4

Changing Shapes: Linear Maps in 2D

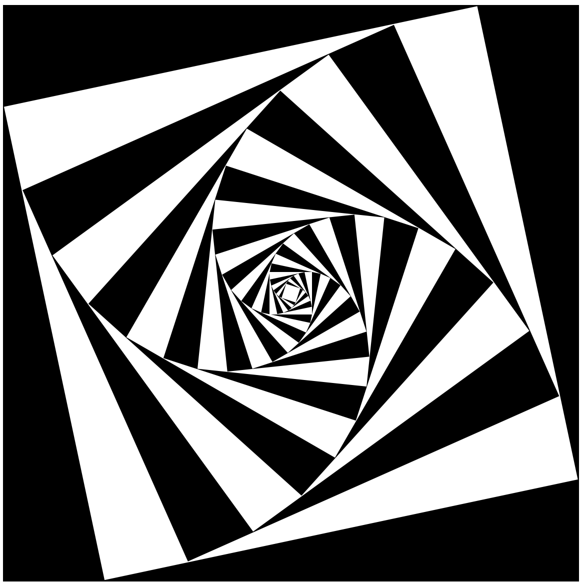


Figure 4.1.
Linear maps in 2D: an interesting geometric figure constructed by applying 2D linear maps to a square.

Geometry always has two parts to it: one part is the description of the objects that can be generated; the other investigates how these objects can be changed (or transformed). Any object formed by several vectors may be mapped to an arbitrarily bizarre curved or distorted object—here, we are interested in those maps that map 2D vectors to 2D vectors and are "benign" in some well-defined sense. All these maps may be described using the tools of matrix operations, or linear maps. An interesting pattern is generated from a simple square in Figure 4.1 by such "benign" 2D linear maps—rotations and scalings.

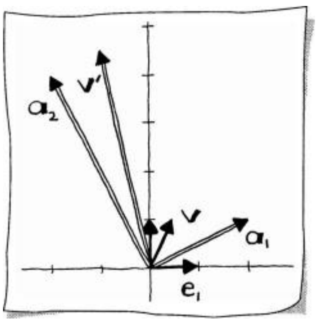
Matrices were first introduced by H. Grassmann in 1844. They became the basis of *linear algebra*. Most of their properties can be studied by just considering the humble 2×2 case, which corresponds to 2D linear maps.

4.1 Skew Target Boxes

In Section 1.1, we saw how to map an object from a unit square to a rectangular target box. We will now look at the part of that mapping that is a linear map.

First, our unit square will be defined by vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Thus, a vector $\mathbf{v}$ in this $[\mathbf{e}_1, \mathbf{e}_2]$-system is defined as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2. \tag{4.1}$$



Sketch 4.1.
A skew target box defined by $\mathbf{a}_1$ and $\mathbf{a}_2$.

If we focus on mapping vectors to vectors, then we will limit the target box to having a lower-left corner at the origin. In Chapter 6 we will reintroduce the idea of a generally positioned target box. Instead of specifying two extreme points for a rectangular target box, we will describe a parallelogram target box by two vectors $\mathbf{a}_1, \mathbf{a}_2$, defining an $[\mathbf{a}_1, \mathbf{a}_2]$-system. A vector $\mathbf{v}$ is now mapped to a vector $\mathbf{v}'$ by

$$\mathbf{v}' = v_1\mathbf{a}_1 + v_2\mathbf{a}_2, \tag{4.2}$$

as illustrated by Sketch 4.1. This simply states that we duplicate the $[\mathbf{e}_1, \mathbf{e}_2]$-geometry in the $[\mathbf{a}_1, \mathbf{a}_2]$-system: The linear map transforms $\mathbf{e}_1, \mathbf{e}_2, \mathbf{v}$ to $\mathbf{a}_1, \mathbf{a}_2, \mathbf{v}'$, respectively. The components of $\mathbf{v}'$ are in the context of the $[\mathbf{e}_1, \mathbf{e}_2]$-system. However, the components of $\mathbf{v}'$ with respect to the $[\mathbf{a}_1, \mathbf{a}_2]$-system are the components of $\mathbf{v}$. Reviewing a definition from Section 2.6, we recall that (4.2) is called a *linear combination*.

Example 4.1

Let's look at an example of the action of the map from the linear combination in (4.2). Let the origin and

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

define a new $[\mathbf{a}_1, \mathbf{a}_2]$-coordinate system, and let

$$\mathbf{v} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

be a vector in the $[\mathbf{e}_1, \mathbf{e}_2]$-system. Applying the components of $\mathbf{v}$ in a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$, results in

$$\mathbf{v}' = \frac{1}{2} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 9/2 \end{bmatrix}. \tag{4.3}$$

Thus $\mathbf{v}'$ has components

$$\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

with respect to the $[\mathbf{a}_1, \mathbf{a}_2]$-system; with respect to the $[\mathbf{e}_1, \mathbf{e}_2]$-system, it has coordinates

$$\begin{bmatrix} -1 \\ 9/2 \end{bmatrix}.$$

See Sketch 4.1 for an illustration.

4.2 The Matrix Form

The components of a subscripted vector will be written with a double subscript as

$$\mathbf{a}_1 = \begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix}.$$

The vector component index precedes the vector subscript.

The components for the vector $\mathbf{v}'$ in the $[\mathbf{e}_1, \mathbf{e}_2]$-system from Example 4.1 are expressed as

$$\begin{bmatrix} -1 \\ 9/2 \end{bmatrix} = \frac{1}{2} \times \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \times \begin{bmatrix} -2 \\ 4 \end{bmatrix}. \tag{4.4}$$

This is strictly an equation between vectors. It invites a more concise notation using *matrix notation*:

$$\begin{bmatrix} -1 \\ 9/2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}. \tag{4.5}$$

The 2×2 array in this equation is called a *matrix*. It has two columns, corresponding to the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. It also has two rows, namely the first row with entries $2, -2$ and the second one with $1, 4$.

In general, an equation like this one has the form

$$\mathbf{v}' = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{4.6}$$

or,

$$\mathbf{v}' = A\mathbf{v}, \tag{4.7}$$

where A is the 2×2 matrix. The vector $\mathbf{v}'$ is called the *image* of $\mathbf{v}$, and thus $\mathbf{v}$ is the *preimage*. The *linear map* is described by the matrix A—we may think of A as being the map's coordinates. We will also refer to the linear map itself by A. Then $\mathbf{v}'$ is in the *range* of the map and $\mathbf{v}$ is in the *domain*.

The elements $a_{1,1}$ and $a_{2,2}$ form the *diagonal* of the matrix. The product $A\mathbf{v}$ has two components, each of which is obtained as a dot product between the corresponding row of the matrix and $\mathbf{v}$. In full generality, we have

$$A\mathbf{v} = \begin{bmatrix} v_1\mathbf{a}_1 + v_2\mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} v_1 a_{1,1} + v_2 a_{1,2} \\ v_1 a_{2,1} + v_2 a_{2,2} \end{bmatrix}.$$

For example,

$$\begin{bmatrix} 0 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 14 \end{bmatrix}.$$

In other words, $A\mathbf{v}$ is equivalent to forming the linear combination $\mathbf{v}$ of the columns of A. All such combinations, that is all such $\mathbf{v}'$, form the *column space* of A.

Another note on notation. Coordinate systems, such as the $[\mathbf{e}_1, \mathbf{e}_2]$-system, can be interpreted as a matrix with columns $\mathbf{e}_1$ and $\mathbf{e}_2$. Thus,

$$[\mathbf{e}_1, \mathbf{e}_2] \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and this is called the 2×2 *identity matrix*.

There is a neat way to write the matrix-times-vector algebra that facilitates manual computation. As explained above, every entry in the resulting vector is a dot product of the input vector and a row of the matrix. Let's arrange this as follows:

$$\begin{array}{cc|c} & & 2 \\ & & 1/2 \\ \hline 2 & -2 & 3 \\ 1 & 4 & 4 \end{array}$$

Each entry of the resulting vector is now at the intersection of the corresponding matrix row and the input vector, which is written as a column. As you multiply and then add the terms in your dot product, this scheme guides you to the correct position in the result automatically! Here we multiplied a 2×2 matrix by a 2×1 vector. Note that the interior dimensions (both 2) must be identical and the outer dimensions, 2 and 1, indicate the resulting vector or matrix size, 2×1. Sometimes it is convenient to think of the vector $\mathbf{v}$ as a 2×1 matrix: it is a matrix with two rows and one column.

