Summer Internship Project

Study of beam dynamics using a lattice-based model

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1 Introduction

Metamaterials are artificially structured materials engineered to have properties not typically found in natural materials. They derive their unique characteristics from their structure rather than their composition.

The erratic properties of metamaterials stem from their dependence on their intricate and often complex internal structures. These properties make their behavior difficult to predict and model accurately using traditional methods. Metamaterials bring up the following complications in modelling:

- **Complexity**: The small-scale structure of metamaterials, often at the microscale or nanoscale, requires detailed modeling that can be computationally expensive.
- **Non-linearity**: Metamaterials can exhibit nonlinear behaviors, further complicating their analysis.
- **Multiscale Modelling**: Accurate models must capture phenomena at various scales, from the microscopic structure to the macroscopic properties.

A lattice based model

The proposed model involves a truss structure of rods arranged in lattice format along the direction of beam.

This kind of abstraction allows modification of individual truss element properties to mimic metamaterials' intrinsic design at the required scale. Moreover, the cost of computation is cheaper because of the assumption of an Euler-Bernoulli beam like structure for each element.

For this project, a study of a homogeneous beam is done using our lattice based abstraction model for easy verification of results with existing analysis.

The repository for the scripts for all calculations can be found at

A homogeneous beam is a structural element with uniform material properties throughout its volume. Despite its apparent simplicity, accurately modeling a homogeneous beam can be challenging due to the differential equations involved in their elasticity and stress analysis, their boundary conditions and cost of computation in high-fidelity models.

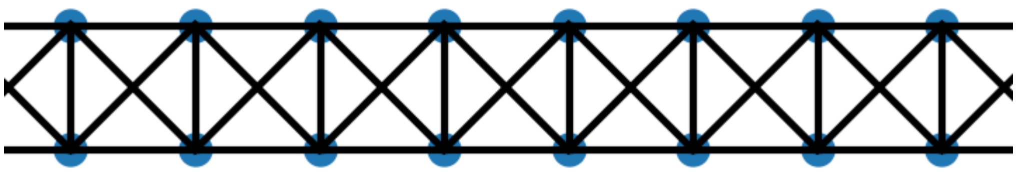

Figure 1: The Lattice Cells

2 Properties of the Beam

For the purpose of standardisation, we will be looking at results for a homogeneous Aluminium beam with the following properties:

- Length: 10 m
- Area of cross section: 0.25 m^2
- Material Density: 2710 kg/m^3
- Young's Modulus: 70 GPa

All dynamic properties of the beam in the longitudinal direction have been calculated using the following equation for the true solution taken from Geradin's Mechanical Vibrations.

$$u(x,t) = \frac{8p_o l}{\pi^2 EA} \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{(2s-1)^2} sin[(2s-1)\frac{\pi x}{2l}][1 - cos[(2s-1)\frac{\pi c}{2l}t]]$$

3 The Implicit Newmark Method

The Newmark method is a numerical integration method used to solve the equations of motion of dynamic systems. It is based on the following equations:

Equations of Motion

Consider the second-order differential equation of motion for a multi-degree of freedom system:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t) \tag{1}$$

where:

- $\mathbf{M}$ is the mass matrix,
- $\mathbf{C}$ is the damping matrix,
- $\mathbf{K}$ is the stiffness matrix,
- $\mathbf{u}(t)$ is the displacement vector of truss nodes,
- $\dot{\mathbf{u}}(t)$ is the velocity vector,
- $\ddot{\mathbf{u}}(t)$ is the acceleration vector,
- $\mathbf{F}(t)$ is the external force vector.

Newmark Integration Formulas

The Newmark method approximates the displacement and velocity at time t_{n+1} as:

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t \dot{\mathbf{u}}_n + \Delta t^2 \left(\frac{1}{2} - \beta\right) \ddot{\mathbf{u}}_n + \beta \Delta t^2 \ddot{\mathbf{u}}_{n+1} \tag{2}$$

$$\dot{\mathbf{u}}_{n+1} = \dot{\mathbf{u}}_n + \Delta t (1-\gamma) \ddot{\mathbf{u}}_n + \gamma \Delta t \ddot{\mathbf{u}}_{n+1} \tag{3}$$

where:

- Δt is the time step size,
- β and γ are Newmark parameters that control the method's accuracy and stability.
- Hereon, we will be using the suggested values of β and γ as $\beta = 0.25$ and $\gamma = 0.5$.

