# 10

# Affine Maps in 3D

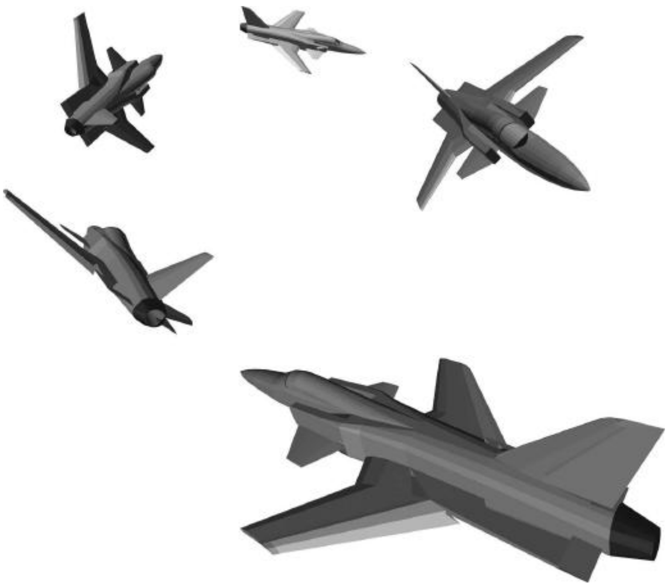
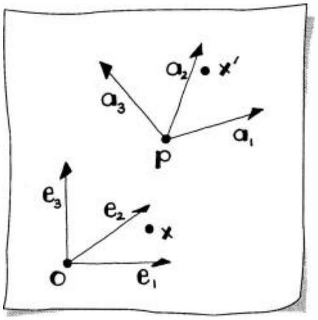


**Figure 10.1.**
Affine maps in 3D: fighter jets twisting and turning through 3D space.

Affine maps in 3D are a primary tool for modeling and computer graphics. Figure 10.1 illustrates the use of various affine maps. This chapter goes a little further than just affine maps by introducing projective maps—the maps used to create realistic 3D images.

## 10.1 Affine Maps




**Sketch 10.1.**
An affine map in 3D.

Linear maps relate vectors to vectors. Affine maps relate points to points. A 3D affine map is written just as a 2D one, namely as

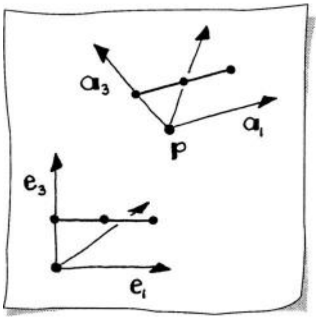
$$\mathbf{x}' = \mathbf{p} + A(\mathbf{x} - \mathbf{o}), \tag{10.1}$$

where $\mathbf{x}, \mathbf{o}, \mathbf{p}, \mathbf{x}'$ are 3D points and $A$ is a $3 \times 3$ matrix. In general, we will assume that the origin of $\mathbf{x}$'s coordinate system has three zero coordinates, and drop the $\mathbf{o}$ term:

$$\mathbf{x}' = \mathbf{p} + A\mathbf{x}. \tag{10.2}$$

Sketch 10.1 gives an example. Recall, the column vectors of $A$ are the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. The point $\mathbf{p}$ tells us where to move the origin of the $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system; again, the real action of an affine map is captured by the matrix. Thus, by studying matrix actions, or linear maps, we will learn more about affine maps.

We now list some of the important properties of 3D affine maps. They are straightforward generalizations of the 2D cases, and so we just give a brief listing.




**Sketch 10.2.**
Affine maps leave ratios invariant. This map is a rigid body motion.

1. Affine maps leave *ratios* invariant (see Sketch 10.2).
2. Affine maps take *parallel planes* to parallel planes (see Figure 10.2).

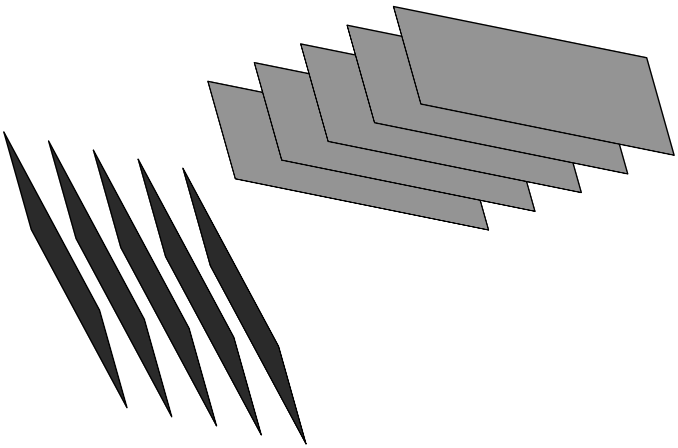


**Figure 10.2.**
Affine map property: parallel planes get mapped to parallel planes via an affine map.

3. Affine maps take *intersecting planes* to intersecting planes. In particular, the intersection line of the mapped planes is the map of the original intersection line.

