MATH

(4)

29.4.1...   Mathematics.

- Probability & statistics → Eng. maths
- Combinatories
- Graph theory
- Set theory & algebra
- Logic

Discrete maths

- Linear algebra
- Num. methods
- Calculus.

Engineering maths

Recommanded Book - (kenneth Rosen)

1. Lectures : Theory & WB
2. WK Book
3. PS - I & II
4. Rosen : DM
5. EM for GATE - By Sundaram Sir.

- Basic probab.
- Probab. Distribution
- Statistics.

Basic probab. -

- Science of uncertain events.

1. Experiment
   - Random - Can't predict before.
   - Non-random - can be predicted before result comes out.

2. Sample space: Set of all possible outcomes.

$$S = \{H, T\}, \quad S = \{1, 2, 3, 4, 5, 6\}$$

$$S = \{(H,H), (H,T), (T,H), (T,T)\}.$$

$$S = \{ (1,1)\ (1,2) \ \cdots \ (1,6),$$
$$(2,1)\ (2,2)\ \cdots$$
$$\cdots \quad (6,6)\}.$$

3. Event

Event $\subseteq$ Sample space

- event is any subset of sample space.
- Sample space is universal set.

Eg. $S = \{H, T\}$ $\quad E_1 = \{\ \}$

$E_2 = \{H\}$

$E_3 = \{T\}$

$E_4 = \{H, T\}$

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$$\Rightarrow \boxed{p(E) = \frac{n(E)}{n(S)}} \rightarrow \text{Classical probability eq}^n$$

Only when all outcomes are equally-likely i.e. all possibility has same chance to come. in dice, (1, - - - 6) no., all have equally-likely property.

- In purely ~~class~~ lottery system, we can use it.

$$p(E_1) = \frac{0}{2} = 0 \rightarrow \text{impossible event}$$

$$p(E_2) = \frac{1}{2} = p(E_3)$$

$$p(E_4) = \text{either head or tail}$$

$$= \frac{2}{2} = 1 \rightarrow \text{Sure event.}$$

Eg. in case of dice – $2^6 = 64$ events are possible.

When dice $\rightarrow$ 2^{36} events will happen.

prob. of max is 3 is $\rightarrow$

$$p(max=3) = \frac{5}{36}$$

(1,2,3,3)
(3, 1,2,3)
(3,3) $\downarrow$ should not be counted twice.

Probability Approaches –

How to calculate

- Classical
- Frequency

- Classical - $p(E) = \frac{n(E)}{n(S)}$

- Frequency - $p(E) = \frac{f(E)}{\sum f} = r.f(E)$

Classical assumption -

- Sample space is finite.
- Outcomes must be equally-likely.

Classical - Analytical / Theoritical → Exact

⇒ Frequency - Practical → Approximate

	f	p = r.f
A	10	.10
B	20	.20
C	60	.60
D	10	.10
	100	

→ 10% chance of A grade.

- In word "BIRD", how many will start by D.

$\boxed{D \mid \; \mid \; \mid \;} \rightarrow 3! = 6$

- $p(\text{start by D}) = \frac{n(\text{start by D})}{n(S)}$

↳ permutation $= \frac{3!}{4!} = \frac{1}{4}$

- 9 Teachers

5 P, 4 Maths ⇒ $5C_2 \times 4C_1$ → combinatory problem.

↓

3

2 Physics, M

Now prob. of 2P & 1M ⇒

$$p(2P+1M) = \frac{{}^5C_2 \times {}^4C_1}{{}^9C_3} \rightarrow \frac{\text{conditional}}{\text{unconditional}}$$

is known as probability basically.

5. Probability Axioms -

1. $0 \leq p \leq 1$
2. $p(S) = 1$
3. $p(A \cup B) = p(A) + p(B)$ → only when A & B are mutually exclusive

Mutually exclusive - Can't happen together.

i.e., $(A \cap B) = \phi$

So, $p(A \cap B) = 0$.

Hence, $p(A \cup B) = p(A) + p(B)$.

In case of cards - Kings & Hearts. They are not mutually exclusive, because hearts also have king.

This case is called - joint probability.

6. Types of events -

- equally likely
- mutually exclusive
- collectively exhaustive
- Independent.

Eg. $A = \{1,2,3\}$, $B = \{4,5,6\}$ → equally likely

Mutually exclusive -

1. $A \cap B = \phi$
2. $p(A \cap B) = \phi$
3. $p(A \cup B) = p(A) + p(B)$.

$A = \{1,2,3\}$, $B = \{3,4,6\}$ → eq^y lik. & not mutually exclusive

Collectively exhaustive -

1. $A \cup B = S$
2. $p(A \cup B) = 1$

Mutually exclusive & collectively exhaustive -

$$\Rightarrow \boxed{p(A) + p(B) = 1}$$

$$\Rightarrow p(B) = 1 - p(A).$$

Q. A & B are running race. $P(A) = 0.1$ then what is $P(B) = ?$

If "only" A & B → collectively exhaustive.

then $p(B) = 1 - p(A)$
$= 1 - 0.1 = 0.9$.

Take table of dice as previously -

$$E(X) = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + \cdots 6\times\frac{1}{6}$$

$$= \frac{21}{6} = 3.5 \text{ due to symmetry}$$

Now,

$$V(X) = \sum x^2 p(x) - \left(\sum X \cdot p(x)\right)^2$$

$$= \left(40^2 \times 0.1 + 50^2 \times 0.5 + 60^2 \times 0.1 + 70^2 \times 0.3\right) - (56)^2$$

$$V(X) = \_\_$$

Bienayme - Chemyshev Rule -

$$p(\mu - k\sigma \le X \le \mu + k\sigma) \ge 1 - \frac{1}{k^2} \quad ; k > 1$$

like

$$p(56 - 20 \le X \le 56 + 20)$$

$$p(36 \le X \le 76) \ge 1 - \frac{1}{2^2}$$

$$= \frac{3}{4} \rightarrow 75\% \text{ chance at least}$$

$$\boxed{\text{prob. of } k\sigma \text{ is always } \ge 1 - \frac{1}{k^2}}$$

⇒ 75% is chance of not loosing money in market
also known as what is risk?

- mi
rule
Σ

Prob. On
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⇒

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$E(X)$

⇒ 1
an exam

=

2. $p(x=5) = \frac{3}{15} = \frac{1}{5}$ / $\frac{1}{6}$

3. $p(x \geq 5) = \frac{2}{6} = \frac{1}{3}$

4. $p(x \leq 5) = \frac{5}{6}$

5. $p(4 \leq x \leq 6) = \frac{3}{6} = \frac{1}{2}$

in case of dice.

6. $E(X) \rightarrow$ Expected value of X.

$$\boxed{E(X) = \mu_x = \bar{X} = \text{Avg. value of } X}$$

$$\Rightarrow \quad E(X) = \sum x \cdot p(x)$$

7. $V(X) \rightarrow$ Variable values $\rightarrow$ Variance.

$$V(X) = \sum x^2 p(x) - \left(\sum x p(x)\right)^2$$

$$\sigma_x^2 = E(X^2) - [E(X)]^2$$

$$\boxed{V(X) = \sigma_x^2 \Rightarrow \sigma_x = \sqrt{V(X)}}$$

standard deviation.

$$E(g(x)) = \sum g(x) \cdot p(x)$$

$$E(X^2) = \sum x^2 p(x)$$

$$E(X^3) = \sum x^3 \cdot p(x)$$

$$\sum (x^2+x+1) = \sum (x^2+x+1) \cdot p(x) \ldots$$

Independent -

1. $p(A/B) = p(A)$ → Given that B already happend.

conditional probab.

Unconditional probab. - marginal probab.

i.e. cond. prob is same as uncond. prob.

2. $p(B/A) = p(B)$.

3. $p(A \cap B) = p(A) \cdot p(B)$.

i.e. if B is happening, it does not effect prob. of A.

As, we know, $\boxed{\begin{array}{l} p(A \cap B) = p(A) * P(B/A) \\ p(A \cap B) = P(B) * P(A/B) \end{array}}$

In general

∴ cond. prob. = uncond. prob, then,

$$p(B) = p\left(\frac{B}{A}\right)$$

Eg.

	A	B
dice	6	6
day	I	II

⇒ independent case.

$$\Rightarrow p\left(\frac{6}{I} \cap \frac{6}{II}\right) = p\left(\frac{6}{I}\right) \times p\left(\frac{6}{II}\right)$$

$$= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}.$$

Again, as, if

A	A
6	6
I	II

→ Not independent case.

Then $\Rightarrow p\left(\frac{A}{I}\right) * p\left(\frac{A}{II} \Big/ \frac{A}{I}\right)$

$$= \frac{4}{52} \times \frac{3}{51}$$

Eg. When take 2 cards with replacement then, becomes independent.

$$\Rightarrow p\left(\frac{A}{I}\right) * p\left(\frac{A}{II} \Big/ \frac{A}{I}\right)$$

$$= \frac{4}{52} \times \frac{4}{52}$$

$p(A \cup B) = A + B - (A \cap B)$

$p(A \cap B) = P(A) \times P(B/A)$

$$\boxed{\begin{array}{l} \Rightarrow p(A \cap B) \leq P(A) \\ \Rightarrow P(A \cup B) \leq p(A) + P(B) \\ \Rightarrow p(A \cap B) \leq p(B/A) \\ \quad P(A/B) \geq P(A \cap B) \end{array}}$$

Rules of probability –

1. $P(A \cup B) = p(A) + p(B) - p(A \cap B)$
2. $p(A \cap B) = P(A) \cdot p(B/A) = p(B) \cdot P(A/B)$
3. $P(A^c) = 1 - P(A)$ → Complementary event

$\Rightarrow A \cup A^c = S$

$p(A^c \cap B^c) = 1 - p(A \cup B)$

$p(A^c \cup B^c) = 1 - p(A \cap B)$

Q.

What

Q. 2

p

let

$\Rightarrow$

Q. $P(A) = 0.1$, $P(B) = 0.2\cancel{8}$, $P(A \cap B) = 0.05$

what is neither condition case?

$$P(A^c \cap B^c) = P((A \cup B)^c)$$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - [0.1 + 0.2 - 0.5]$$

$$= 0.75 \text{ neither of them prob.}$$

Q. 2 dice thrown either of them is not 6.

$$P\left(\underset{\text{I}}{6}\times \cup \underset{\text{II}}{6}\times\right) = ?$$

let us assume 1^{st} for, $P\left(\underset{\text{I}}{6}\right)$

$$\Rightarrow P\left(\underset{\text{I}}{6^c} \cup \underset{\text{II}}{6^c}\right) = P\left(\left(\underset{\text{I}}{6} \cap \underset{\text{II}}{6}\right)^c\right)$$

$$= 1 - P\left(\underset{\text{I}}{6} \cap \underset{\text{II}}{6}\right)$$

$$= 1 - \left(\frac{1}{6} \times \frac{1}{6}\right)$$

$$= 1 - \frac{1}{36} = \frac{35}{36}$$

is prob. that neither ~~8~~ comes with 6.

$$\boxed{\begin{array}{l} A^c \cap B^c \rightarrow \text{NOR} \\ A^c \cup B^c \rightarrow \text{NAND} \end{array}}$$

Q. Let us take all possible words from, "MISSISSIPPI" ?

$p(\text{start without } S) = 1 - p(\text{start with } S)$

$= 1 - \dfrac{n(\text{start with } S)}{n(S)}$

M – 1
I – 4
S – 4
P – 2

11

$\Rightarrow \dfrac{11!}{1!\,4!\,4!\,2!} = n(S)$.

$\Rightarrow \quad = 1 - \dfrac{\dfrac{10!}{4!\,3!\,2!}}{\dfrac{11!}{2!\,4!\,4!}}$

$\underbrace{\boxed{S \,|\,\,|\,\,|\,\,|\,\,|\,\,|\,\,|\,\,|\,\,|\,\,|\,\,}}_{10!}$ → M – 1, I – 4, S – 3, P – 2; total 10

$\dfrac{10!}{1!\,4!\,3!\,2!}$

$= 1 - \dfrac{4}{11} = \dfrac{7}{11}$

4. $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ → when event already happend !

$P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

Q. $p(\text{rain}) = 0.1$

$p(\text{humid} \,\&\, \text{rain}) = 0.05$

What is $p(\text{humid}/\text{rain}) = ?$

Solⁿ-

$p(\text{humid}/\text{rain}) = \dfrac{p(H \cap R)}{P(R)}$

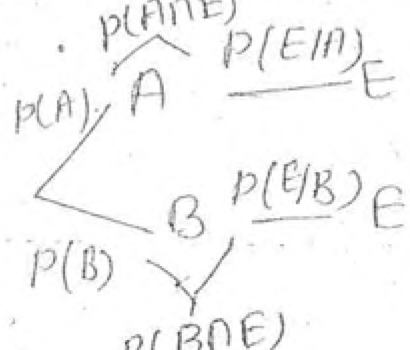
humid on day which is raining

$= \dfrac{0.05}{0.1} = \dfrac{1}{2}$

$p(\text{humid} \,\&\, \text{Rain}) = p(\text{humid}) \times P(\text{Rain}/\text{humid})$

since p(humid) is not given, so, we can't use this further more:

5. Where both can happen at some particular day.



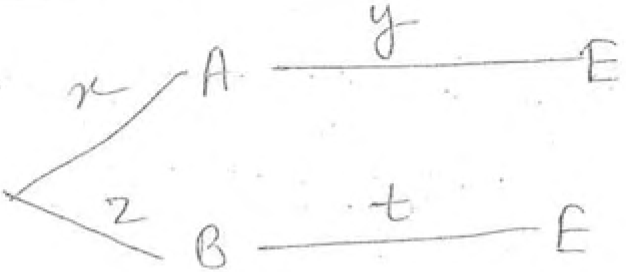
Rules of total probability -

$$p(E) = p(E \cap A) + p(E \cap B)$$

as, E can happen with A as well as B.

$$= p(A) * p(E/A) + p(B) * P(E/B)$$

In general →



$$\Rightarrow \boxed{p(E) = xy + zt}$$

6. Baye's Theorem - Given that E is already happend. Then have to calculate P(A) or P(B) in this case.

i.e. given P(A/E) & P(B/E).

As, $P(A/E) = \dfrac{P(A \cap E)}{P(E)}$

P(E) can be obtain by "Rule of total prob".

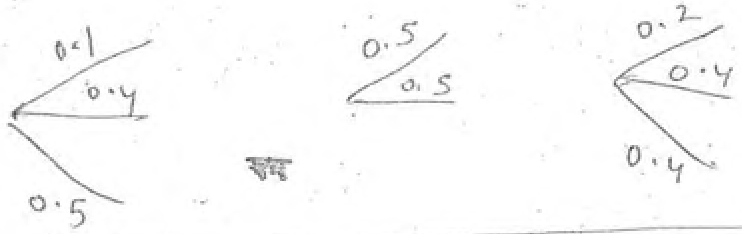
$$\Rightarrow P(A/E) = \frac{xy}{xy + zt}$$

Similarly,

$$P(B/E) = \frac{P(B \cap E)}{P(E)} = \frac{zt}{\underbrace{xy + zt}_{\text{Total prob.}}}$$

always =1

$$\boxed{P(A/E) + P(B/E) + P(C/E) = 1}$$



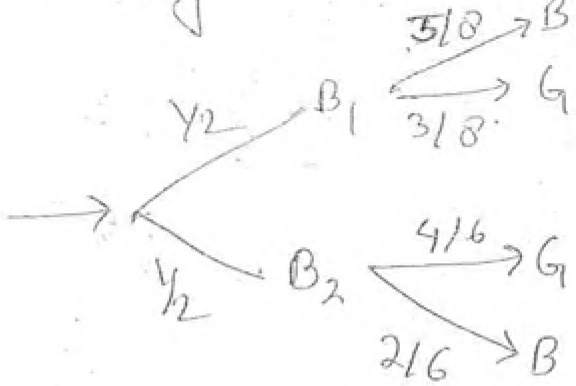
Problems — (on Rule no. 5 & 6)

Prob.

Bag 1	Bag 2
5B, 3G	2B, 4G

pick one bag at random, what is $P(G) = ?$

Sol^n -



$$\Rightarrow \quad P(G) = \frac{1}{2} \times \frac{3}{8} + \frac{1}{2} \times \frac{4}{6}$$

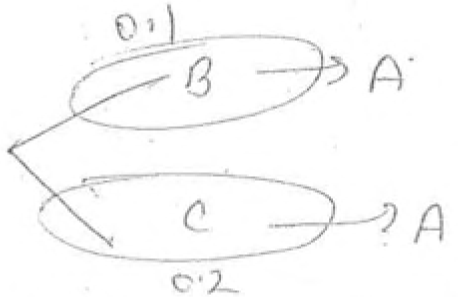
$$= \frac{1}{2}\left(\frac{3}{8} + \frac{4}{6}\right)$$

Q. he founds, ball is green, what is chance it is from bag 1.

$$\Rightarrow P\left(\frac{B_1}{G}\right) = \frac{P(B_1 \cap G)}{P(G)} = \frac{\frac{1}{2} \times \frac{3}{8}}{\frac{1}{2} \times \frac{3}{8} + \frac{1}{2} \times \frac{4}{6}}$$

$=$

Q.



$P(A \cap B) = 0.1$, $P(A \cap C) = 0.2$

what is, $P(B/A) = ?$

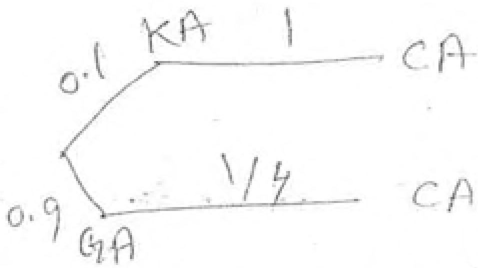
Sol^n -

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{0.1}{P(A)}$$

$\because P(A) = 0.1 + 0.2 = 0.3$

$$P(B/A) = \frac{0.1}{0.3} = \frac{1}{3}$$

Prob.



0.1 Knows Ans.

0.9 Guess Ans.

what is prob. he knows correct ans? as if ques has 4 obtions.

$\Rightarrow$ Find out $P(KA/CA) = ?$

$$\Rightarrow P(KA/CA) = \frac{P(KA \cap CA)}{P(CA)}$$

$$= \frac{0.1 \times 1}{0.1 \times 1 + 0.9 \times \frac{1}{4}} = \frac{0.1}{0.1 + .225}$$

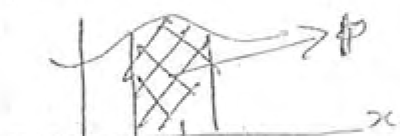
$$= \frac{0.1}{0.325} = \frac{100}{325} = \frac{40}{65} = \frac{8}{13}$$

Probability distribution - dice $\rightarrow X = \{1,2,3,4,5,6\}$

Random variables
- Discrete $\rightarrow$ One value from set of values
- Continuous $\rightarrow$ takes one value from range of values. like weight $\rightarrow 0 \leq X \leq 100$ gm

Discrete distributions (Table form)
- General
- Binomial
- Hypergeometric
- Poission

Continuous distri. (Curve form)
- General
- Uniform
- Normal, standard normal
- Exponential



Area $= p(x_1 \leq x \leq x_2)$

Discrete distributions - We get,

x	1	2	3	4	5	6
$p(x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

whereas in continuous prob.

$$p(x=2) = \frac{1}{6} \rightarrow \text{distrete}$$ 2.

$$p(x=2) = 0 \rightarrow \text{Conti.}$$ 3

4.

1) General Distri. -

X	40	50	60	70
p(x)	0.1	0.5	0.1	0.3

5.

$E(X) = 40\times0.1 + 50\times0.5 + \dots$
$= 56 \quad = 4+25+6+21$

6.

in dice -

X	2	3	4	- - -	10	11	12
p(x)	1/36	2/36	3/36		3/36	2/26	1/36

prob. distri. table.

it is require put all values of x. 7.

1. $\sum p(x) = 1$

eg.

X	3	4	5	6	7
p(x)	k	2k	3k	4k	5k

if above is prob. distri table, then k = ?

Sol^n - $\therefore \sum p(x) = k + 2k + 3k + 4k + 5k = 1$

$$15k = 1$$

$$\boxed{k = \frac{1}{15}}$$ Ans

and $p(x=5) = \frac{3}{15} = \frac{1}{5}$

$$\mu - k\sigma = 36 \qquad \left\{ 1 - \frac{1}{k^2} \text{ always} \right.$$

$$56 - k \times 10 = 36$$

$$k = 2$$

- minimum probability we can find by this rule.

$$\Sigma(x^2+1) = \Sigma(x^2+1) \cdot p(x)$$

$$= 2 \times \frac{1}{6} + 5 \times \frac{1}{6} + 10 \times \frac{1}{16} + 17 \times \frac{1}{6} + 27 \times \frac{1}{6} + 37 \times \frac{1}{6}$$

$$=$$

Prob. One is keep on tossing coin. What is the expected no. of toss, that he gets 2 head continuous and stops the game?

