Assignment No 5.

MATHEMATICAL INDUCTION

Question NO. 1.

Let us suppose that the maximum number of regions $R(n)$ the plane is divided into by n lines is given by the formula:

$$R(n) = \frac{1}{2}(n^2+n+2)$$

Base case:

$$R(0) = \frac{1}{2}(0^2+0+2)$$

$$= \frac{1}{2} \times 2$$

$$= 1$$

$\therefore$ This proves that the plane consists of 1 region.

Inductive Hypothesis:

Let us assume, $n = k$

$$R(k) = \frac{1}{2}(k^2+k+2)$$

Inductive Step:

Adding new lines increases the number of regions by $k+1$.

$$R(k+1) = \frac{1}{2}[(k+1)^2+(k+1)+2]$$

$$= \frac{1}{2}[k^2+2k+1+k+1+2]$$

$$= \frac{1}{2}[k^2+3k+4] \quad \text{---- GOAL}$$

Expanding the formular for $R(k+1)$:

$$\frac{1}{2}(k^2+k+2) + (k+1) = \frac{1}{2}[k^2+k+2+2k+2]$$

$$= \frac{1}{2}[k^2 + 3k + 4]$$

$\therefore$ The maximum number of regions the plane is divided into by n lines is

$$R(n) = \frac{1}{2}(n^2 + n + 2)$$

Question No 2.

Given,

$$3^{(2n+1)} + (-1)^n\, 2 \equiv 0 \ (\text{mod } 5)$$

Base step $(n=0)$

$$3^{[2(0)+1]} + (-1)^0 \cdot 2 \equiv 0 \ (\text{mod } 5)$$

Inductive Hypothesis, For $n = k$

$$3^{2k+1} + (-1)^k \cdot 2 \equiv 0 \ (\text{mod } 5)$$

Inductive Step,

For $n = k+1$

$$3^{2(k+1)+1} + (-1)^{k+1} \cdot 2 \equiv 0 \ (\text{Mod } 5)$$

$$3^{2k+2+1} + (-1)^{k+1} \cdot 2$$

$$3^2 \cdot 3^{2k+1} + (-1)^{k+1} \cdot 2$$

$$9 \cdot 3^{2k+1} + (-1)^{k+1} \cdot 2$$

$$9 \text{ Mod } 5 = 4$$

$$\therefore \quad 4 \cdot 3^{2k+1} + (-1)^{k+1}$$

Now, Using the Inductive Hypothesis, $3^{2k+1} = -(-1)^{k+1} \cdot 2$

$$4 \cdot (-(-1)^k \cdot 2) + (-1)^{k+1} \cdot 2$$

$$(-1)^{k+1} \cdot 8 + (-1)^{k+1} \cdot 2$$

$$(-1)^{k+1}(8+2)$$

$$10(-1)^{k+1}$$

Finally, $10 \equiv 0 \pmod 5$

This confirms the inductive step is correct.

Question No: 3

For all $(n>1)$

Prove: $1+4+7+ \dots + (3n-2) = \frac{n(3n-1)}{2}$

Base case:

$$1+4+7+ \dots (3n-2) = \frac{n(3n-1)}{2}$$

Taking (1),

$$P(1) = \frac{1(3-1)}{2}$$

$$= 1$$

Inductive Hypothesis: Supposing $n=k$

$$1+4+7+ \dots + (3k-2) = \frac{k(3k-1)}{2}$$

Inductive case: for $n=k+1$

$$1+4+7+ \dots + (3k-2) + [3(k+1)-2] = \frac{(k+1)[3(k+1)-1]}{2}$$

$$\frac{k(3k-1)}{2} + \frac{3k+1}{1} = \frac{(k+1)(3k+2)}{2}$$

$$L.H.S: \frac{k(3k+1)+6k+2}{2}$$

$$= \frac{3k^2+5k+2}{2}$$

$$= \frac{(3k+2)(k+1)}{2} = R.H.S \text{ proved}$$

Question No. 4.

Given,

$6^n - 1$ = divisibe by 5 (for any positive integer : n)

Base case

Takeng $n = 1$,

$6 - 1 = 5$ (divisible by 5) [True]

Inductive Hypothesis. (for $n = k$)

$6^{k} - 1$ is divisible by 5 - - - - - (1)

To prove, [Inductive case]

$n \longrightarrow k+1$

$6^{k+1} - 1$

$6^{k} \cdot 6 - 1$

$6^{k} \cdot 6 - (6 - 5)$

$6^{k} \cdot 6 - 6 + 5$

$6(6^{k} - 1) + 5$

↳ divisible by 5 (from equation 1).

$\therefore$ $6^n - 1$ can be divisible by 5 for any positive integer n.

Question No. 5.