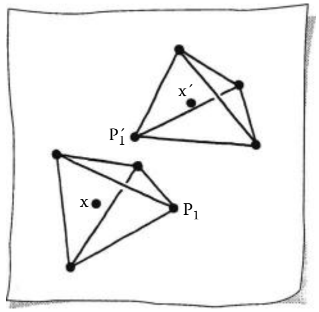
4. Affine maps leave *barycentric combinations* invariant. If

$$\mathbf{x} = c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 + c_4\mathbf{p}_4,$$

where $c_1 + c_2 + c_3 + c_4 = 1$, then after an affine map we have

$$\mathbf{x}' = c_1\mathbf{p}_1' + c_2\mathbf{p}_2' + c_3\mathbf{p}_3' + c_4\mathbf{p}_4'.$$

For example, the *centroid* of a tetrahedron will be mapped to the centroid of the mapped tetrahedron (see Sketch 10.3).




**Sketch 10.3.**
The centroid is mapped to the centroid.

Most 3D maps do not offer much over their 2D counterparts—but some do. We will go through all of them in detail now.

## 10.2 Translations

A translation is simply (10.2) with $A = I$, the $3 \times 3$ identity matrix:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

that is

$$\mathbf{x}' = \mathbf{p} + I\mathbf{x}.$$

Thus, the new $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$-system has its coordinate axes parallel to the $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system. The term $I\mathbf{x} = \mathbf{x}$ needs to be interpreted as a *vector* in the $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system for this to make sense. Figure 10.3 shows an example of repeated 3D translations.

Just as in 2D, a translation is a rigid body motion. The volume of an object is not changed.

## 10.3 Mapping Tetrahedra

A 3D affine map is determined by four point pairs $\mathbf{p}_i \rightarrow \mathbf{p}_i'$ for $i = 1, 2, 3, 4$. In other words, an affine map is determined by a tetrahedron and its image. What is the image of an arbitrary point $\mathbf{x}$ under this affine map?

Affine maps leave *barycentric combinations* unchanged. This will be the key to finding $\mathbf{x}'$, the image of $\mathbf{x}$. If we can write $\mathbf{x}$ in the form

$$\mathbf{x} = u_1\mathbf{p}_1 + u_2\mathbf{p}_2 + u_3\mathbf{p}_3 + u_4\mathbf{p}_4, \tag{10.3}$$

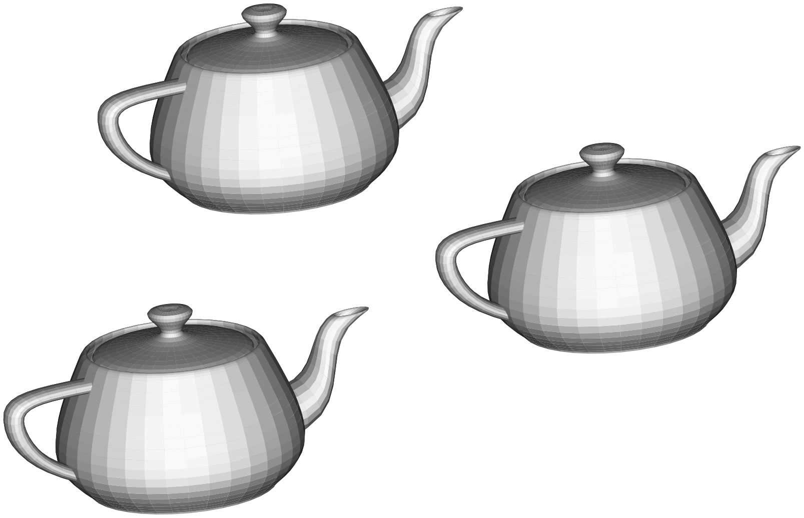


**Figure 10.3.**
Translations in 3D: three translated teapots.

then we know that the image has the same relationship with the $\mathbf{p}_i'$:

$$\mathbf{x}' = u_1\mathbf{p}_1' + u_2\mathbf{p}_2' + u_3\mathbf{p}_3' + u_4\mathbf{p}_4'. \tag{10.4}$$

So all we need to do is find the $u_i$! These are called the *barycentric coordinates* of $\mathbf{x}$ with respect to the $\mathbf{p}_i$, quite in analogy to the triangle case (see Section 6.5).

We observe that (10.3) is short for three individual coordinate equations. Together with the barycentric combination condition

$$u_1 + u_2 + u_3 + u_4 = 1,$$

we have four equations for the four unknowns $u_1, \ldots, u_4$, which we can solve by consulting Chapter 12.