2 - HH ✓
HT ×
TH ×
TT ×

3 - HHH ✓
HHT ✓
HTH
HTT
THH ✓
THT
TTH
TTT

$\Rightarrow$

x	2	3	4	5 - - -
$p(x)$	1/4	1/8	1/8	

$$E(x) = \Sigma\, x \cdot p(x)$$

$$= \frac{1}{4} \times 2 + \frac{1}{8} \times 3 + \frac{1}{8} \times 4 + \cdots$$

$\longrightarrow$ calculate in this manner!

$\Longrightarrow$ let $S = \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots$

an example,

$$\frac{1}{2}S = \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{64} + \cdots$$

$$\Rightarrow S - \frac{1}{2}S = \frac{1}{2}S = \frac{2}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

$$= \frac{1}{2} + \frac{\frac{1}{8}}{1 - \frac{1}{2}} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow S = \frac{6}{4} = \frac{3}{2} = 1.5$$

So, on avg. person will play 1.5 tosses, to get 2 heads.

Prob.

x	2	3	4	5	6	- -	12
$p(x)$							

$$V(x) = \sum x^2 p(x) - \left(\sum x\, p(x)\right)^2$$

Variance of x = ? J.N.U.

Prob.

$$E(ax+b) = a \cdot E(x) + b$$

let $E(x) = 10$, $E(2x+5) = ?$

$$\Rightarrow E(2x+5) = 2 \cdot E(x) + 5$$
$$= 2 \times 10 + 5$$
$$= 25$$

only in case of linear funⁿ.

let if $E(ax^2+b) = \sum (ax^2+b) \cdot p(x)$ → calculate this, not direct formula.

let if $E(ax_1 + bx_2 + c) = a \cdot E(x_1) + b \cdot E(x_2) + c$

$E(x_1) = 5$

$E(x_2) = 2$

What is; $E(3x_1 + 5x_2 + 3) = ?$

$$\Rightarrow 3 \times 5 + 5 \times 7 + 3$$
$$= 53$$ → due to linearity

$$\Rightarrow \quad \mu_{ax+b} = a \cdot \mu_x + b$$

$$\mu_{ax_1+bx_2+c} = a\mu_{x_1} + b\mu_{x_2} + c$$

$a \to$ Scaling
$b \to$ ~~Origin's~~ shifting of origin

$\Rightarrow$ $\pm \Rightarrow$ Shifting
$* \Rightarrow$ Scaling

$$\Rightarrow V(ax+b) = a^2 \cdot V(x)$$

* Variance does not effected by shifting.

$$\sigma^2_{ax+b} = a^2 \cdot \sigma^2_x$$

$\Rightarrow$ standard deviation also does not effected by shifting

$$\Rightarrow \sigma_{ax+b} = a\sigma_x$$

let $E(X) = 50$, $V(X) = 100$

$$E(3x+5) = 3 \times 50 + 5 = 155$$

$$V(3x+5) = 3^2 \times 100 = 900$$

$$\sigma_{3x+5} = 3 \cdot \sigma_x = 30.$$

$$\Rightarrow V(ax_1 + bx_2 + c) = a^2 V(x_1) + b^2 V(x_2) + 2ab \cdot Cov(X_1, X_2)$$

Covariance

let

$V(x_1) = 100$, $V(x_2) = 200$, $Cov(x_1, x_2) = 10$

To find out, $V(3x_1 + 4x_2) = ?$

$$\Rightarrow V(3x_1 + 4x_2) = 3^2 \times 100 + 4^2 \times 200 + 2 \times 3 \times 4 \times 10$$

$$= 900 + 3200 + 240$$

$$= 4340.$$

$$\boxed{\text{if } x_1 \;\&\; x_2 \text{ are indep.} \Rightarrow Cov(x_1, x_2) = 0}$$

$Cov(x_1, x_2)$ measures the dependency b/w x_1 and x_2.

$\Rightarrow$ larger value $Cov(x_1, x_2) \Rightarrow$ larger dependency.

$$\Rightarrow \boxed{-\infty \leq Cov(x_1, x_2) \leq +\infty}$$

* if $Cov(x_1, x_2) = +100 \rightarrow$ direct dependent
  ↓ both moving in same dir^n.

* $-$ve value $\Rightarrow$ inverse dep. $\rightarrow$ in opposite dir^n

* 0 value $\rightarrow$ No connection b/w them.

$$\boxed{Cov(x_1, x_2) = E(XY) - E(X) \cdot E(Y)}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$\Rightarrow \boxed{Cov(X, X) = V(X)}$$ $\rightarrow$ Thats why called covariance.

Two random variable problem –

$Y \backslash X$	0	1	2	$p(Y)$
0	0.1	0.2	0.05	0.35
1	0.3	0.1	0.25	0.65
$p(X) \rightarrow$	0.4	0.3	0.30	

→ Joint prob. distri table.

→ marginal prob.

⇒ Ques. may be ask like –

1. $p(X=1 \cap Y=0)$
2. $p(X \geq 1 \cap Y=1)$
3. $p(X=1 \mid Y=0)$
4. $E(X)$
5. $E(Y)$
6. $V(X)$ & $V(Y)$
7. $cov(X,Y)$
8. $E(X/Y=1)$

⇒ How to answers the above –

1. $p(X=1 \cap Y=0) = 0.2$
2. $p(X \geq 1 \cap Y=1) = 0.1 + 0.25 = 0.35$
3. $p(X=1 \mid Y=0) = \dfrac{p(X=1 \cap Y=0)}{p(Y=0)}$

$$= \frac{0.2}{0.35} = \frac{4}{7}$$

(Uncondi → marginal)

⇒

X	0	1	2
$p(x)$	0.4	0.3	0.3

Y	0	1
$p(Y)$	0.35	0.65

$= 0\times0.1 + 1\times0.3 + 2\times0.03 = 0.3 + 0.6 = 0.9$

4. $E(X) = 0.9$

5. $E(Y) = 0.65$

6. $V(X) = E(X^2) - \Sigma(X \cdot E(X))^2$

$= 0.69$

$= 1.5 - (0.9)^2$

$\text{conditional} = \dfrac{\text{Inside}}{\text{Marginal}}$

7. $V(Y) = 0.65 - (0.65)^2$

$= 0.65 - 0.4 \simeq 0.25$ (Approx.)

8. $\text{cov}(X,Y) = E(X;Y) - E(X)\cdot E(Y)$

$E(X,Y) = \Sigma\, XY \cdot P(X \cap Y)$

$= 1\times1\times0.1 + 2\times1\times0.25$

$= 0.1 + 0.5$

$= 0.6$

$\Rightarrow \text{cov}(X,Y) = 0.6 - 0.9\times0.65$

$= 0.015.$

almost indep. or dep.
we can't say this
accurately.

9.
Condnue
Give
mea
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⇒ Wh
10.
X
p(x)

9. $\boxed{r(X,Y) = \dfrac{cov(X,Y)}{\sigma_X \cdot \sigma_Y}}$ Coincidence

; $-1 \le r \le +1$

Gives how much dependency.
measures linear dependence actually.

$$= \frac{0.015}{\sqrt{0.69} \times \sqrt{0.25}}$$

When r is close to +1 → highly dependent

⇒ When r is close to −1 → least dep.

10. $E(X/Y=1)$ → conditional expectation

X	0	1	2
p(X)	0.4	0.3	0.3

X	0	1	2
p(X/Y=1)	$\frac{0.3}{0.65}$	$\frac{0.1}{0.65}$	$\frac{0.25}{0.65}$

↓

$$p(X=0/Y=1) = \frac{P(X=0 \wedge Y=1)}{P(Y=1)} = \frac{0.3}{0.65}$$

$$E(X|Y=1) = \sum x \cdot P(X|Y=1)$$

$$= \frac{1 \times 0.1}{0.65} + \frac{2 \times 0.25}{0.65}$$

$=$

$\forall\ p(X \cap Y) = p(X) \cdot p(Y)$

only then, we can conclude, that

x & y are independent.

$$\boxed{\text{if } p(X \cap Y) \neq P(X) \cdot P(Y) \Rightarrow \text{Dependent.}}$$

Binomial Distribution -

n trials, x success, $p(\text{success}) = p$.

- n & p are called parameters.
- x is called random variable.

$$\boxed{p(X == x) = {}^nC_x\, p^x (1-p)^{n-x}}$$

Prob- 10 dice, 3 sixes, what prob. of success?

$n = 10$, $p(6) = \frac{1}{6}$, $X = 3$,

$$p(X=3) = {}^{10}C_3 \cdot \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7$$

Prob- let 10 coins, 3 heads?

$$p(X=3) = {}^{10}C_3 \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^7$$

Vari

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Prob 10

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* D

Variations - always; $x = 0, 1, 2, \ldots$

$$p(x \geq 2) = p(x=2) + p(X=3) + \cdots + p(X=6)$$

10 dice
at least two sixes

$$= 1 - p(x \leq 1)$$

$$= 1 - (p(x=0) + p(x=1))$$

$$= 1 - \left[{}^{10}C_0 \left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^{10} + {}^{10}C_1 \left(\frac{1}{6}\right)^1 \cdot \left(\frac{5}{6}\right)^9 \right]$$

$$p(x \leq 2) = p(x=0) + p(x=1) + p(x=2)$$

$$\boxed{\begin{aligned} &\text{In binomial,} \quad E(x) = n \cdot p \\ &V(x) = np(1-p) \end{aligned}}$$

Prob 10 dice, what expected no. of 6.

$n = 10$, $p(6) = \frac{1}{6}$; $E(x) = n \cdot p$
$= 10 \times \frac{1}{6} = 1.66$

$\Rightarrow$ 1.66 of them will be 6. This is on avg. value.

$$V(x) = np(1-p)$$

$$= 10 \times \frac{1}{6} \times \frac{5}{6} = \frac{50}{36} = \frac{25}{18} =$$

Standard deviation, $\sigma_x = \sqrt{\frac{50}{36}} = \frac{5\sqrt{2}}{6}$

* Dice & coin always follows binomial distri.

Assumption for Binomial –

1. Success & failure is always there.

Prob: $P(A) = 0.1$

$P(B) = 0.2$

$P(C) = 0.7$

What is prob. that out of 10, 4 will vote A.

$\Rightarrow \quad n = 10, \; x = 4, \; P(A) = 0.1$

So, using theorem,

$= {}^{10}C_4 \cdot (0.1)^4 \cdot (0.9)^6$

$=$

If prob. (Not vote for A) = ?

$= {}^{10}C_4 \; (0.9)^4 \cdot (0.1)^6$

$=$

2. p should be same from trial to trial.

Prob: 10 cards, 3 Aces, = ?

$\Rightarrow P(3 \text{ Ace}) = {}^{10}C_3 \times \left(\frac{4}{52}\right)^3 \cdot \left(\frac{48}{52}\right)^7$

is wrong.

But if with replacement, then it will be correct.

3. Should not be used, when we use sampling for a FINITE population WITHOUT replacement.

i.e; in infinite ... can use it with no problem.

4. Trial should be statistically independent. i.e result of trial should not effect on consequent trial's result.

Prob: If $\mu = 50$, $\sigma^2 = 25$, $p(X=2) = ?$

$\Rightarrow$ $\mu = np$

$\sigma^2 = np(1-p)$

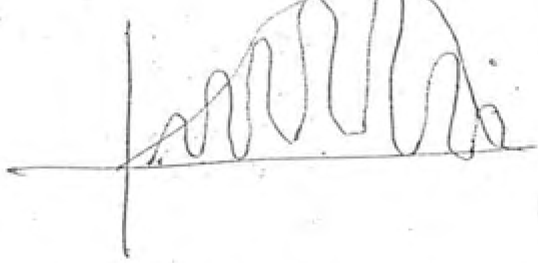
Divide both eqns, $= 1-p = \frac{25}{50} = \frac{1}{2}$

$p = \frac{1}{2}$ $\longrightarrow$ it will remain same.

$50 = \frac{n}{2} \longrightarrow n = 100$

So, $= {}^{100}C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{98} = p(X=2)$

Prob: Binomial distri. for no. of 6 is obtained. What is shape of bin. distri- p(no of sixes).

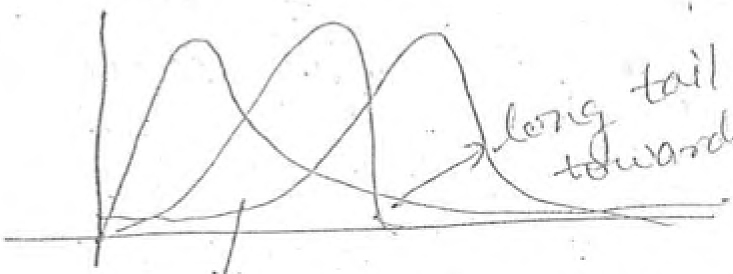


A: Symmetric
B. +ve skew
C. -ve skew
D. None.

$\Rightarrow$ Ans. It will be symmetric shape.

Prob- In case of dice, what will shape?

$\Rightarrow$ +vely skewed curve.

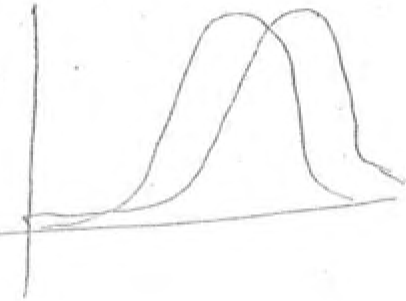


long tail is towards +ve axis.

-vely skewed curve.

Prob if getting no. ≥ 2 is success. Then prob. distri of what shape?

$p(\geq 2) = \frac{5}{6}$



Ans - -vely skewed.

Symmetric $\rightarrow$ $p = q = \frac{1}{2}$

+ve skew $\rightarrow$ $p < q$, $p < \frac{1}{2}$, $q > \frac{1}{2}$

-ve skew $\rightarrow$ $p > \frac{1}{2}$ $q < \frac{1}{2}$; $p > q$

$\Rightarrow$ Remember it!

-ve : mode $\geq$ Median $\geq$ Mean

+ve : Mean $\geq$ Median ~~Mode~~ $\geq$ Mode

Symmetric : Mean = Med = Mode.

$\Rightarrow$ $\boxed{\text{Max. freq} \rightarrow \text{Mode}}$

eg- Mean = 60, Median = 50, Mode = 40,

$\Rightarrow$ ∵ mode is lowest.

Hence +vely skewed distribution.

Hypergeometric Distri -

i.e; finite population without replacement.

Eg - 52 cards, out of which 4 A, 48 NA.

52

4A 48 NA

10 without replacement.

3A 7 NA

$\Rightarrow$ if $X = 3$,

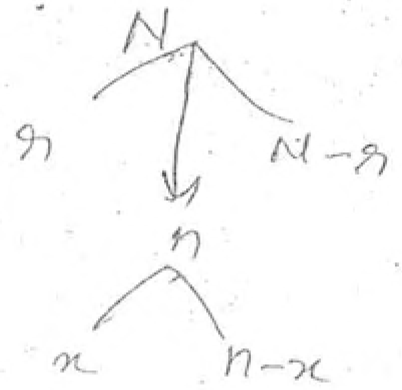
$$p(X=3) = \frac{{}^{4}C_3 \times {}^{48}C_7}{{}^{52}C_{10}} = \frac{n(E)}{n(S)}$$

$$p(X \geq 3) = 1 - p(X \leq 2)$$

$$= 1 - \left[\frac{{}^{4}C_0 \cdot {}^{4}C_{10}}{{}^{52}C_{10}} + \frac{{}^{4}C_1 \times {}^{48}C_9}{{}^{52}C_{10}} + \frac{{}^{4}C_2 \; {}^{48}C_8}{{}^{52}C_{10}} \right]$$

$$\boxed{p(X=x) = \frac{{}^{r}C_x \; {}^{N-r}C_{n-x}}{{}^{N}C_n}}$$

$\Rightarrow$



$\rightarrow$ $\boxed{E(X) = n \cdot \frac{r}{N}}$

expected no. of success.

Poission distri -

$$\boxed{P(X=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}} \; ; \; x = 0, 1, 2, \ldots \infty$$

- Pure poission
- Binomial poission.

① Case - $\lambda = \alpha \cdot \Delta t$

② case - $\lambda = np$.

λ – is avg. no. of success in Observation period.

Binomial $\boxed{n, p}$ $\boxed{x}$

Hypergeo. $\boxed{n, r, N}$, $\boxed{x}$

Poission $\boxed{\lambda}$ $\boxed{x}$

α – is avg. no. of success per unit time.

Δt – Observation period.

* In poission problem only, time factor is present.

Prob Let $\alpha = 40$ A/hrs., $\frac{1}{2}$ hrs, $p(X=10)=?$

$\Rightarrow$ Here, $\alpha = 40$, $\Delta t = \frac{1}{2}$

So, $\lambda = \alpha \cdot \Delta t = 40 \times \frac{1}{2} = 20$

$$\lambda = 20 \text{ A}$$

Now, $p(X=10) = \dfrac{e^{-20} \cdot 20^{10}}{10!}$

$$p(X \geq 1) = 1 - p(X=1)$$
$$= 1 - \frac{e^{-20} \cdot 20^{0}}{0!}$$
$$= 1 - e^{-20}$$

$$p(X \geq 2) = 1 - [p(X=0) + p(X=1)]$$
$$= 1 - \left[\frac{e^{-20} \cdot 20^{0}}{0!} + \frac{e^{-20} \cdot 20^{1}}{1!}\right] =$$

$$\boxed{\begin{array}{l} E(X) = \lambda \\ V(X) = \lambda \end{array}}$$ → Here in case of poission distri. always holds.

Binomial poission -

Prob. Manufacturer is 10,000 tracktors. and on avg. $\frac{1}{2000}$ tracters are defective. Then, what is prob. 4 tracters becomes defective?

Prob if 10 tracters in year, what is P(10) = ? is defective

$$P(X \geq 10) = 1 - P(X \leq 9)$$
$$=$$

Prob. $P(X=2) = ?$

$$= {}^{10,000}C_2 \left(\frac{1}{2000}\right)^2 \left(\frac{1999}{2000}\right)^{9998}$$

$$=$$

*Whenever n is large & p is less, then it creates problem in calculation.

Then we approximate binomial into poission distri.

Then, it comes out as,

$\Rightarrow \lambda = np$

$= 10000 \times \frac{1}{2000} = 5$

then, $p(X=2) = \frac{e^{-5} \cdot 5^2}{2!} =$

*When n & p ~~also~~ both large, we use

Normal distribution.

$\hookrightarrow$ continuous distri.

Recurrance. Relations

Two types
- Determinate order
  - Linear
    - LRCC
      - Homogen
      - Inhom
    - LRVC
  - Non-linear
- Indeterminate order

LRCC - Linear recur. constt coeff

LRVC - " " variable coeff.

Determinate order -

$$a_n = 3a_{n-1} + 4a_{n-2} + 5 \rightarrow 2^{nd} \text{ order}$$

$$a_n = 4a_{n-1} + 10 \rightarrow 1^{st} \text{ order}$$

Indeterminate order -

$$a_n = 4a_{n/2} + 5$$

i.e, order will change according to size of problem.

so, we convert indet into det order by using

two methods -

- Exact solutions
- Master's Theorem

$$a_n = 3a_{n-1}^2 + 4a_{n-2} + 5 \rightarrow \text{Non-linear}$$

$$a_n = 3a_{n-1} + 4a_{n-2} + n^2 \rightarrow \text{Linear}$$

Nonlinear can be converted into Linear prob.

Similarly, $a_n = n \cdot a_{n-1}$

LRVC $\rightarrow$ LRCC.

RCC ⟨ Homogeneous — $a_n - \ldots a_{n-1} \ldots a_{n-2}$
Inhomogeneous → $a_n + 3a_{n-1} = n^2$

RVC

Right side can be polynomial / power fun^n or combination of both, in case of inhomogeneous.

n^2+n+2 ; 4^n

Eg- $a_n - 5a_{n-1} + 6a_{n-2} = 0$

⇒ By characteristic roots method,

let, $a_n = f(n)$.

make a_{n-2} as 1. Then, eq^n become,

$$t^2 - 5t + 6 = 0 \rightarrow \text{called char. } eq^n.$$

⇒ Solving, we get, $t = 2$ and 3.

