COL 106: Data-structures

Course coordinators :

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Data-structures

Teaching assistants:

Two TAs will be available during lab hours.

Make use of TAs for resolving any problems regarding the course :

coding, understanding a particular concept, assignments, etc.

Evaluations components

Quiz : 15%

Minor Exam : 25%

Assignments : 25% (5-6 assignments)

Major exam : 35%

Assignments

You will be expected to program in Python

One programming assignment every 2 weeks

NO late submission (strictly enforced, reasons like illness will not be accepted)

NO COPYING FROM ANY SOURCE

(if caught copying, expect an "F" grade)

Course Information

Make sure you can access the course information from moodle.

Check course web-page for announcements.

Textbook: Data-structures and Algorithms, by Goodrich, Tamassia, Goldwasser.

Topics

Arrays

Lists

Abstract Data Types, object oriented concepts

Stacks, Queues

Trees : Binary trees, Balanced trees, B-trees

Strings : Tries, Matching algorithms

Sorting

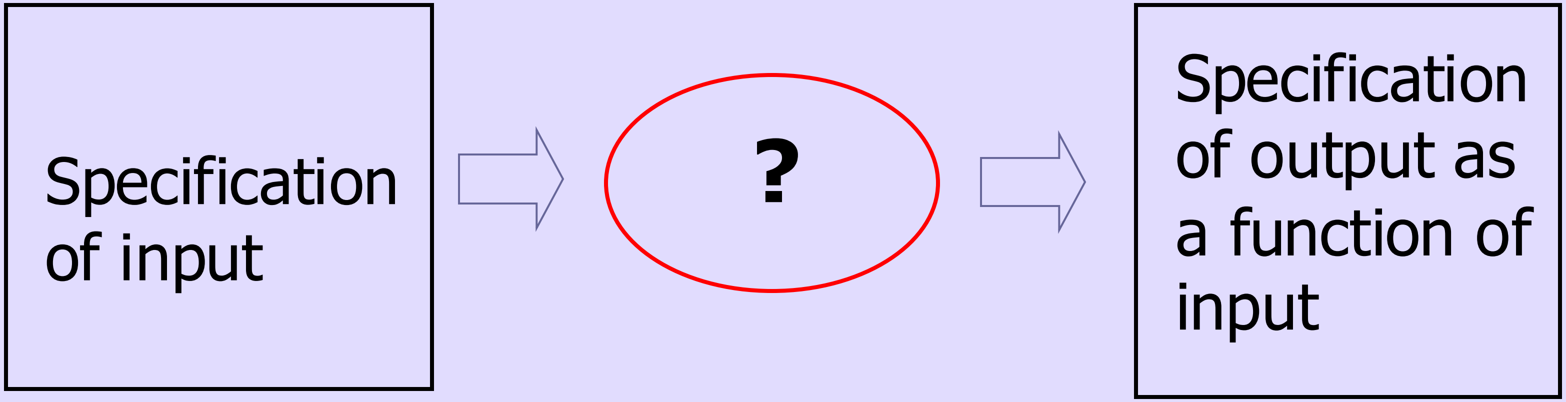
Hashing

Graphs

Data Structures and Algorithms

- Algorithm: Outline, the essence of a computational procedure, step-by-step instructions
- Program: an implementation of an algorithm in some programming language
- Data structure: **Organization** of data needed to solve the problem

Algorithmic problem



- Infinite number of input *instances* satisfying the specification. For eg: A sorted, non-decreasing sequence of natural numbers of non-zero, finite length:
  - 1, 20, 908, 909, 100000, 1000000000.
  - 3.

Algorithmic Solution

Input instance, adhering to the specification

Algorithm

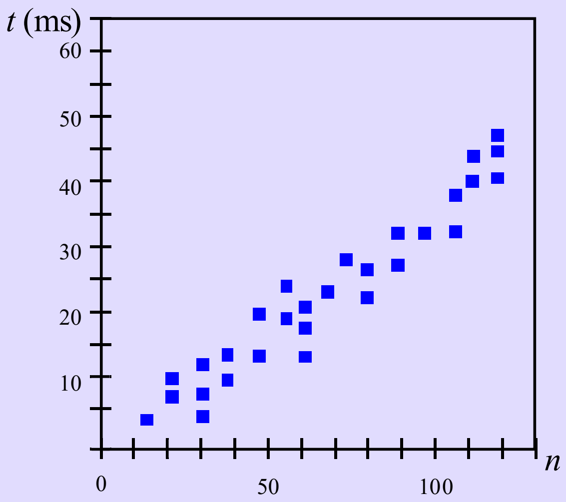
Output related to the input as required

- Algorithm describes actions on the input instance
- Infinitely many correct algorithms for the same algorithmic problem

What is a Good Algorithm?

- Efficient:
  - Running time
  - Space used
- Efficiency as a function of input size:
  - The number of bits in an input number
  - Number of data elements (numbers, points)

Measuring the Running Time



How should we measure the running time of an algorithm?

Experimental Study

- Write a program that implements the algorithm
- Run the program with data sets of varying size and composition.
- Use a system call to get an accurate measure of the actual running time.

Limitations of Experimental Studies

- It is necessary to implement and test the algorithm in order to determine its running time.
- Experiments can be done only on a limited set of inputs, and may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments should be used.

Beyond Experimental Studies

We will develop a general methodology for analyzing running time of algorithms. This approach

- Uses a high-level description of the algorithm instead of testing one of its implementations.
- Takes into account all possible inputs.
- Allows one to evaluate the efficiency of any algorithm in a way that is independent of the hardware and software environment.

Pseudo-Code

- A mixture of natural language and high-level programming concepts that describes the main ideas behind a generic implementation of a data structure or algorithm.
- Eg: **Algorithm** arrayMax(A, n):

  Input: An array A storing n integers.

