(vi) We have $$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 \quad \ldots(1)$$

The general form of the particular solution of recurrence relation (1) is

$$a_r = A_1 + A_2 r + A_3 r^2 \quad \ldots(2)$$

Where A_1, A_2, A_3 be constant which is to be determine. Put this expression in (1), we obtain

$$(A_1 + A_2 r + A_3 r^2) + 5\{(A_1 + A_2 (r-1) + A_3 (r-1)^2)\} + 6\{(A_1 + A_2(r-2) + A_3 (r-2)^2)\} = 3r^2$$

$$\Rightarrow \quad 12A_3 r^2 - (34A_3 - 12A_2) r + (29A_3 + 17A_2 + 12A_1) = 3r^2$$

$$\Rightarrow \quad 12A_3 = 3, \; 34A_3 - 12A_2 = 0, \text{ and } 29A_3 + 17A_2 + 12A_1 = 0$$

On solving these equations, we get

$$A_1 = \frac{115}{288}, A_2 = \frac{17}{24}, A_3 = \frac{1}{4}$$

Therefore, the particular solution of (1) is

$$a_r = \frac{115}{288} + \frac{17}{24} r + \frac{1}{4} r^2$$

(vii) We have $$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2 - 3r + 1 \quad \ldots(1)$$

The particular solution of given recurrence relation is of the form

$$a_r = A_1 + A_2 r + A_3 r^2 \quad \ldots(2)$$

Put this relation (2) in L.H.S. of (1), we find

$$(A_1 + A_2 r + A_3 r^3) + 5\{(A_1 + A_2(r-1) + A_3 (r-1)^2)\} + 6\{(A_1 + A_2(r-2) + A_3 (r-2)^2\} = 3r^2 - 3r + 1$$

$$\Rightarrow \quad 12A_3 r^2 - (34A_3 - 12A_2) r + (29A_3 - 17A_2 + 12A_1) = 3r^2 - 2r + 1$$

Comparing like power of r on both sides, we get

$$12A_3 = 3, \; 34A_3 - 12A_2 = 2, \text{ and } 29A_3 - 17A_2 + 12A_1 = 1$$

On solving these equations, we get $A_1 = \frac{71}{288}, A_2 = \frac{13}{24}, A_3 = \frac{1}{4}$

Hence, the required particular solution of (1) is $a_r = \frac{71}{288} + \frac{13}{24} r + \frac{1}{4} r^2$

Example 6: *Solve the recurrence relation given below:*

(i) $a_r + 6a_{r-1} + 9a_{r-2} = 3$ given that $a_0 = 0$, $a_1 = 1$

[U.P.T.U. (B. Tech.) 2004; R.G.P.V. (B.E.) Bhopal 2004, 2005, 2006, 20

(ii) $a_r - 6a_{r-1} + 8a_{r-2} = r.4^r$ given $a_0 = 8$, $a_1 = 22$ **[U.P.T.U. (B.Tech.) 2009; P.T.U. (B.E.) Punjab 2004, 20**

(iii) $a_{r+2} - 2a_{r+1} + a_r = 2^r$ given that $a_0 = 2$ and $a_1 = 1$ **[U.P.T.U. (B.Tech.) 20**

(iv) What is recursion and recurrence relation. Solve the following recurrence relation using init condition as $s(k) - 9s(k-1) + 8s(k-2) = 9k + 1$, $s(0) = s(1) = 1$. **[U.P.T.U. (B.Tech.) 20**

(v) $a_r - 5a_{r-1} + 6a_{r-2} = 7^r$ **[U.P.T.U. (B.Tech.) 20**

(vi) $a_n = 2a_{n-1} + 3a_{n-2} + 5^n$, $n \geq 2$ with given $a_0 = -2$, $a_1 = 1$

[P.T.U. (B.E.) Punjab 2004, 2008; Kurukshetra (B.E.) 20

(vii) $u_n - 4u_{n-1} + 4_{n-2} = 2^n$ or $a_r - 4a_{r-1} + 4a_{r-2} = 2^r$ **[M.C.A. (Rohtak) 2008; U.P.T.U. (B.Tech.) 20**

(viii) $a_r + 4a_{r-1} + 4a_{r-2} = r^2 - 3r + 5$

(ix) $y_r - 4y_{r-1} + 4y_{r-2} = 3r + 2^r$ or $a_r - 4a_{r-1} + 4a_{r-2} = 3r + 2^r$

(x) $a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r, r \geq 2$ with boundary condition $a_0 = 1$ and $a_1 = 1$.