Then, $a_n = c_1 \cdot 2^n + c_2 \cdot 3^n$

No. of order → No. of constt.

Eg- $a_n - 5a_{n-1} + 6a_{n-2} = 0$; $a_0 = 1$, $a_1 = 2$

⇒ $a_0 = 1$, $a_1 = 2$ → are initial conditions.

By this, ~~b~~ we can get value of constt.

$$a_0 = c_1 \cdot 2^0 + c_2 \cdot 3^0 = 1$$
$$a_1 = c_1 \cdot 2^1 + c_2 \cdot 3^1 = 2.$$

$$c_1 + c_2 = 1$$
$$2c_1 + 3c_2 = 2$$

$c_2 = 0$ & $c_1 = 1$

⇒ $a_n = 1 \cdot 2^n + 0 \cdot 3^n$

$$\boxed{a_n = 2^n}$$ Ans.

When roots are same -

$t^2 - 4t + 2 = 0$

$t = 2, 2$

$$a_n = c_1 \cdot 2^n + c_2 \cdot n \, 2^n$$

$$a_0 = c_1 \cdot 2^0 + c_2 \cdot 0 \cdot 2^0 = 1$$

$$a_1 = c_1 \cdot 2^1 + c_2 \cdot 1 \cdot 2^1 = 2$$

$$\Rightarrow \quad c_1 = 1 \,, \quad 2c_1 + 2c_2 = 2 \Rightarrow c_2 = 0.$$

if three roots are same -

$$a_n = c_1 \cdot 2^n + c_2 \cdot n \cdot 2^n + c_3 \cdot n^2 \cdot 2^n$$

Inhomogeneous -

$$\Rightarrow \quad a_n - 5a_{n-1} + 6a_{n-2} = 5 \quad ; a_0 = 1 , a_1 = 2$$

$$a_n = \underbrace{a_n^H}_{\text{Homogeneous sol}^n} + \overbrace{a_n^P}^{\text{Particular sol}^n}$$

firstly take it as homogeneous eq^n i.e. put 0 at R.H.S. we will get,

$$a_n^H = c_1 \cdot 2^n + c_2 \cdot 3^n$$

Now solve for particular sol^n,

Trial: $a_n^P = d$

So, final sol^n becomes,

	R.H.S	Trial a_n^P
poly.	c	d
	$c_0 + c_1 n$	$d_0 + d_1 n$
	$c_0 + c_1 n + c_2 n^2$	$d_0 + d_1 n + d_2 n^2$
power	$c \cdot a^n$	$d \cdot a^n$
poly × power	$(c_0 + c_1 n) a^n$	$(d_0 + d_1 n) a^n$

2,2

like, $n \cdot 2^n \rightarrow (d_0 + d_1 n) 2^n$

- Terms of a_n^H and a_n^P should be seperate completely always.

If let, $a_n^h = c_1 \cdot 1^n + c_2 \cdot 1^n$ called Double collision;

0.

$\Rightarrow \quad a_n^P = d \cdot n^2$

Since, $\quad a_n^P = d \rightarrow$

$$d - 5d + 6d = 5$$

$$2d = 5$$

$$d = 5/2$$

$a_1 = 2$

$$a_n = c_1 \cdot 2^n + c_2 3^n + 5/2$$

So, $\quad a_0 = c_1 + c_2 - 5/2 = 1$

$\quad a_1 = 2c_1 + 3c_2 + 5/2 = 2$

) at

Eg. $\quad a_n - 5a_{n-1} + 6a_{n-2} = 3n \quad ; a_0 = 1, a_1 = 2$

Trial case, $\quad a_n^P = d_0 + d_1 n$

$$= d_0 + d_1 \cdot n - 5(d_0 + d_1(n-1)) + 6(d_0 + d_1(n-2))$$

$$= 3n$$

$$= (d_0 - 5d_0 + 5d_1 + 6d_0 - 12d_1) - (dn - 5d_1 n + 6d_1 \cdot n) = 3n$$

$$= 2d_0 - 7d_1 \quad \leftarrow (1)$$

$$= 2d_1 = 3 \quad — (2)$$

$$d_1 = 3/2$$

al

n

$+ d_2 n^2$

$) a^n$

$2d_0 = \frac{21}{2}, \therefore d_0 = \frac{21}{4}$

$\Rightarrow$ So, $\boxed{a_0 = c_1 \cdot 2^n + c_3 \cdot 3^n + \frac{21}{4} + \frac{31}{2} \cdot n}$

Ans.

Power function:

$$a_n - 5a_{n-1} + 6a_{n-2} = 2^n$$

$$\Rightarrow \quad a_n = a_n^H + a_n^P$$

$$a_n^n = c_1 \cdot 2^n + c_2 \cdot 3^n$$

$$a_n^P = dn \cdot 2^n$$

$$a_n - 5a_{n-1} + 6a_{n-2} = 5^n$$

$$\Rightarrow \quad a_n^H = c_1 \cdot 2^n + c_2 \cdot 3^n$$

$$a_n^P = d \cdot 5^n$$

$$\frac{d \cdot 5^n - 5 \cdot d \cdot 5^{n-1} + 6 \cdot d \cdot 5^{n-2}}{5^{n-2}} = \frac{5^n}{5^{n-2}}$$

$$\Rightarrow \quad 25d - 25d + 6d = 25$$

$$\Rightarrow \quad 6d = 25$$

$$d = \frac{25}{6}$$

So, we get,

$$a_n^P = \frac{25}{6} \times 5^n$$

So, complete answer

$$a_n = c_1 \cdot 2^n + c_2 \cdot 3^n + \frac{25}{6} \cdot 5^n$$

NOTE – When multiply by n, it will effect all others.

$$a_n - 5a_{n-1} + 6a_{n-2} = 5^n + 2\cdot 7^n$$

$$\Rightarrow \quad a_n = a_n^H + a_n^{P_1} + a_n^{P_2}$$

Now put only 5^n, then solve for d.

Then put $2\cdot 7^n$, then solve for d.

So, $\quad a_n^{P_2} = d\cdot 7^n = \frac{25}{6}\cdot 5^n$

[Only LRCC is possible] → only possible.

LRVC → LRCC

$$n\cdot a_n - n\cdot a_{n-1} + a_{n-2} = 5$$

$$n\cdot a(n) - n\cdot a(n-1) + a(n-2) = 5$$

Let $b_n = n\cdot a_n$ so, we have,

$$b_{n-1} = (n-1)a_{n-1}$$

$$n\,a_n - n\,a_{n-1} + a_{n-1} = 0$$

$$n\cdot a_n + n\cdot a_{n-1} = 5$$

$$b_n - b_{n-1} = 5 \rightarrow \text{inhomo.}$$

$$b_n - b_{n-1} = 0 \Rightarrow t - 1 = 0 \quad t = 1$$

So, $b_n^H = c_1 1^n = c_1 \Rightarrow$

$$\boxed{b_n^H = c}$$

$$b_n - b_{n-1} = 5$$

$$b_n^p = dn = 5n$$

$$dn - d(n-1) = 5$$

$$d = 5$$

So, $b_n = b_n^H + b_n^P$

$= c + 5n$

So, as $b_n = n \cdot a_n = \frac{c+5n}{n}$ let $a_1 = 2$

$$a_1 = \frac{c+5}{1} = 2 \rightarrow c = -3$$

$$a_n = \frac{-3+5n}{n} = 5 - \frac{3}{n}$$

$\Rightarrow$ ∵ condition is given on a_n not on b_n.

Non-Linear → Linear -

$$a_n^2 - 3a_{n-1}^2 = 5 \quad ; \; a_0 = 2$$

$\Rightarrow$ let $b_n = a_n^2$

$$\Rightarrow \quad b_n - 3b_{n-1} = 5$$

$$t - 3 = 0.$$

$$t = 3,$$

So, $b_n = b_n^H + b_n^P$

$$b_n^H = c \cdot 3^n$$

$$b_n^P = d.$$

$$d - 3d = 5$$

$$d = -5/2$$

$$b_n = b_n^h + b_n^p = c \cdot 3^n - 5/2$$

As, $a_n = \pm\sqrt{c \cdot 3^n - 5/2}$

(+ only because initial cond. is +ve)

$= 2$

So, $a_0 = \sqrt{c - 5/2} = 2$

$$c - 5/2 = 4$$

$$c = 4 + \frac{5}{2} = \frac{13}{2}$$

So, $a_n = \sqrt{\frac{13}{2} \cdot 3^n - \frac{5}{2}} = \sqrt{\frac{13 \cdot 3^n - 5}{2}}$

Indeterminate order

$$a_n = 5a_{n/3} + 7$$

let $n = 3^k$, $n/3 = 3^{k-1}$

$$a_{3^k} = 5 \cdot a_{3^{k-1}} + 7$$

let $b_k = a_{3^k}$, so,

$$b_k = 5b_{k-1} + 7$$

$$b_k - 5b_{k-1} = 7$$

$$b_k - 5b_{k-1} = 0$$

$$t - 5 = 0$$
$$t = 5 \Rightarrow b_k^H = c \cdot 5^k$$

eg-

$$b_k - 5b_{k-1} = 7$$

$$b_k^p = d = -7/4$$

$$d - 5d = 7 \Rightarrow d = -\frac{7}{4}$$

So, $b_k = c \cdot 5^k - \frac{7}{4}$

$$a_n = c \cdot 5^k - \frac{7}{4}$$

$\because k = \log_3 n$, Therefore,

$$a_n = c \cdot 5^{\log_3 k} - \frac{7}{4}$$

$$\boxed{\because a^{\log_c b} = b^{\log_c a}}$$

$$\Rightarrow a_n = c \cdot n^{\log_3 5} - \frac{7}{4} \quad ; \ a_1 = 5,$$

$$a_1 = c - \frac{7}{4} = 5$$

$$c = \frac{7}{4} + 5 = \frac{27}{4}$$

$$a_n = \frac{27}{4} \cdot n^{\log_3 5} - \frac{7}{4}$$

By master's theorem,

$$a_n = O(n^{\log_3 5}).$$

Eg -

$$a_n = 5 \cdot a_{n/3} + n$$

$n = 3^k$,

$$a_{3^k} = 5 a_{3^{k-1}} + 3^k$$

$$b_k = 5 \cdot b_{k-1} + 3^k$$

$$b_n - 5 b_{k-1} = 3^k$$

18.10.10 # Set Theory & Algebra

– Relations –

- Types of relations : Closures
- Operations on relations
- Representation of rel^{ns}
- Equivalence & Partial order rel^n : Poset, Lattice, Boolean Algebra
- Properties of equivalence rel^n

– Functions

- Types of functions : Domain & Range
- Composition of fun^n : $f \circ g$
- Identity, inverse : f^{-1}, I

– Algebra –

- Semigroup
- Monoid
- Group
- Abelian gp.
- Gp. example
- Gp. properties : Order of gp.
  - Order of elements
  - Cycle of gp.
  - subgp.
  - Normal subgp.
  - Lagrange's Theorem
  - Homo & Isomorphism of gp.

– Poset, Lattice & BA –

1. Poset, Toset, Woset
2. Product partial order
3. Hasse diagram
4. Extreme elements of posets

– Se
– Th
{
⇒ (
A
I. ϕ
II. ϕ
III. {a
IV. {a
A X B
|A| =
Cartesian product

5. Types of lattice

6. Sublattice, Semilattice

7. Boolean Algebra

lattice, Algebra

Relations

- Set is a well defined collection of elements.
- There is no order.

$\{ \} \rightarrow$ No sequence

$\Rightarrow$ $( ) \rightarrow$ sequence.

$$A = \{ \phi, \{a,b\}, (1,2), 3 \}$$

I. $\phi \in A$ — TRUE

II. $\phi \subseteq A$ — TRUE

III. $\{a,b\} \in A$ — TRUE

IV. $\{a,b\} \subseteq A \rightarrow$ FALSE $\Rightarrow$ if $\{\{a,b\}\} \subseteq A \rightarrow$ TRUE.

$$A - B = A - (A \cap B) = A \cap B^c = ab'$$

$$A \oplus B = (A-B) \cup (B-A)$$

$$= AB^c + \cancel{B}A'B$$

$$= ab' + a'b$$

$$A \times B = \{(x,y) \mid x \in A \ \& \ y \in B\}$$

$|A| = m, |B| = n \quad |A \times B| = mn$

Cartesian product $\quad A \times B \neq B \times A \Rightarrow$ Not Commutative.

$\Rightarrow$ $A = \{1, 2, 3\}$, $B = \{a, b\}$

$A \times B = \{(1,a)\ (1,b)\ (2,a)\ (2,b)\ (3,a)\ (3,b)\}$

$\Rightarrow$ $\boxed{R \subseteq A \times B}$

$\{\}$ is smallest relⁿ b/w A & B.

Biggest relⁿ is $|A| \cdot |B| = |A \times B|$

and 2^{mn} relⁿ will be there is set.

$S = \{(1,a), (3,b)\}$

$1 S a$ $\quad$ $3 S b$

$(1,a) \in S$ $\quad$ $(3,b) \in S$

Prob. $R = \{(1,a)\ (1,b)\ (2,a)\ (3,a)\}$

R - relⁿ set, $R(1) = ?$

$R(1) = \{a, b\}$ → i.e. all are related to 1.

$R(2) = \{a\}$ → all are related to 2.

in equivalence relⁿ ⇒

$[1] = \{a, b\}$

$[2] = \{a\}$

called equivalence class

Domain – 1st element } in paired sets
Range – 2nd element

,3)}

$\Rightarrow$ $\boxed{\begin{array}{c} D(R) \subseteq A \\ \text{Range}(R) \subseteq B \end{array}}$ defined in $A \times B$

$\downarrow$

Codomain

So, Codomain can be bigger than range.

<u>Operations on relⁿ</u> – $\cup, \cap, L^c, R-S, S-R, R \oplus S$.

$\rightarrow$ $R \circ S$, $S \circ R$, R^{-1}, S^{-1} can be done only in relⁿ because of ordered pair.

<u>Prob.</u> $A = \{1,2,3\}$

let $R = \{(1,2)\,(1,3)\,(2,2)\,(3,2)\}$

$S = \{(2,1)\,(2,3)\,(1,3)\,(1,1)\}$

$\Rightarrow$ $R \cup S = \{(1,2)\,(1,3)\,(2,2)\,(3,2)\,\cancel{(2,1)}\}$

$R \cap S = \{(1,3)\}$

$R^c = \bar{R} = U - R = A \times A - R$

$= \{(1,1)\,(2,1)\,(2,3)\,(3,1)\,(3,3)\}$

As, $R \cup \bar{R} = A \times A$

R in $|A| = n$, $|R| = m$, then

Cardinality of $\bar{R}$ $\rightarrow$ $\boxed{|\bar{R}| = n^2 - m}$

$\bar{S} = \{(1,2)(2,2)(3,1)(3,2)(3,3)\}$

$R - S = \{(1,2)\ (2,2)\ (3,3)\}$

$S - R = \{(2,1)\ (2,3)\ (1,1)\}$

$R \oplus S = \{(1,2)\ (2,2)\ (3,2)\ (2,1)\ (2,3)\ (1,1)\}$

$R \circ S = RS = \{(x,z) \mid (x,y) \in S \ \& \ (y,z) \in R\}$

$= \{(2,2)\ (2,3)\ (1,2)\ (1,3)\}$

$S \circ R = SR = \{(1,1), (1,3)\ (2,1)\ (2,3)\ (3,1)\ (3,3)\}$

$R \circ S \neq S \circ R$ → Not commutative

But $\boxed{R \circ (S \circ T) = (R \circ S) \circ T}$

$R^{-1} \neq \bar{R}$

$R^{-1} = \{(y,x) \mid (x,y) \in R\}$

$\Rightarrow R^{-1} = \{(2,1)\ (3,1)\ (2,2)\ (2,3)\}$

$\boxed{R^{-1} = R} \Rightarrow R$ is symmetric

Representation of Rel^n - There are several ways -

- Listing → it is finite
- Statement
- Set Builder

- Matrix rep.
- Digraph
- Arrow diagram
- Table
- Graph
- formula method.

① Listing -

$A = \{1, 2, 3\}$

$R = \{(1,2)\ (2,3)\ (2,2)\ (3,1)\}$

② Set Builder -

$R = \{(x, y) \mid x \le y\}$ on A

in Listing, $R = \{(1,1)(1,2)(1,3)(2,2)(3,3)(2,3)\}$

③ Statement -

$x R y$ iff $x \parallel y$

like on line on planes.

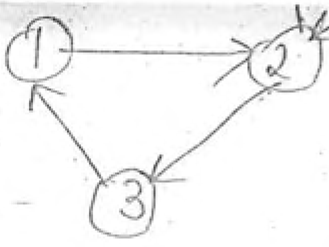
$x R y$ iff $x \perp y$

$x R y$ iff $x \le y$

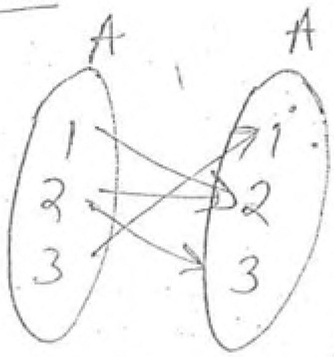
④ Matrix -

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

⑤ Digraph - Can be used only for AXA. It can't be used for AXB.



⑥ Arrow Diagram -

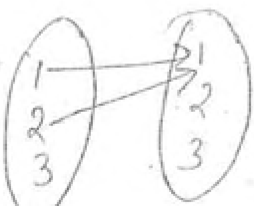


fun^n is such a rel^n where one is related to only one element.

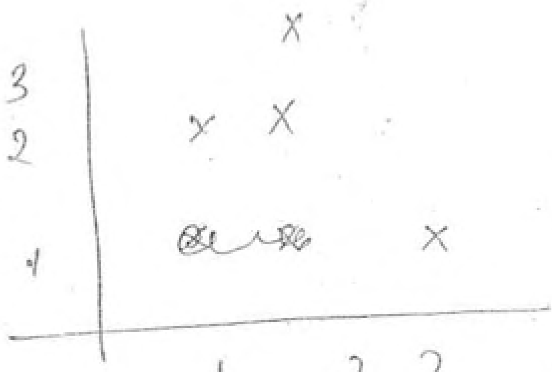
$f(x) = x^2 \longrightarrow fun^n$

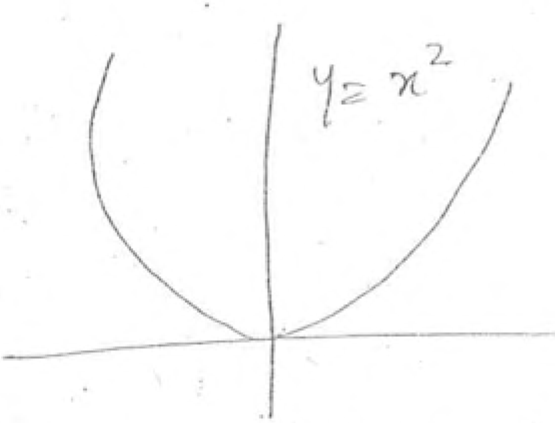
$(4,2)\ (4,-2) \Rightarrow rel^n$

fun^n -



$\Rightarrow$ for given i/p, there should be only one o/p.





⑦ Formula method - It is particularly used for fun^{ns}.

Types of Rel

Reflexive	$\forall x\ (x,x) \in R$; $\forall x,\ xRx$ → Selfloops
Symmetric	$(x,y) \in R \Rightarrow (y,x) \in R$ $xRy \Rightarrow yRx$
Transitive	$(x,y) \in R$ & $(y,z) \in R \Rightarrow (x,z) \in R$ xRy & $yRz \Rightarrow xRz$
Irreflexive	$\forall x\ (x,x) \notin R$; $\forall x,\ x \not{R} x$
Antisymmetric	$(x,y) \in R \Rightarrow (y,x) \notin R$ (unless $x=y$) $xRy \Rightarrow y \not{R} x$ i.e. self loops are allowed.
Asymmetric	$(x,y) \in R \Rightarrow (y,x) \notin R$

$R = \{(1,1)\ (1,2)\ (3,1)\ (3,3)\}$ → Not Reflexive → Also not irreflexive

$R = \{(1,1)\ (2,2)\ (1,3)\ (3,3)\}$ → Reflexive

Largest reflexive set size $= n^2$

Minimum — — $= n$

$T = \{(1,3)\ (3,2)\}$ → Irreflexive

i.e. no self-loop should exists.