  Output: The maximum element in A.

  currentMax ← A[0]

  for i ← 1 **to** n-1 **do**

    if currentMax < A[i] **then** currentMax ← A[i]

  return currentMax

Pseudo-Code

It is more structured than usual prose but less formal than a programming language

- Expressions:
  - use standard mathematical symbols to describe numeric and boolean expressions
  - use $\leftarrow$ for assignment ("=" in Java)
  - use = for the equality relationship ("==" in Java)
- Method Declarations:
  - **Algorithm** name(param1, param2)

Pseudo Code

- Programming Constructs:
  - decision structures: **if ... then ... [else ...]**
  - while-loops: **while ... do**
  - repeat-loops: **repeat ... until ...**
  - for-loop: **for ... do**
  - array indexing: **A[i], A[i,j]**
- Methods:
  - calls: object method(args)
  - returns: **return** value

Analysis of Algorithms

- **Primitive Operation:** Low-level operation independent of programming language. Can be identified in pseudo-code. For eg:
  - Data movement (assign)
  - Control (branch, subroutine call, return)
  - arithmetic an logical operations (e.g. addition, comparison)
- By inspecting the pseudo-code, we can count the number of primitive operations executed by an algorithm.

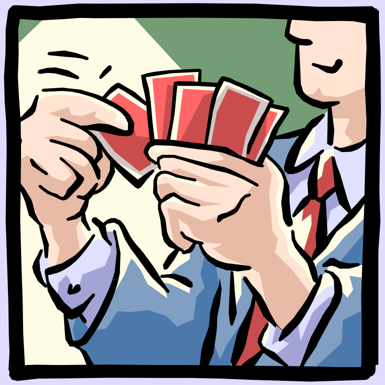
Example: Sorting

INPUT

sequence of numbers

$a_1, a_2, a_3, \ldots, a_n$

2 5 4 10 7



OUTPUT

a permutation of the sequence of numbers

$b_1, b_2, b_3, \ldots, b_n$

2 4 5 7 10

Correctness (requirements for the output)

For any given input the algorithm halts with the output:

- $b_1 < b_2 < b_3 < \ldots < b_n$
- $b_1, b_2, b_3, \ldots, b_n$ is a permutation of $a_1, a_2, a_3, \ldots, a_n$

Running time

Depends on

- number of elements (n)
- how (partially) sorted they are
- algorithm

Insertion Sort

A	3	4	6	8	9	7	2	5	1
	1					j→			n

← i

Strategy

- Start “empty handed”
- Insert a card in the right position of the already sorted hand
- Continue until all cards are inserted/sorted

INPUT: A[1..n] – an array of integers
OUTPUT: a permutation of A such that A[0]≤ A[1]≤ …≤A[n-1]

```
for j←0 to n-1 do
    key ←A[j]
    Insert A[j] into the sorted sequence
    A[1..j-1]
    i←j-1
    while i>=0 and A[i]>key
        do A[i+1]←A[i]
           i--
    A[i+1]←key
```

Analysis of Insertion Sort

	cost	times
for j←1 **to** *n-1* **do**	c_1	n
key←A[j]	c_2	n-1
Insert A[j] into the sorted sequence A[1..j-1]	0	n-1
i←j-1	c_3	n-1
while i>=0 **and** A[i]>key	c_4	$\sum_{j=2}^{n} t_j$
do A[i+1]←A[i]	c_5	$\sum_{j=2}^{n} (t_j - 1)$
i--	c_6	$\sum_{j=2}^{n} (t_j - 1)$
A[i+1] ← key	c_7	n-1

Total time $= n(c_1+c_2+c_3+c_7) + \sum^{n}_{j=2} t_j \; (c_4+c_5+c_6)$

$- (c_2+c_3+c_5+c_6+c_7)$

Best/Worst/Average Case

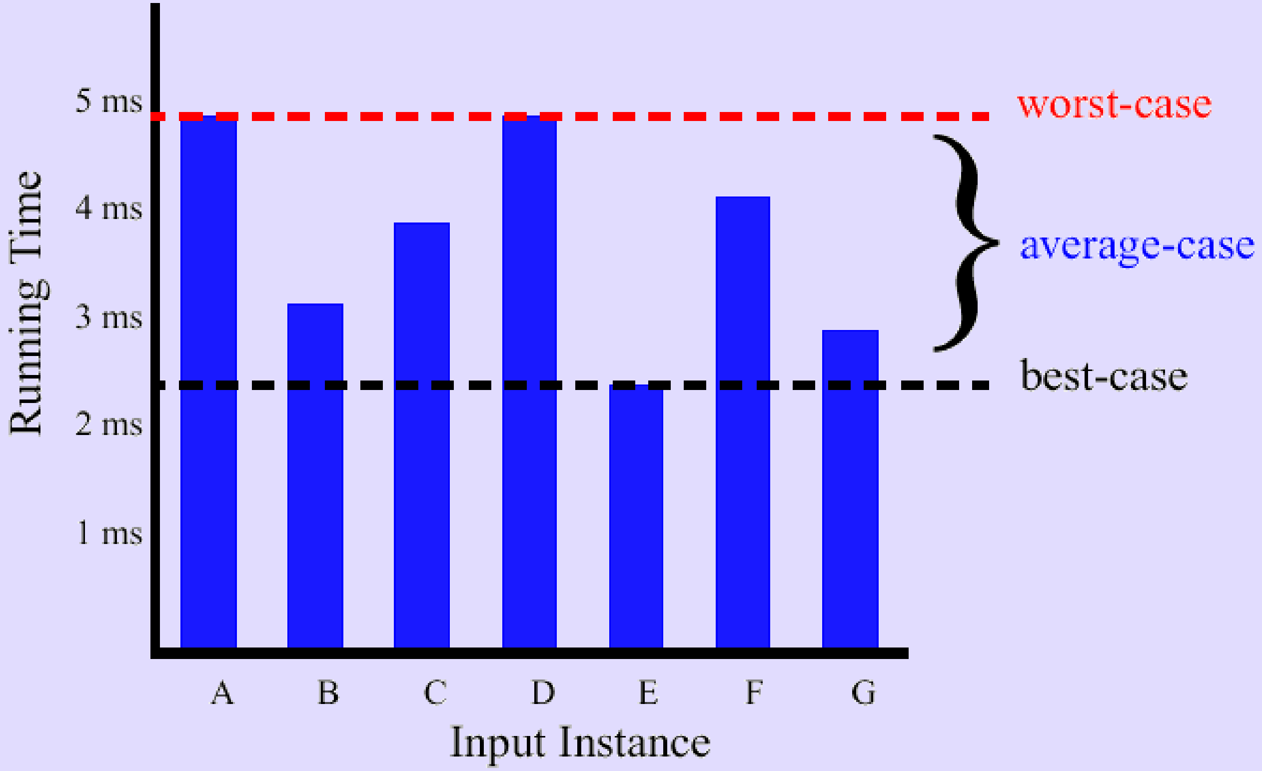
Total time = $n(c_1+c_2+c_3+c_7) + \sum^{n-1}_{j=1} t_j \; (c_4+c_5+c_6)$