### Example 10.1

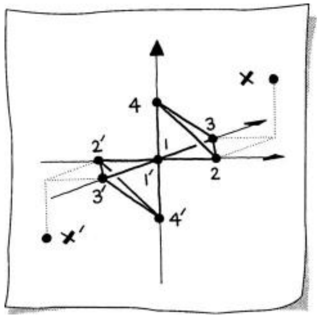
Let the original tetrahedron be given by the four points $\mathbf{p}_i$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let's assume we want to map this tetrahedron to the four points $\mathbf{p}'_i$

$$\begin{bmatrix}0\\0\\0\end{bmatrix}, \quad \begin{bmatrix}-1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\-1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\0\\-1\end{bmatrix}.$$

This is a pretty straightforward map if you consult Sketch 10.4.




**Sketch 10.4.**

An example tetrahedron map.

Let's see where the point

$$\mathbf{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

ends up. First, we find that

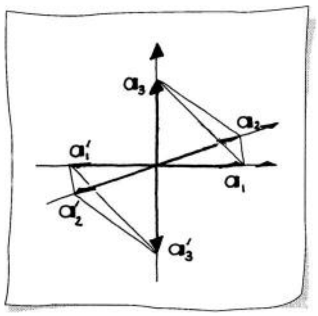
$$\begin{bmatrix}1\\1\\1\end{bmatrix} = -2\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix},$$

i.e., the barycentric coordinates of $\mathbf{x}$ with respect to the original $\mathbf{p}_i$ are $(-2, 1, 1, 1)$. Note how they sum to one. Now it is simple to compute the image of $\mathbf{x}$; compute $\mathbf{x}'$ using the same barycentric coordinates with respect to the $\mathbf{p}'_i$:

$$\mathbf{x}' = -2\begin{bmatrix}0\\0\\0\end{bmatrix} + \begin{bmatrix}-1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\-1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\-1\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}.$$

A different approach would be to find the $3 \times 3$ matrix $A$ and point $\mathbf{p}$ that describe the affine map. Construct a coordinate system from the $\mathbf{p}_i$ tetrahedron. One way to do this is to choose $\mathbf{p}_1$ as the origin[1] and the three axes are defined as $\mathbf{p}_i - \mathbf{p}_1$ for $i = 2, 3, 4$. The coordinate system of the $\mathbf{p}'_i$ tetrahedron must be based on the same indices. Once we have defined $A$ and $\mathbf{p}$ then we will be able to map $\mathbf{x}$ by this map:

$$\mathbf{x}' = A[\mathbf{x} - \mathbf{p}_1] + \mathbf{p}'_1.$$




**Sketch 10.5.**

The relationship between tetrahedra.

Thus, the point $\mathbf{p} = \mathbf{p}'_1$. In order to determine $A$, let's write down some known relationships. Referring to Sketch 10.5, we know

$$A[\mathbf{p}_2 - \mathbf{p}_1] = \mathbf{p}'_2 - \mathbf{p}'_1,$$
$$A[\mathbf{p}_3 - \mathbf{p}_1] = \mathbf{p}'_3 - \mathbf{p}'_1,$$
$$A[\mathbf{p}_4 - \mathbf{p}_1] = \mathbf{p}'_4 - \mathbf{p}'_1,$$

[1]Any of the four $\mathbf{p}_i$ would do, so for the sake of concreteness, we choose the first one.

which may be written in matrix form as

$$A\begin{bmatrix}\mathbf{p}_2-\mathbf{p}_1 & \mathbf{p}_3-\mathbf{p}_1 & \mathbf{p}_4-\mathbf{p}_1\end{bmatrix} = \begin{bmatrix}\mathbf{p}_2'-\mathbf{p}_1' & \mathbf{p}_3'-\mathbf{p}_1' & \mathbf{p}_4'-\mathbf{p}_1'\end{bmatrix}. \tag{10.5}$$

Thus,

$$A = \begin{bmatrix}\mathbf{p}_2'-\mathbf{p}_1' & \mathbf{p}_3'-\mathbf{p}_1' & \mathbf{p}_4'-\mathbf{p}_1'\end{bmatrix}\begin{bmatrix}\mathbf{p}_2-\mathbf{p}_1 & \mathbf{p}_3-\mathbf{p}_1 & \mathbf{p}_4-\mathbf{p}_1\end{bmatrix}^{-1}, \tag{10.6}$$

and $A$ is defined.