Implicit Formulation

Rearranging the above equations to solve for $\ddot{\mathbf{u}}_{n+1}$:

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \Delta t \dot{\mathbf{u}}_n + \Delta t^2 \left(\frac{1}{2} - \beta\right) \ddot{\mathbf{u}}_n + \beta \Delta t^2 \ddot{\mathbf{u}}_{n+1} \tag{4}$$

$$\Rightarrow \ddot{\mathbf{u}}_{n+1} = \frac{1}{\beta \Delta t^2} \left(\mathbf{u}_{n+1} - \mathbf{u}_n - \Delta t \dot{\mathbf{u}}_n - \Delta t^2 \left(\frac{1}{2} - \beta\right) \ddot{\mathbf{u}}_n\right) \tag{5}$$

Similarly, for the velocity:

$$\dot{\mathbf{u}}_{n+1} = \dot{\mathbf{u}}_n + \Delta t (1-\gamma) \ddot{\mathbf{u}}_n + \gamma \Delta t \ddot{\mathbf{u}}_{n+1} \tag{6}$$

$$\Rightarrow \ddot{\mathbf{u}}_{n+1} = \frac{1}{\gamma \Delta t} \left(\dot{\mathbf{u}}_{n+1} - \dot{\mathbf{u}}_n - \Delta t (1-\gamma) \ddot{\mathbf{u}}_n\right) \tag{7}$$

Solving the System of Equations

Substitute $\ddot{\mathbf{u}}_{n+1}$ into the equation of motion at time t_{n+1}:

$$\mathbf{M}\ddot{\mathbf{u}}_{n+1} + \mathbf{C}\dot{\mathbf{u}}_{n+1} + \mathbf{K}\mathbf{u}_{n+1} = \mathbf{F}_{n+1} \tag{8}$$

Using the expressions for $\ddot{\mathbf{u}}_{n+1}$ and $\dot{\mathbf{u}}_{n+1}$, we get:

$$\begin{aligned}\left(\frac{1}{\beta \Delta t^2}\mathbf{M} + \frac{\gamma}{\beta \Delta t}\mathbf{C} + \mathbf{K}\right)\mathbf{u}_{n+1} = \mathbf{F}_{n+1} &+ \mathbf{M}\left(\frac{1}{\beta \Delta t^2}\mathbf{u}_n + \frac{1}{\beta \Delta t}\dot{\mathbf{u}}_n + \frac{1-2\beta}{2\beta}\ddot{\mathbf{u}}_n\right) \\ &+ \mathbf{C}\left(\frac{\gamma}{\beta \Delta t}\mathbf{u}_n + \frac{\gamma - 1}{\beta}\dot{\mathbf{u}}_n + \Delta t \left(\frac{\gamma - 2\beta}{2\beta}\right)\ddot{\mathbf{u}}_n\right)\end{aligned} \tag{9}$$

Solve the above system of equations for $\mathbf{u}_{n+1}$, and then compute $\ddot{\mathbf{u}}_{n+1}$ and $\dot{\mathbf{u}}_{n+1}$ using the Newmark integration formulas.

4 Natural Frequency and addition of layers

In this section we look at the effect of additional layers in the model to our results. Here, we compare the natural frequency of our beam to the abstraction's natural frequency. To ensure that our beam has uniform elastic properties, irrespective of the number of layers, the following conversion was used for every individual element's elastic properties: (Where property is denoted by E)

$$E_e = \frac{E_{beam}}{2n+1}$$

Where n = the number of layers.