One fundamental matrix operation is *matrix addition.* Two matrices A and B may be added by adding corresponding elements:

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} \end{bmatrix}. \tag{4.8}$$

Notice that the matrices must be of the same dimensions; this is not true for matrix multiplication, which we will demonstrate in Section 4.10.

Using matrix addition, we may write

$$A\mathbf{v} + B\mathbf{v} = (A + B)\mathbf{v}.$$

This works because of the very simple definition of matrix addition. This is also called the *distributive law.*

Forming the *transpose matrix* is another fundamental matrix operation. It is denoted by A^{T} and is formed by interchanging the rows and columns of A: the first row of A^{T} is A's first column, and the second row of A^{T} is A's second column. For example, if

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}, \quad \text{then} \quad A^{\mathrm{T}} = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}.$$

Since we may think of a vector $\mathbf{v}$ as a matrix, we should be able to find $\mathbf{v}$'s transpose. Not very hard: it is a vector with one row and two columns,

$$\mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v}^{\mathrm{T}} = \begin{bmatrix} -1 & 4 \end{bmatrix}.$$

It is straightforward to confirm that

$$[A+B]^{\mathrm{T}} = A^{\mathrm{T}} + B^{\mathrm{T}}. \tag{4.9}$$

Two more identities are

$$A^{\mathrm{T}^{\mathrm{T}}} = A \quad \text{and} \quad [cA]^{\mathrm{T}} = cA^{\mathrm{T}}. \tag{4.10}$$

A *symmetric matrix* is a special matrix that we will encounter many times. A matrix A is symmetric if $A = A^{\mathrm{T}}$, for example

$$\begin{bmatrix} 5 & 8 \\ 8 & 1 \end{bmatrix}.$$

There are no restrictions on the diagonal elements, but all other elements are equal to the element about the diagonal with reversed indices. For a 2×2 matrix, this means that $a_{2,1} = a_{1,2}$.

With matrix notation, we can now continue the discussion of independence from Section 2.6. The columns of a matrix define an $[\mathbf{a}_1, \mathbf{a}_2]$-system. If the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent then the matrix is said to have *full rank*, or for the 2×2 case, the matrix has *rank* 2. If $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly dependent then the matrix has rank 1. These two statements may be summarized as: the rank of a 2×2 matrix equals the number of linearly independent column (row) vectors. Matrices that do not have full rank are called *rank deficient* or *singular*. We will encounter an important example of a rank deficient matrix, a projection matrix, in Section 4.8. The only matrix with rank zero is the *zero matrix*, a matrix with all zero entries. The 2×2 zero matrix is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

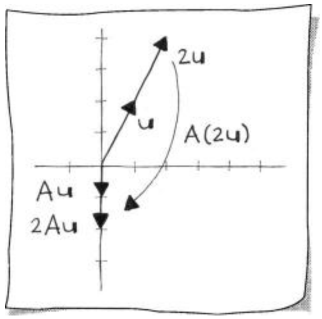
The importance of the rank equivalence of column and row vectors will come to light in later chapters when we deal with $n \times n$ matrices. For now, we can observe that this fact means that the ranks of A and A^{T} are equal.

4.3 Linear Spaces

2D linear maps act on vectors in 2D *linear spaces*, also known as 2D *vector spaces*. Recall from Section 2.1 that the set of all ordered pairs, or 2D vectors $\mathbf{v}$ is called $\mathbb{R}^2$. In Section 2.8, we encountered the concept of a *subspace* of $\mathbb{R}^2$ in finding the orthogonal projection of a vector $\mathbf{w}$ onto a vector $\mathbf{v}$. (In Chapter 14, we will look at more general linear spaces.)

The standard operations in a linear space are addition and scalar multiplication, which are encapsulated for vectors in the linear combination in (4.2). This is called the *linearity property.* Additionally, linear maps, or matrices, are characterized by preservation of linear combinations. This statement can be encapsulated as follows

$$A(a\mathbf{u} + b\mathbf{v}) = aA\mathbf{u} + bA\mathbf{v}. \tag{4.11}$$



Sketch 4.2.
Matrices preserve scalings.

Let's break this statement down into the two basic elements: scalar multiplication and addition. For the sake of concreteness, we shall use the example

$$A = \begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

We may multiply all elements of a matrix by one factor; we then say that we have multiplied the matrix by that factor. Using our example, we may multiply the matrix A by a factor, say 2:

$$2 \times \begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}.$$

When we say that matrices preserve multiplication by scalar factors we mean that if we scale a vector by a factor c, then its image will also be scaled by c:

$$A(c\mathbf{u}) = cA\mathbf{u}.$$

Example 4.2

Here are the computations that go along with Sketch 4.2:

$$\begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \left(2 \times \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 2 \times \begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

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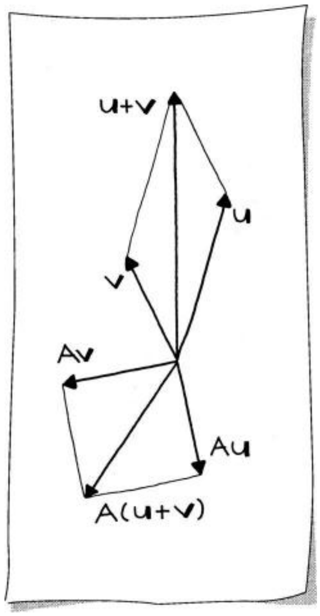
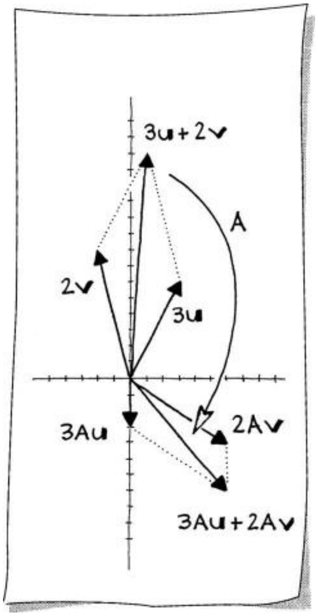


Sketch 4.3.
Matrices preserve sums.

Matrices also preserve sums:

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}.$$

This is also called the *distributive law.* Sketch 4.3 illustrates this property (with a different set of $A, \mathbf{u}, \mathbf{v}$).



Sketch 4.4.
Matrices preserve linear combinations.

Now an example to demonstrate that matrices preserve linear combinations, as expressed in (4.11).

Example 4.3

$$A(3\mathbf{u}+2\mathbf{v}) = \begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \left(3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right)$$
$$= \begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 14 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}.$$

$$3A\mathbf{u} + 2A\mathbf{v} = 3 \begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 & 1/2 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}.$$

Sketch 4.4 illustrates this example.

Preservation of linear combinations is a key property of matrices—we will make substantial use of it throughout this book.

4.4 Scalings

Consider the linear map given by

$$\mathbf{v}' = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \mathbf{v} = \begin{bmatrix} v_1/2 \\ v_2/2 \end{bmatrix}. \tag{4.12}$$

This map will shorten $\mathbf{v}$ since $\mathbf{v}' = 1/2\mathbf{v}$. Its effect is illustrated in Figure 4.2. That figure—and more to follow—has two parts. The left part is a Phoenix whose feathers form rays that correspond to a sampling of unit vectors. The right part shows what happens if we map the Phoenix, and in turn the unit vectors, using the matrix from (4.12). In this figure we have drawn the $\mathbf{e}_1$ and $\mathbf{e}_2$ vectors,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

but in future figures we will not. Notice the positioning of these vectors relative to the Phoenix. Now in the right half, $\mathbf{e}_1$ and $\mathbf{e}_2$ have

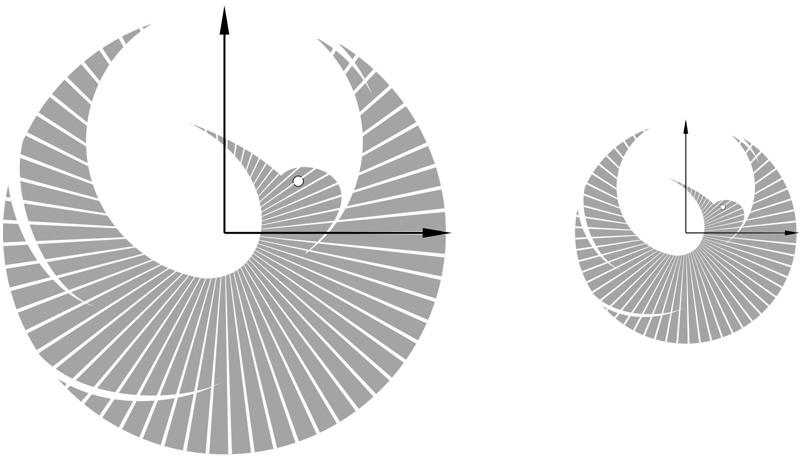


Figure 4.2.
Scaling: a uniform scaling.

been mapped to the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. These are the column vectors of the matrix in (4.12),

$$\mathbf{a}_1 = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}.$$

The Phoenix's shape provides a sense of orientation. In this example, the linear map did not change orientation, but more complicated maps will.