### Example 10.2

Revisiting Example 10.1, we now want to construct the matrix $A$. By selecting $\mathbf{p}_1$ as the origin for the $\mathbf{p}_i$ tetrahedron coordinate system there is no translation; $\mathbf{p}_1$ is the origin in the $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$-system and $\mathbf{p}_1' = \mathbf{p}_1$. We now compute $A$. ($A$ is the product matrix in the bottom right position):

$$\begin{array}{rrr|rrr}
 & & & 1 & 0 & 0\\
 & & & 0 & 1 & 0\\
 & & & 0 & 0 & 1\\
\hline
-1 & 0 & 0 & -1 & 0 & 0\\
0 & -1 & 0 & 0 & -1 & 0\\
0 & 0 & -1 & 0 & 0 & -1
\end{array}$$

In order to compute $\mathbf{x}'$, we have

$$\mathbf{x}' = \begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}.$$

This is the same result as in Example 10.1.

## 10.4 Parallel Projections

We looked at orthogonal parallel projections as basic linear maps in Sections 4.8 and 9.7. Everything we draw is a projection of necessity—paper is 2D, after all, whereas most interesting objects are 3D. Figure 10.4 gives an example. Here we will look at projections in the context of 3D affine maps that map 3D points onto a plane.

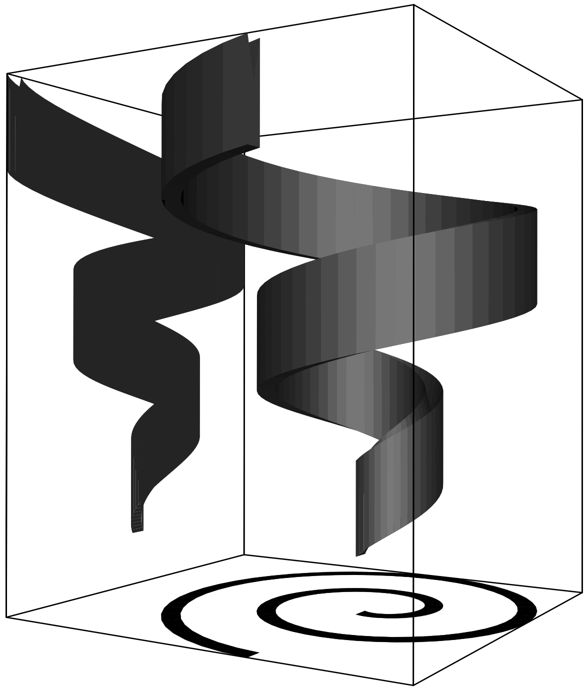
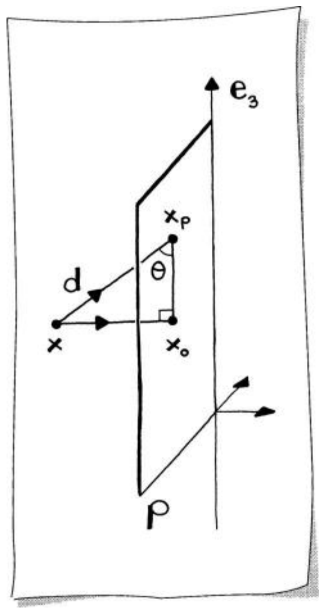


**Figure 10.4.**
Projections in 3D: a 3D helix is projected into two different 2D planes.

As illustrated in Sketch 10.6, a parallel projection is defined by a *direction* of projection $\mathbf{d}$ and a *projection plane* $P$. A point $\mathbf{x}$ is projected into $P$, and is represented as $\mathbf{x}_p$ in the sketch. This information in turn defines a *projection angle* $\theta$ between $\mathbf{d}$ and the line joining the perpendicular projection point $\mathbf{x}_o$ in $P$. This angle is used to categorize parallel projections as *orthogonal* or *oblique*.

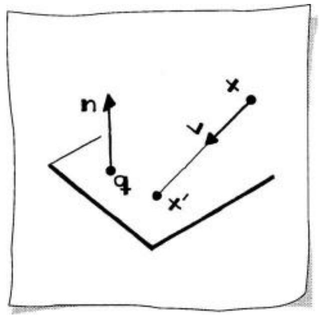



**Sketch 10.6.**
Oblique and orthographic parallel projections.

Orthogonal (also called orthographic) projections are special; their projection direction is perpendicular to the plane. There are special names for many particular projection angles; see a computer graphics text such as [10] for more details.