⇒ Reflexive ⇒ Not Irreflexive
Irreflexive ⇒ Not Reflexive } TRUE

⇒ Not reflexive ⇒ Irreflexive
Not irreflexive ⇒ Reflexive } FALSE

$\Rightarrow R = \{(x, y) \mid x \le y\}$ on $R \times R$ (by default)

it is reflexive ?

$\Rightarrow$ This is not irreflexive but reflexive.

$R = \{(x, y) \mid x < y\}$

↳ Not reflexive but irreflexive.

$R = \{(x, y) \mid x + y = 10\}$

↳ Neither reflexive nor irreflexive

$R = \{(x, y) \mid x = y^2,\ z \text{ is integer}\}$

↳ is reflexive.

$R = \{(x, y) \mid x \text{ is one inch from } y\}$

(pt in plane)

↳ is irreflexive rel^n.

$R = \{(x, y) \mid x \text{ is brother of } y\}$

↳ is irreflexive.

$R = \{(x, y) \mid x = y\}$

↳ is reflexive.

$R \Rightarrow (x_1, y_1) R (x_2, y_2)$ iff ~~$x_1 + x_2 = y_1 + y_2$~~ $x_1 + y_1 = x_2 + y_2$

as, $R = \{(1,2)(3,0), \ldots\ (1,2)(2,1), \ldots\ (1,2)(1,2) \ldots\}$

is reflexive.

ault)

→ $\{(x,y) \mid x \leq y\}$ → Not Symmetric

→ $\{(x,y) \mid x+y=10\}$ → Symmetric

→ $\{(x,y) \mid x=y^2; z \text{ is int}\}$ → Not symmetric

→ $\{(x,y) \mid x$ is brother of $y\}$ → Not Symmetric

→ $\{(x_1,y_1),(x_2,y_2) \mid x_1+y_1=x_2+y_2\}$ → Symmetric

$R=\{\ \}$ → Symmetric (By default)

* Self loops are always symmetric.

$R=\{(1,1)(1,2)(3,1)(2,2)\}$ → Antisymmetric

↳ Not asymmetric

$R=\{(1,3)(3,2)\}$ → Antisym. & asymmetric

$\boxed{xRy \ \& \ yRx \Rightarrow x=y}$ → Antisymmetric defn

$xRy \ \& \ yRx = \phi$ → Asymmetric defn

$R=\{(x,y) \mid x \leq y\}$ → Antisymmetric but not asymmetric

$R=\{(x,y) \mid x<y\}$ → Asymmetric & antisymmetric

$\boxed{\text{* Asymmetric always be Antisymmetric.}}$

$R = \{(1,2)\ (2,1)\ (3,1)\} \rightarrow$ Not symmetric
also not asymmetric.
also not antisymmetric.

$R = \{\ \} \Rightarrow$ Both symmetric & Antisymmetric
& symmetric.

"In { } empty Rel" Only Reflexive property does not hold true."

$R = \{(1,1)\ (2,2)\ (3,3)\} \rightarrow$ symmetric &
antisymmetric both.

$\downarrow$

(largest set)

$R = \{(x,y) \mid x+y=10\}$

as, $x+y=10$ & $y+x=10 \not\Rightarrow x=y$

so, it is not anti symmetric also
not asymmetric.

$\{(x,y) \mid x=y^2\}$ $\quad x=y^2$ & $y=x^2 \not\Rightarrow x=y$
(as $x=1$)

$\Rightarrow$ is antisymmetric but not
asymmetric.

$\{(x,y) \mid x$ is one inch from $y\}$

– Not antisymmetric also
not asymmetric.

ric. $\{(x,y) \mid x$ is brother of $y\}$

ric. $\rightarrow$ Not antisymmetric not asymmetric.

ric $\Rightarrow (x_1,y_1) R (x_2,y_2)$ & $(x_2,y_2) R (x_1,y_1) \Rightarrow (x_1,y_1) = (x_2,y_2)$

$x_1 + y_1 = x_2 + y_2$

+y $\Rightarrow R = \{(x,y) \mid x = y\} \rightarrow$ Antisymmetric but not asymmetric.

th. $R = \{(x,y) \mid x \perp y\}$

xRy & $yRz \Rightarrow x \perp y$ & $y \perp z \not\Rightarrow x \perp z$

So, Not transitive.

* In checking transitive, ignore self-loop always.

$\{(x,y) \mid x+y = 10\} \rightarrow$ Not transitive.

$= \{(x,y) \mid x = y^2\} \rightarrow$

$x = y^b$ & $y = z^a \Rightarrow x = (z^a)^b = z^{ab}$

So, it is transitive.

=y

ob

$= (x,y) \mid x < y \rightarrow$ Transitive.

$= \{x$ is brother of $y\} \rightarrow$ Transitive

$= \{x_1 + y_1 = x_2 + y_2\} \rightarrow$ Transitive.

$(x_1,y_1) R (x_2 y_2)$ & $x_2 y_2 R x_3 y_3 \Rightarrow x_1 y_1 R x_3 y_3$

$x_1 + y_1 = x_2 + y_2$ & $x_2 + y_2 = x_3 + y_3 \Rightarrow x_1 + y_1 = x_3 + y_3$

$\{(x,y) \mid x \parallel y\} \rightarrow$ Transitive & Reflexive.

Equivalence Rel^n (Ref, Sym., Trans)

Partial order (Ref, Antisym, Trans.)

Prob $(x,y) \mid$

		R	S	AS	T	
$x \parallel y$	$\rightarrow$	✓	✓	✗	✓	Equi.
$x < y$	$\rightarrow$	✗	✗	✓	✓	Not equi nor Par.
$x \leq y$	$\rightarrow$	✓	✗	✓	✓	Partial

Word "Same" $\Rightarrow$ Always equivalence rel^n.

$\{(x,y) \mid x \equiv y \bmod m\}$ $\rightarrow$ Congruence modulo m rel^n on $Z \times R$.
$m=5$

$\downarrow$ equivalent rel^n because residue is same always.

So, $\{(x,y) \mid x - y = 5k\}$ $\rightarrow$ Integral multiple.

$$\boxed{\text{If Only self Loops} \rightarrow \text{Identity Rel}^n \Rightarrow \text{Equivalence \& Partial Order Both.}}$$

Clo
— R
— S
— T
Refle
Then
sma
i.e., S
Sym
Trans
S
Prob
S
WARSH
B
— It is

Closures of Rel -

- Ref. Clousure
- Symm. "
- Transitive ".

Reflexive Closure - Let $A = \{1,2,3\}$,

$$R = \{(1,1)(2,2)(1,2)(3,2)\}$$

- Then ref. closure, may be defined as, "it is smallest rel^n that contain R is reflexive".

i.e, $S = \{(1,1)(2,2)(1,2)(3,2)(3,3)\}$ is ref. closure of R.

Sym. Closure -

$$S' = \{(1,1)(2,2)(1,2)(2,1)(3,2)(2,3)\}$$

Trans. Closure -

$$S'' = \{(1,1)(2,2)(1,2)(3,2)\} \cancel{(1,2)(}$$

Prob - $R = \{(1,1)(2,2)(1,2)(3,1)\}$ $R = \{1,2,3,4\}$

$$S' = \{(1,1)(2,2)(1,2)(3,1)(3,2)(4,3)(4,1),(4,2)\}$$

WARSHALL'S ALGO - $O(n^3) \rightarrow$ intelligent method.

Brute force method - $O(n^4)$

- It is used for Transitive Closure finding.

R^{∞} is symbol for trans. closure.

As,

$$\boxed{R^{\infty} = R \cup R^2 \cup R^3 \cup \cdots \cup R^{\infty}}$$

$$= R \cup R^2 \cup \cdots\cdots R^n$$

- as it is proved that beyond n, there is nothing new interesting thing happens.

<u>Prob.</u> $R = \{(x,y) \mid x < y\}$ defined on $R \times R$

$\Rightarrow$ a. $(x,y) \mid x = y$

b. $(x,y) \mid x > y$

c. $(x,y) \mid x \leq y$ $\rightarrow$ Ref. Closure of R ✓

d. $(x,y) \mid x \geq y$

e. None of these

f. $(x,y) \mid x \neq y$ $\rightarrow$ Sym. Closure of R

g. $(x,y) \mid x < y$ $\rightarrow$ Trans. Closure as ~~it~~ it is already

<u>Restrictions on set R</u> -

$R \cap (B \times B)$ $\rightarrow$ Restriction of R to set B.

Let $A = \{1, 2, 3, 4\}$

$R = \{(1,3)\ (2,1)\ (1,2)\ (3,3)\ (4,3)\}$ $B \subseteq A$

$B = \{1, 2\}$

So, $B \times B = \{(1,1)\ (1,2)\ (2,1)\ (2,2)\}$

$\Rightarrow R' = \{(2,1)\ (1,2)\} \rightarrow$ Restriction of R to set B.

Powers of Rel^n -

$$R^2 = R \circ R$$

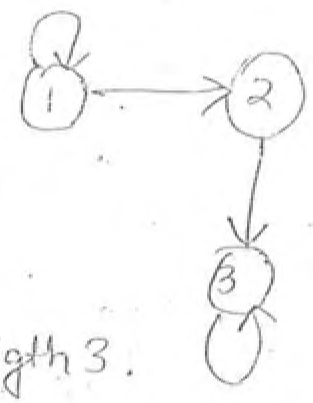
$$R^3 = R \circ R^2 = R^2 \circ R = R \circ R \circ R$$

$\Rightarrow$ Let $A = \{1, 2, 3\}$,

$R = \{(1,1)\ (1,2)\ (2,3)\ (3,3)\}$

$\Rightarrow R^2 = \{(1,1)\ (1,2)\ (1,3)\ (2,3)\ (3,3)\}$

if some $(x, y) \in R^2$ then $\rightarrow$ path of length 2.



Similarly,

R^3 = contains all path of length 3.

Transitive Closure -

$$R^\infty = R \cup R^2 \cup \text{-----}$$

$\rightarrow (x, y) \in R^\infty$ iff $\exists$ a path b/w x & y of any length.

Reachability relⁿ – R^*

– It is almost same as R^∞, except it contains self-loops also.

i.e.,
$$\boxed{R^* = R^\infty \cup I}$$

i.e. connected as well as self-loops.

if we have M_S & M_R, then.

$$\boxed{M_{R \circ S} = M_S \odot M_R}$$

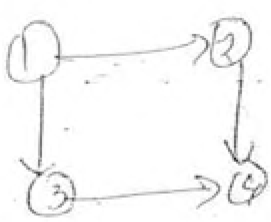
→ Boolean Multiplication

$$\Rightarrow \quad M_{R^2} = M_R \odot M_R$$

if $A = \{1, 2, 3, 4\}$, and $R^5 = \begin{array}{c} \\ 3 \end{array} \begin{array}{c} 4 \\ \begin{bmatrix} & | \\ - - & 1 \end{bmatrix} \end{array}$

then means, $\exists$ path of length 5 from 3 to 4.

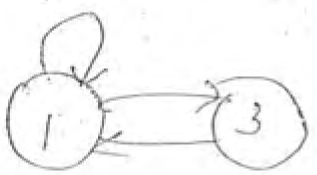
$R^5 = \begin{array}{c} \\ 3 \end{array} \begin{array}{c} 4 \\ \begin{pmatrix} & | \\ - - & 6 \end{pmatrix} \end{array}$ $\xrightarrow{\text{gives how many paths?}}$ $\Rightarrow \exists$ 6 paths of length 5 from 3 to 4.



$(1, 4) \rightarrow 2$

R^2 ?

⇒ self loop can be counted as path of any length

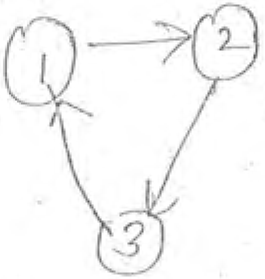


$(3,3) \in R^{100}$

but $(3,3) \notin R^{101}$

Whereas, $(1,1) \in R^{100}$ as well as $(1,1) \in R^{101}$

⇒ Let



Now $R^{300} = \{(1,1)\,(2,2)\,(3,3)\}$

path of length 2

$R^{197} = R^{3n+2} = \{(1,3)\,(2,1)\,(3,2)\}$

$R^{3n+1} = \{(1,2)\,(2,3)\,(3,1)\}$

<u>Theorem</u> – $\boxed{\text{if } R \text{ is reflexive} \Rightarrow R^{-1} \text{ is also reflexive.}}$

Similarly for sym. & transitive also.

$(x,y)\,(y,x) \Rightarrow (y,x)\,(x,y)$

$(x,y)\,(y,z)\,(x,z) \Rightarrow (y,x)\,(z,y) \Rightarrow (z,x)$

i.e., $\boxed{\text{if } R \text{ is equivalence rel}^n \Rightarrow R^{-1} \text{ also equivalence rel}^n.}$

if R & S are equivalence rel^n, then

I. $R \cap S$ also eq^n rel^n → TRUE i.e. closed under intersection

II. $R \cup S$ also eq^n rel^n. → FALSE.

Because both contains loops, so $\cap$ will also be contain loops.

$\Rightarrow (x,y) \in R \cap S,$

$\Rightarrow (x,y) \in R$ & $(x,y) \in S.$

$\Rightarrow (y,x) \in R$ & $(y,x) \in S$ (because sym.)

$\rightarrow (y,x) \in R \cap S.$

let $(x,y), (y,z) \in R \cap S$

$\Rightarrow (x,y)$ & $(y,z) \in R$ — — and $\in S$.

$\because$ R is transitive, so,

$(x,z) \in R$ — and $(x,z) \in S$

$\Rightarrow (x,z) \in R \cap S.$

$\Rightarrow$ If R & S are equivalence, then $R \cap S$ will also be equivalence.

But not true for $R \cup S$.

If R & S are both reflexive & symmetric, then $R \cap S$ & $R \cup S$ will also be reflexive & symm. but it is not true in case of trans.

Theorem - If R & S are equivalence then

$\boxed{R \cap S \text{ is largest eq}^n \text{ set.}}$

Similarly smallest set, $R \cup S$, but it is not necessarily eq^n rel^n set.

Because $R \cup S$ may not be eq^n set, due to transitive property.

But $\boxed{(R \cup S)^{\infty} \text{ is guaranteed to be smallest equivalence set.}}$

$\left\{ \begin{array}{l} \text{largest eq}^n \text{ set size} = n^2 \\ \text{Smallest eq}^n \text{ set size} = n \end{array} \right.$

Theorem Every eq^n rel^n creates a quotient set.

$\Rightarrow A = \{1, 2, 3\}$,

$R = \{(1,1)\ (1,2)\ (2,1)\ (2,2)\ (3,3)\}$ is eq^n rel^n.

Now,

A/R = Quotient set of A induced by R.

$\hookrightarrow A/R$ = set of all eq^n classes of all elements of A.

$[1] = R(1) = \{1, 2\}$

$[2] = \{1, 2\}$ So, $A/R = \{\{1,2\}, \{3\}\}$

$[3] = \{3\}$ ↳ set of distinct eq^n classes.

- A/R is always a partion of A by R.
- for every eq^n rel^n $\exists$ unique quotient set.
- for every quotient set $\exists$ unique eq^n rel^n set.

<u>Theorem</u>- Corresponding to any partition π of A, $\exists$ an unique eq^n rel^n R, such that,

$$\boxed{A/R = \pi}$$

$\Rightarrow$ Let $\pi = \{\{1,2\},\{3\}\}$, $A = \{1,2,3\}$

find out eq^n rel^n set.

$\Rightarrow$ $\pi = \overline{12}\ \overline{3}$ → Blocks.

<u>Method</u>- $\because \pi = \{\underset{A_1}{\{1,2\}}, \underset{A_2}{\{3\}}\}$

$$R = \{A_1 \times A_1, A_2 \times A_2 \ldots\}$$
$$= A_1 \times A_1 \cup A_2 \times A_2 \cup \ldots$$

$$R = \{(1,1)(1,2)(2,1)(2,2)(3,3)\}$$

$\Rightarrow$ Let $A = \{1,2,3,4\}$, $\pi = \overline{12}\ \overline{34}$

What is eq^n rel^n?

eq^n rel^n $\{(1,1)(2,2)(1,2)(2,1)(3,3)(4,4)$
$(3,4)(4,3)\}$.

$\therefore A/R = \{ \{1,2\} \{3,4\} \}$ → to check answer!

* [No. of blocks at least = 1
  at most = n]

Properties of eq^n class - Assume $A \times A$ is R.

if $[x]$ & $[y]$ are any eq^n classes, then

① $\forall x \in A,\ x \in [x]$ → due to reflexivity

② $[x] \cap [y] = \phi$; if $[x] \neq [y]$

③ $\bigcup_{\forall x \in A} [x] = A$

Properties of partition class -

Let $A = \{1,2,3\}$,

$\pi = \{ \{1,2\} \{2,3\} \}$ → Is not partition set.

① $A_i \cap A_j = \phi$; if $i \neq j$

② $\bigcup_i A_i = A$

Prob. $W = \{ \text{bat}, \text{ball}, \text{cat}, \text{call}, \text{catch} \}$

$R = \{ (x,y) \mid x$ & y starts with same letters$\}$

$\Rightarrow W/R = ?$

$W/R = \{\{bat, ball\}, \{cat, call, catch\}\}$

$[bat] = \{bat, ball\}$

$[ball] = \{bat, ball\}$

$[cat] = \{cat, call, catch\}$

$[call] = \{ \quad " \quad \}$

$[catch] = \{ \quad " \quad \}$

Rel^n

- Eq^n Rel^n (Filtering) - Grouping
- Partial order Rel^n (Asending order) - Sorting

$\Rightarrow$ Prob. $R = \{(x,y) \mid |x| = |y|\}$

So, $W/R = \{\{bat, cat\}, \{ball, call\}, \{catch\}\}$

Prob. $R = \{(x,y) \mid x \equiv y \bmod m\}$ on $Z \times Z$.

$\Rightarrow$ There will be exactly m different residue.

So, there will be exact m distinct eq^n classes.

$$Z/R = \{ \underbrace{\{\pm 0, \pm 5, \pm 10, \ldots\}}_{5n}, \underbrace{\{\ldots \pm 6, \pm 11, \pm 16, \ldots\}}_{5n+1}, \underbrace{\{\ldots \pm 7, \pm 12, \ldots\}}_{5x+2}, \underbrace{\{\ldots \pm 8, \pm 13, \ldots\}}_{5n+3}, \underbrace{\{\ldots \pm 9, \pm 14, \ldots\}}_{5n+4} \}$$

$\downarrow$

y mod 5

Prob: if $x \equiv y \bmod 5$ then $[7] = ?$

$[7] = 5x+2$ —

$[4] = 5x+4$ —

Functions

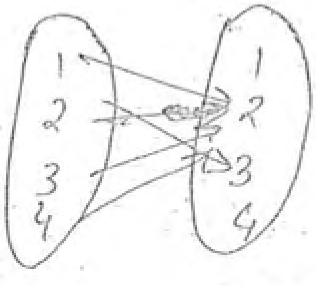
- Definition
- Domain & Range
- Mapping
- One to one, Many to many fun^n
- Onto, Into fun^n
- fog
- f^{-1}
- Identity
- Equality functions, Symmetric fun^n

Definition - Unique valued rel^n is called fun^n for every I/P.

i.e. for every I/P, $\exists$ only one O/P.

$\Rightarrow A = \{1,2,3,4\}, \quad R = \{(1,2)(2,3)(3,2)(4,2)\}$

↳ This is funⁿ as for every I/P $\exists$ only one O/P.



$f(x) = x^2$

$f = \{(x,y) \mid y = x^2\}$

↳ also funⁿ

$f(x) = \sqrt{x} \rightarrow$ Not funⁿ

$f(x) = \sin x \rightarrow$ Not funⁿ

Domain & Range of funⁿ –

Domain –

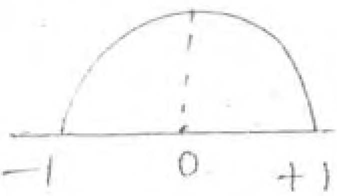
– All those I/Ps which gives meaningful O/P.

$\Rightarrow \quad f(x) = x^2 \quad$ on $R \times R$

So, $\text{Dom}(f) = R$

$\Rightarrow \quad f(x) = \sqrt{1-x^2} \quad$ on $R \times R$

So, $\text{Dom}(f) = \{-1 \le x \le 1\}$



$\Rightarrow$ $\boxed{\text{If Domain is equal to } 1^{st} \text{ set} \Rightarrow \text{Total fun}^n}$

i.e., if $A \times B$, then $\text{Dom}(f) = A \Rightarrow$ Mapping

$f(x) = x^2 \rightarrow$ mapping

$f(x) = \sqrt{4-x^2} \rightarrow$ Partial funⁿ on $R \times R$.