Next, consider

$$\mathbf{v}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{v}.$$

Now, $\mathbf{v}$ will be "enlarged."

In general, a scaling is defined by the operation

$$\mathbf{v}' = \begin{bmatrix} s_{1,1} & 0 \\ 0 & s_{2,2} \end{bmatrix} \mathbf{v}, \tag{4.13}$$

thus allowing for nonuniform scalings in the $\mathbf{e}_1$- and $\mathbf{e}_2$-direction. Figure 4.3 gives an example for $s_{1,1} = 1/2$ and $s_{2,2} = 2$.

A scaling affects the *area* of the object that is scaled. If we scale an object by $s_{1,1}$ in the $\mathbf{e}_1$-direction, then its area will be changed by a factor $s_{1,1}$. Similarly, it will change by a factor of $s_{2,2}$ when we

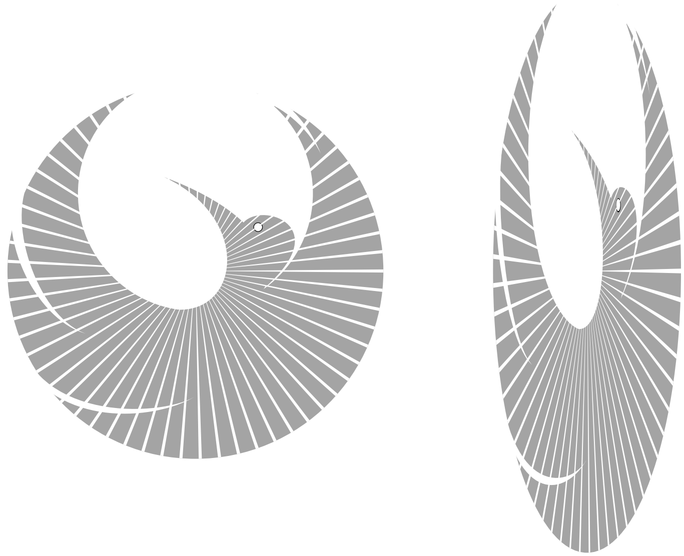


Figure 4.3.
Scaling: a nonuniform scaling.

apply that scaling to the $\mathbf{e}_2$-direction. The total effect is thus a factor of $s_{1,1}s_{2,2}$.

You can see this from Figure 4.2 by mentally constructing the square spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ and comparing its area to the rectangle spanned by the image vectors. It is also interesting to note that, in Figure 4.3, the scaling factors result in no change of area, although a distortion did occur.

The distortion of the circular Phoenix that we see in Figure 4.3 is actually well-defined—it is an ellipse! In fact, all 2×2 matrices will map circles to ellipses. (In higher dimensions, we will speak of ellipsoids.) We will refer to this ellipse that characterizes the action of the matrix as the *action ellipse*.[1] In Figure 4.2, the action ellipse is a scaled circle, which is a special case of an ellipse. In Chapter 16, we will relate the shape of the ellipse to the linear map.

[1]We will study ellipses in Chapter 19. An ellipse is symmetric about two axes that intersect at the center of the ellipse. The longer axis is called the *major axis* and the shorter axis is called the *minor axis*. The semi-major and semi-minor axes are one-half their respective axes.

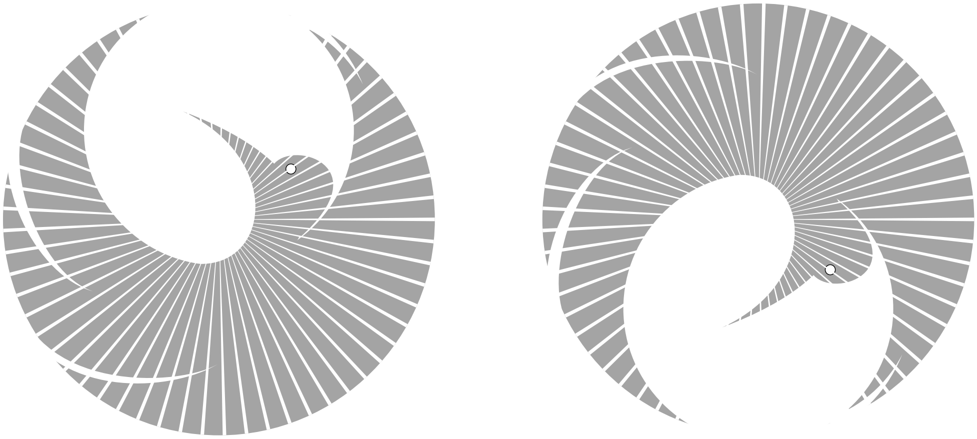


Figure 4.4.
Reflections: a reflection about the $\mathbf{e}_1$-axis.

4.5 Reflections

Consider the scaling

$$\mathbf{v}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v}. \tag{4.14}$$

We may rewrite this as

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}.$$

The effect of this map is apparently a change in sign of the second component of $\mathbf{v}$, as shown in Figure 4.4. Geometrically, this means that the input vector $\mathbf{v}$ is reflected about the $\mathbf{e}_1$-axis, or the line $x_1 = 0$.

Obviously, reflections like the one in Figure 4.4 are just a special case of scalings—previously we simply had not given much thought to negative scaling factors. However, a reflection takes a more general form, and it results in the mirror image of the vectors. Mathematically, a reflection maps each vector about a line through the origin.

The most common reflections are those about the coordinate axes, with one such example illustrated in Figure 4.4, and about the lines $x_1 = x_2$ and $x_1 = -x_2$. The reflection about the line $x_1 = x_2$ is

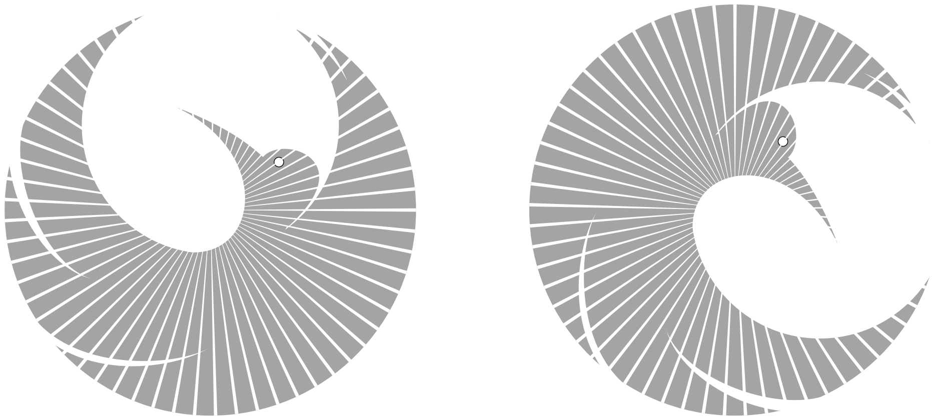


Figure 4.5.
Reflections: a reflection about the line $x_1 = x_2$.

achieved by the matrix

$$\mathbf{v}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{v} = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}.$$

Its effect is shown in Figure 4.5; that is, the components of the input vector are interchanged.

By inspection of the figures in this section, it appears that reflections do not change areas. But be careful—they do change the *sign* of the area due to a change in orientation. If we rotate $\mathbf{e}_1$ into $\mathbf{e}_2$, we move in a counterclockwise direction. Now, rotate

$$\mathbf{a}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{into} \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and notice that we move in a clockwise direction. This change in orientation is reflected in the sign of the area. We will examine this in detail in Section 4.9.