Let $\mathbf{x}$ be the 3D point to be projected, let $\mathbf{v}$ indicate the projection direction, and the projection plane is defined by point $\mathbf{q}$ and normal $\mathbf{n}$, as illustrated in Sketch 10.7. For some point $\mathbf{x}'$ in the plane, the plane equation is $[\mathbf{x}' - \mathbf{q}] \cdot \mathbf{n} = 0$. The intersection point is on the line defined by $\mathbf{x}$ and $\mathbf{v}$ and it is given by $\mathbf{x}' = \mathbf{p} + t\mathbf{v}$. We need to find $t$, and this is achieved by inserting the line equation into the plane equation and solving for $t$,

$$\begin{aligned}[\mathbf{x} + t\mathbf{v} - \mathbf{q}] \cdot \mathbf{n} &= 0,\\ [\mathbf{x} - \mathbf{q}] \cdot \mathbf{n} + t\mathbf{v} \cdot \mathbf{n} &= 0,\\ t &= \frac{[\mathbf{q} - \mathbf{x}] \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}.\end{aligned}$$




**Sketch 10.7.**
Projecting a point on a plane.

The intersection point $\mathbf{x}$ is now computed as

$$\mathbf{x}' = \mathbf{x} + \frac{[\mathbf{q} - \mathbf{x}] \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}\mathbf{v}. \tag{10.7}$$

How do we write (10.7) as an affine map in the form $A\mathbf{x} + \mathbf{p}$? Without much effort, we find

$$\mathbf{x}' = \mathbf{x} - \frac{\mathbf{n} \cdot \mathbf{x}}{\mathbf{v} \cdot \mathbf{n}}\mathbf{v} + \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}\mathbf{v}.$$

We know that we may write dot products in matrix form (see Section 4.11):

$$\mathbf{x}' = \mathbf{x} - \frac{\mathbf{n}^{\mathrm{T}}\mathbf{x}}{\mathbf{v} \cdot \mathbf{n}}\mathbf{v} + \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}\mathbf{v}.$$

Next, we observe that

$$\left[\mathbf{n}^{\mathrm{T}}\mathbf{x}\right]\mathbf{v} = \mathbf{v}\left[\mathbf{n}^{\mathrm{T}}\mathbf{x}\right].$$

Since matrix multiplication is associative (see Section 4.12), we also have

$$\mathbf{v}\left[\mathbf{n}^{\mathrm{T}}\mathbf{x}\right] = \left[\mathbf{v}\mathbf{n}^{\mathrm{T}}\right]\mathbf{x}.$$

Notice that $\mathbf{v}\mathbf{n}^{\mathrm{T}}$ is a $3 \times 3$ matrix. Now we can write

$$\mathbf{x}' = \left[I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v} \cdot \mathbf{n}}\right]\mathbf{x} + \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}}\mathbf{v}, \tag{10.8}$$

where $I$ is the $3 \times 3$ identity matrix. This is of the form $\mathbf{x}' = A\mathbf{x} + \mathbf{p}$ and hence is an affine map.[2]

Let's check the properties of (10.8). The projection matrix $A$, formed from $\mathbf{v}$ and $\mathbf{n}$ has rank two and thus reduces dimensionality, as designed. From the derivation of projection, it is intuitively clear

[2]Technically, we should replace $\mathbf{x}$ with $\mathbf{x} - \mathbf{o}$ to have a vector and replace $\alpha\mathbf{v}$ with $\alpha\mathbf{v} + \mathbf{o}$ to have a point, where $\alpha = (\mathbf{q} \cdot \mathbf{n})/(\mathbf{v} \cdot \mathbf{n})$.

that once $\mathbf{x}$ has been mapped into the projection plane, to $\mathbf{x}'$, it will remain there. We can also show the map is idempotent algebraically,

$$\begin{aligned} A^2 &= \left(I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}}\right)\left(I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}}\right) \\ &= I^2 - 2\frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}} + \left(\frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}}\right)^2 \\ &= A - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}} + \left(\frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}}\right)^2 \end{aligned}$$

Expanding the squared term, we find that

$$\frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}\mathbf{v}\mathbf{n}^{\mathrm{T}}}{(\mathbf{v}\cdot\mathbf{n})^2} = \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\cdot\mathbf{n}}$$

and thus $A^2 = A$. We can also show that repeating the affine map is idempotent as well:

$$\begin{aligned} A(A\mathbf{x} + \mathbf{p}) + \mathbf{p} &= A^2\mathbf{x} + A\mathbf{p} + \mathbf{p} \\ &= A\mathbf{x} + A\mathbf{p} + \mathbf{p}. \end{aligned}$$

Let $\alpha = (\mathbf{q}\cdot\mathbf{n})/(\mathbf{v}\cdot\mathbf{n})$, and examining the middle term,

$$\begin{aligned} A\mathbf{p} &= \left(I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{v}\mathbf{n}}\right)\alpha\mathbf{v} \\ &= \alpha\mathbf{v} - \alpha\mathbf{v}\left(\frac{\mathbf{n}^{\mathrm{T}}\mathbf{v}}{\mathbf{v}\mathbf{n}}\right) \\ &= 0 \end{aligned}$$

Therefore, $A(A\mathbf{x}+\mathbf{p})+\mathbf{p} = A\mathbf{x}+\mathbf{p}$, and we have shown that indeed, the affine map is idempotent.