[R.G.P.V. (B.E.) Bhopal 2002, 2009]

(xi) $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r$, $a_0 = 1, a_1 = 2, a_2 = 3$

[R.G.P.V. (B.E.); Raipur 2003, 2008; Rohtak (B.E.) 2004, 2007]

(xii) $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)^2, r \geq 2$ [R.G.P.V. (B.E.) Bhopal 2003]

(xiii) $6a_r - 7a_{r-1} - 20a_{r-2} = 3r^2 - 2r + 8$ [M.K.U. (B.E.) 2005, 2009]

(xiv) $a_{r+2} - 6a_{r+1} + 8a_r = 3r^2 + 2 - 5.3^r$ [Osmonia (B.E.) 2006, 2008; R.G.P.V. (B.E.) Raipur 2009]

(xv) $a_{r+3} - 3a_{r+2} + 3a_{r+1} - a_r = 24(r+2)$ [Kurukshetra (B.E.) 2005, 2008; Delhi (B.E.) 2009]

Solution: (i) The given recurrence relation is

$$a_r + 6a_{r-1} + 9a_{r-2} = 3 \quad \text{...(1)}$$

The characteristic equation of (1) is given as

$$\alpha^2 + 6\alpha + 9 = 0 \quad \Rightarrow \quad (\alpha+3)^2 = 0 \quad \text{or } \alpha = -3, -3$$

Therefore, the general homogeneous solution is

$$a_r = (A_1 + A_2 r)(-3)^r \quad \text{...(2)}$$

Since (-3) is a characteristic root of multiplicity 2, the general form of particular solution of (1) is given as

$$a_r = A_3 + A_4 r + A_5 r^2 \quad \text{...(3)}$$

Put this expression in (1), we get

$$\{A_3 + A_4 r + A_5 r^2\} + 6\{A_3 + A_4(r-1) + A_5(r-1)^2\} + 9\{A_3 + A_4(r-2) + A_5(r-2)^2\} = 3$$

$$\Rightarrow \quad 16A_5 r^2 + (16A_4 - 48A_5)r + (16A_3 - 24A_4 + 42A_5) = 3$$

Comparing like terms on both sides, we get

$$16A_5 = 0 \Rightarrow A_5 = 0,\ 16A_4 - 48A_5 = 0 \Rightarrow A_4 = 0, \text{ and } 16A_3 - 24A_4 + 42A_5 = 3 \Rightarrow A_3 = \frac{3}{16}$$

Then from (3) $a_r = \frac{3}{16}$

Hence, total solution as (i) is

$$a_r = \text{homogeneous solution + particular solution}$$

$$\Rightarrow \quad a_r = (A_1 + A_2 r)(-3)^r + \frac{3}{16} \quad \text{...(4)}$$

The given boundary condition is $a_0 = 0,\ a_1 = 1$

Put $r = 0, 1$ in (4), we get

$$a_0 = A_1 + \frac{3}{16},\quad a_1 = (A_1 + A_2)(-3) + \frac{3}{16}$$

or

$$o = A_1 + \frac{3}{16},\quad 1 = (A_1 + A_2)(-3) + \frac{3}{16}$$

$$\Rightarrow \quad A_1 = \frac{-3}{16} \text{ and } A_2 = \frac{-1}{12}$$

Put these value in (4), we get required solution

$$a_r = \left(\frac{-3}{16} - \frac{1}{12}r\right)(-3)^r + \frac{3}{16}$$

(ii) The given recurrence relation is

$$a_r - 6a_{r-1} + 8a_{r-2} = r.4^r \quad ...(1)$$

given $$a_0 = 8,\ a_1 = 22$$

The characteristic equation of (1) is

$$\alpha^2 - 6\alpha + 8 = 0 \text{ or } (\alpha - 2)(\alpha - 4) = 0 \Rightarrow \alpha = 2, 4$$