But on $A \times R$, it is mapping as Total funⁿ.

fun itself as mapping otherwise, we define its domain.

$\Rightarrow A \times B \longrightarrow$ Codomain

$\rightarrow$ Range is not necessarily codomain.

But $\Rightarrow$ $\boxed{\text{Range}(f) \subseteq B}$ always.

$\Rightarrow f(x) = x^2$; $R \times R$

$\Rightarrow$ Range $(f) = R^+ \cup \{0\}$ $\rightarrow$ Nonnegative Int.

$y = x^2$

$\sqrt{y} = x$

So R = only +ve no. + $\{0\}$

$\Rightarrow f(x) = 3x + 1$ on $R \times R$

$\Rightarrow$ Domain $(f) = R$

Range $(f) = R$.

Put $y = 3x + 1$

$x = \frac{y-1}{3}$ on $R \times R$

So, Range is always be real.

Domain of fun^n $\rightarrow$ Mapping $\rightarrow$ Total

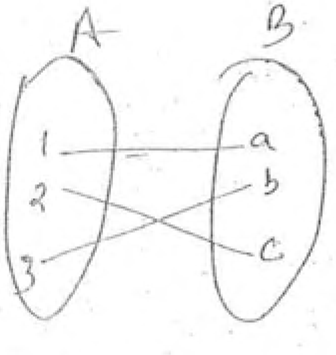
Range of fun^n $\longrightarrow$ onto fun^n $\rightarrow$ Partial

<u>Onto fun^n</u> – $\forall y \in B, \exists x \in A, x R y$.

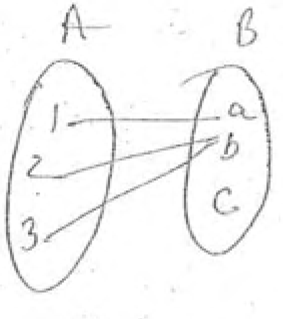
and Range $(f) = B$.

Mapping – $\forall x \in A, \exists y \in B, \; xRy$

Into funⁿ – $\text{Range}(f) \subset B$.



Onto



Into

$\Rightarrow$ ① $f(x) = e^x$ on $R \times R$

② $f(x) = \sin x$ on $R \times R$

Solⁿ

① $y = e^x \rightarrow x = \log_e y$ on $R \times R$

$\text{Range}(f) = R^+ \rightarrow$ Not codomain
so, not Onto.

② $y = \sin x$

$x = \sin^{-1}(y)$ $R \times R$

$(-1 \leq y \leq 1) \rightarrow$ Not codomain
so, not onto.

$\Rightarrow$ $f(x) = 3x + 1$ on $Z \times Z$

Solⁿ – $y = 3x+1$ $x = \dfrac{y-1}{3}$ on $Z \times Z$

∴ For any int y, x may get be fra
i.e. not integer, Hence not onto. It is into.
But on R×R, it is onto funⁿ.

One to One – f is one to one, iff

$$\boxed{f(x_1) = f(x_2) \Longrightarrow x_1 = x_2}$$

$\Rightarrow f(x) = x^2$ on R×R
$f(x) = e^x$ – One to one
$f(x) = 3x+1$ – One to One

Solⁿ – let $f(x_1) = f(x_2)$

① $x_1^2 = x_2^2 \Rightarrow x_1 = \pm x_2$ $\quad \begin{matrix} -2 \\ +2 \end{matrix} \rangle 4$

Not On-to. It is Many-to One.

② $e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2$

→ One to one

③ $3x_1 + 1 = 3x_2 + 1 \Rightarrow x_1 = x_2$

→ One to One.

Also Onto.

$\Rightarrow f(x) = x^3$ on R×R

Solⁿ $x_1^3 = x_2^3 \Rightarrow x_1 = x_2, x_2\omega, x_2\omega^2$

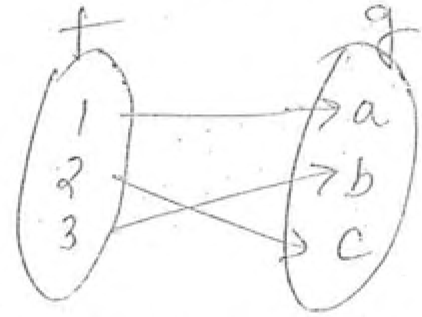
$\Downarrow$ complex No.

in Real, $x_1 = x_2$

So → One to One.

on $C \times C$, it will be Many to One.

<u>fog</u> (Composition of f & g)



⇒ One to One correspondance fun^n

(One to One)

if 1^{st} element repeated → Not even fun^n.

⇒ <u>One to One</u> - (Injection)

⇒ <u>Onto</u> (Surjective fun^n)

→ Both One to One + Onto + Mapping } Bijective fun^n - mapping is (One to One correspondance fun^n)

fog ⇒ $\boxed{\text{Bijection exists b/w } A \text{ \& } B \text{ iff } |A| = |B|}$

~~let~~,

→ No bijection b/w N ~~&~~ R because R is uncountable & N is countable.

$\Rightarrow f = \{(1,2)(2,3)(3,3)\}$

$g = \{(1,3)(3,1)(3,2)\}$

$fog = \{(1,3)(3,2)(3,3)\}$

$gof = \{(2,1)(2,2)(3,1)(3,2)\}$

$\Rightarrow f(x) = 3x+1 \; ; \; g(x) = \sin x$

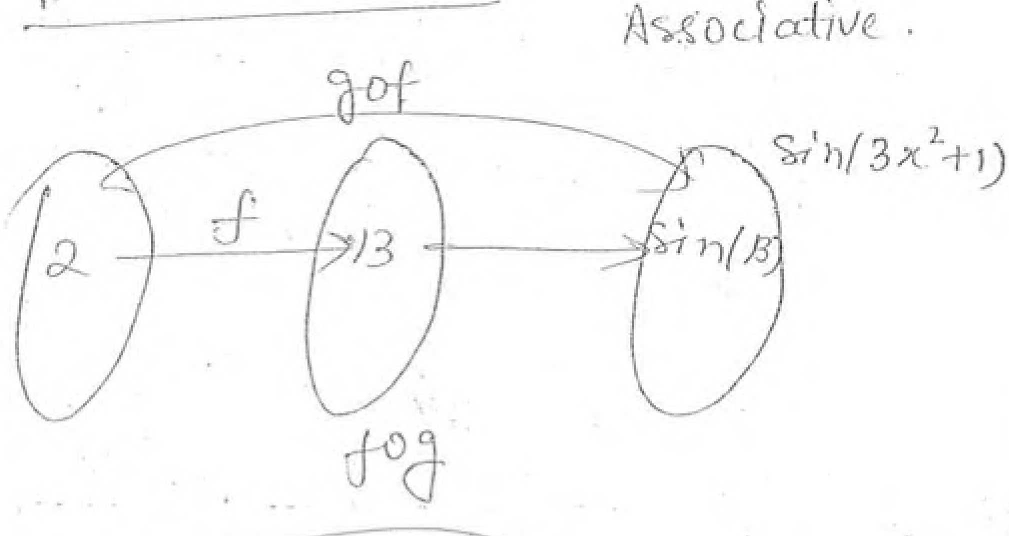
Sol^n: $fog(x) = f(g(x))$

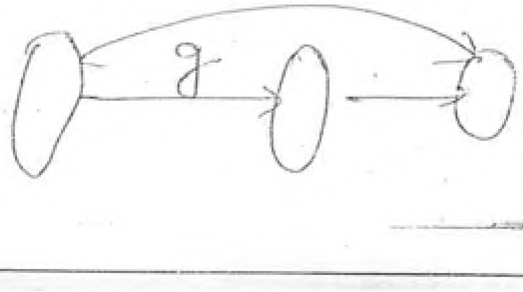
$= f(\sin x) = 3\sin^2 x + 1$

$gof(x) = g(f(x))$

$= g(3x^2+1) = \sin(3x^2+1)$

$\Rightarrow \boxed{fog \neq gof}$ → Not commutative But Associative.





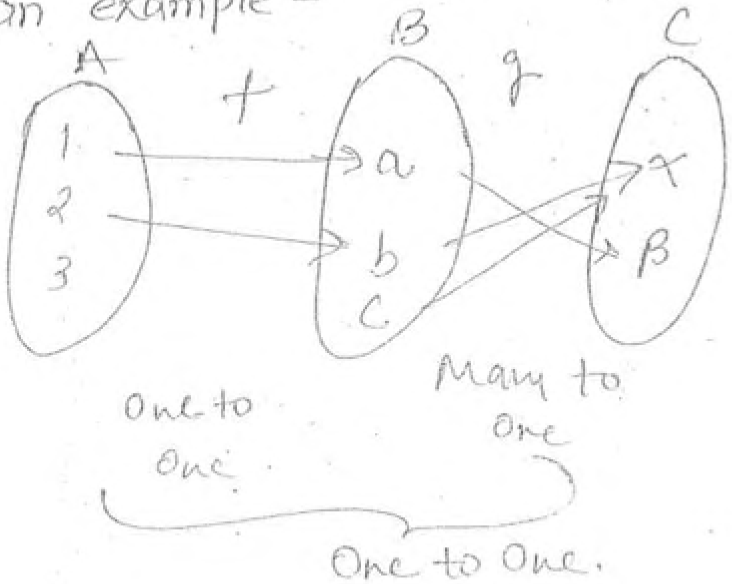
→ f is 1-1 & g is 1-1 ⇒ gof is 1-1

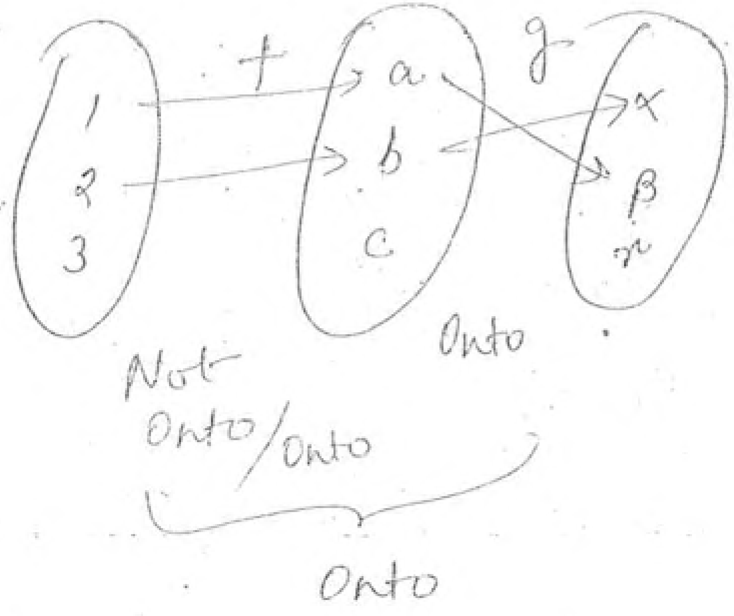
→ f is onto & g is onto ⇒ gof is Onto

→ f is 1-1 & onto i.e,
Bijection & g is bijection ⇒ gof is bijection

i.e, injection funⁿ & surjective funⁿ are closed under composition.

⇒ Take an example -





into

bijection

Inverse of fun -

$$f^{-1} = \{(y,x) \mid (x,y) \in f\}$$

$\Rightarrow$ $f = \{(2,3)\ (3,5)\ (5,2)\}$

Sol^n: $f^{-1} = \{(3,2)\ (5,3)\ (2,5)\}$

$\Rightarrow$ $f(x) = 3x+1$ $\qquad \because x = f^{-1}(y)$

Sol^n $y = 3x+1 \Rightarrow x = \frac{y-1}{3}$

$\Rightarrow$ $f^{-1}(y) = \frac{y-1}{3}$ — put x in place of y to get inverse.

So, $\boxed{f^{-1}(x) = \frac{x-1}{3}}$ Ans

$\Rightarrow$ $f(x) = e^x$

Sol^n $y = e^x \Rightarrow f^{-1}(y) = x = \log_e y$

$f^{-1}(y) = \log_e y \Rightarrow \boxed{\log_e x = f^{-1}(x)}$

inverse of f.

Imp.

$\Rightarrow$ $\boxed{f \text{ is invertible, iff } f^{-1} \text{ is one to one function.}}$

$\Rightarrow$ $f(x) = x^2 \longrightarrow$ not invertible

$= e^x \longrightarrow$ invertible.

$= 3x+1$

* if f is bijective $\Rightarrow f^{-1}$ will also be bijective.

if $f^{-1} = g$ & $g^{-1} = f$, then

$\hookrightarrow \boxed{f \circ g = g \circ f = \text{Identity}}$

Equivalence of funⁿ – Two funⁿ f & g are equal, iff

$\forall x \in \text{Dom}(f) \Rightarrow f(x) = g(x)$.

Eg. $f(x) = \frac{x^2-1}{x-1}$, $g(x) = x+1$

$\Rightarrow f(x) = g(x)$. as domain is same.

Prob. $f(x) = x \Rightarrow f^{-1}(x) = x = f(x)$.

$\Rightarrow f(x) = \{(1,2)(2,1)\}$

$f^{-1}(x) = \{(1,2)(2,1)\} = f(x)$

Binary operations on funⁿˢ –

$\Rightarrow \boxed{a * b = c \ \ ; c \text{ is unique.}}$

Also, $f(a,b) = c$

Eg. $a * b = a^2 + b^2$ on $R \times R$

$\rightarrow$ it is binary funⁿ.

Eg.

Bina

can be

like

$\Rightarrow$

th

$\Rightarrow$

jective. Eg. $a * b = \sqrt{ab}$ → Not Binary operation

$2 \times 2 = \sqrt{2 \times 2} = \pm 2$ ⇒ Not unique.

re

Binary operations -

Can be $a * b = a + b$, ab, a^b, $ab + a - b$, a like,

⇒ $a * b = ab + a - b$

*	a	b	c
a	b	c	a
b	a	b	a
c	a	a	b

⇒ Only useful for finite no. of elements.

if instead of b, we write d, then it will be binary opⁿ but not closed under operation *.

⇒ $a * b = a + b$ → closed under $R \times R$
$= a - b$ → Not closed under $N \times N$.

$N = \{0, 1, 2, \ldots\}$, $\bar{Z} = \{-1, -2, \ldots\}$

$Z^+ = \{1, 2, \ldots\}$

$Z = Z^+ \cup \bar{Z} \cup \{0\}$

$= \bar{Z} \cup N$

So, for binary op we have to check-

- Closure
- Associative
- Identity
- Inverse
- Commutative

} 5 properties of binary operation.

① <u>Closure</u> - $\forall a, b \in S$,
$a * b \in S$.

② <u>Associative</u> - $\forall a, b \in S$,

$$a*(b*c) = (a*b)*c$$

③ <u>Identity</u> - $\forall a \in S$,

$$a * e = e * a = a$$
$$a + e = e + a = a$$

; $e \in S$ unique. It is identity for entire set.

④ <u>Inverse</u> - $\forall a \in S$,

$$a * a^{-1} = a^{-1} * a = e$$

also iff $a^{-1} \in S$, a^{-1} is unique.

⑤ <u>Commutative</u> - $\forall a, b \in S$,

$$a * b = b * a$$

<u>Closure Table</u> -

*	a	b	c
a	a	b	c
b	b	a	b
c	c	b	a

identity

Associativity - Not so feasible to check with the help of table

Identity - Row header and column header, if repeated, both simultaneously.

	a	b	c
a	a	a	b
b	b	a	b
c	c	b	a

→ Not repeated row element.

→ Identity element does not exists.

Inverse - i.e., $a * a$ should be a and unique.

	a	b	c
a	a	a	b
b	a	b	c
c	b	c	a

$$\Rightarrow a^{-1} = c$$
$$b^{-1} = b$$
$$c^{-1} = a$$

Commutative - There should be mirror image. B/w upper triangular & lower triangular matrix.

⇒ Formula Method -

$$a * b = a + b \quad \text{on } R \times R$$

$$\Rightarrow \quad a + (b + c) = (a + b) + c$$

$$\Rightarrow \quad a * e = e * a = a$$
$$a + e = e + a = a \quad \Rightarrow e = 0 \rightarrow \text{Real No.}$$

$$\Rightarrow \forall a \in R, \quad a * a^{-1} = a^{-1} * a = 0$$
$$a + a^{-1} = a^{-1} + a = 0 \quad \Rightarrow a^{-1} = -a \ \in R$$

ie, inverse of $a = -a$ in R.

Prob $a * b = a + b - ab$ on $R \times R$

$\Rightarrow$ $a + b - ab$

$a * (b * c) = (a * b) * c$

$a * (b + c - bc) = a + (b + c - bc) - a(b + c - bc)$

$= a + b + c - bc - ab - ac + abc$

$(a * b) * c = (a + b - ab) * c$

$= ((a + b) - ab) + c - (a + b - ab)c$

$= a + b - ab + c - ac - bc + abc$

$\Rightarrow$ Associativity

$a * e = e * a \quad \forall a \in R$

$\Rightarrow a + e - ae = e + a - ea = a$

$a + e - ae = a$

$e(1 - a) = 0$

$e = 0 \in R$

$\Rightarrow a * a^{-1} = a^{-1} * a = 0$

$a + a^{-1} - aa^{-1} = a^{-1} + a - a^{-1}a = 0$

$a + a^{-1} - aa^{-1} = 0$

$$a^{-1}(1-a) = -a$$

$$a^{-1} = \frac{-a}{1-a} = -\left(\frac{a}{1-a}\right)$$

So, every element has inverse except 1.
Hence it has no inverse property.

bc)

bc

$$a+b-ab = b+a-ba$$

$\Rightarrow$ Commutative.

c

bc

$(S, *)$
- Semigp. — Closure + Ass. along with $*$.
- Monoid — C + A + Identity
- Gp. — Monoid + Inverse.
- Abelian gp. — Gp + Commutative.

$\downarrow$ Abelian gp.

$(R, *) \Rightarrow$ Monoid but not gp.

$(R-\{0\}, *) \Rightarrow$ Abelian gp.

Group Theory

$(Z, +) \Rightarrow$ Abelian gp.

$(Q-\{0\}, *) \Rightarrow$ Abelian gp.

$(Z-\{0\}, *) \Rightarrow$ Monoid not gp.

Basic examples of gp.-

1. $\{0,1\}, \oplus$

$\oplus$	0	1
0	0	1
1	1	0

id = 0,

inv. of 0, = 0,

1 = 1

$\Rightarrow$ Abelian gp.

2. $(Z_m, +_m)$ → addition modulo m

$Z_m = \{0, 1, 2, \ldots m-1\}$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

4n+3

4n+2 →

gives same result, due to modulo operator.

$0^{-1} = 0$

$1^{-1} = 3$

$2^{-1} = 2$

$3^{-1} = 1$

$\Rightarrow$ Abelian gp.

3. $((1, 2, 3, \ldots p-1), *_p)$ → prime no.

$\Rightarrow$ Abelian gp.

$*_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$1^{-1} = 1$

$2^{-1} = 3$

$3^{-1} = 2$

$4^{-1} = 4$

Take

$*_4$	1	2	3
1	1	2	3
2	2	0	2
3	3	2	1

4 is not prime.

→ 0 comes, Hence, not closed.

4. $(S_n, o) \longrightarrow$ (Symmetric gp. of permutation)

$\downarrow$

One-to One correspondance fun^n

let $S = \{1, 2, 3\}$

Now, $S_3 = p_1 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $p_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

So, S_3 will have exactly $3! = 6$ factorial fun^ns.

$S_3 = \{p_1, p_2 - - - p_6\}$

So, $(S_3, o) \longrightarrow$ Gp. Not Abelian

as composition is associative not commutative.

$\Rightarrow$

	p_1	p_2	p_3	p_4	p_5	p_6
p_1	p_1	p_2	p_3	p_4	p_5	p_6
p_2	p_2	—		—		—
p_3	p_3	p_5	-		—	
p_4	p_4	—		—		
p_5	p_5	—		—		
p_6	p_6	—			—	

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = p_5$

$\longleftarrow$

$\Rightarrow p_2 \circ p_3 = p_5$

inverse, $p_3^{-1} = ?$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\text{so, } p_3^{-1} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = p_3$$

$$\Rightarrow p_3^{-1} = p_3$$

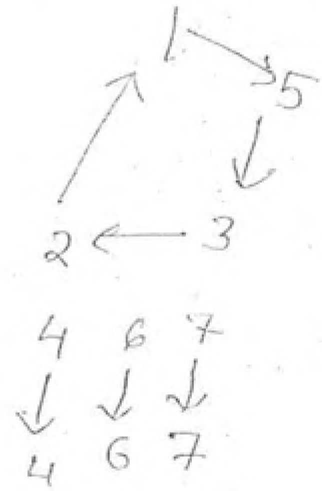
so, $(S_n, \circ)$ is gp, not Abelian gp.