$- (c_2+c_3+c_5+c_6+c_7)$

- **Best case**: elements already sorted; $t_j=1$, running time = f(n), i.e., *linear* time.
- **Worst case**: elements are sorted in inverse order; $t_j=j$, running time = $f(n^2)$, i.e., *quadratic* time
- **Average case**: $t_j=j/2$, running time = $f(n^2)$, i.e., *quadratic* time

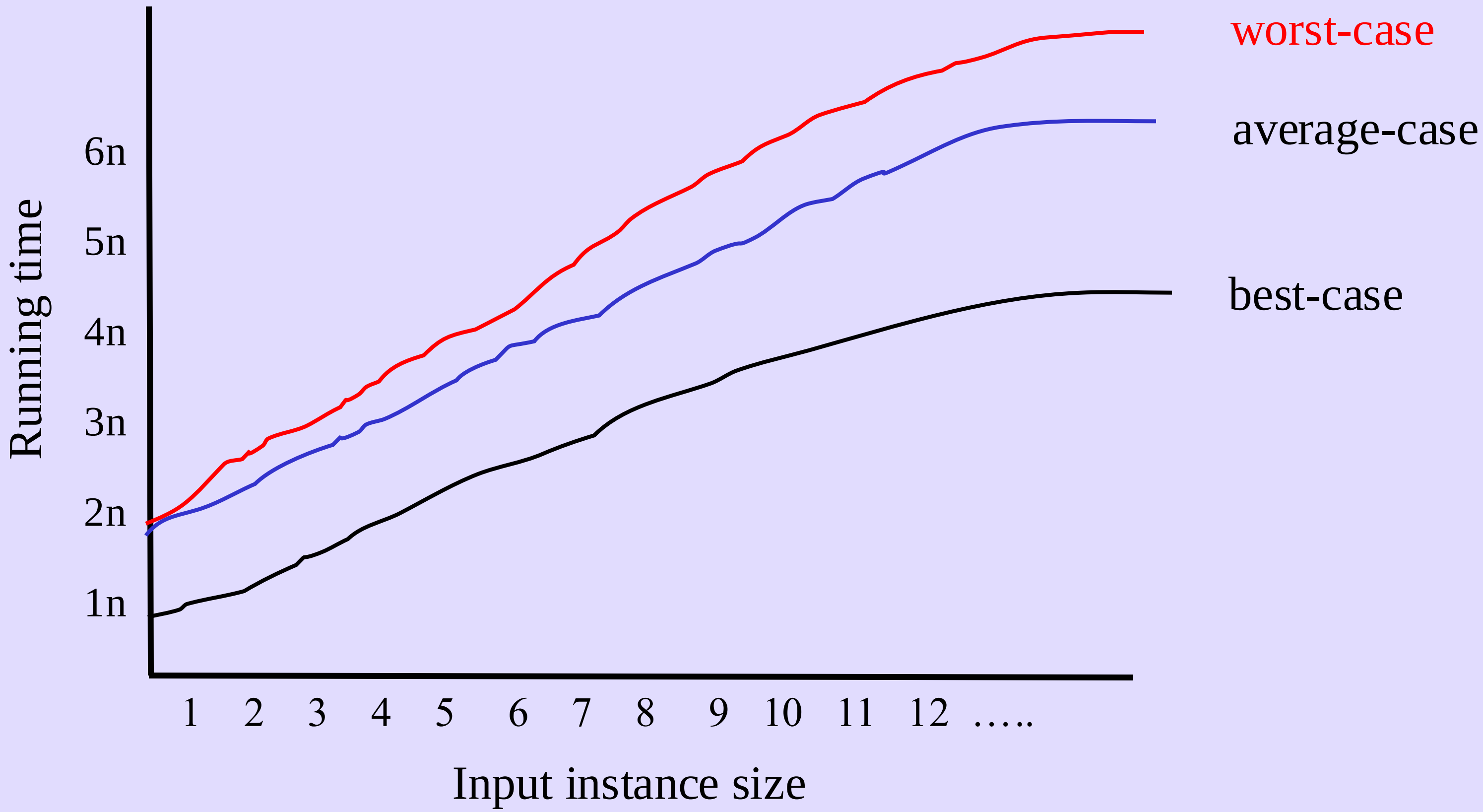
Best/Worst/Average Case (2)

- For a specific size of input n, investigate running times for different input instances:



Best/Worst/Average Case (3)

For inputs of all sizes:



Best/Worst/Average Case (4)

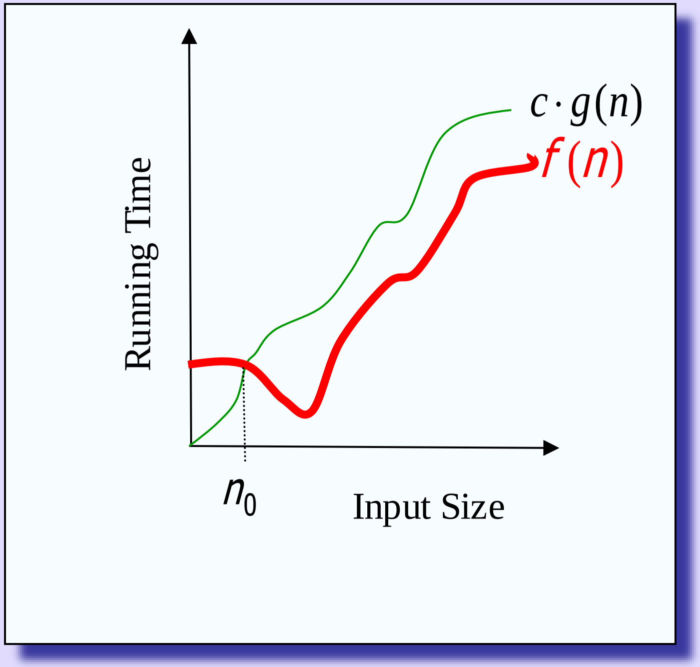
- **Worst case** is usually used: It is an upper-bound and in certain application domains (e.g., air traffic control, surgery) knowing the **worst-case** time complexity is of crucial importance
- For some algorithms **worst case** occurs fairly often
- **Average case** is often as bad as the **worst case**
- Finding **average case** can be very difficult

