Total No. of Questions: 6 *Total No. of Printed Pages:3*

Enrollment No......................................

Faculty of Engineering

End Sem (Odd) Examination Dec-2018

EN3BS01 Engineering Mathematics-I

Programme: B.Tech. Branch/Specialisation: All

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. A system of linear homogeneous equations is having non trivial solution if: $where, n = no. of\ unknowns, r = rank\ of\ coefficient\ matrix$ **1**

(a) $n = r$ (b) $n > r$ (c) $n < r$ (d) None of these

ii. Characteristics roots for the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ are: **1**

(a) 1 and 6 (b) -1 and 6 (c) -6 and 1 (d) -1 and -6

iii. If $u = \tan^{-1}\frac{y}{x}$, then the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ will be: **1**

(a) 0 (b) u (c) 3u (d) None of these

iv. If the real valued function f is continuous on a closed interval (a, b), differentiable on the open interval (a, b) and f(a) = f(b), then their exist a point c in (a, b) such that f '(c)=0 is statement of **1**

(a) Taylor's theorem

(b) Rolle's theorem

(c) Lagrang's mean value theorem

(d) None of these

v. The value of $\Gamma\frac{3}{2}$ is **1**

(a) $\frac{\sqrt{\pi}}{2}$ (b) $\frac{\pi}{2}$. (c) $\frac{2\sqrt{2}}{\pi}$ (d) None of these

vi. The curve $x^2 + y^2 = a^2$ is symmetrical about **1**

(a) x axis (b) y axis

(c) Both (a) and (b) (d) None of these

P.T.O.

vii. The general solution of the differential equation $xdx + ydy = 0$ is

(a) $xy = c$ (b) $x + y = c$

(c) $x^2 + y^2 = c$ (d) None of these

viii. The differential equation $Mdx + Ndy = 0$ is an exact differential equation if and only if **1**

(a) $\frac{\partial M}{\partial Y} \neq \frac{\partial N}{\partial X}$ (b) $\frac{\partial M}{\partial Y} = \frac{\partial N}{\partial X}$ (c) $\frac{\partial M}{\partial X} \neq \frac{\partial N}{\partial Y}$ (d) $\frac{\partial M}{\partial X} = \frac{\partial N}{\partial Y}$

ix. If $1 + P + Q = 0$, then which of the following will be a part of complementary function of $D^2y + PDy + Qy = R$? Where $D = \frac{d}{dx}$ **1**

(a) $y = x$ (b) $y = \frac{1}{x}$ (c) $y = e^{-x}$ (d) $y = e^x$

x. Complementary Function of equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = \frac{e^{3x}}{x^2}$ is: **1**

(a) $e^{3x}(C_1\, cos2x + C_2\, sin2x)$

(b) $C_1e^{2x} + C_2e^{3x}$

(c) $(C_1 + C_2x)e^{3x}$

(d) $e^{2x}(C_1\, cos3x + C_2\, sin3x)$

Q.2 Attempt any two:

i. Find echelon form of the given matrix and hence find its rank and nullity: **5**

$$\begin{bmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{bmatrix}$$

ii. Examine the following system of equations for consistency, and if consistent, find the complete solution: **5**

$$5x + 3y + 14z = 4$$
$$y + 2z = 1$$
$$x - y + 2z = 0$$
$$2x + y + 6z = 2$$

iii. Verify Cayley Hamilton Theorem for the given matrix, and hence find its inverse: **5**

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

Q.3 Attempt any two:

i. Using Taylor's series expand $\tan\left(x + \frac{\pi}{4}\right)$ as far as the term x^4 and hence evaluate $\tan 46.5°$. **5**

ii. If $u = \tan^{-1}(y^2/x)$, then prove that- **5**

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin^2 u \sin 2u.$$

iii. Discuss the maxima and minima of the function: **5**

$$x^3 + y^3 - 3axy$$

Q.4 Attempt any two:

i. Evaluate: $\lim_{n\to\infty} \left[\left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right) \dots \dots . \left(1 + \frac{n}{n}\right)\right]^{\frac{1}{n}}$ **5**

ii. Change the order of integration in $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dy\, dx$ and hence evaluate **5**

iii. Prove that: $\Gamma\frac{1}{2} = \sqrt{\pi}$ **5**

Q.5 Attempt any two:

i. Solve: $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x$, Where $D = \frac{d}{dx}$ **5**

ii. Solve: $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$ **5**

iii. Solve: $\frac{dx}{dt} + 5x + y = e^t$, $\frac{dy}{dt} - x + 3y = e^{2t}$ **5**