### Example 10.3

Suppose we are given the projection plane $x_1 + x_2 + x_3 - 1 = 0$, a point $\mathbf{x}$ (not in the plane), and a direction $\mathbf{v}$ given by

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

If we project $\mathbf{x}$ along $\mathbf{v}$ onto the plane, what is $\mathbf{x}'$? Sketch 10.8 illustrates this geometry. First, we need the plane's normal direction. Calling it $\mathbf{n}$, we have

$$\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Now, choose a point $\mathbf{q}$ in the plane. Let's choose

$$\mathbf{q} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for simplicity. Now we are ready to calculate the quantities in (10.8):

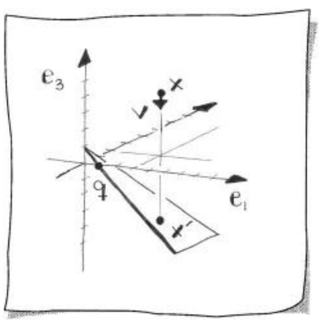
$$\mathbf{v} \cdot \mathbf{n} = -1,$$

$$\mathbf{v}\mathbf{n}^{\mathrm{T}} = \begin{array}{r|rrr} & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{array},$$

$$\frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{v} \cdot \mathbf{n}} \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Putting all the pieces together:

$$\mathbf{x}' = \left[ I - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right] \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}.$$




**Sketch 10.8.**
A projection example.

Just to double-check, enter $\mathbf{x}'$ into the plane equation

$$3 + 2 - 4 - 1 = 0,$$

and we see that

$$\begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + 8 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix},$$

which together verify that this is the correct point. Sketch 10.8 should convince you that this is indeed the correct answer.

Which of the two possibilities, (10.7) or the affine map (10.8) should you use? Clearly, (10.7) is more straightforward and less involved. Yet in some computer graphics or CAD system environments, it may be desirable to have all maps in a unified format, i.e., $A\mathbf{x} + \mathbf{p}$. We'll revisit this unified format idea in Section 10.5.

## 10.5 Homogeneous Coordinates and Perspective Maps

There is a way to condense the form $\mathbf{x}' = A\mathbf{x} + \mathbf{p}$ of an affine map into just one matrix multiplication

$$\underline{\mathbf{x}}' = M\underline{\mathbf{x}}. \tag{10.9}$$

This is achieved by setting

$$M = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & p_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & p_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\underline{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}, \quad \underline{\mathbf{x}}' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ 1 \end{bmatrix}.$$

The 4D point $\underline{\mathbf{x}}$ is called the *homogeneous form* of the affine point $\mathbf{x}$. You should verify for yourself that (10.9) is indeed the same affine map as before.

The homogeneous representation of a vector $\underline{\mathbf{v}}$ must have the form,

$$\underline{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{bmatrix}.$$

This form allows us to apply the linear map to the vector,

$$\underline{\mathbf{v}}' = M\underline{\mathbf{v}}.$$

By having a zero fourth component, we disregard the translation, which we know has no effect on vectors. Recall that a vector is defined as the difference of two points.

This method of condensing transfomation information into one matrix is implemented in the popular computer graphics *Application*

*Programmer's Interface* (*API*), OpenGL [15]. It is very convenient and efficient to have all this information (plus more, as we will see), in one data structure.

The homogeneous form is more general than just adding a fourth coordinate $x_4 = 1$ to a point. If, perhaps as the result of some computation, the fourth coordinate does not equal one, one gets from the homogeneous point $\underline{\mathbf{x}}$ to its affine counterpart $\mathbf{x}$ by dividing through by $x_4$. Thus, one affine point has infinitely many homogeneous representations!

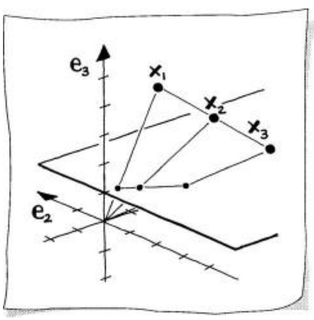
### Example 10.4

This example shows two homogeneous representations of one affine point. (The symbol $\approx$ should be read "corresponds to.")