Therefore, the homogeneous solution is

$$a_r = A_1 2^r + A_2 4^r \quad ...(2)$$

The general form of particular solution is

$$a_r = r(A_3 + A_4 r)4^r \quad ...(3)$$

Put this relation (1), we find

$$\{r(A_3 + A_4 r)4^r\} - 6\{(r-1)\{(A_3 + A_4(r-1)\}4^{r-1}\} + 8\{(r-2)\{A_3 + A_4(r-2)\}\ 4^{r-2}\} = r.\ 4^r$$

This expression holds for all values of r and in particular for $r = 0$, we have

$$\frac{3}{2}(-A_4 + A_3) - (-2A_4 + A_3) = 0 \text{ or } A_3 + A_4 = 0 \quad ...(4)$$

For $r = 1$, we get

$$4(A_4 + A_3) - 2(-A_4 + A_3) = 4$$

or $$3A_4 + A_3 = 2 \quad ...(5)$$

On solving (4) and (5), we find $A_4 = 1, A_3 = -1$

Therefore, the particular solution of (1) is given from (3) $a_r = r(r-1)4^r$

Hence, the general (total) solution is

$$a_r = \text{homogeneous solution + particular solution}$$

$$\Rightarrow \quad a_r = A_1\ 2^r + A_2\ 4^r + r(r-1)4^r \quad ...(6)$$

The initial condition $a_0 = 8,\ a_1 = 22$

Put $r = 0, 1$ in (6), we get

$$a_0 = A_1 + A_2 \Rightarrow A_1 + A_2 = 8 \quad ...(7)$$

$$a_1 = 2A_1 + 4A_2 \Rightarrow 2A_1 + 4A_2 = 22 \quad ...(8)$$

Solve (7) and (8), we find $A_1 = 5,\ A_2 = 3$

Hence, required total solution is $a_r = 5.2^r + 3.4^r + r(r-1)4^r$

(iii) The given recurrence relation is $a_{r+2} - 2a_{r+1} + a_r = 2^r$ ...(1)

The characteristic equation of (1) is $\alpha^2 - 2\alpha + 1 = 0$ or $(\alpha - 1)^2 = 0$ or $\alpha = 1, 1$

Therefore, the general homogeneous solution is $a_r = (A_1 + A_2 r)(1)^r$ ...(2)

The general form of particular solution of (1) is given as $a_r = A_3 2^r$ ...(3)

Put this in the given recurrence relation

$$A_3 2^{r+2} - 2A_3 2^{r+1} + A_3 2^r = 2^r$$

$$\Rightarrow \quad 4A_3 - 4A_3 + A_3 = 1 \quad \text{or} \quad A_3 = 1$$

Put in (3) $\Rightarrow a_r = 2^r$ ...(4)

Hence, general (total) solution is

$$a_r = (A_1 + A_2 r) + 2^r \quad ...(5)$$

Now, applying the initial condition $a_0 = 2,\ a_1 = 1$ in (5)

Put $r = 0, 1$ in (5), we find

$$a_0 = A_1 + 1 \Rightarrow A_1 + 1 = 2 \Rightarrow A_1 = 1$$
$$a_1 = A_1 + A_2 + 2 \Rightarrow A_1 + A_2 + 2 = 1 \Rightarrow A_2 = -2$$
$$\therefore \quad a_r = (1 - 2r)(1)^r + 2^r$$

(iv) We have the recurrence relation

$$s(k) - 9s(k-1) + 8s(k-2) = 9k + 1, \; s(0) = s(1) = 1$$

or

$$a_k - 9a_{k-1} + 8a_{k-2} = 9k + 1, \; a_0 = a_1 = 1 \quad \ldots(1)$$

The characteristic equation of (1)

$$(\alpha)^2 - 9\alpha + 8) = 0 \quad \text{or} \quad (\alpha - 8)(\alpha - 1) = 0 \quad \text{or} \quad \alpha = 1, 8$$

Therefore, the general homogeneous solution is $a_k = A_1(1)^k + A_2(8)^k$

The general form of particular solution of (1) is given as $\quad a_k = A_1 k + A_2 k^2 \quad \ldots(2)$

Put this value in (1), we get

$$(A_1 k + A_2 k^2) - 9(A_1(k-1) + A_2(k-1)^2) + 8\{A_1(k-2) + A_2(k-2)^2\} = 9k + 1$$