Asymptotic Analysis

- Goal: to simplify analysis of running time by getting rid of ”details”, which may be affected by specific implementation and hardware
  - like “rounding”: 1,000,001 $\approx$ 1,000,000
  - $3n^2 \approx n^2$
- Capturing the essence: how the running time of an algorithm increases with the size of the input *in the limit*.
  - Asymptotically more efficient algorithms are best for all but small inputs

Asymptotic Notation

- The “big-Oh” *O*-Notation
  - asymptotic upper bound
  - f(n) is O(g(n)), if:



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- The "big-Oh" *O*-Notation
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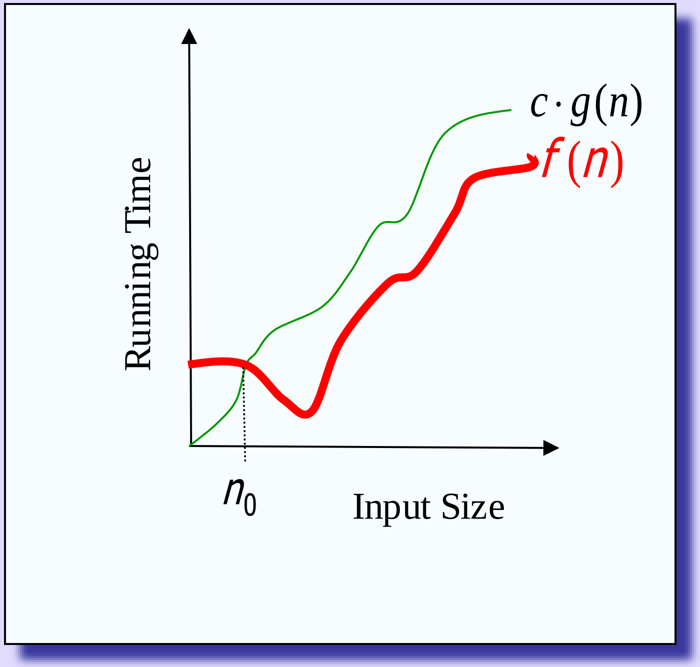

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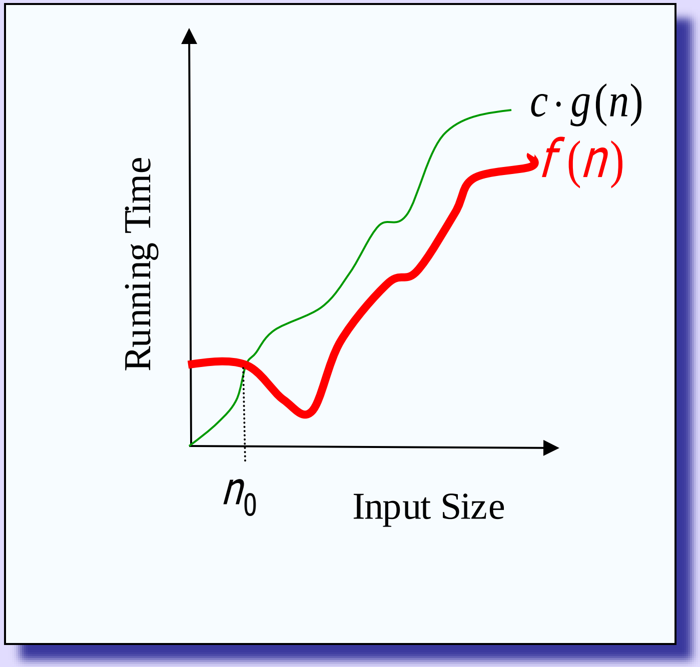
  - f(n) is O(g(n)), if there exists constants *c* and n_0, *s.t.* **f(n) ≤ c g(n)** for n $\geq n_0$
  - f(n) and g(n) are functions over non-negative integers
- Used for *worst-case* analysis



Asymptotic Notation *(terminology)*

- Special classes of algorithms:
  - Logarithmic:
  - Linear:
  - Quadratic:
  - Polynomial:
  - Exponential:
- "Relatives" of the Big-Oh
  - Ω (f(n)): Big Omega -asymptotic *lower* bound
  - Θ (f(n)): Big Theta -asymptotic *tight* bound

Asymptotic Notation *(terminology)*

- Special classes of algorithms:
  - Logarithmic: $O(\log n)$
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- "Relatives" of the Big-Oh
  - $\Omega(f(n))$: Big Omega -asymptotic *lower* bound
  - $\Theta(f(n))$: Big Theta -asymptotic *tight* bound

Example

For functions f(n) and g(n) there are positive constants c and n_0 such that: $f(n) \leq c\ g(n)$ for $n \geq n_0$

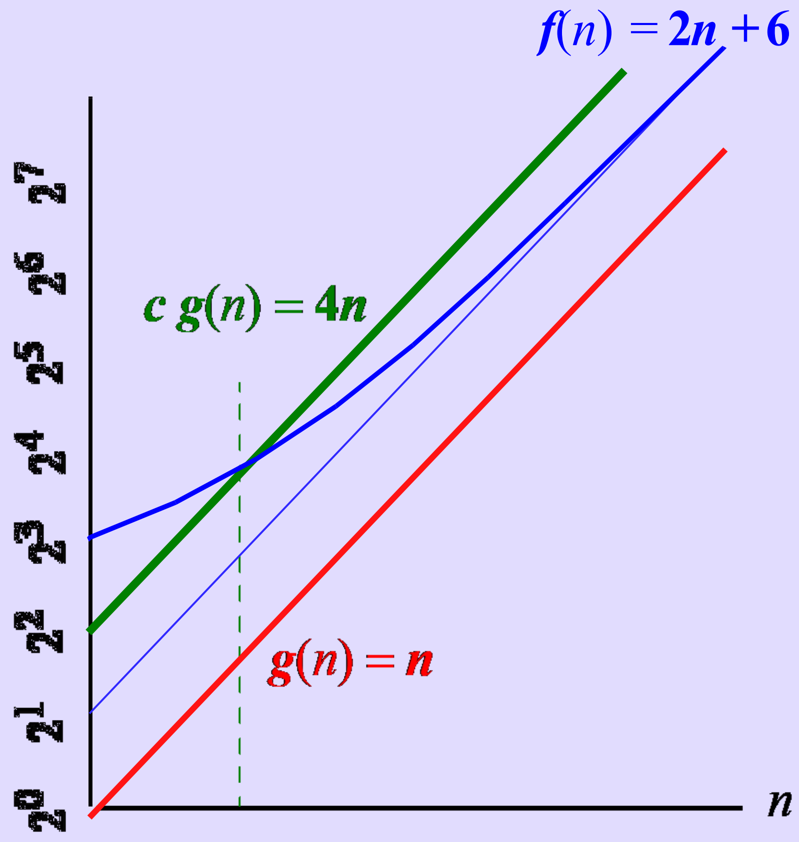
2n+6 is ______

Example

For functions f(n) and g(n) there are positive constants c and n_0 such that: $f(n) \le c\, g(n)$ for $n \ge n_0$

<u>conclusion</u>:

2n+6 is O(n).