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \approx \begin{bmatrix} 10 \\ -10 \\ 30 \\ 10 \end{bmatrix} \approx \begin{bmatrix} -2 \\ 2 \\ -6 \\ -2 \end{bmatrix}.$$

Using the homogeneous matrix form of (10.9), the matrix $M$ for the point into a plane projection from (10.8) becomes

$$\left[\begin{array}{c|c} \begin{bmatrix} \mathbf{v}\cdot\mathbf{n} & 0 & 0 \\ 0 & \mathbf{v}\cdot\mathbf{n} & 0 \\ 0 & 0 & \mathbf{v}\cdot\mathbf{n} \end{bmatrix} - \mathbf{v}\mathbf{n}^{\mathrm{T}} & (\mathbf{q}\cdot\mathbf{n})\mathbf{v} \\ \hline 0 \quad\quad 0 \quad\quad 0 & \mathbf{v}\cdot\mathbf{n} \end{array}\right]$$




**Sketch 10.9.**
Perspective projection.

Here, the element $m_{4,4} = \mathbf{v}\cdot\mathbf{n}$. Thus, $\underline{x}_4 = \mathbf{v}\cdot\mathbf{n}$, and we will have to divide $\underline{\mathbf{x}}$'s coordinates by $\underline{x}_4$ in order to obtain the corresponding affine point.

A simple change in our equations will lead us from parallel projections onto a plane to *perspective projections*. Instead of using a constant direction $\mathbf{v}$ for all projections, now the direction depends on the point $\mathbf{x}$. More precisely, let it be the line from $\mathbf{x}$ to the origin of our coordinate system. Then, as shown in Sketch 10.9, $\mathbf{v} = -\mathbf{x}$, and (10.7) becomes

$$\mathbf{x}' = \mathbf{x} + \frac{[\mathbf{q} - \mathbf{x}]\cdot\mathbf{n}}{\mathbf{x}\cdot\mathbf{n}}\mathbf{x},$$

which quickly simplifies to

$$\mathbf{x}' = \frac{\mathbf{q} \cdot \mathbf{n}}{\mathbf{x} \cdot \mathbf{n}} \mathbf{x}. \tag{10.10}$$

In homogeneous form, this is described by the following matrix

$$M : \begin{array}{|ccc|c|} \hline & I[\mathbf{q} \cdot \mathbf{n}] & & \mathbf{o} \\ \hline 0 & 0 & 0 & \mathbf{x} \cdot \mathbf{n} \\ \hline \end{array}.$$

Perspective projections are not affine maps anymore! To see this, a simple example will suffice.

### Example 10.5

Take the plane $x_3 = 1$; let

$$\mathbf{q} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

be a point on the plane. Now $\mathbf{q} \cdot \mathbf{n} = 1$ and $\mathbf{x} \cdot \mathbf{n} = x_3$, resulting in the map

$$\mathbf{x}' = \frac{1}{x_3} \mathbf{x}.$$

Take the three points

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}.$$

This example is illustrated in Sketch 10.9. Note that $\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_1 + \frac{1}{2}\mathbf{x}_3$, i.e., $\mathbf{x}_2$ is the midpoint of $\mathbf{x}_1$ and $\mathbf{x}_3$.

Their images are

$$\mathbf{x}'_1 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}'_2 = \begin{bmatrix} 1 \\ -1/3 \\ 1 \end{bmatrix}, \quad \mathbf{x}'_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

The perspective map destroyed the midpoint relation! Now,

$$\mathbf{x}'_2 = \frac{2}{3}\mathbf{x}'_1 + \frac{1}{3}\mathbf{x}'_3.$$

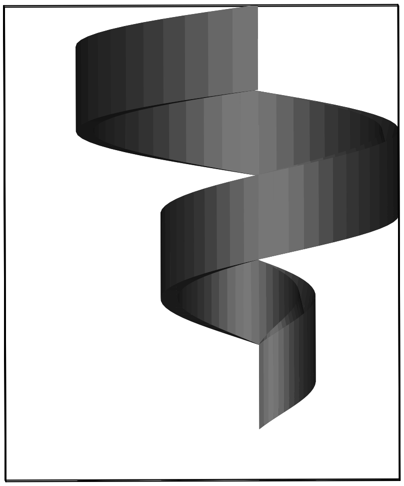


**Figure 10.5.**
Parallel projection: a 3D helix and two orthographic projections on the left and bottom walls of the bounding cube—not visible due to the orthographic projection used for the whole scene.

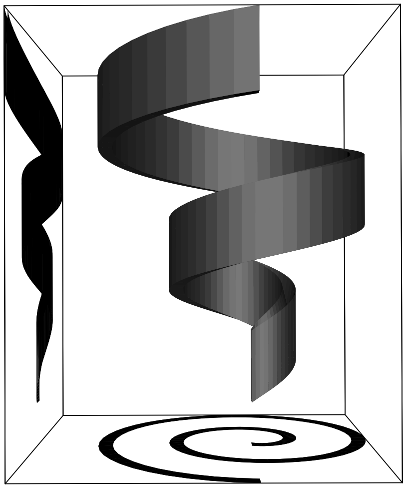


**Figure 10.6.**
Perspective projection: a 3D helix and two orthographic projections on the left and bottom walls of the bounding cube—visible due to the perspective projection used for the whole scene.