Comparing the like term on both sides we get

$$A_1 = \frac{-221}{98}, \; A_2 = \frac{-9}{14}$$

Put in (2)

$$a_k = \frac{-221}{98}k - \frac{9}{14}k^2$$

Hence, general solution is $\quad \ldots(3)$

$$a_k = A_1(1)^k + A_2(8)k - \frac{221}{98}k - \frac{9}{14}k^2 \quad \ldots(4)$$

(v) The given recurrence relation is $a_r - 5a_{r-1} + 6a_{r-2} = 7^r$

The characteristic equation of (1) is given $\quad \ldots(1)$

$$\alpha^2 - 5\alpha + 6 = 0 \quad \Rightarrow \quad (\alpha - 3)(\alpha - 2) = 0 \quad \text{or} \quad \alpha = 2, 3.$$

Therefore, the homogeneous solution is $a_r = A_1 2^r + A_2 3^r$

The general form of particular solution of (1) is given as $a_r = A7^r \quad \ldots(2)$

Where A is constant $\quad \ldots(3)$

Put (3) in (1), we get

$$A7^r - 5A7^{r-1} + 6A7^{r-2} = 7^r \quad \text{or} \quad 49A - 35A + 6A = 49 \text{ or } A = \frac{49}{20}$$

Put in (3)

$$a_r = \frac{49}{20}7^r$$

Hence, general solution is $\quad \ldots(4)$

$$a_r = A_1 2^r + A_2 3^r + \frac{49}{20}7^r$$

(vi) $a_n = 2a_{n-1} + 3a_{n-2} + 5^n, n \geq 2 \quad$ and $\quad a_0 = -2, \; a_1 = 1 \quad \ldots(1)$

The characteristic equation of (1) is

$$\alpha^2 - 2\alpha - 3 = 0 \quad \Rightarrow \quad (\alpha - 3)(\alpha + 1) = 0 \text{ or } \alpha = -1, \alpha = 3$$

Therefore, the homogeneous solution is $a_n = A_1(-1)^n + A_2(3)^n$ ...(2)

The general form of particular solution of (1) is $a_n = A5^n$ ...(3)

Where A is constant, Put (3) in (1), we obtain

$$A5^n - 2A5^{n-1} - 3A5^{n-2} = 5^n \Rightarrow 25A - 10A - 3A = 25 \Rightarrow 12A = 25,\ A = \frac{25}{12}$$

Put in (3) $a_n = \frac{25}{12}5^n$ ...(4)

Hence, general solution is $a_n = A_1(-1)^n + A_2(3)^n + \frac{25}{12}5^n$ ...(5)

Applying the initial conditions $a_0 = -2, a_1 = 1$. Put $n = 0, 1$ in (5)

$$a_0 = A_1 + A_2 + \frac{25}{12} \quad \text{and} \quad a_1 = -A_1 + 3A_2 + \frac{125}{12}$$

or $$A_1 + A_2 + \frac{25}{12} = -2 \quad \text{and} \quad -A_1 + 3A_2 + \frac{125}{12} = 1$$

$$\Rightarrow \quad A_1 = \frac{-17}{24},\ A_2 = \frac{-27}{8}$$

Put in (5), we get

$$a_n = \frac{-17}{24}(-1)^n - \frac{27}{8}(3)^n + \frac{25}{12}5^n$$

(vii) The given recurrence relation $a_r - 4a_{r-1} + 4a_{r-2} = 2^r$ ...(1)

The characteristic equation is $\alpha^2 - 4\alpha + 4 = 0$ $(\alpha - 2)^2 = 0$ or $\alpha = 2, 2$

Therefore, the homogeneous solution is $a_r = (A_1 + A_2 r)2^r$ ...(2)

Since 2 is a characteristic root of multiplicity 2, the general form of the particular solution of (1) is

$$a_r = Ar^2 2^r \quad ...(3)$$

Put this in (1), we find

$$Ar^2 2^r - 4A(r-1)^2 2^{r-1} + 4A(r-2)^2 2^{r-2} = 2^r$$

$$\Rightarrow \quad A = 1/2$$

Put in (3) $a_r = \frac{1}{2}r^r 2^r$

The general solution $a_r = (A_1 + A_2 r)2^r + \frac{1}{2}r^2 2^r$ ...(1)