Another Example

On the other hand…

n^2 is not $O(n)$ because there is

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$\boldsymbol{n^2}$ is not O($\boldsymbol{n}$) because there is no $\boldsymbol{c}$ and $\boldsymbol{n_0}$ such that:

$$\boldsymbol{n^2} \leq \boldsymbol{cn} \text{ for } n \geq n_0$$

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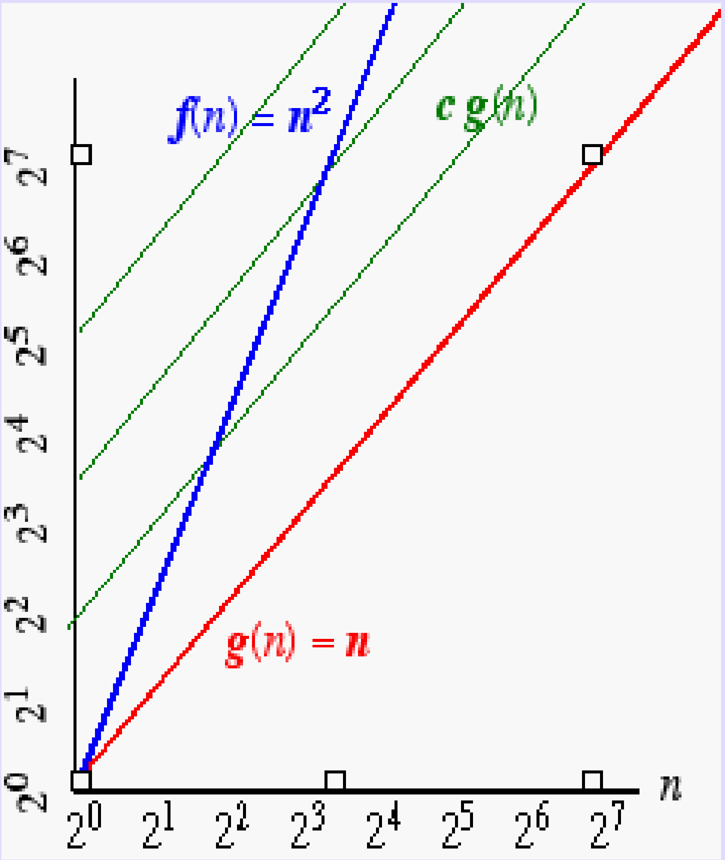
Another Example

On the other hand…

$\boldsymbol{n^2}$ is not O($\boldsymbol{n}$) because there is no $\boldsymbol{c}$ and $\boldsymbol{n_0}$ such that:

$$\boldsymbol{n^2} \leq \boldsymbol{cn} \text{ for } n \geq n_0$$

The graph to the right illustrates that no matter how large a c is chosen there is an n big enough that $\boldsymbol{n^2} > \boldsymbol{cn}$)



Asymptotic Notation

- Simple Rule: Drop lower order terms and constant factors.
  - $50\ n \log n$ is __________
  - $7n - 3$ is ___________
  - $8n^2 \log n + 5n^2 + n$ is _________

Asymptotic Notation

- Simple Rule: Drop lower order terms and constant factors.
  - $50\, n \log n$ is $O(n \log n)$
  - $7n - 3$ is $O(n)$
  - $8n^2 \log n + 5n^2 + n$ is $O(n^2 \log n)$
- Note: Even though $(50\, n \log n)$ is $O(n^5)$, it is expected that such an approximation be of as small an order as possible

Asymptotic Analysis of Running Time

- Use *O*-notation to express number of primitive operations executed as function of input size.
- Comparing asymptotic running times
  - an algorithm that runs in $O(n)$ time is better than one that runs in $O(n^2)$ time
  - similarly, $O(\log n)$ is better than $O(n)$
  - hierarchy of functions: $\log n < n < n^2 < n^3 < 2^n$

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  - similarly, $O(\log n)$ is better than $O(n)$
  - hierarchy of functions: $\log n < n < n^2 < n^3 < 2^n$
- Caution! Beware of very large constant factors. An algorithm running in time 1,000,000 n is still $O(n)$ but might be less efficient than one running in time $2n^2$, which is $O(n^2)$

Example of Asymptotic Analysis

Algorithm prefixAverages1(X):

Input: An n-element array X of numbers.

Output: An n-element array A of numbers such that A[i] is the average of elements X[0], ... , X[i].

for i $\leftarrow$ 0 **to** n-1 **do**

    a $\leftarrow$ 0

return array A

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for i $\leftarrow$ 0 **to** n-1 **do**

    a $\leftarrow$ 0

    for j $\leftarrow$ 0 **to** i **do**

        a $\leftarrow$ a + X[j]   1 step

    A[i] $\leftarrow$ a/(i+1)

n iterations

i iterations with i=0,1,2...n-1

return array A

Analysis: running time is $O(n^2)$

A Better Algorithm ?

Algorithm prefixAverages1(X):

Input: An n-element array X of numbers.

Output: An n-element array A of numbers such that A[i] is the average of elements X[0], ... , X[i].

for i $\leftarrow$ 0 **to** n-1 **do**

a $\leftarrow$ 0

return array A

A Better Algorithm

Algorithm prefixAverages2(X):

Input: An *n*-element array X of numbers.