The following is a table of the first five natural frequencies of our beam, calculated for the homogeneous beam, a 2-layered abstraction and a 4-layered abstraction: The beam, under constant

	Homogeneous beam	**2 layered lattice**	**4 layered lattice**
First Natural Frequency (f1)	133 Hz	130 Hz	130 Hz
Second Natural Frequency (f2)	357 Hz	400 Hz	385 Hz
Third Natural Frequency (f3)	608 Hz	690 Hz	640 Hz
Fourth Natural Frequency (f4)	687 Hz	910 Hz	890 Hz

Table 1: Natural Frequencies

force of 1 GN, shows the following end deflection in the time domain:

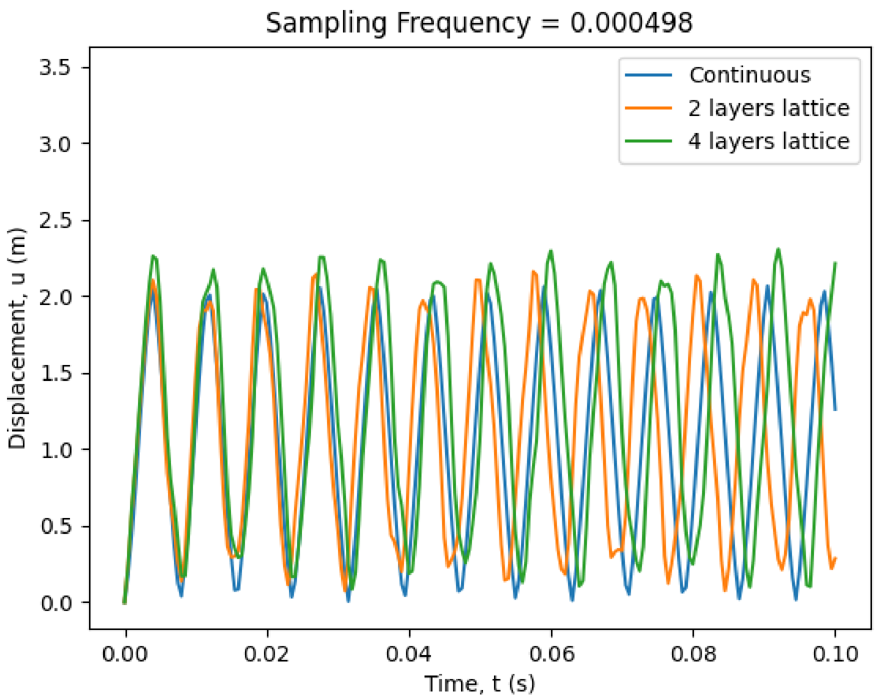

Figure 2: Comparing different layered distributions in the time domain

The same data represented in the frequency domain (using a fast fourier transform):

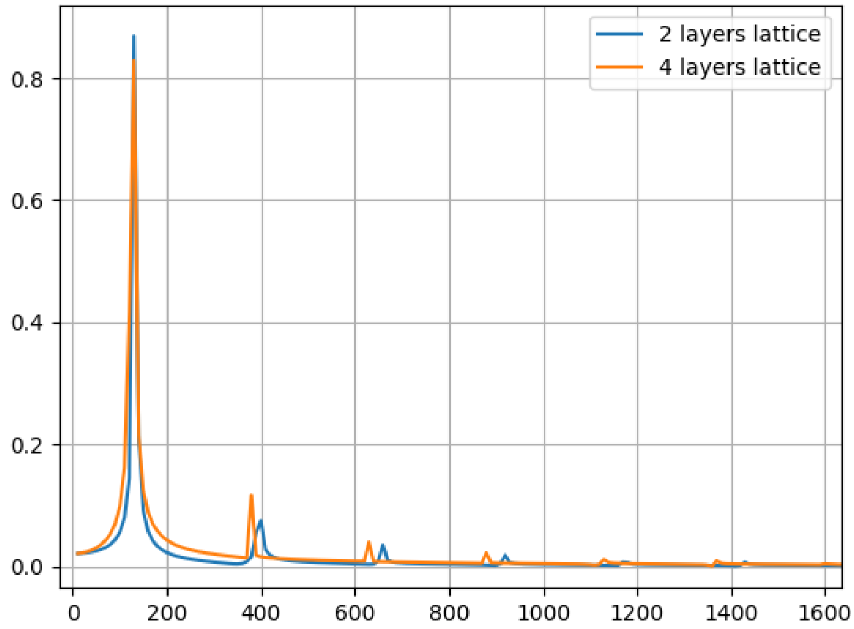

Figure 3: Comparing different layered distributions in the time domain

5 Response to Harmonic Forces

To validate resonance at the first natural frequency, we observe the output vibrations of the beam to different harmonic forces.