(viii) The given recurrence relation is $a_r + 4a_{r-1} + 4a_{r-2} = r^2 - 3r + 5$ ...(1)

The characteristic equation of (1) is

$$\alpha^2 + 4\alpha + 4 = 0 \quad \Rightarrow \quad (\alpha + 2)^2 = 0 \quad \text{or} \quad \alpha = -2, -2$$

Therefore, the homogeneous solution is $a_r = (A_1 + A_2 r)(-2)^r$ ...(2)

The general form of particular solution of (1) is $a_r = (A_3 + A_4 r + A_5 r^2)$ ...(3)

Put this value in (1), we find

$$(A_3 + A_4 r + A_5 r^2) + 4\{A_3 + A_4(r-1) + A_5(r-1)^2\} + 4\{A_3 + A_4(r-2) + A_5(r-2)^2\} = r^2 - 3r + 5$$

Comparing like power of r on both sides then solve, we get $A_3 = \frac{7}{9}$, $A_4 = \frac{13}{27}$, $A_5 = \frac{1}{9}$

Then put in (3), we get
$$a_r = \frac{1}{9}\left(7 - \frac{13}{3}r + r^2\right)$$

Therefore, the total solution is $a_r = (A_1 + A_2 r)(-2)^r + \frac{1}{9}\left(7 - \frac{13}{3}r + r^2\right)$

(ix) This part is similar to part (x)

(x) The given recurrence relation is $a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r$ ...(1)

with boundary conditions $a_0 = 1$, $a_1 = 1$.

Therefore, the characteristic equation of (1) is
$$\alpha^2 - 5\alpha + 6 = 0 \quad \text{or} \quad (\alpha - 3)(\alpha - 2) = 0 \text{ or } \alpha = 2, 3$$

Therefore, the general homogeneous solution of given equation (1) is
$$a_r = A_1 2^r + A_2 3^r$$ ...(2)

The general form of particular solution is
$$a_r = A_3 r 2^r + A_4 r + A_5$$ ...(3)

Put this relation in (1), We get
$$(A_3 r 2^r + A_4 r + A_5) - 5\{A_3(r-1)2^{r-1} + A_4(r-1) + A_5\} + 6\{A_3(r-2)\,2^{r-2} + A_4(r-2) + A_5\} = 2^r + r$$

$$\Rightarrow \left(A_3 - \frac{5}{2}A_3 + \frac{3}{2}A_3\right) r2^r + \left[\frac{5}{2}A_3 - 3A_3\right]2^r + [A_5 + 5A_5 + 5A_4 + 6A_5 - 12A_4]$$
$$+ [A_4 - 5A_4 + 6A_4]\, r = 2^r + r$$

or
$$\frac{-1}{2}A_3 2^r + (2A_5 - 7A_4) + 2A_4 r = 2^r + 1$$

Comparing like power on both sides, we get
$$-\frac{1}{2}A_3 = 1,\ 2A_5 - 7A_4 = 0,\ 2A_4 = 1$$

$$\Rightarrow \quad A_3 = -2,\ A_4 = \frac{1}{2},\ A_5 = \frac{7}{4}$$

Put these values in (3), we get particular solution.
$$a_r = -2r2^r + \frac{1}{2}r + \frac{7}{4}$$

Hence, the total solution of (1) is
$$a_r = A_1 2^r + A_2 3^r - 2r.2^r + \frac{1}{2}r + \frac{7}{4}$$ ...(4)

Put $r = 0, 1$ in (4) then used $a_0 = 1$, $a_1 = 1$, we find
$$1 = A_1 + A_2 + \frac{7}{4} \quad \text{and} \quad 1 = 2A_1 + 3A_2 - 4 + \frac{1}{2} + \frac{7}{4}$$

or
$$A_1 + A_2 = -\frac{3}{4}$$ ...(5)

$$2A_1 + 3A_2 = \frac{11}{4}$$ ...(6)

On solving (5) and (6), we get $A_1 = -5$, $A_2 = \frac{17}{4}$

Hence the required solution of (1) is $a_r = -5.2^r + \frac{17}{4}.3^r - 2r.2^r + \frac{1}{2}r + \frac{7}{4}$

(xi) The given recurrence relation is $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)2^r$ ...(1

with boundary conditions $a_0 = 1, a_1 = 2, a_2 = 3.$

The characteristic equation of the given recurrence relation is

$$\alpha^2 - 4\alpha + 4 = 0 \quad \Rightarrow \quad (\alpha - 2)^2 = 0 \quad \text{or} \quad \alpha = 2, 2$$

Hence, the homogeneous solution is $a_r = (A_1 r + A_2)2^r$ ...(

Since 2 is a characteristic root of multiplicity 2, the general form of particular solution is

$$a_r = r^2(A_3 r + A_4).2^r$$ ...