Output: An *n*-element array A of numbers such that A[*i*] is the average of elements X[0], ... , X[*i*].

s $\leftarrow$ 0

for i $\leftarrow$ 0 **to** n **do**

    s $\leftarrow$ s + X[i]

    A[i] $\leftarrow$ s/(i+1)

return array A

Analysis: Running time is O(n)

Asymptotic Notation *(terminology)*

- Special classes of algorithms:
  - Logarithmic
  - Linear:
  - Quadratic:
  - Polynomial:
  - Exponential:

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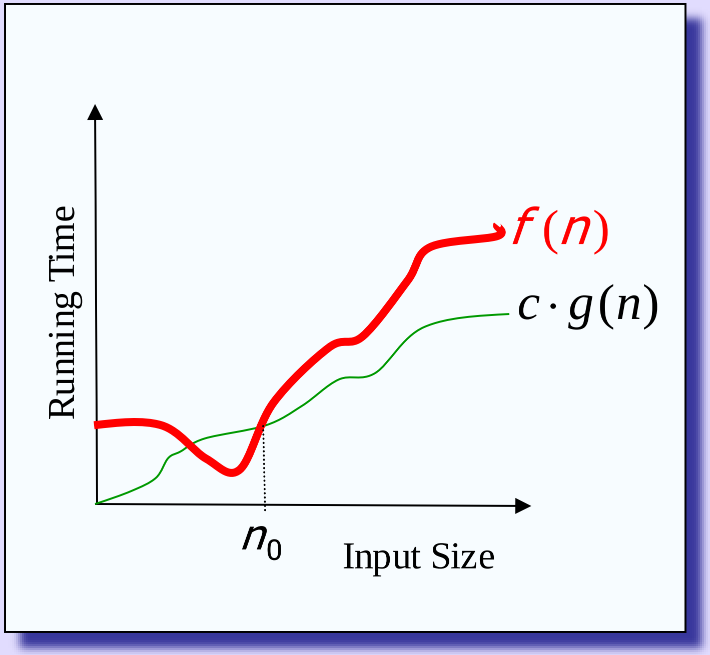
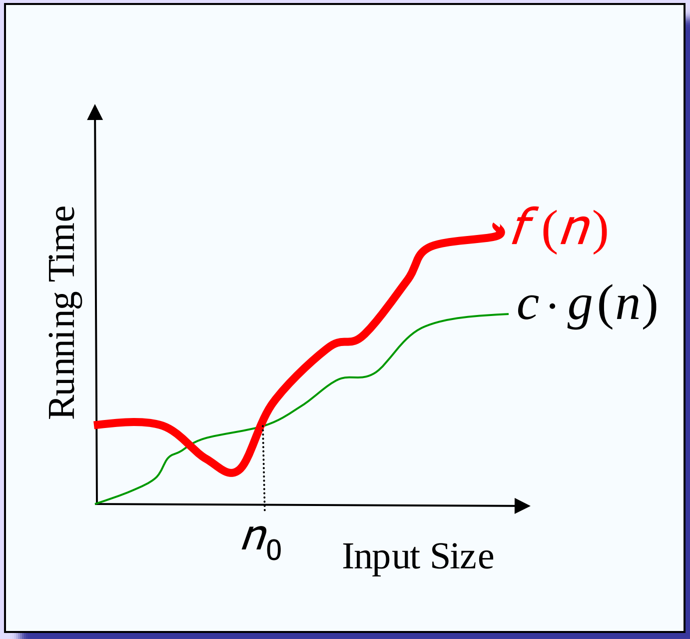
Asymptotic Notation

- The “big-Omega” Ω–Notation



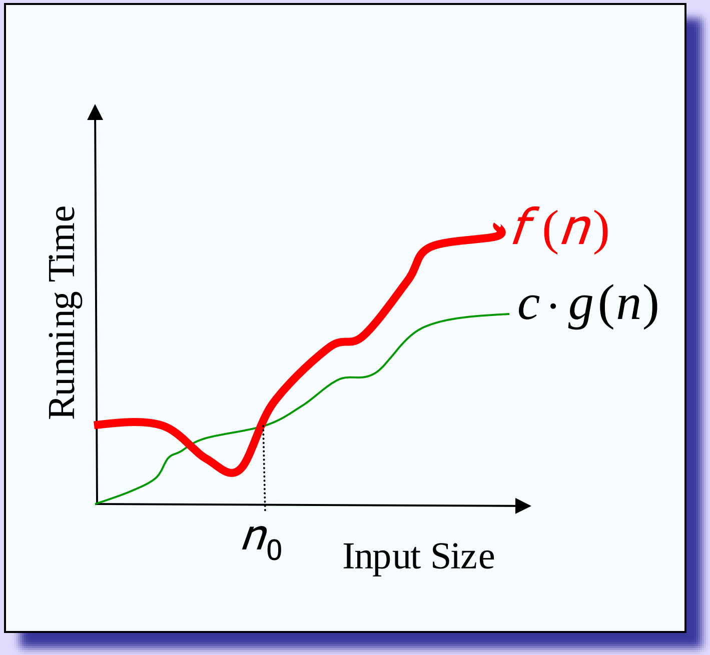
Asymptotic Notation

- The "big-Omega" Ω– Notation
  - asymptotic lower bound
  - *f(n)* is $\Omega(g(n))$ if there exists constants *c* and n_0, *s.t.* $\mathbf{c\ g(n) \le f(n)}$ for $n \ge n_0$



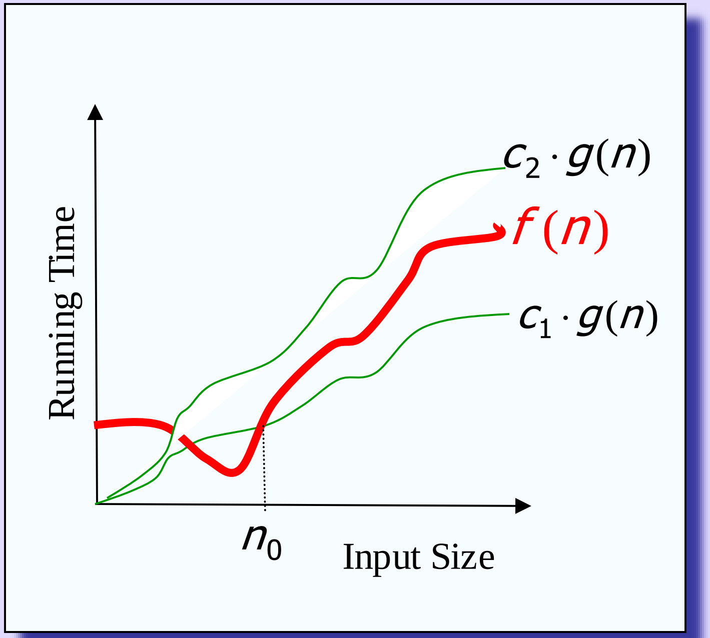
Asymptotic Notation

- The "big-Omega" Ω–Notation
  - asymptotic lower bound
  - *f(n)* is $\Omega(g(n))$ if there exists constants *c* and n_0, *s.t.* **c g(n) ≤ f(n)** for $n \geq n_0$
- Used to describe *best-case* running times or lower bounds for algorithmic problems
  - E.g., lower-bound for searching in an unsorted array is $\Omega(n)$.