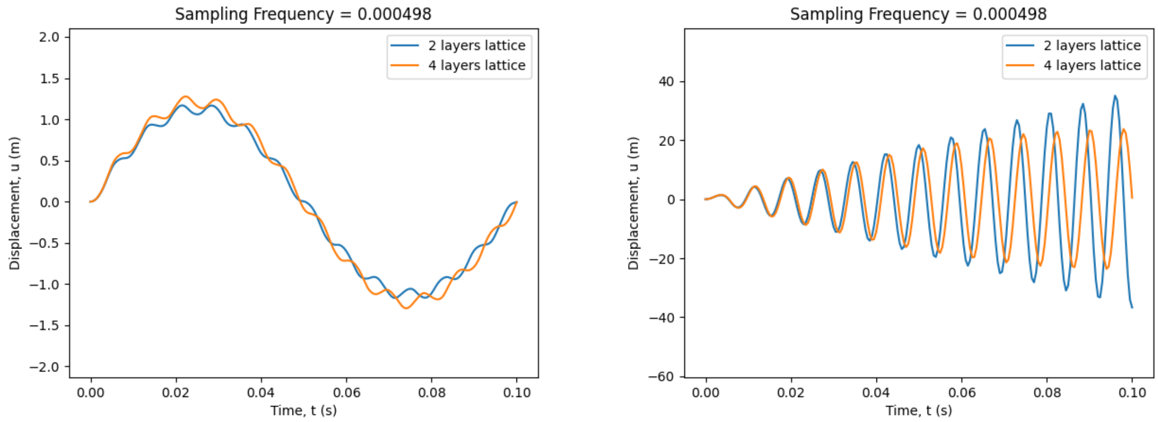

Figure 4: Response to harmonic forces of 10 Hz frequency [left] and force of 130 Hz frequency [right]

We also look at the frequency response of the same:

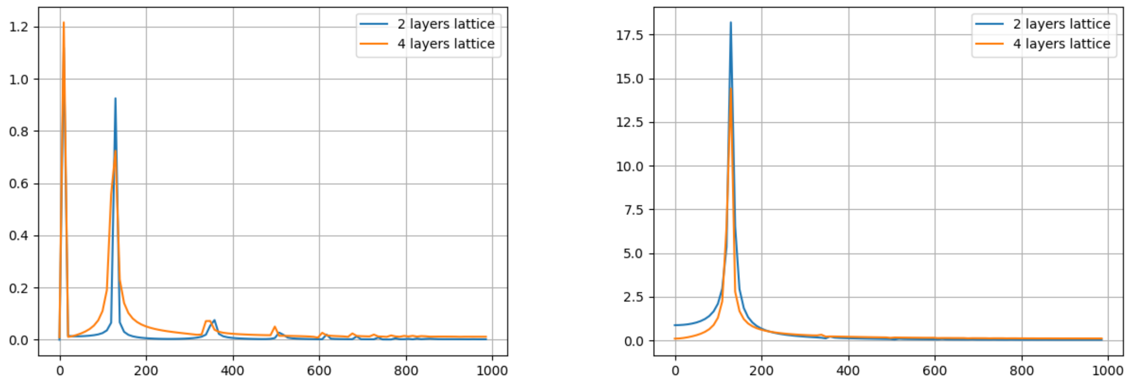

Figure 5: Frequency response to harmonic forces of 10 Hz frequency [left] and force of 130 Hz frequency [right]

This further validates our model for more complex calculations.

6 Energy transmission across the beam

In context of metamaterials, an important characteristic is the transmission of energy across the beam.

Being able to control this energy transmission is a key aspect of the material's functionality- it can go on to control propagation of electromagnetic waves, customising impedances and for finding possibly new phenomena.