Put this relation in (1), we find

$$r^2(A_3 r + A_4)2^r - 4\{(r-1)^2(A_3(r-1) + A_4)2^{r-1}\} + 4\{(r-2)^2(A_3(r-2) + A_4)\} = (r+1)2^r$$

$$\Rightarrow \quad A_3\{r^3 - 2(r-1)^3 + (r-2)^3\}2^r + A_4\{r^2 - 2(r-1)^2 + (r-2)^2\}2^r = (r+1)2^r$$

$$\Rightarrow \quad A_3(6r-6)\, 2^r + 2A_4 2^r = (r+1)\, 2^r$$

$$\Rightarrow \quad 6A_3 r 2^r + (-6A_3 + 2A_4)2^r = r2^r + 2^r$$

Comparing the like terms on both sides

$$6A_3 = 1 \text{ and } -6A_3 + 2A_4 = 1 \quad \Rightarrow A_3 = 1/6 \quad \text{and } A_4 = 1$$

Put these value in (3)

$$a_r = r^2\left(\frac{r}{6} + 1\right) 2^r$$

Hence, the required total (general) solution is $a_r = (A_1 r + A_2)2^r + r^2\left(\frac{r}{6} + 1\right) 2^r$

(xii) The given recurrence relation is $a_r - 4a_{r-1} + 4a_{r-2} = (r+1)^2$

The characteristic equation of (1) is

$$\alpha^2 - 4\alpha + 4 = 0 \quad \Rightarrow \quad (\alpha - 2)^2 = 0 \quad \text{or} \quad \alpha = 2, 2$$

Therefore, the homogeneous solution of (1) is $a_r = (A_1 + A_2 r)2^r$

The general form of particular solution is $a_r = A_3 + A_4 r + A_5 r^2$

Put this relation in, we find

$$\{A_3 + A_4 r + A_5 r^2\} - 4\{A_3 + A_4(r-1) + 5(r-1)^2\} + 4\{A_3 + A_4(r-2) + 5(r-2)^2] = 1 + 2r + r^2$$

$$\Rightarrow \quad (A_3 - 4A_3 + 12A_5) + (A_4 - 8A_5)r + A_5 r^2 = 1 + 2r + r^2$$

Comparing like terms on both sides, we obtain

$$A_3 - 4\, A_4 + 12A_5 = 1,\ A_4 - 8\, A_5 = 2,\ A_5 = 1$$

On solving these equations, we obtain $A_3 = 29,\ A_4 = 10,\ A_5 = 1$

Put these value in (3), we get particular solution $a_r = 29 + 10\, r + r^2$

Hence, the required total or general solution of (1) is $a_r = (A_1 + A_2 r)2^r + 29 + 10r + r^2$

(xiii) The given recurrence relation is $6a_r - 7a_{r-1} - 20a_{r-2} = 3r^2 - 2r + 8$

The characteristic equation of given recurrence relation is

$$6\alpha^2 - 7\alpha - 20 = 0 \quad \text{or } (2\alpha - 5)(3\alpha + 4) = 0 \quad \text{or } \alpha = \frac{5}{2}, \frac{-4}{3}$$

Therefore, the homogeneous solution is, $a_r = A_1\left(\frac{5}{2}\right)^r + A_2\left(\frac{-4}{3}\right)^r$

The particular solution of given equation (1) is $a_r = A_3 r^2 + A_4 r + A_5$

Put this expression in (1), we obtain

$$6\{A_3r^2 + A_4r + A_5\} - 7\{A_3(r-1)^2 + A_4(r-1) + A_5\} - 20\{A_3(r-2)^2 + A_4(r-2) + A_5\} = 3r^2 - 2r + 8$$

$$\Rightarrow \quad -21A_3r^2 + (94A_3 - 21A_4)r + (-87A_3 + 47A_4 - 21A_5) = 3r^2 - 2r + 8$$

