Sardar Patel Institute of technology

Design and Analysis of Algorithms

Problem Statement:

4.5-4

Can the master method be applied to the recurrence $T(n) = 4T(n/2) + n^2 \lg n$? Why or why not? Give an asymptotic upper bound for this recurrence.

Solution

$$T(n) = 4T(n/2) + n^2 \lg n$$

In the given recurrence, $a = 4$ and $b = 2$.

Hence, $n^{\log_b a} = n^{\log_2 4} = n^2$ and $f(n) = \Theta(n^2 \lg n)$.

Now, asymptotically $f(n) = n^2 \lg n$ is definitely larger than n^2, but it is not polynomially larger than n^2. So, we cannot apply master method to this recurrence.

Solution

$$T(n) = 4T(n/2) + n^2 \lg n$$

So, solving by expansion:

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2 \log n$$

$$= 16T\left(\frac{n}{2^2}\right) + n^2 \log n + n^2 \log\left(\frac{n}{2}\right)$$

$$\vdots$$

$$T(n) = 4^k T\left(\frac{n}{2^k}\right) + n^2 \left[\log n + \cdots \log \frac{n}{2^{k-1}}\right]$$

$$\frac{n}{2^k} = 1$$

Solution

$$T(n) = 4T(n/2) + n^2 \lg n$$

$$\frac{n}{2^k} = 1 \qquad k = \log_2 n$$

$$T(n) = n^2 + n^2 \left[\log \frac{n}{2^{k-1}} + \log \frac{n}{2^{k-2}} + \dots \log n \right]$$

$$\frac{n}{2^k} = 1 \qquad \therefore \quad \frac{n}{2^{k-1}} = 2 \, , \quad \frac{n}{2^{k-2}} = 4$$

$$T(n) = n^2 + n^2 \left[\log 2 + \log 4 + \dots + \log n \right]$$

$$= n^2 + n^2 \left[\log_2 2 + \log_2 2^2 + \dots + \log_2 n \right]$$

Solution

$$T(n) = 4T(n/2) + n^2 \lg n$$

$$= n^2 + n^2 \left[\log_2 2 + \log_2 2^2 + \dots + \log_2 n \right]$$

$$= n^2 + n^2 \left[1 + 2 + 3 + \dots + \log \frac{n}{2} + \log n \right]$$

$$= n^2 + n^2 \left[1 + 2 + 3 + \dots + \log n - 1 + \log n \right]$$

$$\therefore \; 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Solution

$$T(n) = 4T(n/2) + n^2 \lg n$$

$$T(n) = n^2 + n^2 \left[\frac{(\log n)^2 + \log n}{2} \right]$$

$$\therefore \; T(n) = O\left(n^2 \log^2 n\right)$$

$$T(n) = \Theta\left(n^2 \log^2 n\right)$$