For all $n \geq 1$,

$$1^2 + 2^2 + 3^2 + \ldots\ldots + (2n)^2 = \frac{[n(2n+1)(4n+1)]}{3}$$

Base case:

Taking $n = 1$,

$$1^2 + 2^2 = \frac{1(2+1)(4+1)}{3}$$

$$5 = \frac{1 \times \cancel{3} \times 5}{\cancel{3}} \qquad 5 = 5 \text{ [True]}$$

<u>Inductive Hypothesis</u> $n = k$

$$1^2 + 2^2 + 3^2 + \ldots + (2k)^2 = \frac{k(2k+1)(4k+1)}{3}$$

<u>Inductive Step</u>: For $n = k+1$

$$1^2 + 2^2 + \ldots\ldots + (2(k+1))^2$$

$$\Rightarrow \underbrace{1^2 + 2^2 + \ldots\ldots + (2k)^2} + (2k+1)^2 + (2k+2)^2$$

$$\Rightarrow \frac{k(2k+1)(4k+1)}{3} + 8k^2 + 12k + 5$$

$$\Rightarrow \frac{(2k^2+k)(4k+1) + 24k^2 + 36k + 15}{3}$$

$$\Rightarrow \frac{8k^3 + 30k^2 + 37k + 15}{3}$$

By factorization,

$k = (-1)$

$2(-1)^3 + 30(-1)^2 + (37)(-1) + 15 = 0$

$(k+1)$ is a factor of this:

$$\begin{array}{r|l}
k+1 & 8k^3 + 30k^2 + 37k + 15 \;(\; 8k^2 + 22k + 15 \\
 & \underline{\overset{-}{+}8k^3 \overset{-}{+} 8k^2} \\
 & 22k^2 + 37k + 15 \\
 & \underline{\overset{-}{+}22k^2 \overset{-}{+} 22k} \\
 & 15k + 15 \\
 & \underline{15k \pm 15} \\
 & \times
\end{array}$$

$8k^2 + 22k + 15$

or, $8k^2 + 12k + 10k + 15$

or, $4k(2k+3) + 5(2k+3)$

$(4k+5)(2k+3)$ are also factor's of the cubic equation.

$$\Rightarrow \frac{(k+1)(4k+5)(2k+3)}{3}$$

$$\Rightarrow \frac{(k+1)[4(k+1)+1][2(k+1)+1]}{3}$$

$= R.H.S.$

Thus, the formula holds for $n = k+1$.

Question NO.6.

For n, any positive integer, $2^{2n}-1$ is divisible by 3.

Base case For $(n=1)$

$2^{2n}-1$

$2^{2(1)}-1$

$4-1=3$ (which is divisible by 3) [True]

Inductive Hypothesis, for $n=k$

$2^{2k}-1$ is divisible by 3 ---- (1)

To prove, Inductive Case [By Mathematical Induction]

$n \rightarrow k+1$

$2^{2(k+1)}-1$

$2^{2k}.2^{2}-1$

$2^{2k}\,4-(4-3)$

$2^{2k}.4-4+3$

$4(2^{2k}-1)+3$

└ which is divisible by 3. From eq(1)

proved,,

Question No. 7

$2n+1 < 2^n$, for all natural numbers $n \geq 3$.

Base Case (~~From~~ $n=3$)

$2(3)+1 < 2^3$

$7 < 8$ (True)

Inductive Hypothesis.

Assume inequality holds for some natural number $k \geq 3$

$2k+1 < 2^k$

Inductive Case.

For $n = k+1$

$2(k+1)+1 = 2k+2+1$

$= (2k+1)+2$

From Inductive Hypothesis,

$2k+1 < 2^k$

Adding 2 on both sides:

$(2k+1)+2 < 2^k+2$

We can even replace 2 with 2^k, because 2^k is greater than or equal to 2.

$\Rightarrow (2k+1)+2 < 2^k+2^k$

$\Rightarrow (2k+1)+2 < 2^{k+1}$

$\therefore$ This shows that if the inequality is true for k, it is also true for k+1

Question No 8:

Let $P(n)$ denote the statement 'There is a suivivor (who is not hit) in the odd pie, fight with $2n+1$ people.

Base case: Taking $P(1)$.

There are only three people. Out of the three people, Let's suppose that the closest pair is 'A&B' and C is the third pair. Since distances between people are different, the distances between A and C, and B and C are greater than that between A and B. Therefore, A and B throws pie at each other, and C survive

Inductive Case:

Suppose that $P(K)$ is True, that is in the pie fight with $2k+1$ people, there is a survivor. Consider the fight with $2(k+1)+1$ people. Let A and B be the closest pair of people in this group of $2k+3$ people. Then they throw pies at each other. If someone else throws a pie at one of them, then for the remaining $2k+1$ people, there are only $2k$ pies and one of them servives.

Otherwise, the remaining $2k+1$ people throw pies at each other, playing the pie fight with $2k+1$ people.

By the Inductive Hypothesis, there is a survivor in such a fight.