We now observe this energy transmission by controlling the motion of one end of the beam and observing how this affects the motion at the other end.

The transfer function for this transmission is calculated in the s-domain using the Laplace and inverse Laplace transforms.

Step-by-Step Methodology

We have the equations as:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t) \tag{10}$$

Taking the Laplace transform of this equation, we get (assuming initial conditions as nil):

$$(\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K})\mathbf{u}(s) = \mathbf{F}(s) \tag{11}$$

Let A be represented as:

$$\mathbf{A} = s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} \tag{12}$$

We know that the input will be the first n elements of u(s) (where n is the degrees of freedom per node), and that the output will be the last n elements of u(s). We can then represent it as:

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} R \\ 0 \\ 0 \end{Bmatrix} \tag{13}$$

where R is the reaction force at input, u1 is the input vector, u3 is the output vector. We need the direct transformation of u3 from u1.

On solving the equations, we get the following:

We define the effective input matrix $\mathbf{B}_{\text{in}}$ and output matrix $\mathbf{B}_{\text{out}}$ as:

$$\mathbf{B}_{\text{in}} = \mathbf{A}_{31} - \mathbf{A}_{32}\mathbf{A}_{22}^{-1}\mathbf{A}_{21} \tag{14}$$

$$\mathbf{B}_{\text{out}} = \mathbf{A}_{33} - \mathbf{A}_{32}\mathbf{A}_{22}^{-1}\mathbf{A}_{23} \tag{15}$$

The output motion $u_3(s)$ at the other end of the beam is given by:

$$u_3(s) = -\mathbf{B}_{\text{out}}^{-1}\mathbf{B}_{\text{in}}u_1(s) \tag{16}$$

Finally, the time domain signal $v(t)$ is obtained by taking the inverse Laplace transform of $u_3(s)$:

$$v(t) = \mathcal{L}^{-1}\{u_3(s)\} \tag{17}$$

Using the python library Sympy for calculations in the s-domain was turning out computationally expensive, so we use the Fourier transform to move to the frequency domain, where the conversion lies in

$$s = i\omega$$

Where i is the complex imaginary number $\sqrt{-1}$.

We assume no damping and use Scipy's Fast Fourier Transform and the inverse, we get a curve of the amplitude ratio of output u_o with respect to the input u_i as we sweep across different frequencies of the input wave.

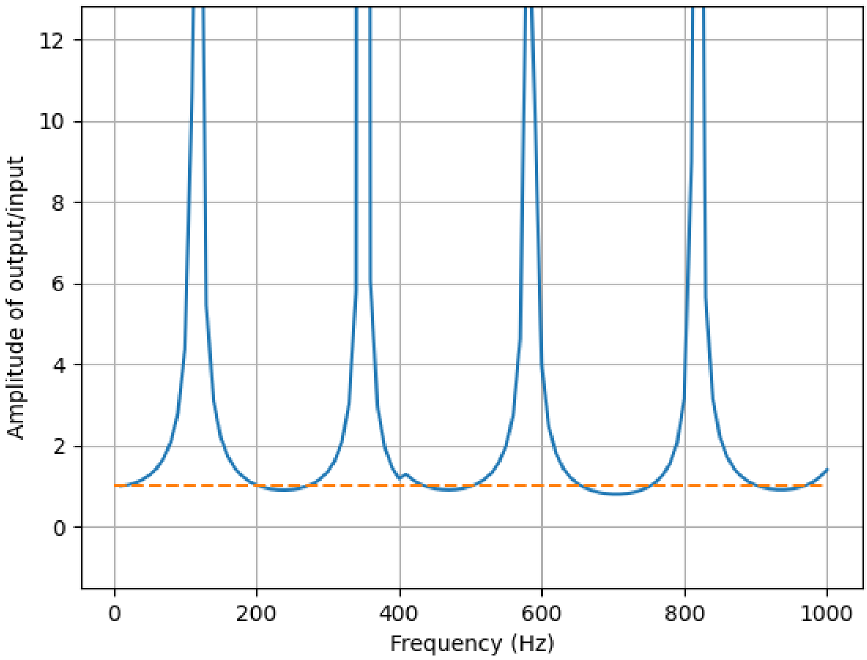

Figure 6: Ratio of amplitudes v/s Frequency

The smooth curve suggests that, in the lattice model, the energy transmission does not encounter the specific conditions required for antiresonance. This highlights the differences in dynamic behavior between homogeneous and lattice structures.