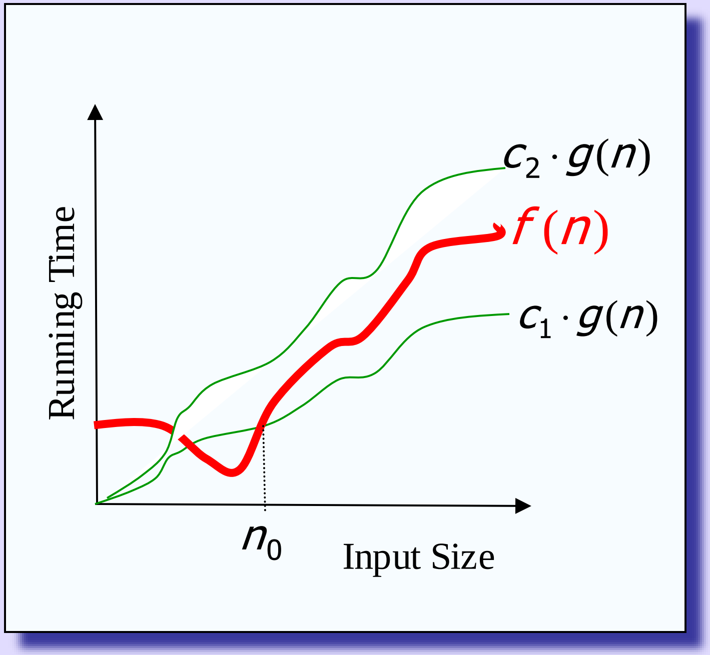
Asymptotic Notation

- The “big-Theta” Θ–Notation
  - asymptotically tight bound



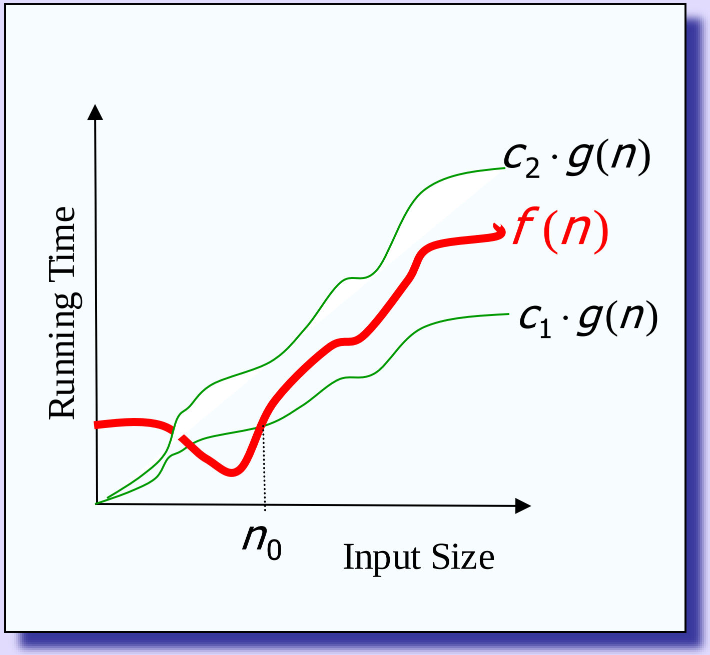
Asymptotic Notation

- The "big-Theta" Θ–Notation
  - asymptotically tight bound
  - $f(n) = \Theta(g(n))$ if there exists constants c_1, c_2, and n_0, *s.t.* $\mathbf{c_1\ g(n) \le f(n) \le c_2\ g(n)}$ for $n \ge n_0$



Asymptotic Notation

- The "big-Theta" Θ–Notation
  - asymptotically tight bound
  - *f(n)* = Θ*(g(n))* if there exists constants c_1, c_2, and n_0, *s.t.* $\mathbf{c_1\ g(n) \le f(n) \le c_2\ g(n)}$ for $n \ge n_0$
- *f(n)* is Θ*(g(n))* if and only if *f(n)* is *O(g(n))* and *f(n)* is Ω*(g(n))*
- *O(f(n))* is often misused instead of Θ*(f(n))*



Asymptotic Notation

Two more asymptotic notations

- "Little-Oh" notation *f(n)* is *o(g(n))* non-tight analogue of Big-Oh
  - For every *c*, there should exist n_0, s.t. $\mathbf{f(n) \le c\, g(n)}$ for $n \ge n_0$
  - Used for **comparisons** of running times. If *f(n)=o(g(n)),* it is said that *g(n) dominates f(n).*
- "Little-omega" notation *f(n)* is $\omega(g(n))$ non-tight analogue of Big-Omega

Asymptotic Notation

- Analogy with real numbers
  - $f(n) = O(g(n)) \quad \cong \quad f \le g$
  - $f(n) = \Omega(g(n)) \quad \cong \quad f \ge g$
  - $f(n) = \Theta(g(n)) \quad \cong \quad f = g$
  - $f(n) = o(g(n)) \quad \cong \quad f < g$
  - $f(n) = \omega(g(n)) \quad \cong \quad f > g$
- Abuse of notation: $f(n) = O(g(n))$ actually means $f(n) \in O(g(n))$

Comparison of Running Times

Running Time	Maximum problem size (n)		
	1 second	1 minute	1 hour
$400n$	2500	150000	9000000
$20n \log n$	4096	166666	7826087
$2n^2$	707	5477	42426
n^4	31	88	244
2^n	19	25	31