Comparing like term on both sides, we get

$$-21A_3 = 3, \quad 94A_3 - 21A_4 = -2, \quad \text{and} \quad -87A_3 + 47A_5 = 8$$

On solving these equations, we get

$$A_3 = -\frac{1}{7}, A_4 = -\frac{80}{147}, A_5 = -\frac{3109}{3087}$$

Thus, the particular solution is $a_r = -\frac{r^2}{7} - \frac{80r}{147} - \frac{3109}{3087}$

Hence, the required general solution is $a_r = A_1\left(\frac{5}{2}\right)^r + A_2\left(-\frac{4}{3}\right)^r - \frac{r^2}{7} - \frac{80r}{147} - \frac{3109}{3087}$

(xiv) The given recurrence relation is $a_{r+2} - 6a_{r+1} + 8a_r = 3r^2 + 2 - 5.3^r$ ...(1)

The characteristic equation of given recurrence relation is

$$\alpha^2 - 6\alpha + 8 = 0 \quad \text{or} \quad (\alpha-2)(\alpha-4) = 0 \quad \text{or} \quad \alpha = 2, 4$$

Thus, the homogeneous solution is $a_r = A_1 2^r + A_2 4^r$ ...(2)

The general form of particular solution of (1) is $a_r = (A_3r^2 + A_4r + A_5) + A_6 3^r$ ...(3)

Put this value in (1), we find

$$\{A_3(r+2)^2 + A_4(r+2) + A_5\} + A_6 3^{(r+2)} - 6\{A_3(r+1)^2 + A_4(r+1) + A_5 + A_6 3^{r+1}\}$$
$$+8\{A_3r^2 + A_4r + A_5 + A_6 3^r\} = 3r^2 + 2 - 5.3^r$$

Comparing coefficients of like terms on both sides, we get

$$3A_3 = 3, \; 3A_4 - 8A_3 = 0, \; 3A_5 - 4A_4 - 4A_4 - 2A_3 = 2, \; 9A_6 - 18A_6 + 8A_6 = -5$$

or $\quad A_3 = 1, \; A_4 = 8/3, \quad A_5 = \frac{44}{9}, \; A_6 = 5$

Then, particular solution is

$$a_r = r^2 + \frac{8}{3}r + \frac{44}{9} + 5.3^r$$

Hence, required total (general) solution is $a_r = A_1 2^r + A_2 4^r + r^2 + \frac{8}{3}r + \frac{44}{9} + 5.3^r$

(xv) The given recurrence relation is $a_{r+3} - 3a_{r+2} + 3a_{r+1} - a_r = 24(r+2)$ ...(1)

The characteristic equation of given recurrence relation is

$$\alpha^3 - 3\alpha^2 + 3\alpha - 1 = 0 \Rightarrow (\alpha-1)^3 = 0 \text{ or } \alpha = 1, 1, 1$$

Hence, the general homogeneous solution is $a_r = A_1r^2 + A_2r + A_3$ ...(2)

Since (1) is the characteristic root of multiplicity 3, the general form of particular solution of (1) is given as

$$a_r = A_4r^4 + A_5r^3 \quad \text{...(3)}$$

Put this relation in (1), we find.

$$\{A_4(r+3)^4 + A_5(r+3)^3\} - 3\{A_4(r+2)^4 + A_5(r+2)^3\} + 3\{A_4(r+1)^4 + A_5(r+1)^3\} - \{A_4r^4 + A_5r^3\} = 24r + 48$$

Then $\quad 24A_4r + 36A_4 + A_5 = 24r + 48$

or $\quad 24A_4 = 24, \; 36A_4 + A_5 = 48$

On solving, we get $\quad A_4 = 1, A_5 = 12$

Then from (3), we get $$a_r = r^4 + 12r^3$$ ...

Hence total solution of (1) is

$$a_r = \text{homogeneous solution} + \text{particular solution}$$

$$\Rightarrow \quad a_r = A_1 r^3 + A_2 r + A_3 + r^4 + 12r^3.$$