Thus, the ratio of three points is changed by perspective maps. As a consequence, two parallel lines will not be mapped to parallel lines. Because of this effect, perspective maps are a good model for how we perceive 3D space around us. Parallel lines do seemingly intersect in a distance, and are thus not *perceived* as being parallel! Figure 10.5 is a parallel projection and Figure 10.6 illustrates the same geometry with a perspective projection. Notice in the perspective image, the sides of the bounding cube that move into the page are no longer parallel.

As we saw above, $m_{4,4}$ allows us to specify perspective projections. The other elements of the bottom row of $M$ are used for *projective maps*, a more general mapping than a perspective projection. The topic of this chapter is affine maps, so we'll leave a detailed discussion of these elements to another source: A mathematical treatment of this map is supplied by [6] and a computer graphics treatment is supplied by [14]. In short, these entries are used in computer graphics for mapping a *viewing volume*[3] in the shape of a frustum to one in the shape of a cube, while preserving the perspective projection effect.

[3]The dimension and shape of the viewing volume defines *what* will be displayed and *how* it will be displayed (orthographic or perspective).

**Figure 10.7.**
Perspective maps: an experiment by A. Dürer.

Algorithms in the graphics pipeline are very much simplified by only dealing with geometry known to be in a cube.

The study of perspective goes back to the fourteenth century—before that, artists simply could not draw realistic 3D images. One of the foremost researchers in the area of perspective maps was A. Dürer.[4] See Figure 10.7 for one of his experiments.

[4]From *The Complete Woodcuts of Albrecht Dürer,* edited by W. Durth, Dover Publications Inc., New York, 1963.

- affine map
- translation
- affine map properties
- barycentric combination
- invariant ratios
- barycentric coordinates
- centroid
- mapping four points to four points
- parallel projection
- orthogonal projection
- oblique projection
- line and plane intersection
- idempotent
- dyadic matrix
- homogeneous coordinates
- perspective projection
- rank

## 10.6 Exercises

We'll use four points

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

four points

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{y}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and also the plane through $\mathbf{q}$ with normal $\mathbf{n}$:

$$\mathbf{q} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{n} = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}.$$

1. What are the two parts of an affine map?
2. An affine map $\mathbf{x}_i \rightarrow \mathbf{y}_i; i = 1, 2, 3, 4$ is uniquely defined. What is it?
3. What is the image of
$$\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
under the map from Exercise 2? Use two ways to compute it.
4. What are the geometric properties of the affine map from Exercises 2 and 3?
5. An affine map $\mathbf{y}_i \rightarrow \mathbf{x}_i; i = 1, 2, 3, 4$ is uniquely defined. What is it?

6. Using a direction

$$\mathbf{v} = \frac{1}{4}\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix},$$

what are the images of the $\mathbf{x}_i$ when projected in this direction onto the plane defined at the beginning of the exercises?

7. Using the same $\mathbf{v}$ as in Exercise 6, what are the images of the $\mathbf{y}_i$?

8. What are the images of the $\mathbf{x}_i$ when projected onto the plane by a perspective projection through the origin?

9. What are the images of the $\mathbf{y}_i$ when projected onto the plane by a perspective projection through the origin?

10. Compute the centroid $\mathbf{c}$ of the $\mathbf{x}_i$ and then the centroid $\mathbf{c}'$ of their perspective images (from Exercise 8). Is $\mathbf{c}'$ the image of $\mathbf{c}$ under the perspective map?

11. We claimed that (10.8) reduces to (10.10). This necessitates that

$$\left[I - \frac{\mathbf{v}\mathbf{n}^{\mathrm{T}}}{\mathbf{n}\cdot\mathbf{v}}\right]\mathbf{x} = \mathbf{0}.$$

Show that this is indeed true.

12. What is the affine map that rotates the point $\mathbf{q}$ (defined above) 90° about the line defined as

$$\mathbf{l}(t) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}?$$

*Hint: This is a simple construction, and does not require (9.10).*

13. Suppose we have the unit cube with "lower-left" and "upper-right" points

$$\mathbf{l} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

respectively. What is the affine map that scales this cube uniformly by two, rotates it $-45°$ around the $\mathbf{e}_2$-axis, and then positions it so that $\mathbf{l}$ is mapped to

$$\mathbf{l}' = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}?$$

What is $\mathbf{u}'$?

14. Suppose we have a diamond-shaped geometric figure defined by the following vertices,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix},$$

that is positioned in the $x_3 = 2$ plane. We want to rotate the diamond about its centroid $\mathbf{c}$ (with a positive angle) so it is positioned in the $x_1 = c_1$ plane. What is the affine map that achieves this and what are the mapped vertices $\mathbf{v}'_i$?