The matrix

$$\mathbf{v}' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v}, \tag{4.15}$$

as seen in Figure 4.6, appears to be a reflection, but it is really a rotation of 180°. (Rotations are covered in Section 4.6.) If we rotate $\mathbf{a}_1$ into $\mathbf{a}_2$ we move in a counterclockwise direction, confirming that this is not a reflection.

Notice that all reflections result in an action ellipse that is a circle.

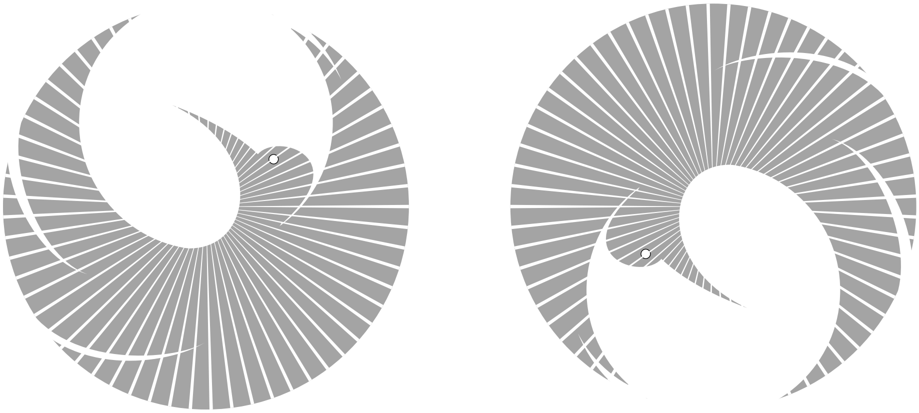


Figure 4.6.
Reflections: a reflection about both axes is also a rotation of 180°.

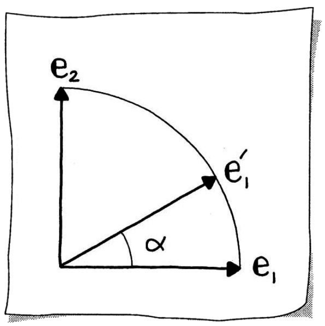
4.6 Rotations

The notion of rotating a vector around the origin is intuitively clear, but a corresponding matrix takes a few moments to construct. To keep it easy at the beginning, let us rotate the unit vector

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

by α degrees counterclockwise, resulting in a new (rotated) vector

$$\mathbf{e}_1' = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}.$$



Sketch 4.5.
Rotating a unit vector.

Notice that $\cos^2\alpha + \sin^2\alpha = 1$, thus this is a rotation. Consult Sketch 4.5 to convince yourself of this fact.

Thus, we need to find a matrix R that achieves

$$\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Additionally, we know that $\mathbf{e}_2$ will rotate to

$$\mathbf{e}_2' = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix}.$$

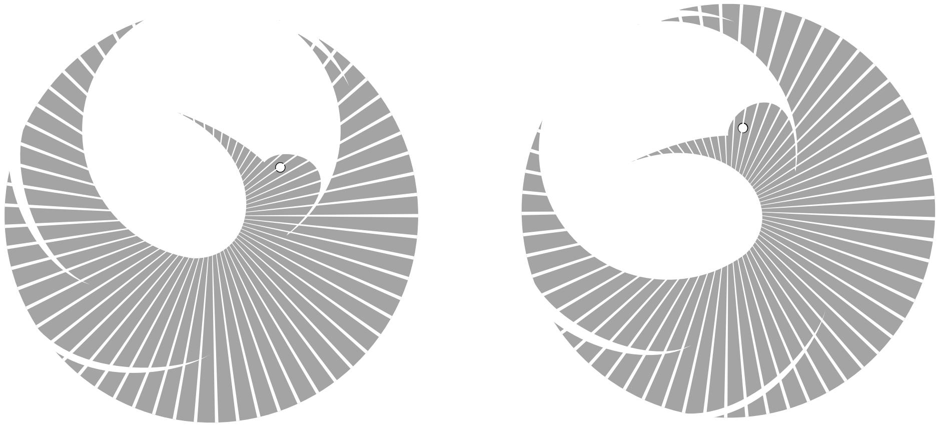


Figure 4.7.
Rotations: a rotation by 45°.

This leads to the correct *rotation matrix*; it is given by

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}. \tag{4.16}$$

But let's verify that we have already found the solution to the general rotation problem.

Let $\mathbf{v}$ be an arbitrary vector. We claim that the matrix R from (4.16) will rotate it by α degrees to a new vector $\mathbf{v}'$. If this is so, then we must have

$$\mathbf{v} \cdot \mathbf{v}' = \|\mathbf{v}\|^2 \cos\alpha$$

according to the rules of dot products (see Section 2.7). Here, we made use of the fact that a rotation does not change the length of a vector, i.e., $\|\mathbf{v}\| = \|\mathbf{v}'\|$ and hence $\|\mathbf{v}\| \cdot \|\mathbf{v}'\| = \|\mathbf{v}\|^2$.

Since

$$\mathbf{v}' = \begin{bmatrix} v_1 \cos\alpha - v_2 \sin\alpha \\ v_1 \sin\alpha + v_2 \cos\alpha \end{bmatrix},$$

the dot product $\mathbf{v} \cdot \mathbf{v}'$ is given by

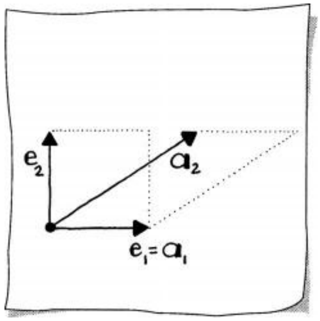
$$\begin{aligned} \mathbf{v} \cdot \mathbf{v}' &= v_1^2 \cos\alpha - v_1 v_2 \sin\alpha + v_1 v_2 \sin\alpha + v_2^2 \cos\alpha \\ &= (v_1^2 + v_2^2) \cos\alpha \\ &= \|\mathbf{v}\|^2 \cos\alpha, \end{aligned}$$

and all is shown! See Figure 4.7 for an illustration. There, $\alpha = 45°$, and the rotation matrix is thus given by

$$R = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

Rotations are in a special class of transformations; these are called *rigid body motions*. (See Section 5.9 for more details on these special matrices.) The action ellipse of a rotation is a circle. Finally, it should come without saying that rotations do not change areas.

4.7 Shears



Sketch 4.6.

A special shear.

What map takes a rectangle to a parallelogram? Pictorially, one such map is shown in Sketch 4.6.

In this example, we have a map:

$$\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \mathbf{v}' = \begin{bmatrix} d_1 \\ 1 \end{bmatrix}.$$

In matrix form, this is realized by

$$\begin{bmatrix} d_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4.17}$$

Verify! The 2×2 matrix in this equation is called a *shear matrix*. It is the kind of matrix that is used when you generate italic fonts from standard ones.

A shear matrix may be applied to arbitrary vectors. If $\mathbf{v}$ is an input vector, then a shear maps it to $\mathbf{v}'$:

$$\mathbf{v}' = \begin{bmatrix} 1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 d_1 \\ v_2 \end{bmatrix},$$

as illustrated in Figure 4.8. Clearly, the circular Phoenix is mapped to an elliptical one.

We have so far restricted ourselves to shears along the $\mathbf{e}_1$-axis; we may also shear along the $\mathbf{e}_2$-axis. Then we would have

$$\mathbf{v}' = \begin{bmatrix} 1 & 0 \\ d_2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_1 d_2 + v_2 \end{bmatrix},$$

as illustrated in Figure 4.9.

Since it will be needed later, we look at the following. What is the shear that achieves

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \longrightarrow \mathbf{v}' = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}?$$

Figure 4.8.
Shears: shearing parallel to the $\mathbf{e}_1$-axis.

Figure 4.9.
Shears: shearing parallel to the $\mathbf{e}_2$-axis.

It is obviously a shear parallel to the $\mathbf{e}_2$-axis and is given by the map

$$\mathbf{v}' = \begin{bmatrix} v_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -v_2/v_1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{4.18}$$

Shears do not change areas. In Sketch 4.6, we see that the rectangle and its image, a parallelogram, have the same area: both have the same base and the same height.