7 Resonance and Antiresonance

Antiresonance is a phenomenon observed in mechanical and structural systems where, at certain frequencies, the amplitude of vibration at a particular point in the system becomes significantly reduced or even zero. This occurs despite the presence of continuous excitation, leading to a dip in the amplitude response curve at these specific frequencies.

In a homogeneous beam, antiresonance occurs due to the constructive and destructive interference of waves propagating through the structure. The key reasons for antiresonance in such systems are:

- **Natural Frequencies and Mode Shapes:** The homogeneous beam has well-defined natural frequencies and mode shapes. When the excitation frequency coincides with one of these natural frequencies, resonance occurs, leading to large amplitude vibrations. At specific frequencies between these resonances, the phase difference between different parts of the beam causes destructive interference, leading to antiresonance.

- **Boundary Conditions:** The boundary conditions of the beam (fixed, free, or a combination) affect the distribution of natural frequencies and the occurrence of antiresonance. These conditions ensure that certain frequencies result in minimal or no displacement at particular points.

Lack of Antiresonance in Lattice Models

In the case of the lattice model, the absence of antiresonance can be attributed to several factors:

- **Discretization and Heterogeneity:** Unlike a homogeneous beam, a lattice model consists of discrete elements and nodes. The heterogeneity and the discrete nature of the model lead to a more complex distribution of natural frequencies and mode shapes. This complexity can obscure the clear antiresonance points that are evident in homogeneous structures.

- **Localized Modes:** In a lattice structure, modes can be highly localized due to the discrete nature and possible imperfections in the lattice. These localized modes do not propagate energy through the structure in the same way as the continuous modes in a homogeneous beam, thus reducing the likelihood of destructive interference at specific frequencies.

- **Energy Transmission Characteristics:** The energy transmission in a lattice model is different from that in a homogeneous beam. The presence of joints, connections, and the arrangement of lattice elements affect the transmission path and phase relationships, leading to a smoother amplitude response curve without distinct dips characteristic of antiresonance.

Antiresonance is a distinct phenomenon in homogeneous beams due to well-defined natural frequencies and boundary conditions, leading to destructive interference at certain frequencies. In contrast, lattice models, with their discrete and heterogeneous nature, exhibit different energy transmission characteristics, resulting in a smoother frequency response without clear antiresonance dips.

8 Conclusion

In this work, we explored various advanced concepts in the field of structural dynamics and metamaterials, focusing on their unique properties and analytical methods for understanding energy transmission.

The Newmark method, an implicit numerical integration technique, was presented for solving the equations of motion in dynamic systems. We provided a detailed mathematical formulation using displacement vectors and matrices to represent mass, damping, and stiffness. This method is crucial for accurately predicting the response of structures under dynamic loads, such as vibrations and seismic waves.

We applied both the Laplace and Fourier transforms to analyze energy transmission across a material. Using the Laplace transform, we derived the transfer function for the system in the s-domain. The Fast Fourier Transform (FFT) and its inverse were employed to study the amplitude ratio of output to input as a function of frequency. This analysis revealed that in a lattice model, the amplitude ratio plot showed no dips below 1, indicating the absence of antiresonance.

The phenomenon of antiresonance, where the amplitude of vibration at certain frequencies becomes minimal or zero, was examined. In homogeneous beams, antiresonance occurs due to constructive and destructive interference of wave modes. However, in lattice models, the discrete and heterogeneous nature leads to a different energy transmission characteristic, resulting in a smoother amplitude response curve without distinct antiresonance dips.

Our study provided a comprehensive understanding of the unique properties of metamaterials, the importance of energy transmission, and the advanced analytical methods used in structural dynamics. Future work could involve further exploration of layered metamaterials, optimization of lattice structures for specific applications, and the development of more sophisticated numerical methods for dynamic analysis.

9 Bibliography

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