Theorem -

1. Every permutation fun^n can be broken down into product of ~~distinct~~ disjoint cycles.
2. - - - - - - into product of transpositions.

(1) $\Rightarrow$ let $S = \{1, 2, \cdots 7\}$ ~~8, 9~~

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 2 & 4 & 3 & 6 & 7 \end{pmatrix}$$



$= (1, 5, 3, 2) \rightarrow$ called as cycles, remaining goes to itself

$$(3, 6, 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 6 & 3 & 5 & 4 \end{pmatrix}$$

Only if single cycle ⇒ Permutation cyc

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 4 & 2 & 5 & 1 & 7 \end{pmatrix} = \underbrace{(1,6)\circ(2,3,4)}_{\text{product of disjoint cycles.}}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 2 & 3 & 4 & 5 & 1 & 7 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 2 & 5 & 6 & 7 \end{pmatrix}$$

$$\longleftarrow$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 4 & 2 & 5 & 1 & 7 \end{pmatrix}$$

- This breakdown is unique. only order may differ, and also should be in cyclic order.

(2) ⇒ Transposition - is cycle permutation ~~with~~ which contains only 2 positions.

$$(x, y, z, t) = (x,t)\circ(x,z)\circ(x,y)$$

$$= (1,6)\circ(2,4)\circ(2,3)$$

$$(2,3,4)$$

$(1,5,3,2) = (1,2)\circ(1,3)\circ(1,5)$

If No. of transpositions in breakdown is Even $\rightarrow$ Even Permutation.

If No. is Odd $\rightarrow$ Odd Permutation.

$\Rightarrow$

Composition of 2 Even $\rightarrow$ Even Permut.
comp. of Even & Odd $\rightarrow$ Odd Permut.
Comp. of 2 Odd $\rightarrow$ Even Permut.

Abelian gp properties -

① A gp. $(G, *)$ is Abelian, iff,
(Two way)

$$\boxed{(g*h)^2 = g^2 * h^2} \quad \forall g, h \in G$$

② If in a gp. $(G,*)$, $\forall g \in G$, and
(One way)
$g \in g^{-1}$ then, G is abelian.

(i.e. every element has its own inverse $\rightarrow$ Abelian

- It is not true vice-versa.

$a^2 = a*a$

Proof -

① L.H.S, $= (g*h)^2 = (g*h)(g*h)$
$= g*(h*g)*h.$
$= g*(g*h)*h$
$= (g*g)*(h*h)$
$= g^2 * h^2 = R.H.S.$

$\Rightarrow \quad a * b = a + b$

$(a+b)^2 = a^2 + b^2$ is true multi gp.

$(a+b)^2 = (a+b) + (a+b) = (a+a) + (b+b)$
$= a^2 + b^2$

$\Rightarrow (R - \{0\}, *)$

$\hookrightarrow (a * b)^2 = a^2 * b^2$

$(a \times b)^2 = a^2 \times b^2$

$(a \times b) \times (a \times b) = (a \times a) \times (b \times b) = a^2 \times b^2$

Proof-

② $\Rightarrow$

$\oplus$	0	1
0	0	1
1	1	0

$\Rightarrow$ Every element is its own inverse

$\Downarrow$

Abelian gp.

i.e. diagonals will be identity element, e.

Abelian But converse is not true, as, $(z, +)$ is Abelian but each element is not its own inverse.

$a * a$ **Q.** G is abelian iff -

a. $g = g^{-1} \ \forall g \in G$

b. $g^2 = g \ \forall g \in G$

c. $(g o h)^2 = g^2 o h^2$ ✓ is correct answer.

d. None

5-

→"Abelian gp also called as commutative gp."

Properties of Gps – (Imp)

1. Order of gp.
2. finite & infinite gp.
3. Basic properties of gps.
4. Powers of an element of a gp.
5. Order of an element of a gp.
6. Cyclic gp.
7. Subgp.
8. Normal subgps.
9. Lagrange's Theorem
10. Homomorphism & Isomorphism of gps.

(1) Order of gp – In G, G(G, *),

$$\boxed{O(G) = |G|}$$

$\oplus$	0	1
0	0	1
1	1	0

$\Rightarrow |G| = 2$

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\Rightarrow |G| = 3$

$(Z, +) \Rightarrow |G| = \infty$

$\boxed{\text{Minimum order of gp} = 1}$ → due to identity element.

	e
e	e

(2) Finite & Infinite gps –

Order is finite → Infinite order.

$(Z_m, +_m) \to$ finite

$(\{1,2,3,\dots p-1\}, \times_p) \to$ finite

$(R, +) \to$ Infinite.

(3) →

– Id. is unique.

– Inverse is unique for given element

– $(a^{-1})^{-1} = a$

– $(ab)^{-1} = b^{-1} * a^{-1}$

$\hookrightarrow (a*b)^{-1}$

let $(Z, +)$ → $(2*3)^{-1} = 3^{-1} * 2^{-1}$

$2+3 \downarrow$; $5^{-1} = 5$; $-3 + -2 \to -5$

(4) – $ax = ay \Rightarrow x = y \to$ Left Cancellation

– $xa = ya \Rightarrow x = y \to$ Right Canc.

As, In monoid it is restricted. In $(R, \times)$, 0 does not have inverse. $0 \times 5 = 0 \times 4$

$5 = 4$ → Not True.

– $ax = b$ has unique sol^n.

$\hookrightarrow \boxed{x = a^{-1} * b}$ → because inverse is unique. and in binary op^n sol^n is always unique

$ya = b$ also has unique sol^n as,

$$\boxed{y = b * a^{-1}}$$

Consequence of this, in gp every coloumn & row must be permutation of G.

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

→ each element is distinct.

i.e. if $aa = a$, $ab = a$ } not unique. → Clearly not allowed in gp.

$b * a = b$, $c * a = b$ } → Not allowed.

$\boxed{\text{i.e. gp op}^n \text{ table can't have repeatition.}}$

*In case of distinct table, may or may not be gp.

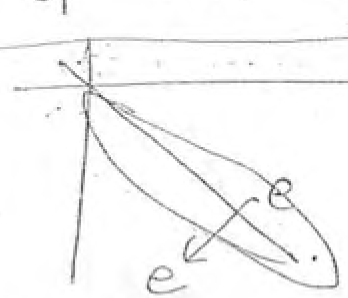
⇒

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Q. Fill the blanks.

↘ due to should not repeat.

→ $\boxed{\text{Gp Op}^n \text{ Table of finite gp} \rightarrow \text{Cayley Table.}}$



if in upper of diagonal ∃ any e, then its mirror place should also have e

	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

Shift everything to LF

$\Downarrow$

always be gp.

Shortcut method to make gp.

* When matrix is symmetric in Cayley Table $\rightarrow$

Always be Gp.

Powers of an element - $\forall a \in G$,

$a^0 = e \;,\; a^1 = a,\; a^2 = a * a$

$a^3 = a * a^2 = a^2 * a = a * a * a$

In $(Z, +) \rightarrow \; 2^0 = 0$ $\qquad a^{-1} = a^{-1}$

$3^2 = 3 + 3 = 6$ $\qquad a^{-2} = a^{-1} * a^{-1}$

$2^{-1} = -2$

$2^{-2} = -2 + -2 = -4$

$2^{-3} = -2 + -2 + -2 = -6$

$a^m * a^n = a^{m+n}$

$(a^m)^n = a^{mn}$

Order of gp - $\forall a \in G$,

$\Rightarrow O(a) =$ Smallest +ve integer which satisfies $a^n = e$ is order of a.

$\Rightarrow$ In $(Z, +)$,

$o(2) = \infty \rightarrow$ because not able to find any sol^n.

$2^n = 0 \rightarrow$ Never get

$2^1 = 2, \; 2^2 = 2+2 = 4 \; \ldots$

$o(0) = 1$ because, $0^n = 0 \rightarrow 0^1 = 0$.

$\Rightarrow \boxed{o(e) = 1 \text{ in any gp.}}$ True always.

as, other may have any order.

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\Rightarrow$ $o(a) = 1$

$o(b) = 3$

$o(c) = 3$

b is inverse of c. Hence, same

$b^1 = b$

$b^2 = b * b = c$

$b^3 = b * b^2 = b * c = a \rightarrow$ identity element

$c^1 = c$

$c^2 = c * c = b$

$c^3 = c * b = a$

$\Rightarrow \quad o(a) \leq o(G)$

$\leq 3.$

Properties of $o(a)$ –

1. $o(a) \leq o(G)$
2. $o(a) = o(a^{-1})$

sol, 3. $O(a*b) = O(b*a)$

get 4. $O(xax^{-1}) = O(a)$; $x, a \in G$

let $O(a) = t$, $a^t = e$

$\Rightarrow$ $O(xax^{-1}) = t$

$(xax^{-1})^t = e$

$= (xax^{-1})\cdot(xax^{-1})\cdot(xax^{-1}) \cdots\cdots$ t times

$= x(a(x^{-1}x)a(x^{-1}x)ax^{-1} \cdots$

$= x\, aeaea \cdots$

$= xa^t x^{-1} = xex^{-1} = xx^{-1} = e$

rse
c
e, same

$\Rightarrow$ $O(x^{-1}ax) = O(a)$

5. if $O(a) = m$, then $a^x = e$, iff m/x.

let $O(a) = m$, $a^m = e$

$\Rightarrow$
$a^x = e \rightarrow$ x should be multiple of m.

like- $O(b) = 3$, $b^3 = a$ $b^6 = a$ $b^9 = a$

as, $b^5 \neq a$ because 5 is not multiple of 3.

like- $b^{mn} = (b^m)^n = (e)^m = e$

eg- if $O(a) = 5$, then which can't be e -

a. a^6 b. a^7 c. a^8 d. a^{10}

Ans. a^{10}. as 10 is multiple of 5.

6. $O(a) = m \Rightarrow O(a^{x}) = m$, if x is relatively prime to m.

if $O(a) = 5$, $O(a^3) = ?$

$O(a^3) = 5$ because 3 & 5 are relatively prime.

$O(a^7) = 5$, 5 & 7 are relatively prime.

6. <u>Cyclic Gp:-</u>

$\Rightarrow (G, *)$ is cyclic gp. iff $\exists\, a \in G$, such that

$$\boxed{\forall g \in G, \quad g = a^n}$$

a = generator of gp.

*A cyclic gp can have more than one generator.

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

- a can't be gen.

b is gen.

$\Rightarrow b^0 = a$
$b^1 = b$
$b^2 = b * b = c$

c also gen.

$\Rightarrow c^0 = a$
$c^1 = c$
$c^2 = c * c = b$

* Identity can never be generator.

$\Rightarrow$ b & c both are generator in this case.

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$(\{1,2,3,4\}, \times_5)$ modulo 5

$$\left.\begin{array}{l} 2^0=1 \\ 2^1=2 \\ 2^2=4 \\ 2^3=3 \end{array}\right\} 2 \text{ is gen.}$$

$$\left.\begin{array}{l} 3^0=1 \\ 3^1=3 \\ 3^2=4 \\ 3^3=2 \end{array}\right\} 3 \text{ is gen.}$$

$$\left.\begin{array}{l} 4^0=1 \\ 4^1=4 \\ 4^2=1 \\ 4^3=4 \end{array}\right\} \text{can't gen. hence not gen.}$$

$\Rightarrow$ There are two generators 3 & 2.

① If a is gen. of cyclic gp., a^{-1} also is gen. of cyclic gp.

As, 2 & 3 are inverse of each other. Hence, They both are gen.

$(Z, +)$ & $1^{-1}=-1$ $1^{-2}=-2$ $1^0=0$ $1^1=1$ $1^2=1+1=2$ $\Rightarrow$ is cyclic gp. as 1 is gen.

$\Downarrow$

So, (-1) is also gen.

$$\left.\begin{array}{l} 2^0=1 \\ 2^1=2 \\ 2^2=4 \end{array}\right\} \text{It can't gen.}$$

That, is,

* cyclic gp. always have two generators.

$(-1)^{0} = 0$ $\qquad (-1)^{-1} = 1$

$(-1)^{1} = -1$ $\qquad (-1)^{-2} = (-1)^{-1} + (-1)^{-1} = 1+1 = 2$

$(-1)^{2} = -2$

Generates gp.

$\Rightarrow (R - \{0\}, \times)$ is Not cyclic gp.

$2^2 = 4$, $3^2 = 9$, $1^2 = 1$, $0^2 = 0$ $\Rightarrow$ No one is generating gp.

$2^3 = 8$, $1^3 = 1$

$\Rightarrow (R, +)$ is Not cyclic gp.

(2) If a finite gp. of order n, contains an element of order n, then gp. is cycle.

$\Rightarrow$

	a	b	c
a			
b			
c			

if $\exists$ any element of order of gp. as same size of gp., then those elements will be generators.

Prob- Gp. of 5 elements, then $O(a) = 5$ then what type of gp.

$\Rightarrow$ cyclic Gp.

3) Every gp. of prime order is cyclic.

$\Rightarrow$ 3, 5, 7, 11 ...... always be cyclic.

4) Cyclic gp. is always Abelian Gp.

* Converse is not true.

Like $(R,+)$ is abelian but not cyclic gp.

$\Rightarrow \forall a,b \in G, \quad a*b = b*a$

L.H.S., $\quad a*b = g^m * g^n = g^{m+n} = g^{n+m}$

$= g^n * g^m$

$= b*a = R.H.S.$

7. Subgp :-

Every subgp of cyclic gp. is always cyclic gp.

Every (n, n^{th} roots of unity, x) is a cyclic gp.

5) The (n, n^{th} roots of unity, x) is a cyclic gp.

$\Rightarrow$ Take, $\sqrt{1}$, $(\{1,-1\}, x)$

x	1	-1
1	1	-1
-1	-1	1

id = 1

gen = -1

$(-1)^0 = 1$

$(-1)^1 = -1$

$\Rightarrow (\{1, \omega, \omega^2\}, x) \rightarrow$ cyclic gp.

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$\Rightarrow$ Gen = ω, ω^2

$\omega^0 = 1$

$\omega^1 = 1$

$\omega^2 = \omega^2$

$(\omega^2)^0 = 1$

$(\omega^2)^1 = \omega^2$

$(\omega^2)^2 = \omega$

$\Rightarrow (1, -1, i, -i)$

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

<u>Subgp</u> - $\boxed{(H, *) \in (G, *) \text{ iff } H \subseteq G \text{ and } H \text{ must be a gp.}}$

i.e., id. should be same in H & G.

$\Rightarrow (Z, +)$ has subgp. $(E, +)$

I. $(E, +) \rightarrow$ Subgp.

II. $(O, +) \rightarrow$ Not gp. because of closure prop. not closed under +.

III. $(3Z, +) \rightarrow$ Subgp.

$\hookrightarrow (KZ, +)$ also subgp, where K is integer.

$(Z, +)$ also subgp. of itself.

<u>Always be Subgp</u>:-

I. $(G, *)$ } Trivial subgp.

II. $(\{e\}, *)$

$(Z, +) \rightarrow \begin{cases} (Z, +) \\ (\{0\}, +) \end{cases}$ } Trivial

- Other than trivial is known as Proper Subg

In case of finite sets - $\{\{0,1,2,3\}, +_4\}$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\Rightarrow$ Subgp.

I. $\{0,1\}, +_4$

II. $\{0,2\}, +_4$ $\rightarrow$ is only subgp of above

III. $\{1,2\}, +_4$ * 0 will always be present due to identity.

IV. $\{0,1,2\}, +_4$

as I, III, IV are not closed under $+_4$.

NOTE -

$\Rightarrow$ Order of subgp divides the order of Gp.

ie in this case for 4,

1, 2, 4 are divisiable of 4.

$(\{0\}, +)$, $(\{0,1,2,3\}, +)$

Trivial

$2+2=0$
$0+0=0$

inverse of $0 \rightarrow 0$, 2 is 2.

$\rightarrow$ So, No. of Subgroups $= 3$

No. of Proper subgps $= 1$

Subgrps are closed under $\cap$, not under union.

prop.

eger.

Subg

Q. How many subgp of order of gp 17 ?

⇒ Only 2 subgps. Due to 17 is prime no. Because only trivial subgps will be there.

Normal Subgps –

(left) ↓ columns, (Right) → rows

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$+_4$	0	2
0	0	2
2	2	0

→ Subgp is Normal, iff $aH = Ha \quad \forall a \in G$.

Where aH = left coset of H determined by a; Ha = Right coset of H determined by a.

⇒
$$0H = \{0,2\} \qquad H0 = \{0,2\}$$
$$1H = \{1,3\} \qquad H1 = \{1,3\}$$
$$2H = \{2,0\} \qquad H2 = \{2,0\}$$
$$3H = \{3,1\} \qquad H3 = \{3,1\}$$

Cosets –

$$\Rightarrow \left[\begin{array}{l} aH = \{a * x \mid x \in H\} \\ Ha = \{x * a \mid x \in H\} \end{array}\right]$$

⇒ $\{0,2\}$ is Normal subgp.

↳ Also proper normal subgp.

→ Normal subgp need not to be prop;

⇒

$+_4$	0
0	0

→ Also trivial Normal subgp.

$0H = \{0\}$

$1H = \{1\}$

$2H = \{2\}$

$3H = \{3\}$

→ Every subgp has to be Abelian, if it is Normal subgp.

So, if any gp is Normal, its subgp will always be Normal subgp. Only check for Non-abelian gp.

Lagrange's Theorem – for any finite gp;

1. $O(H) / O(G)$
2. $O(a) / O(G)$
3. In any finite gp, if $|G| = n$, then $\forall a \in G$,

$$a^n = e.$$

Also,

$$\Rightarrow \boxed{O(HK) = \frac{O(H) \cdot O(K)}{O(H \cap K)}}$$

H & K be any two subgp of finite gp G,

$$HK = \{x \in G \mid x \in hk;\ h \in H,\ k \in K\}$$

= O(HK) basically determines how many elements in HK.

Homomorphism & Isomorphism -

Homomorphism - $(G, *_1)$ & $(G', *_2)$

f : Mapping from $G \to G'$, defined as,
$G \to G'$

$\Rightarrow \boxed{f(a *_1 b) = f(a) *_2 f(b)}$

$\Rightarrow$ i.e., if $\overset{G}{a} \to \overset{G'}{f(a)}$ & $\overset{G}{b} \to \overset{G'}{f(b)}$

then, $\Rightarrow a *_1 b \to f(a) *_2 f(b)$

Let, $(R, +) \to (R^+, \times)$

Now, $f(x) = e^x$ make fun^n. Check whether homomorphism or not?

$\hookrightarrow$ Check, $f(a *_1 b) = f(a) *_2 f(b)$

$$f(a+b) = f(a) * f(b)$$
$$e^{a+b} = e^a \cdot e^b = e^{a+b}$$

$\Rightarrow$ Homomorphism.

$\Rightarrow$ Check $(R^+, \times) \to (R^+, \times)$

$$f(a \times b) = f(a) \times f(b)$$
$$e^{ab} = e^a \cdot e^b = e^{a+b}$$
$$e^{ab} \neq e^{a+b} \Rightarrow \text{Not Homomorphism}$$

$\Rightarrow$ Check, $(R^{1}, \times) \longrightarrow (R^{1}, \times)$

$f(x) = x^2$?

$$\Rightarrow \quad f(a *_1 b) = f(a) *_2 f(b)$$

$$f(a \times b) = f(a) \times f(b)$$

$$(ab)^2 = a^2 \cdot b^2$$

$\Rightarrow$ Homomorphism.

$\Rightarrow (Z, +) \longrightarrow (\{1, -1\}, \times)$, ~~f(x)~~

$$f(x) = 1 \quad ; x \text{ is even}$$
$$= -1 \quad ; x \text{ is odd}$$

$$f(a *_1 b) = f(a) *_2 f(b)$$

$$f(a+b) = f(a) \times f(b)$$

a b	a+b	f(a+b)
E E	E	+1
E O	O	−1
O E	O	−1
O O	E	+1

So,

f(a)	f(b)	f(a)·f(b)
1	1	1
1	−1	−1
−1	1	−1
−1	−1	1

$\Downarrow$

Homomorphism.