4.8 Projections

Projections—parallel projections, for our purposes—act like sunlight casting shadows. Parallel projections are characterized by the fact that all vectors are projected in a parallel direction. In 2D, all vectors are projected onto a line. If the angle of incidence with the line is ninety degrees then it is an *orthogonal projection*, otherwise it is an *oblique projection*. In linear algebra, orthogonal projections are very important, as we have already seen in Section 2.8, they give us a best approximation in a particular subspace. Oblique projections are important to applications in fields such as computer graphics and architecture. On the other hand, in a *perspective projection*, the projection direction is not constant. These are not linear maps, however; they are introduced in Section 10.5.

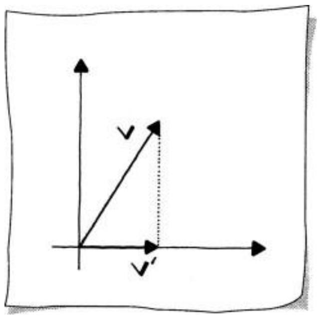
Let's look at a simple 2D orthogonal projection. Take any vector $\mathbf{v}$ and "flatten it out" onto the $\mathbf{e}_1$-axis. This simply means: set the v_2-coordinate of the vector to zero. For example, if we project the vector

$$\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

onto the $\mathbf{e}_1$-axis, it becomes

$$\mathbf{v}' = \begin{bmatrix} 3 \\ 0 \end{bmatrix},$$

as shown in Sketch 4.7.



Sketch 4.7.
An orthogonal, parallel projection.

What matrix achieves this map? That's easy:

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

This matrix will not only project the vector

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

onto the $\mathbf{e}_1$-axis, but in fact *every vector*! This is so since

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}.$$

While this is a somewhat trivial example of a projection, we see that this projection does indeed feature a main property of a projection: it *reduces dimensionality*. In this example, every vector from 2D

Figure 4.10.
Projections: all vectors are "flattened out" onto the $\mathbf{e}_1$-axis.

space is mapped into 1D space, namely onto the $\mathbf{e}_1$-axis. Figure 4.10 illustrates this property and that the action ellipse of a projection is a straight line segment that is covered twice.

To construct a 2D orthogonal projection matrix, first choose a unit vector $\mathbf{u}$ to define a line onto which to project. The matrix is defined by $\mathbf{a}_1$ and $\mathbf{a}_2$, or in other words, the projections of $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively onto $\mathbf{u}$. From (2.21), we have

$$\mathbf{a}_1 = \frac{\mathbf{u} \cdot \mathbf{e}_1}{\|\mathbf{u}\|^2}\mathbf{u} = u_1\mathbf{u},$$

$$\mathbf{a}_2 = \frac{\mathbf{u} \cdot \mathbf{e}_2}{\|\mathbf{u}\|^2}\mathbf{u} = u_2\mathbf{u},$$

thus

$$A = \begin{bmatrix} u_1\mathbf{u} & u_2\mathbf{u} \end{bmatrix} \tag{4.19}$$

$$= \mathbf{u}\mathbf{u}^T. \tag{4.20}$$

Forming a matrix as in (4.20), from the product of a vector and its transpose, results in a *dyadic matrix*. Clearly the columns of A are linearly dependent and thus the matrix has rank one. This map reduces dimensionality, and as far as areas are concerned, projections take a lean approach: whatever an area was before application of the map, it is zero afterward.

Figure 4.11 shows the effect of (4.20) on the $\mathbf{e}_1$ and $\mathbf{e}_2$ axes. On the left side, the vector $\mathbf{u} = [\cos 30^\circ \;\; \sin 30^\circ]^{\mathrm{T}}$ and thin lines show the projection of $\mathbf{e}_1$ (black) and $\mathbf{e}_2$ (dark gray) onto $\mathbf{u}$. On the right side, many $\mathbf{u}$ vectors are illustrated: every 10°, forming 36 arrows or $\mathbf{u}_i$ for $i = 1, 36$. The black circle of arrows is formed by the projection of $\mathbf{e}_1$ onto each $\mathbf{u}_i$. The gray circle of arrows is formed by the projection of $\mathbf{e}_2$ onto each $\mathbf{u}_i$.

In addition to reducing dimensionality, a projection matrix is also *idempotent*: $A = AA$. Geometrically, this means that once a vector has been projected onto a line, application of the same projection will leave the result unchanged.

Example 4.4

Let the projection line be defined by

$$\mathbf{u} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$

Then (4.20) defines the projection matrix to be

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.$$

This $\mathbf{u}$ vector, corresponding to a 45° rotation of $\mathbf{e}_1$, is absent from the right part of Figure 4.11, so find where it belongs. In this case, the projection of $\mathbf{e}_1$ and $\mathbf{e}_2$ are identical.

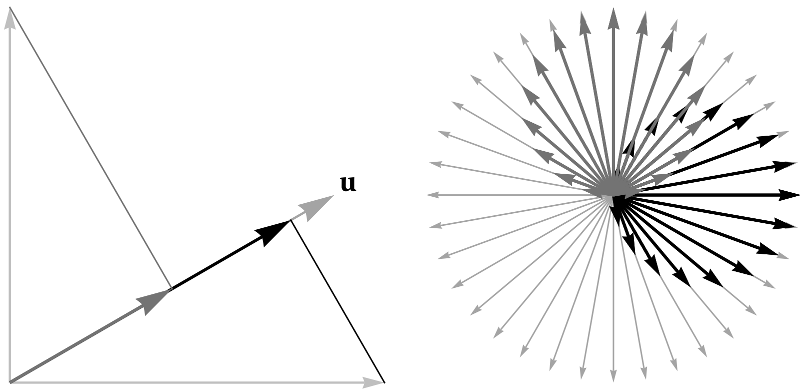


Figure 4.11.

Projections: $\mathbf{e}_1$ and $\mathbf{e}_2$ vectors orthogonally projected onto $\mathbf{u}$ results in $\mathbf{a}_1$ (black) and $\mathbf{a}_2$ (dark gray), respectively. Left: vector $\mathbf{u} = [\cos 30° \ \sin 30°]^{\mathrm{T}}$. Right: vectors $\mathbf{u}_i$ for $i = 1, 36$ are at 10° increments.

Try sketching the projection of a few vectors yourself to get a feel for how this projection works. In particular, try

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Sketch $\mathbf{v}' = A\mathbf{v}$. Now compute $\mathbf{v}'' = A\mathbf{v}'$. Surprised?

Let's revisit the orthogonal projection discussion from Section 2.8 by examining the action of A in (4.20) on a vector $\mathbf{x}$,

$$A\mathbf{x} = \mathbf{u}\mathbf{u}^{\mathrm{T}}\mathbf{x} = (\mathbf{u} \cdot \mathbf{x})\mathbf{u}.$$

We see this is the same result as (2.21). Suppose the projection of $\mathbf{x}$ onto $\mathbf{u}$ is $\mathbf{y}$, then $\mathbf{x} = \mathbf{y} + \mathbf{y}^{\perp}$, and we then have

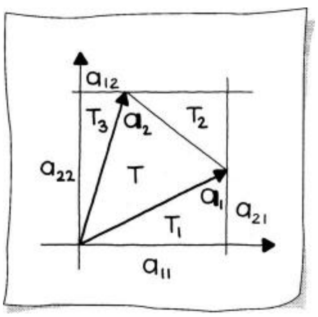
$$A\mathbf{x} = \mathbf{u}\mathbf{u}^{\mathrm{T}}\mathbf{y} + \mathbf{u}\mathbf{u}^{\mathrm{T}}\mathbf{y}^{\perp},$$

and since $\mathbf{u}^{\mathrm{T}}\mathbf{y} = \|\mathbf{y}\|$ and $\mathbf{u}^{\mathrm{T}}\mathbf{y}^{\perp} = 0$,

$$A\mathbf{x} = \|\mathbf{y}\|\mathbf{u}.$$

Projections will be revisited many times, and some examples include: homogeneous linear systems in Section 5.8, 3D projections in Sections 9.7 and 10.4, creating orthonormal coordinate frames in Sections 11.8, 14.4, and 20.7, and least squares approximation in Section 12.7.

4.9 Areas and Linear Maps: Determinants



Sketch 4.8.
Area formed by $\mathbf{a}_1$ and $\mathbf{a}_2$.

