 - b_1|}{||\mathbf{w}||_2} = \frac{|b_1 - b_2|}{||\mathbf{w}||_2}$

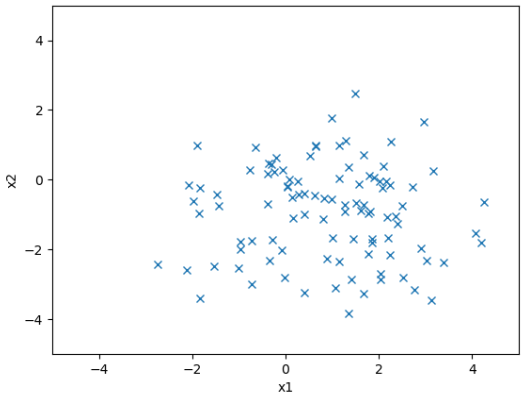
8 Programming Skills [3 pts]

Sampling from a distribution. For each question, submit a scatter plot (you will have 5 plots in total). Make sure the axes for all plots have the same ranges.

1. Draw 100 items $\mathbf{x} = [x_1, x_2]^\top$ from a 2-dimensional Gaussian distribution $N(\mu, \Sigma)$ with mean $\mu = (0, 0)^T$ and identity covariance matrix $\Sigma = I$, i.e., $p(\mathbf{x}) = \frac{1}{2\pi} \exp\left(-\frac{||\mathbf{x}||^2}{2}\right)$, and make a scatter plot (x_1 vs. x_2).



2. Make a scatter plot by drawing 100 items from $N(\mu + (1, -1)^\top, 2\Sigma)$.



3. Make a scatter plot by drawing 100 items from a mixture distribution $0.3N\left((1,0)^\top, \begin{pmatrix} 1 & 0.2 \\ 0.2 & 1 \end{pmatrix}\right) + 0.7N\left((-1,0)^\top, \begin{pmatrix} 1 & -0.2 \\ -0.2 & 1 \end{pmatrix}\right)$.