Properties of Gp. Homomorphism – $G \to G'$, $e \to e'$

1. $f(e) = e'$
2. $f(a^{-1}) = [f(a)]^{-1}$
3. If $S \subseteq G$, then $f(S) \leq G'$

Isomorphism –

1. Epimorphism → Homo + Onto
2. Monomorphism → Homo + One to One
3. Isomorphism → Homo + Bijection

$\Rightarrow (R, +) \to (R^+, \times)$; $f(x) = e^x$

Put $y = e^x \to x = \log_e y \Rightarrow$ Onto
$\forall$ any +ve y, log value always defined.

let $e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2 \Rightarrow$ One to One

$\Rightarrow$ finally it is Isomorphism.

$\Rightarrow (R^+, \times) \to (R^+, \times)$, $f(x) = x^2$

Put $y = x^2 \to x = \sqrt{y} \Rightarrow$ Onto
→ Always real.
for –ve undefined but R^+ doesnot contain –ve.

$x_1^2 = x_2^2 \Rightarrow x_1 = x_2$ (because doesnot have –ve)

$\Rightarrow$ finally Isomorphism.

$\Rightarrow (Z, +) \rightarrow (\{1, -1\}, \times)$

$f(x) = 1$; if x is even
$= -1$; if x is odd

→ Onto funⁿ
→ Many to One because all even & odd go to ~~one~~ +ve 1 as well as -ve 1.

So, Not One to one.
⇒ This is Epimorphism.

<u>Kernel of ~~H~~omomorphism</u> — $f_{G \rightarrow G'}$

Kernel $(f) = \{x \in G \mid f(x) = e'\}$

$\Rightarrow (R^+, \times) \rightarrow (R^+, +)$; $f(x) = x^2$.

$f(x) = 0 \Rightarrow x^2 = 0 \rightarrow x = 0$

So, Kernel $(f) = \{0\}$.

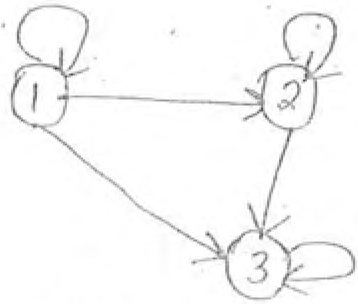
In case of Isomorphism, x will always be Identity.

$\Rightarrow (Z, +) \rightarrow (\{1, -1\}, \times)$; $f(x) = 1$, even
$= -1$, odd

Kernel $(f) = \{x \mid x = 2z, z \in Z\}$

Poset, Lattice & Boolean Alzebra

Partial order Total order well order
- Poset, Toset, Woset
- Hasse Diagram
- External elements of poset
- Dual poset
- lattice
- Types of lattice
- Sublattices, Semilattices
- Boolean alzebra

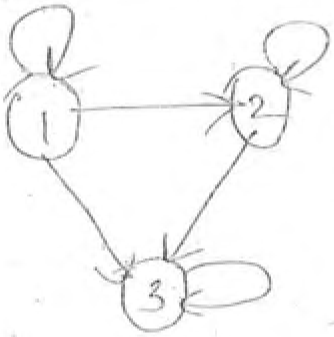


Poset – $(S, \leq)$

$\Rightarrow$ x / y is Poset.
$\downarrow$
Reflexive & Antisymm. : Poset
and fully transitive.

Toset – Has to be Poset + every pair has to be comparable. $\forall x, y \in S$.

$$\boxed{x \leq y \text{ or } y \leq x.}$$

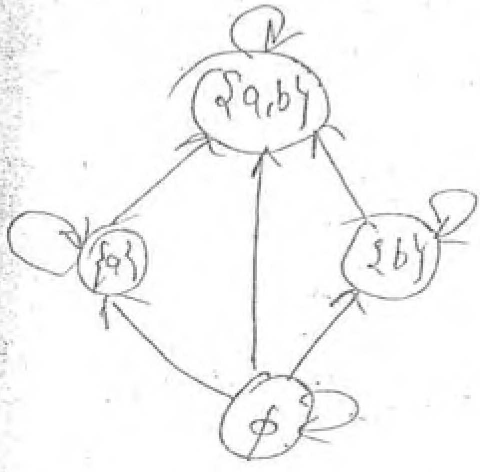
$\Rightarrow$  $\Rightarrow$ Toset

$\Rightarrow$ $(P(S), \leq) \Rightarrow$ Poset
Not Toset because $\{a\} \not\subseteq \{b\}$
a & b called incomparible.

$\Rightarrow \{\{\phi, \{a\}, \{a,b\}\} \Rightarrow$ Toset

In poset, it is not necessary, that every element should be related.

Toset will be always straight chain.



Poset

Toset

$\Rightarrow (Z, \leq) \rightarrow$ Toset & Poset

$\Rightarrow (Z, /) \rightarrow$ Poset, Not Toset because 2 does not divides 3 as 'int'

$\Rightarrow (\{1,2,4,8\}, /) \rightarrow$ Toset

$\Rightarrow (\{1,2,3,4,8\}, /) \rightarrow$ Poset

Woset – Has to be Toset + every subset of S must have least element.

$\Rightarrow (Z, \leq) \longrightarrow$ Not Woset

$\Rightarrow (Z^+, \leq) \rightarrow$ Woset, as least element is 1.

$\Rightarrow (R, \leq) \rightarrow$ Not Woset.

$\Rightarrow (R^+, \leq) \rightarrow$ Not Woset

$\Rightarrow (1 \leq x \leq 2, \leq) \rightarrow$ Not Woset

because even if set has least element as 1, but its subset $1 < x < 2$ does not have any least element.

It is uncountably infinite set.

So, Check (Poset + Comparable) $\xrightarrow{\text{ess}^n \text{ set}}$

Toset + Least Element + Discrete sets $\Rightarrow$ Woset.

Real, Complex, Rational $\rightarrow$ does not have least elements.

$\Rightarrow (D_n, /)$

$D_n = \{$all +ve integral ~~devices~~ dividers of any int. $n\}$ $n \in Z$.

$D_{20} = (\{1, 2, 4, 5, 10, 20\}, /) \rightarrow$ Not Toset

$\Rightarrow ((1, 2, 20), /) \rightarrow$ Woset

$(Z^-, \geq) \Rightarrow$ Woset

least $\rightarrow$ does not mean smallest no. it is like starting pt.

$(Z^-, \leq) \Rightarrow$ Not Woset, because least element $= -\infty$

Hasse Diagram -

Product Partial Order -

$(x_1, y_1) \, R \, (x_2, y_2)$ iff $x_1 \le x_2$ & $y_1 \le y_2$

$\Rightarrow (1,2) \, R \, (2,3)$ ✓

$\Rightarrow (1,2) \, R \, (0,1)$ ✗

Anti-Symm:

$(x_1, y_1) \, R \, (x_2, y_2)$ & $(x_2, y_2) \, R \, (x_1, y_1) \Rightarrow (x_1, y_1) \Rightarrow (x_2, y_2)$

$\downarrow$ $x_1 \le x_2$, $y_1 \le y_2$; $\downarrow$ $x_2 \le x_1$, $y_2 \le y_1$ $\Rightarrow$ $\downarrow$ $x_1 = x_2$, $y_1 = y_2$

Transitivity:

$(x_1, y_1) \, R \, (x_2 y_2)$ & $(x_2, y_2) \, R \, (x_3 y_3) \Rightarrow (x_1, y_1) \, R \, (x_3, y_3)$

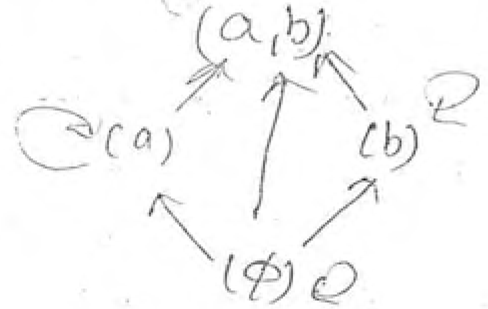
$\downarrow$ $x_1 \le x_2$, $y_1 \le y_2$; $\downarrow$ $x_2 \le x_3$, $y_2 \le y_3$; $\downarrow$ $x_1 \le x_3$, $y_1 \le y_3$

if AND → Only then holds transitivity
OR → No Transitivity will hold.

⇒ This is Poset, not Toset.

Hasse Diagram - Diagram of Poset can be reduced into simplified form, called Hasse diagram.

⇒ $(P(S), \subseteq)$



Steps -

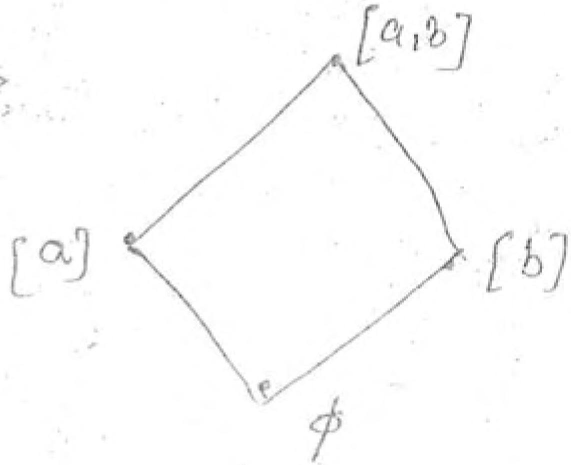
1. ○ ⟶ ·

2. Remove ○ loops.

3. All arrows pointing upwards.
   remove head of arrow ∧.

4. Remove all transitive arrows.

⟹

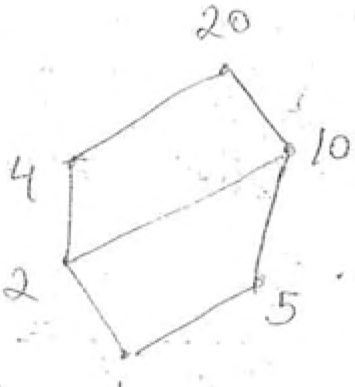


Prob. $S = \{a\}$, $P(S) = \{\phi, \{a\}\}$ $(P(S), \subseteq)$

⟹ Hasse Diagram,

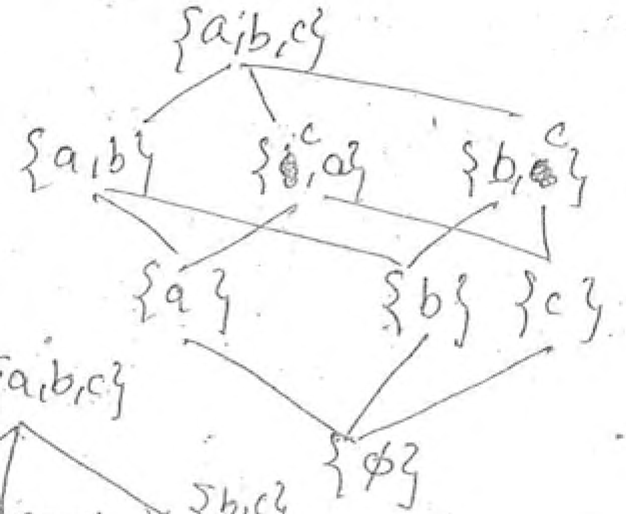
Prob. Draw H-Diagram $D_{20} = \{(1, 2, 4, 5, 10, 20), |\}$

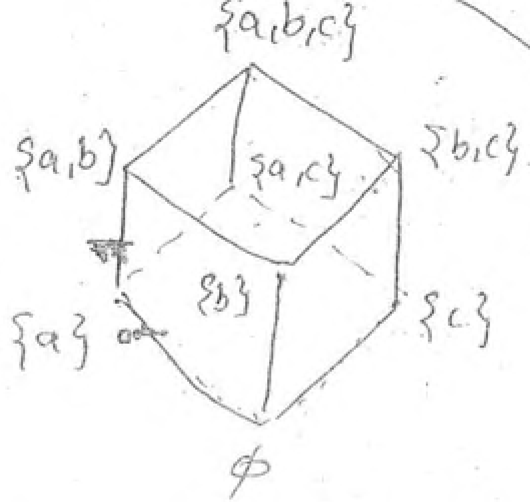
⟹



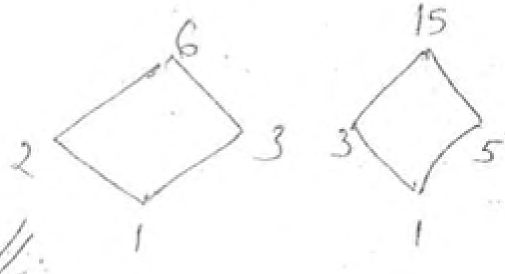
Prob $S = \{a,b,c\}$, $(P(S), \subseteq)$

$\Rightarrow$





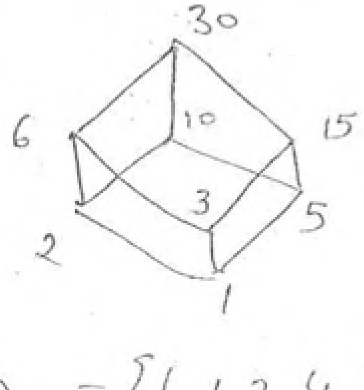
$\Rightarrow$ $D_6 = (\{1,2,3,6\}, /)$



called as

Homomorphic H-Diagram.

$\Rightarrow$ $D_{30} = \{(1,2,3,5,6,10,15,30), /\}$



$\Rightarrow$ $D_{20} = \{(1,2,4,5,10,20), /\}$

* D_{15} & $D_6 \Rightarrow$ Isomorphic Diagram.

Prob. How many edge in D_{30} ?

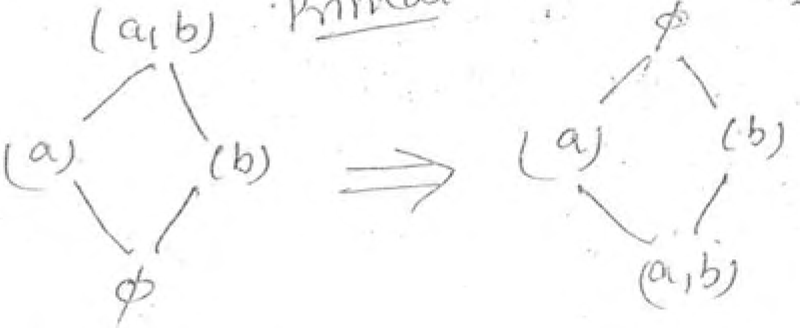
$\Rightarrow$ 12 edges; as due to cube structure.

<u>Dual Poset</u> – Inverse Rel^n exists.

like, if $(S, \leq) \rightarrow$ dual poset $(S, \geq)$

$(S, \subseteq) \rightarrow (S, \supseteq)$

Primal Dual

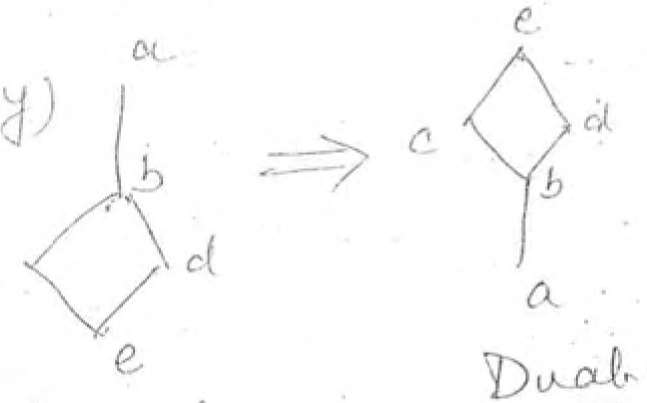


* Dual of dual $\Rightarrow$ Primal Poset.

$\Rightarrow (D_n, /) \rightarrow$ dual $(D_n$, x is divisible by y)

↓

(x divides y)



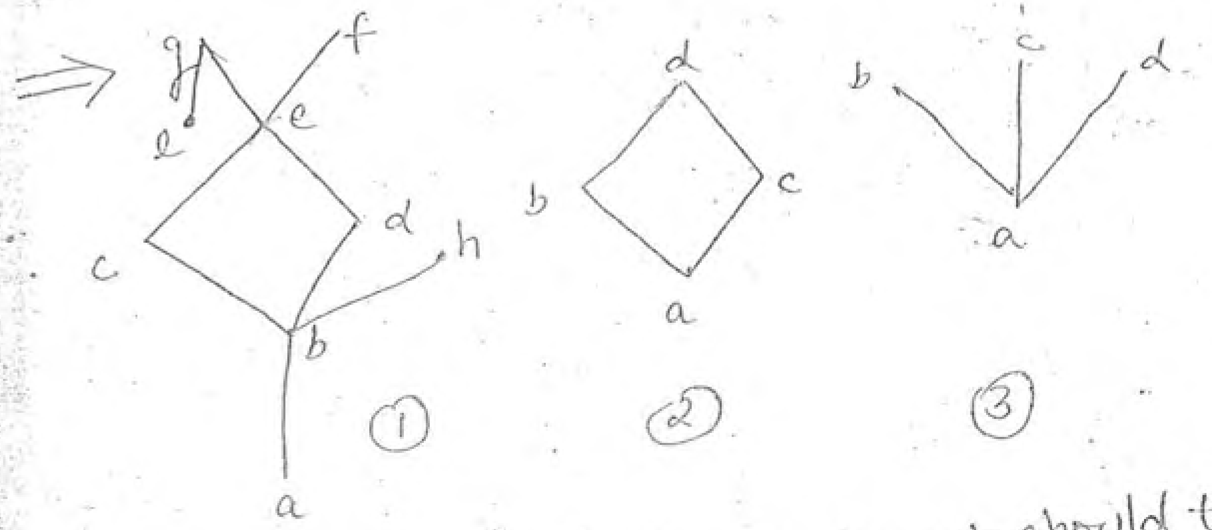
Primal Dual

<u>External elements of Poset</u> –

1. Find maximal elements
2. " minimal "
3. " greatest elements
4. least "
5. UpperBounds

6. Lower Bounds
7. LUB (Least Upper Bound)
8. GLB (Greatest Lower Bound)

$\Rightarrow$



Maximal → Above which no element should there.

i.e., $x \in S$ maximal iff $\nexists\, y \in S \mid x \le y$

① — $\{g, f, h\}$
② — $\{d\}$ maximal
③ — $\{b, c, d\}$

Minimal → Nothing below.

i.e., $x \in S$ minimal iff $\nexists\, y \in S \mid y \le x$

~~①~~, ②, ③ → $\{a\}$ → minimal.

① → $\{a, l\}$ → minimal

Greatest → Only one unique element. May not exist.

i.e., $x \in S$ greatest iff $\forall\, y \in S \mid y \le x$

Greatest always be from one of maximal.

① → No greatest element

②, ~~③~~ → d

③ → No.

Least →

$x \in S$ least iff $\forall y \in S \mid x \leq y$

if we have several minimal or maximum then we would never get least or greatest elements.

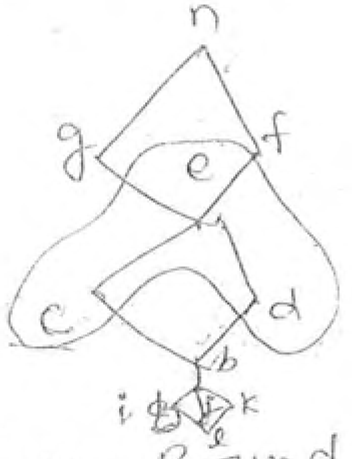
$$\left.\begin{array}{l} ① \to \text{No element} \\ ② \to a \\ ③ \to a \end{array}\right\} \text{least}$$

* In Boolean, least & greatest are defined by 0 and 1.

Upper Bound - let S, and $A \subseteq S$, Then

$UB(A) = \{ x \text{ is } UB(A) \text{ iff } \forall y \in A \mid y \leq x \}$

① $A = \{c, d, e\}$, $U(A) = \{e, g, f, n\}$



$L(A) = \{b, a, i, j, k, l\}$

Lower Bound - x is $LB(A)$ iff $\forall y \in A \mid x \leq y$

* LUB & GLB are always unique.

Least upper Bound -

$x \in S$ is LUB where $A \subseteq S$, iff

① $x \in UB(A)$

② $\forall c \in UB(A)$, $x \leq c$

* Nearest one.

Greatest Lower Bound - $x \in S$ in $GLB(A)$, iff

① $x \in LB(A)$

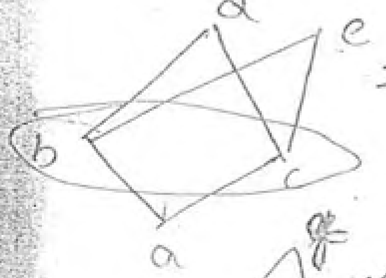
② $\forall c \in LB(A)$.