As you might have noticed, we discussed one particular aspect of linear maps for each type: how areas are changed. We will now discuss this aspect for an arbitrary 2D linear map. Such a map takes the two vectors $[\mathbf{e}_1, \mathbf{e}_2]$ to the two vectors $[\mathbf{a}_1, \mathbf{a}_2]$. The area of the square spanned by $[\mathbf{e}_1, \mathbf{e}_2]$ is 1, that is area$(\mathbf{e}_1, \mathbf{e}_2) = 1$. If we knew the area of the parallelogram spanned by $[\mathbf{a}_1, \mathbf{a}_2]$, then we could say how the linear map affects areas.

How do we find the area P of a parallelogram spanned by two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$? Referring to Sketch 4.8, let us first determine the area T of the triangle formed by $\mathbf{a}_1$ and $\mathbf{a}_2$. We see that

$$T = a_{1,1}a_{2,2} - T_1 - T_2 - T_3.$$

We then observe that

$$T_1 = \frac{1}{2}a_{1,1}a_{2,1}, \tag{4.21}$$

$$T_2 = \frac{1}{2}(a_{1,1} - a_{1,2})(a_{2,2} - a_{2,1}), \tag{4.22}$$

$$T_3 = \frac{1}{2}a_{1,2}a_{2,2}. \tag{4.23}$$

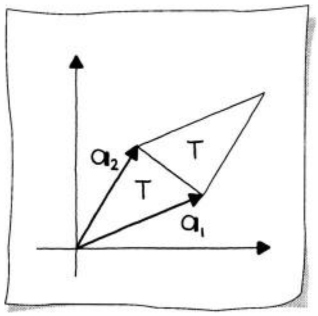
Working out the algebra, we arrive at

$$T = \frac{1}{2}a_{1,1}a_{2,2} - \frac{1}{2}a_{1,2}a_{2,1}.$$

Our aim was not really T, but the parallelogram area P. Clearly (see Sketch 4.9),

$$P = 2T,$$

and we have our desired area.



Sketch 4.9.
Parallelogram and triangles.

It is customary to use the term *determinant* for the (signed) area of the parallelogram spanned by $[\mathbf{a}_1, \mathbf{a}_2]$. Since the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ form the columns of the matrix A, we also speak of the determinant of the matrix A, and denote it by $\det A$ or $|A|$:

$$|A| = \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}. \tag{4.24}$$

Since A maps a square with area one onto a parallelogram with area $|A|$, the determinant of a matrix characterizes it as follows:

- If $|A| = 1$, then the linear map does not change areas.
- If $0 \le |A| < 1$, then the linear map shrinks areas.
- If $|A| = 0$, then the matrix is rank deficient.
- If $|A| > 1$, then the linear map expands areas.
- If $|A| < 0$, then the linear map changes the orientation of objects. (We'll look at this closer after Example 4.5.) Areas may still contract or expand depending on the magnitude of the determinant.

Example 4.5

We will look at a few examples. Let

$$A = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix},$$

then $|A| = (1)(1) - (5)(0) = 1$. Since A represents a *shear*, we see again that those maps do not change areas.

For another example, let

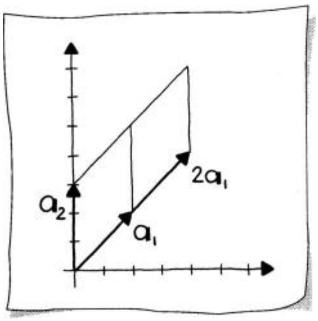
$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

then $|A| = (1)(-1) - (0)(0) = -1$. This matrix corresponds to a *reflection*, and it leaves areas unchanged, except for a *sign change*.

Finally, let

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix},$$

then $|A| = (0.5)(0.5) - (0.5)(0.5) = 0$. This matrix corresponds to the *projection* from Section 4.8. In that example, we saw that projections collapse any object onto a straight line, i.e., to an object with zero area.



Sketch 4.10.
Resulting area after scaling one column of *A*.

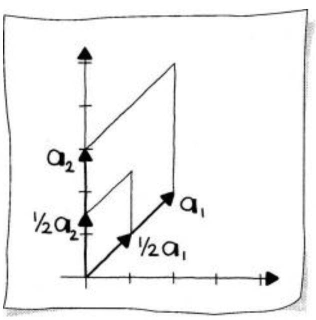
There are some rules for working with determinants:

If $A = [\mathbf{a}_1, \mathbf{a}_2]$, then

$$|c\mathbf{a}_1, \mathbf{a}_2| = c|\mathbf{a}_1, \mathbf{a}_2| = c|A|.$$

In other words, if one of the columns of A is scaled by a factor c, then A's determinant is also scaled by c. Verify that this is true from the definition of the determinant of A. Sketch 4.10 illustrates this for the example, $c = 2$. If *both* columns of A are scaled by c, then the determinant is scaled by c^2:

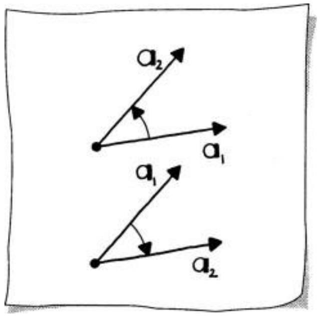
$$|c\mathbf{a}_1, c\mathbf{a}_2| = c^2|\mathbf{a}_1, \mathbf{a}_2| = c^2|A|.$$



Sketch 4.11.
Resulting area after scaling both columns of *A*.

Sketch 4.11 illustrates this for the example $c = 1/2$.

If $|A|$ is positive and c is negative, then replacing $\mathbf{a}_1$ by $c\mathbf{a}_1$ will cause $c|A|$, the area formed by $c\mathbf{a}_1$ and $\mathbf{a}_2$, to become negative. The notion of a negative area is very useful computationally. Two 2D vectors whose determinant is positive are called *right-handed*. The standard example is the pair of vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Two 2D vectors whose determinant is negative are called *left-handed*.[2] Sketch 4.12 shows a pair of right-handed vectors (top) and a pair of left-handed ones (bottom). Our definition of positive and negative area is not totally arbitrary: the triangle formed by vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ has area $1/2 \times \sin(\alpha)\|\mathbf{a}_1\|\|\mathbf{a}_2\|$. Here, the angle α indicates how much we have to rotate $\mathbf{a}_1$ in order to line up with $\mathbf{a}_2$. If we interchange the two vectors, the sign of α and hence the sign of $\sin(\alpha)$ also changes!



Sketch 4.12.
Right-handed and left-handed vectors.

There is also an area sign change when we *interchange* the columns of A:

$$|\mathbf{a}_1, \mathbf{a}_2| = -|\mathbf{a}_2, \mathbf{a}_1|. \tag{4.25}$$

This fact is easily verified using the definition of a determinant:

$$|\mathbf{a}_2, \mathbf{a}_1| = a_{1,2}a_{2,1} - a_{2,2}a_{1,1}.$$

4.10 Composing Linear Maps

Suppose you have mapped a vector $\mathbf{v}$ to $\mathbf{v}'$ using a matrix A. Next, you want to map $\mathbf{v}'$ to $\mathbf{v}''$ using a matrix B. We start out with

$$\mathbf{v}' = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a_{1,1}v_1 + a_{1,2}v_2 \\ a_{2,1}v_1 + a_{2,2}v_2 \end{bmatrix}.$$

Next, we have

$$\begin{aligned} \mathbf{v}'' &= \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \begin{bmatrix} a_{1,1}v_1 + a_{1,2}v_2 \\ a_{2,1}v_1 + a_{2,2}v_2 \end{bmatrix} \\ &= \begin{bmatrix} b_{1,1}(a_{1,1}v_1 + a_{1,2}v_2) + b_{1,2}(a_{2,1}v_1 + a_{2,2}v_2) \\ b_{2,1}(a_{1,1}v_1 + a_{1,2}v_2) + b_{2,2}(a_{2,1}v_1 + a_{2,2}v_2) \end{bmatrix}. \end{aligned}$$

Collecting the terms in v_1 and v_2, we get

$$\mathbf{v}'' = \begin{bmatrix} b_{1,1}a_{1,1} + b_{1,2}a_{2,1} & b_{1,1}a_{1,2} + b_{1,2}a_{2,2} \\ b_{2,1}a_{1,1} + b_{2,2}a_{2,1} & b_{2,1}a_{1,2} + b_{2,2}a_{2,2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

The matrix that we have created here, let's call it C, is called the *product matrix* of B and A:

$$BA = C.$$

[2]The reason for this terminology will become apparent when we revisit these definitions for the 3D case (see Section 8.2).