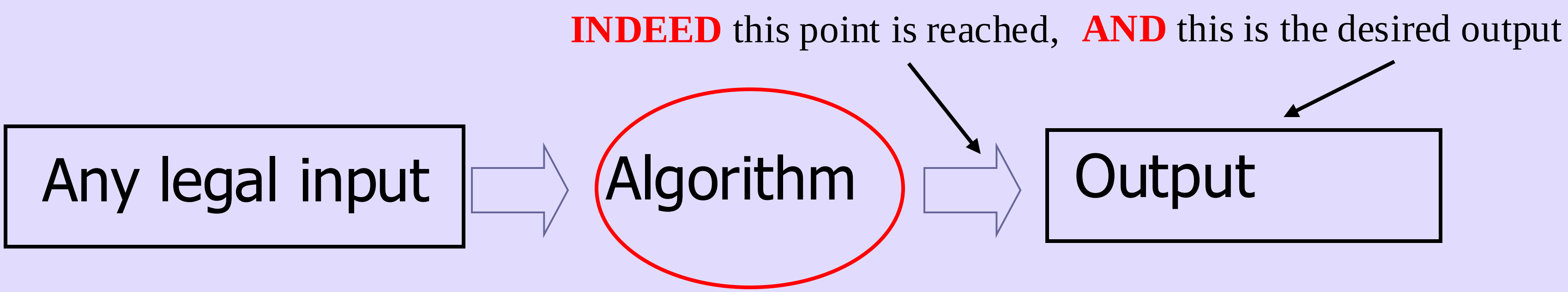
Correctness of Algorithms

- The algorithm is *correct* if for any legal input it terminates and produces the desired output.
- Automatic proof of correctness is not possible
- But there are practical techniques and rigorous formalisms that help to reason about the correctness of algorithms

Partial and Total Correctness

- Partial correctness

- Total correctness



Loop Invariants

```
for j ← 1 to length(A)-1
   do
      key ← A[j]
      i ← j-1
      while i>0 and A[i]>key
        do A[i+1] ← A[i]
           i--
      A[i+1] ← key
```

Example of Loop Invariants (1)

- **Invariant**: *at the start of each **for** loop, A[0…j-1] consists of elements originally in A[0…j-1] but in sorted order*

```
for j ← 1 to length(A)-1
    do
        key ←A[j]
        i ← j-1
        while i>0 and A[i]>key
          do A[i+1]←A[i]
             i--
        A[i+1] ← key
```

Example of Loop Invariants (2)

- **Invariant**: *at the start of each **for** loop, A[0…j-1] consists of elements originally in A[1…j-1] but in sorted order*

```
for j ← 1 to length(A)-1
   do
      key ←A[j]
      i ← j-1
      while i>0 and A[i]>key
        do A[i+1]←A[i]
           i--
      A[i+1] ← key
```

- **Initialization**: *j = 1,* the invariant trivially holds because *A*[0] is a sorted array ☺

Example of Loop Invariants (3)

- **Invariant**: *at the start of each* ***for*** *loop, A[0…j-1] consists of elements originally in A[0…j-1] but in sorted order*

```
for j ← 1 to length(A)-1
   do
      key ←A[j]
      i ← j-1
      while i>0 and A[i]>key
        do A[i+1]←A[i]
           i--
      A[i+1] ← key
```

- **Maintenance**: the inner **while** loop moves elements $A[j-1]$, $A[j-2]$, ..., $A[j-k]$ one position right without changing their order. Then the former $A[j]$ element is inserted into k-th position so that $A[k-1] \leq A[k] \leq A[k+1]$.

$$A[0...j-1] \text{ sorted} + A[j] \longrightarrow A[0...j] \text{ sorted}$$

Example of Loop Invariants (4)

- **Invariant**: *at the start of each **for** loop, A[0…j-1] consists of elements originally in A[0…j-1] but in sorted order*

```
for j ← 1 to length(A)-1
    do
       key ←A[j]
       i ← j-1
       while i>0 and A[i]>key
         do A[i+1]←A[i]
            i--
       A[i+1] ← key
```

- **Termination**: the loop terminates, when *j=n*. Then the invariant states: *"A[0…n-1] consists of elements originally in A[0…n-1] but in sorted order"* ☺

Assertions

- To prove correctness we associate a number of **assertions** (statements about the state of the execution) with specific checkpoints in the algorithm.
  - E.g., A[1], …, A[k] form an increasing sequence
- **Preconditions** – assertions that must be valid *before* the execution of an algorithm or a subroutine
- **Postconditions** – assertions that must be valid *after* the execution of an algorithm or a subroutine

Loop Invariants

- **Invariants** – assertions that are valid any time they are reached (many times during the execution of an algorithm, e.g., in loops)
- We must show three things about loop invariants:
  - **Initialization** – it is true prior to the first iteration
  - **Maintenance** – if it is true before an iteration, it remains true before the next iteration
  - **Termination** – when loop terminates the invariant gives a useful property to show the correctness of the algorithm

Math You Need to Review

- Properties of logarithms:

  $\log_b(xy) = \log_b x + \log_b y$

  $\log_b(x/y) = \log_b x - \log_b y$

  $\log_b x^a = a \log_b x$

  $\log_b a = \log_x a / \log_x b$

- Properties of exponentials:

  $a^{(b+c)} = a^b a^c$; $a^{bc} = (a^b)^c$

  $a^b / a^c = a^{(b-c)}$; $b = a^{\log_a b}$

- Floor: $\lfloor x \rfloor$ = the largest integer $\leq x$
- Ceiling: $\lceil x \rceil$ = the smallest integer $\geq x$

Math Review

- Geometric progression
  - given an integer n_0 and a real number *0*< $a \neq 1$

$$\sum_{i=0}^{n} a^i = 1 + a + a^2 + ... + a^n = \frac{1 - a^{n+1}}{1 - a}$$

  - geometric progressions exhibit exponential growth
- Arithmetic progression

$$\sum_{i=0}^{n} i = 1 + 2 + 3 + ... + n = \frac{n^2 + n}{2}$$

Summations

- The running time of insertion sort is determined by a nested loop

```
for j←1 to length(A)-1
    key←A[j]
    i←j-1
    while i>=0 and A[i]>key
        A[i+1]←A[i]
        i←i-1
    A[i+1]←key
```

- Nested loops correspond to summations

$$\sum_{j=1}^{n-1} j = O(n^2)$$