$(c, d\ b) \rightarrow LU = e$
$GL = b$

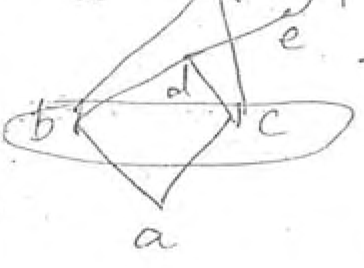
* First meeting pt always be LU & GL bounds. [Upward - LUB / Downward - GLB.]

$\{b, c, e, g\} \Rightarrow LU = g$
$GL = b$

* LUB & GLB may not exists.



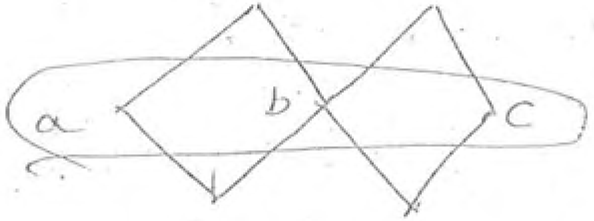
$\Rightarrow$ Upper bound = {d, e} due to incomparible. But not LUB.



$\Rightarrow$ Not LUB, but upper bounds = {e, f, d, g}

If we can't compare any two lower bounds, there, greatest lower bound will not exists.

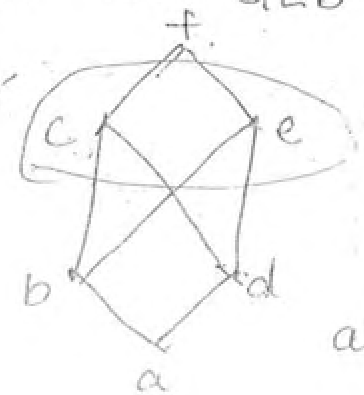
Prob.



LUB(a,b,c) → No LUB.

also no GLB of this.

Prob.

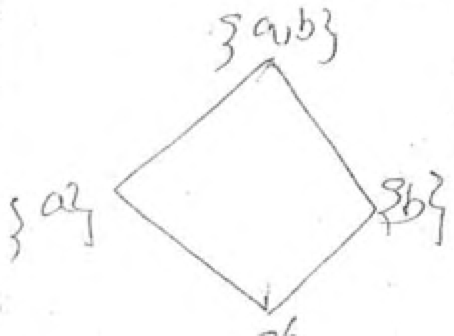


LUB = f

GLB = does not exists.

as, a can't be GLB, because we have to consider only nearest one.

Topological Sorting - By which Poset can be converted into Toset.



$\phi \leq \{a\} \leq \{a,b\}$

$\Rightarrow \phi \leq \{b\} \leq \{a,b\}$

convert it without violating exists order.

Also called compatible Toset → Topological Sorting of Poset.

- $\phi \le \{a\} \le \{a,b\} \le \{b\}$ ×
- $\phi \le \{b\} \le \{a\} \le \{a,b\}$ ✓
- $\phi \le \{a\} \le \{b\} \le \{a,b\}$ ✓
- $\{a\} \le \{b\} \le \phi \le \{a,b\}$ ×

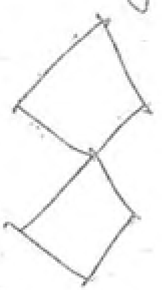
Steps of Sorting -

1. Select a minimal element
2. Delete that element.
3. Goto step 1.

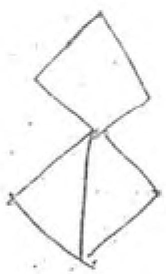
So, we get, $\phi \le \{b\} \le \{a\} \le \{a,b\}$.

or $\phi \le \{a\} \le \{b\} \le \{a,b\}$.

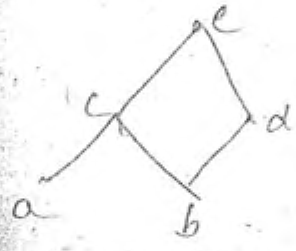
Prob: How many Topological sorting exists in following diagram -



$\Rightarrow 4$



$\Rightarrow 3! \times 2!$

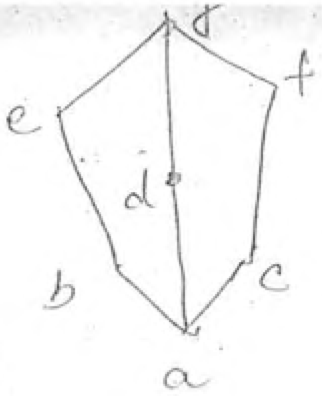


$\Rightarrow a \le 2!$

$b \le a \left.\begin{matrix} c\,d\,e \\ d\,c\,e \end{matrix}\right\} 2$

$b \le d \le a\,c\,e \} 1$

$\Rightarrow 2+2+1 = 5$ Topologic order.



$a \le b \le e \le d \le c\, f\, g$

$a \le c \le$

$a \le d$

$a \le b \le e \le c \le \begin{matrix} d\,f\,g \\ f\,d\,g \end{matrix} \Big] \; 2$

$a\,b\,d\,e\,c\,f\,g \to 1$ $\qquad$ $a\,b\,c \to 6$

$a\,b\,d\,c \le \begin{matrix} e\,f\,g \\ f\,g\,e \end{matrix} \Big\} \to 2$ $\qquad$ $a\,b\,e \to 3$

$a\,b\,d \to 3$

ac ⇒ same as ab.

$a\,d \to 2! \times 2! \Rightarrow \begin{matrix} a\,d\,b \le e \to 1 \\ a\,d\,b\,c \to 2 \end{matrix} \Big] \Rightarrow 3 \Big] \; 6$

$\Big] \Rightarrow 3$

$\Rightarrow 12 + 12 + 6 \Rightarrow 30$ Topological sorting

Theorem – The diagraph of Poset can't have any cycle of length more than one, other than self loop.

i.e. only allowed loop is self loop.

→ It is due to trans. & antisymm. property.

Proof let $a_1 \le a_2 \le a_3 \ldots \le a_n \le a_1$

using trans $\Rightarrow$ $a_1 \le a_n \le a_1$

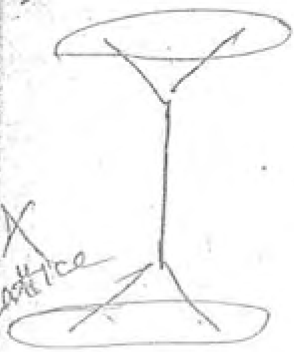
Now using ~~trans~~, antisymm.,

$a_1 \leq a_1 \leq - - - \leq a_1 \Rightarrow$ Self Loops only.

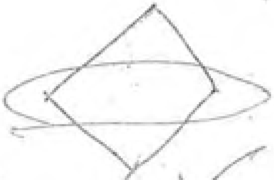
Lattice – A Poset is Lattice, iff

$\forall a, b \in S$, $LUB(a,b)$ & $GLB(a,b)$ must exists and should belong to S.

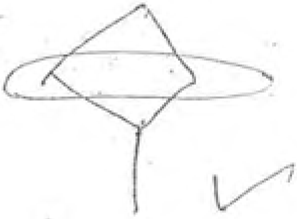
Prob: Which of poset is lattice?



X lattice



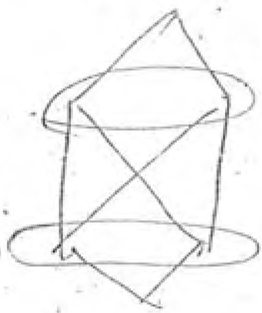
Lattice

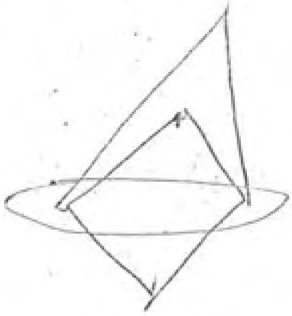


lattice

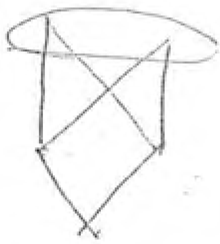
* Test only for incomparible Digram.
* open at anyside will never be lattice.
* But it is not necessary that closed will be lattice, we have example,

No LUB & GLB. ⇐

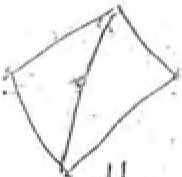




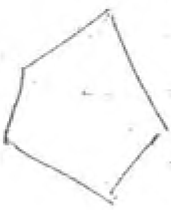
Not Lattice



Not lattice



⇓ Lattice



→ Pentagon Lattice.

⇓ lattice

Properties of Lattice $\Rightarrow$ (doesn't have distri prop.)

1. Commutative
2. Associative
3. Idompotent
4. Law of Absorption
5. Closure

$$LUB(a,b) = a \vee b \Rightarrow OR \rightarrow Join(a,b)$$
$$GLB(a,b) = a \wedge b \Rightarrow AND \rightarrow Meet(a,b)$$

LUB = Supremum (a,b) = Union
GLB = Infimum (a,b) = Intersection

(1) Commutative -
$a \vee b = b \vee a$; $\forall a,b \in S$
$a \wedge b = b \wedge a$

(2) Associative -
$a \vee (b \vee c) = (a \vee b) \vee c$; $\forall a,b \in S$
$a \wedge (b \wedge c) = (a \wedge b) \wedge c$;

(3) Idempotent -
$a \vee a = a$
$a \wedge a = a$; $\forall a \in S$

(4) Law of absorption -
$a \vee (a \wedge b) = a$; $\forall a,b \in S$
$a \wedge (a \vee b) = a$

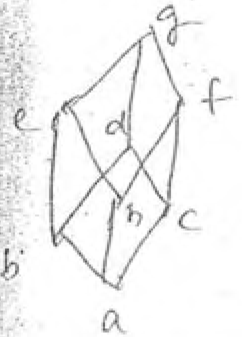
Need not be exists some properties -
- Bounded
- Distributed
- Complemented

$\rightarrow$ if we add these prop., Lattice becomes Boolean Algebra.

* $\boxed{a \le b \text{ iff } a \vee b = b \text{ \& } a \wedge b = a}$

$\left.\begin{array}{l} \Rightarrow a \le a \vee b \text{ \& } b \le a \vee b \\ \Rightarrow a \wedge b \le a \text{ \& } a \wedge b \le b . \end{array}\right]$

$\Rightarrow$ if $a \le b$ & $c \le d \Rightarrow a \vee c \le b \vee d$.
$a \wedge c \le b \wedge d$.



* complemented is only possible if set is bounded.

so, Complemented + Distributed $\Rightarrow$ Boolean Alzebra.

in lattice $\rightarrow (S, +, \cdot)$

$(S, +, \cdot, ')$
$\vee \quad \wedge \quad '/\sim$

6 properties of Boolean Alzebra -

- Closures
- Comm.
- Assoc.
- Distri
- Identity
- complement

in boolean $\Rightarrow$ idempotent & law of absorption also exists, but it is derived prop., so we don't include directly.

Types of lattice – $(S, \leq)$

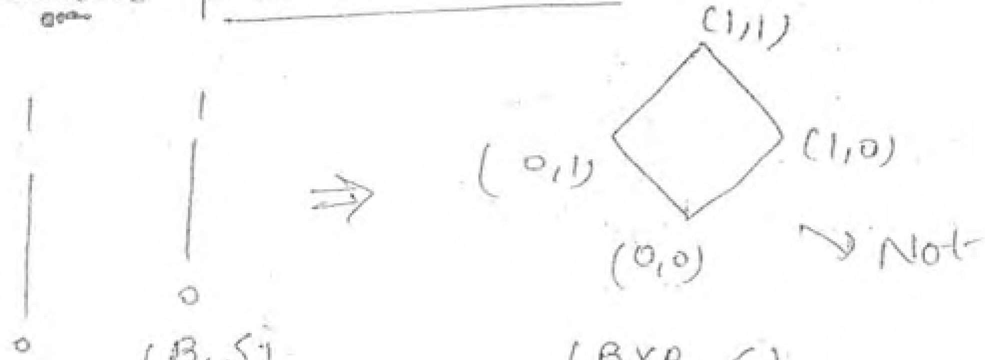
– Bounded lattice – iff $0, 1 \in S$ for entire set.

– Complemented

– Distributive

Lattice Results –

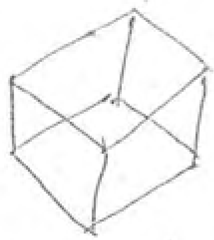
1. Dual of any lattice always a lattice.
2. Product of two lattice also a lattice, called product lattice.



$(B, \leq)$ $(B, \leq)$ $(B \times B, \leq)$

(Product partial order)

$(B, \leq)$ Boolean ⇒



$(B \times B \times B, \leq)$

These types of digraph are called n-cubes in BA.

3. Some standard lattices –

$(S, \leq)$, $(P(S), \leq)$, $(D_n, /)$, $(B^n, \leq)$, $(Z^+, /)$

$(S, \leq)$ unbounded; $(P(S), \leq)$ [ϕ, S], $(D_n, /)$ [$1, n$] Bounded; $(Z^+, /)$ also lattice but unbounded

$(B^n, \leq)$ is isomorphic to $(P(S), \leq)$

$n \Rightarrow |S|$

$(B \times B, /)$ $\rightarrow (00, 11)$

$(S, \leq)$ is not bounded, so not BA.

$(D_n, |)$ is always lattice but not true for all for BA.

i.e. $(D_n, |)$ is BA iff $n = 2^3 3^2 \ldots$

i.e. prime no. breakdown, all numbers should be unique.

so, $30 = 2 \times 3 \times 5$ → due to distinct, it is BA.

$20 = \underline{2} \times 2 \times 5$ → since repeation, so not BA.

Q. Which is BA ⇒ ?

D_{30} = B.A.

D_6 = B.A.

D_{27} = Not B.A.

D_{42} = B.A.

$3 \,|\, 27$, $3 \,|\, 9$, 3

$2 \,|\, 42$, $3 \,|\, 21$, 7

$2 \,|\, 6$, $3 \,|\, 3$, 1

$2 \,|\, 30$, $3 \,|\, 15$, 5

$(P(S), \subseteq)$ → Not BA

$(D_n, |)$ → sometimes BA.

Toset is always a lattice. Toset also known as Chain.

$(Z^+, |)$ ↓ also lattice but bound

∴ glb = gcd & lub = lcm

Every finite lattices are Bounded.

If we have, $(a_1, a_2, a_3 \ldots a_n)$

$LUB = a_1 \vee a_2 \vee \ldots \vee a_n$

$GLB = a_1 \wedge a_2 \wedge \ldots \wedge a_n$

Prop. of Bounded Lattice -

1- Identity prop. $\longrightarrow$ $a \vee 0 = a$, $a \wedge 1 = a$

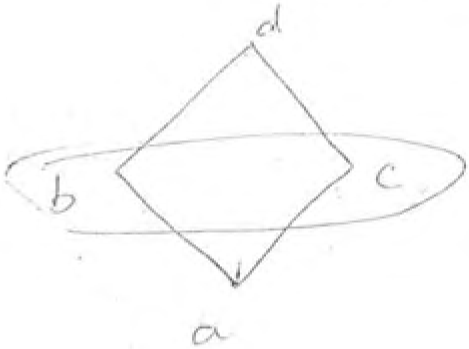
2- Dominating prop. $\longrightarrow$ $a \vee 1 = 1$, $a \wedge 0 = 0$

3- $0 \leq a \leq 1$, $\forall a \in S$

* In bounded, minimum 2 elements are required always.
* In lattice, minimum 1 element is sufficient.

Complemented Lattice - Lattice L is complemented, iff $\forall a \in L$, a must have at least one complement.

$$\Rightarrow \begin{array}{l} a \vee a' = 1 \\ a \wedge a' = 0 \end{array} \qquad 0' = 1 \ \& \ 1' = 0$$

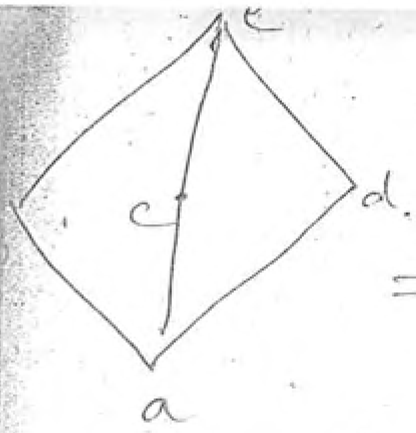


$\Rightarrow$

	comp.
a	d
b	c
c	b
d	a

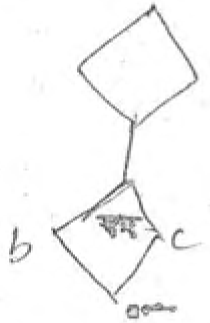
$\left.\begin{array}{l} b \ \ c \\ c \ \ b \end{array}\right\}$ as, complement never be related. other than 0 & 1.

i.e. only incomparible can be complement.



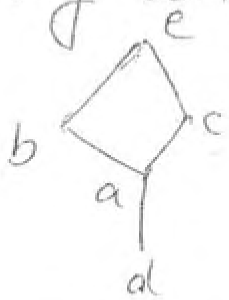
$\Rightarrow$

a	e
b	c, d
c	b, d
d	b, c
e	a



$\Rightarrow$ Does not have complement.

* In Neck type digram, there will not be any complemented.



$\Rightarrow$ as $b \vee c \Rightarrow a$ but we should get d.

also a have no complement.

$\Rightarrow$ Not complement.

* A Toset can be complemented, only if it has exactly two element.

<u>Distributive Lattice</u> – Lattice L is distri. iff

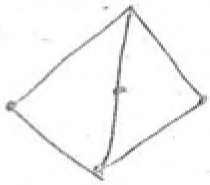
$\forall a, b, c \in S$,

$\Rightarrow$

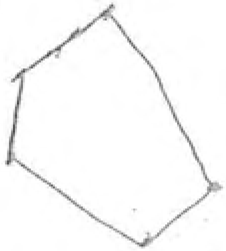
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

- A Lattice L is distri. iff it does not contain sublattice isomorphic to one (I) or two (II).



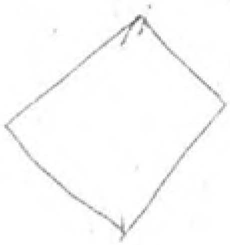
(I)



(II)

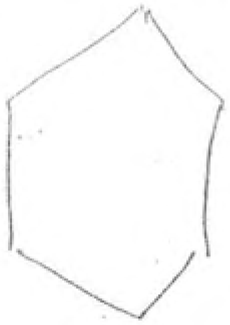
- These two are culplets, which destroys the prop. of distribution.

Eg-



⇒ ~~Not~~ distributive.

* Less than 5 pts. will always be distri.

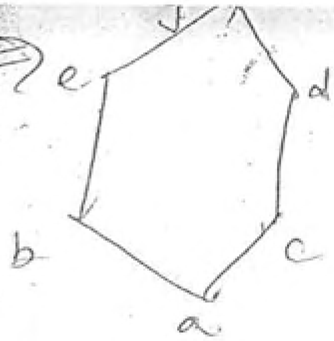


⇒ Not distr. because pentagon is sublattice of this.

Sublattice - Same LUB & GLB must exists.

* Every ~~ct~~ cycle with more than 5 pts. will not distri.

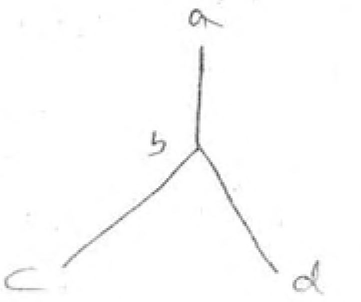
- If a complemented lattice is distri. then complement will be unique always.

(iv) 

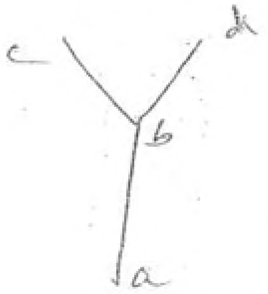
⇒ ∵ this complement but not have unique comp. hence it is not distri.

b's comp. ⇒ {c, d}

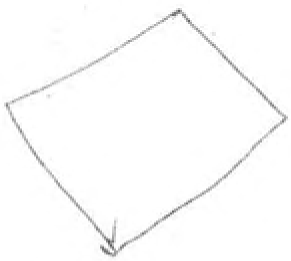
Semi-Lattice — Meet- Semi lattice / Join - Semi Lattice



Join semi latt.



Meet-semi latt.



⇒ Join & Meet Semilattice both.