In more detail,

$$\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} b_{1,1}a_{1,1} + b_{1,2}a_{2,1} & b_{1,1}a_{1,2} + b_{1,2}a_{2,2} \\ b_{2,1}a_{1,1} + b_{2,2}a_{2,1} & b_{2,1}a_{1,2} + b_{2,2}a_{2,2} \end{bmatrix}. \tag{4.26}$$

This looks messy, but a simple rule puts order into chaos: the element $c_{i,j}$ is computed as the dot product of B's ith row and A's jth column.

We can use this product to describe the *composite map*:

$$\mathbf{v}'' = B\mathbf{v}' = B[A\mathbf{v}] = BA\mathbf{v}.$$

Example 4.6

Let

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}.$$

Then

$$\mathbf{v}' = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

and

$$\mathbf{v}'' = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}.$$

We can also compute $\mathbf{v}''$ using the matrix product BA:

$$C = BA = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 3 & -3 \end{bmatrix}.$$

Verify for yourself that $\mathbf{v}'' = C\mathbf{v}$.

Analogous to the matrix/vector product from Section 4.2, there is a neat way to arrange two matrices when forming their product for manual computation (yes, that is still encountered!). Using the matrices of Example 4.6, and highlighting the computation of $c_{2,1}$, we write

$$\begin{array}{rr|rr} & & \mathbf{-1} & 2 \\ & & \mathbf{0} & 3 \\ \hline 0 & -2 & & \\ \mathbf{-3} & \mathbf{1} & \mathbf{3} & \end{array}$$

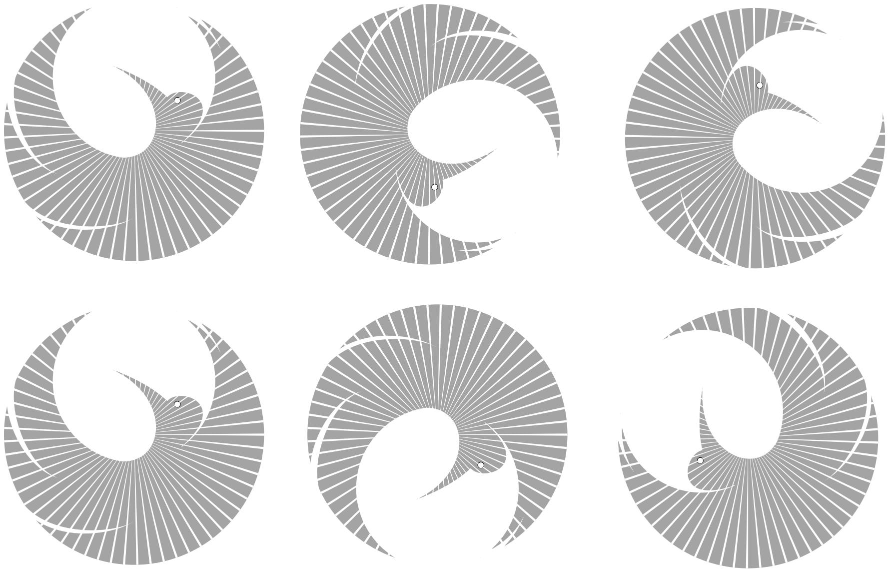


Figure 4.12.
Linear map composition is order dependent. Top: rotate by –120°, then reflect about the (rotated) $\mathbf{e}_1$-axis. Bottom: reflect, then rotate.

You see how $c_{2,1}$ is at the intersection of column one of the "top" matrix and row two of the "left" matrix.

The complete multiplication scheme is then arranged like this

$$\begin{array}{rr|rr} & & -1 & 2 \\ & & 0 & 3 \\ \hline 0 & -2 & 0 & -6 \\ -3 & 1 & 3 & -3 \end{array}$$

While we use the term "product" for BA, it is very important to realize that this kind of product differs significantly from products of real numbers: it is not *commutative.* That is, in general

$$AB \neq BA.$$

Matrix products correspond to linear map compositions—since the products are not commutative, it follows that it matters in which order we carry out linear maps. *Linear map composition is order dependent.* Figure 4.12 gives an example.

Example 4.7

Let us take two very simple matrices and demonstrate that the product is not commutative. This example is illustrated in Figure 4.12. A rotates by $-120°$, and B reflects about the $\mathbf{e}_1$-axis:

$$A = \begin{bmatrix} -0.5 & 0.866 \\ -0.866 & -0.5 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We first form AB (reflect and then rotate),

$$AB = \begin{bmatrix} -0.5 & 0.866 \\ -0.866 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -0.5 & -0.866 \\ -0.866 & 0.5 \end{bmatrix}.$$

Next, we form BA (rotate and then reflect),

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -0.5 & 0.866 \\ -0.866 & -0.5 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.866 \\ 0.866 & 0.5 \end{bmatrix}.$$

Clearly, these are not the same!

Of course, *some* maps *do* commute; for example, the 2D rotations. It does not matter if we rotate by α first and then by β or the other way around. In either case, we have rotated by $\alpha + \beta$. In terms of matrices,

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} =$$

$$\begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}.$$

Check for yourself that the other alternative gives the same result! By referring to a trigonometry reference, we see that this product matrix can be written as

$$\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix},$$

which corresponds to a rotation by $\alpha+\beta$. As we will see in Section 9.9, rotations in 3D do not commute.

What about the rank of a composite map?

$$\operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}. \tag{4.27}$$

This says that matrix multiplication does not increase rank. You should try a few rank 1 and 2 matrices to convince yourself of this fact.

One more example on composing linear maps, which we have seen in Section 4.8, is that projections are idempotent. If A is a projection matrix, then this means

$$A\mathbf{v} = AA\mathbf{v}$$

for any vector $\mathbf{v}$. Written out using only matrices, this becomes

$$A = AA \quad \text{or} \quad A = A^2. \tag{4.28}$$

Verify this property for the projection matrix

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.$$

Excluding the identity matrix, only rank deficient matrices are idempotent.

4.11 More on Matrix Multiplication

Matrix multiplication is not limited to the product of 2×2 matrices. In fact, when we multiply a matrix by a vector, we follow the rules of matrix multiplication! If $\mathbf{v}' = A\mathbf{v}$, then the first component of $\mathbf{v}'$ is the dot product of A's first row and $\mathbf{v}$; the second component of $\mathbf{v}'$ is the dot product of A's second row and $\mathbf{v}$.

In Section 4.2, we introduced the transpose A^{T} of a matrix A and a vector $\mathbf{v}$. Usually, we write $\mathbf{u} \cdot \mathbf{v}$ for the dot product of $\mathbf{u}$ and $\mathbf{v}$, but sometimes considering the vectors as matrices is useful as well; that is, we can form the product as

$$\mathbf{u}^{\mathrm{T}}\mathbf{v} = \mathbf{u} \cdot \mathbf{v}. \tag{4.29}$$

For examples in this section, let

$$\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}.$$

Then it is straightforward to show that the left- and right-hand sides of (4.29) are equal to 15.

If we have a product $\mathbf{u}^{\mathrm{T}}\mathbf{v}$, what is $[\mathbf{u}^{\mathrm{T}}\mathbf{v}]^{\mathrm{T}}$? This has an easy answer:

$$(\mathbf{u}^{\mathrm{T}}\mathbf{v})^{\mathrm{T}} = \mathbf{v}^{\mathrm{T}}\mathbf{u},$$

as an example will clarify.

Example 4.8

$$[\mathbf{u}^{\mathrm{T}}\mathbf{v}]^{\mathrm{T}} = \left(\begin{bmatrix}3 & 4\end{bmatrix}\begin{bmatrix}-3\\ 6\end{bmatrix}\right)^{\mathrm{T}} = [15]^{\mathrm{T}} = 15$$

$$\mathbf{v}^{\mathrm{T}}\mathbf{u} = \begin{bmatrix}-3 & 6\end{bmatrix}\begin{bmatrix}3\\ 4\end{bmatrix} = [15] = 15.$$

The results are the same.

We saw that addition of matrices is straightforward under transposition; matrix multiplication is not that straightforward. We have

$$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}. \tag{4.30}$$

To see why this is true, recall that each element of a product matrix is obtained as a dot product. In the matrix products below, we show the calculation of one element of the left- and right-hand sides of (4.30) to demonstrate that the dot products for this one element are identical. Let transpose matrix elements be referred to as $b^{\mathrm{T}}_{i,j}$.

$$(AB)^{\mathrm{T}} = \left(\begin{bmatrix}a_{1,1} & a_{1,2}\\ \mathbf{a_{2,1}} & \mathbf{a_{2,2}}\end{bmatrix}\begin{bmatrix}\mathbf{b_{1,1}} & b_{1,2}\\ \mathbf{b_{2,1}} & b_{2,2}\end{bmatrix}\right)^{\mathrm{T}} = \begin{bmatrix}c_{1,1} & \mathbf{c_{1,2}}\\ c_{2,1} & c_{2,2}\end{bmatrix},$$

$$B^{\mathrm{T}}A^{\mathrm{T}} = \begin{bmatrix}\mathbf{b^{T}_{1,1}} & \mathbf{b^{T}_{1,2}}\\ b^{\mathrm{T}}_{2,1} & b^{\mathrm{T}}_{2,2}\end{bmatrix}\begin{bmatrix}a^{\mathrm{T}}_{1,1} & \mathbf{a^{T}_{1,2}}\\ a^{\mathrm{T}}_{2,1} & \mathbf{a^{T}_{2,2}}\end{bmatrix} = \begin{bmatrix}c_{1,1} & \mathbf{c_{1,2}}\\ c_{2,1} & c_{2,2}\end{bmatrix}.$$

Since $b_{i,j} = b^{\mathrm{T}}_{j,i}$, we see the identical dot product is calculated to form $c_{1,2}$.

What is the *determinant* of a product matrix? If $C = AB$ denotes a matrix product, then

$$|AB| = |A||B|, \tag{4.31}$$

which tells us that B scales objects by $|B|$, A scales objects by $|A|$, and the composition of the maps scales by the product of the individual scales.

Example 4.9

As a simple example, take two scalings

$$A = \begin{bmatrix}1/2 & 0\\ 0 & 1/2\end{bmatrix}, \quad B = \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}.$$

We have $|A| = 1/4$ and $|B| = 16$. Thus, A scales down, and B scales up, but the effect of B's scaling is greater than that of A's. The product

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

thus scales up: $|AB| = |A||B| = 4$.

Just as for real numbers, we can define *exponents* for matrices:

$$A^r = \underbrace{A \cdot \ldots \cdot A}_{r \text{ times}}.$$

Here are some rules.

$$\boxed{\begin{gathered} A^{r+s} = A^r A^s \\ A^{rs} = (A^r)^s \\ A^0 = I \end{gathered}}$$

For now, assume r and s are positive integers. See Sections 5.9 and 9.10 for a discussion of A^{-1}, the *inverse matrix*.

4.12 Matrix Arithmetic Rules

We encountered some of these rules throughout this chapter in terms of matrix and vector multiplication: a vector is simply a special matrix. The focus of this chapter is on 2×2 and 2×1 matrices, but these rules apply to matrices of any size. In the rules that follow, let a, b be scalars.

Importantly, however, the matrix sizes must be compatible for the operations to be performed. Specifically, matrix addition requires the matrices to have the same dimensions and matrix multiplication requires the "inside" dimensions to be equal: suppose A's dimensions are $m \times r$ and B's are $r \times n$, then the product $C = AB$ is permissible since the dimension r is shared, and the resulting matrix C will have the "outer" dimensions, reading left to right: $m \times n$.

Commutative Law for Addition
$$A + B = B + A$$

Associative Law for Addition
$$A + (B + C) = (A + B) + C$$

No Commutative Law for Multiplication
$$AB \neq BA$$

Associative Law for Multiplication
$$A(BC) = (AB)C$$

Distributive Law
$$A(B + C) = AB + AC$$

Distributive Law
$$(B + C)A = BA + CA$$

Rules involving scalars:

$$\boxed{\begin{aligned} a(B + C) &= aB + aC \\ (a + b)C &= aC + bC \\ (ab)C &= a(bC) \\ a(BC) = (aB)C &= B(aC) \end{aligned}}$$

Rules involving the transpose:

$$\boxed{\begin{aligned} (A + B)^{\mathrm{T}} &= A^{\mathrm{T}} + B^{\mathrm{T}} \\ (bA)^{\mathrm{T}} &= bA^{\mathrm{T}} \\ (AB)^{\mathrm{T}} &= B^{\mathrm{T}}A^{\mathrm{T}} \\ A^{\mathrm{T}^{\mathrm{T}}} &= A \end{aligned}}$$

Chapter 9 will introduce 3×3 matrices and Chapter 12 will introduce $n \times n$ matrices.

- linear combination
- matrix form
- preimage and image
- domain and range
- column space
- identity matrix
- matrix addition
- distributive law
- transpose matrix
- symmetric matrix
- rank of a matrix
- rank deficient
- singular matrix
- linear space or vector space
- subspace
- linearity property
- scalings
- action ellipse
- reflections
- rotations
- rigid body motions
- shears
- projections
- parallel projection
- oblique projection
- dyadic matrix
- idempotent map
- determinant
- signed area
- matrix multiplication
- composite map
- noncommutative property of matrix multiplication
- transpose of a product or sum of matrices
- rules of matrix arithmetic

4.13 Exercises

For the following exercises, let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1/2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

1. What linear combination of

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

results in

$$\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}?$$

Write the result in matrix form.

2. Suppose $\mathbf{w}' = w_1\mathbf{c}_1 + w_2\mathbf{c}_2$. Express this in matrix form.
3. Is the vector $\mathbf{v}$ in the column space of A?
4. Construct a matrix C such that the vector

$$\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is *not* in its column space.

5. Are either A or B symmetric matrices?
6. What is the transpose of A, B, and $\mathbf{v}$?
7. What is the transpose of the 2×2 identity matrix I?
8. Compute $A^{\mathrm{T}} + B^{\mathrm{T}}$ and $[A + B]^{\mathrm{T}}$.
9. What is the rank of B?
10. What is the rank of the matrix $C = [\mathbf{c}\ 3\mathbf{c}]$?
11. What is the rank of the zero matrix?
12. For the matrix A, vectors $\mathbf{v}$ and $\mathbf{u} = [1\ 0]^{\mathrm{T}}$, and scalars $a = 4$ (applied to $\mathbf{u}$) and $b = 2$ (applied to $\mathbf{v}$), demonstrate the linearity property of linear maps.
13. Describe geometrically the effect of A and B. (You may do this analytically or by using software to illustrate the action of the matrices.)
14. Compute $A\mathbf{v}$ and $B\mathbf{v}$.
15. Construct the matrix S that maps the vector $\mathbf{w}$ to $3\mathbf{w}$.
16. What scaling matrix will result in an action ellipse with major axis twice the length of the minor axis?
17. Construct the matrix that reflects about the $\mathbf{e}_2$-axis.
18. What is the shear matrix that maps $\mathbf{v}$ onto the $\mathbf{e}_2$-axis?
19. What type of linear map is the matrix
$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}?$$
20. Are either A or B a rigid body motion?
21. Is the matrix A idempotent?
22. Construct an orthogonal projection onto
$$\mathbf{u} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}.$$
23. For an arbitrary unit vector $\mathbf{u}$, show that $P = \mathbf{u}\mathbf{u}^{\mathrm{T}}$ is idempotent.
24. Suppose we apply each of the following linear maps to the vertices of a unit square: scaling, reflection, rotation, shear, projection. For each map, state if there is a change in area and the reason.
25. What is the determinant of A?
26. What is the determinant of B?
27. What is the determinant of $4B$?
28. Compute $A + B$. Show that $A\mathbf{v} + B\mathbf{v} = (A + B)\mathbf{v}$.
29. Compute $AB\mathbf{v}$ and $BA\mathbf{v}$.

30. Compute $B^{\mathrm{T}}A$.

31. What is A^2?

32. Let M and N be 2×2 matrices and each is rank one. What can you say about the rank of $M + N$?

33. Let two square matrices M and N each have rank one. What can you say about the rank of MN?

34. Find matrices C and D, both having rank greater than zero, such that the product CD has rank zero.

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