Q.6 Attempt any two:

i. Solve by method of variation of parameters: **5**

$$(x^2D^2 + xD - 1)y = x^2e^x, \text{ Where } D = \frac{d}{dx}$$

ii. Solve by changing the independent variable: **5**

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3 \sin x^2$$

iii. Solve: $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = e^x \sec x$ **5**

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MEDICAPS UNIVERSITY

End Sem Examination 2018

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Q1

Ans. i) b $(n > r)$ 1

ii) a (1 and 6) 1

iii) a (0) 1

iv) b (Rolle's theorem) 1

v) a $(\sqrt{\pi}/2)$ 1

vi) c (Both (a) and (b)) 1

vii) c $(x^2+y^2=c)$ 1

viii) b $\left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right)$ 1

ix) d $(y = e^{x})$ 1

x) a $(e^{3x}(c_1\cos 2x + c_2 \sin 2x))$ 1

Q 2

Ans. i) Given $A = \begin{bmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{bmatrix}$

By Operating

$R_4 \to R_4 - R_3,\ R_3 \to R_3 - R_2,\ R_2 \to R_2 - R_1$

$A \sim \begin{bmatrix} 1 & 4 & 9 & 16 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 11 \\ 7 & 9 & 11 & 13 \end{bmatrix}$ 1

$R_4 \to R_4 - R_3,\ R_3 \to R_3 - R_2,\ R_2 \to R_2 - R_1$

$A \sim \begin{bmatrix} 1 & 4 & 9 & 16 \\ 2 & 1 & -2 & -7 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$ 1

$R_4 \to R_4 - R_3,\ R_3 \to R_3 - R_2$

$A \sim \begin{bmatrix} 1 & 4 & 9 & 16 \\ 2 & 1 & -2 & -7 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ 1

$R_2 \to R_2 - 2R_1$

$$A \sim \begin{bmatrix} 1 & 4 & 9 & 16 \\ 0 & -7 & -20 & -39 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ 1

$R_2 \leftrightarrow R_3$ and $R_3 \to R_3 + 7R_2$

$$A \sim \begin{bmatrix} 1 & 4 & 9 & 16 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 8 & 24 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is in echelon form 1

Hence, the rank of A i.e $\rho(A)$ = Number of non zero rows = 3.

Nullity = 4-3 = 1.

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ii) Given equations

$$5x + 3y + 14z = 4$$
$$y + 2z = 1$$
$$x - y + 2z = 0$$
$$2x + y + 6z = 2$$

Writing the given system in the matrix form

$$AX = B$$ 1

Where

$$A = \begin{bmatrix} 5 & 3 & 14 \\ 0 & 1 & 2 \\ 1 & -1 & 2 \\ 2 & 1 & 6 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$\therefore$ Augmented Matrix

$$[A:B] = \left[\begin{array}{ccc:c} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 2 & -1 & 2 & 0 \\ 2 & 1 & 6 & 2 \end{array}\right]$$

Operating $R_1 \leftrightarrow R_3$ we get

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 3 & 14 & 4 \\ 2 & 1 & 6 & 2 \end{array}\right]$$ 1

$R_3 \to R_3 - 5R_1$, $R_4 \to R_4 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 8 & 4 & 4 \\ 0 & 3 & 2 & 2 \end{array}\right]$$ 1

Operating $R_3 \to R_3 - 8R_2$; $R_4 \to R_4 - 3R_2$

$$\sim \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & -12 & : & -4 \\ 0 & 0 & -4 & : & -1 \end{bmatrix}$$

$R_4 \to 3R_4$, we get

$$\sim \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & -12 & : & -4 \\ 0 & 0 & -12 & : & -3 \end{bmatrix}$$ 1

$R_4 \to R_4 - R_3$, we get

$$[A : B] \sim \begin{bmatrix} 1 & -1 & 2 & : & 0 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & -12 & : & -4 \\ 0 & 0 & 0 & : & 1 \end{bmatrix}$$

Which is in echelon form 1

$\therefore \quad \rho(A:B) = 4, \; \rho(A) = 3$

$\Rightarrow \quad \rho(A:B) \neq \rho(A)$

$\Rightarrow$ The system is inconsistent

$\Rightarrow$ System has no solution

iii) The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 3 & 1 \\ 2 & 1-\lambda & -2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$(4-\lambda)\{(1-\lambda)^2 + 4\} - 3\{2 - 2\lambda + 2\} + 1(4 - 1 + \lambda) = 0$

$(4-\lambda)(1 + \lambda^2 - 2\lambda + 4) - 3(4 - 2\lambda) + (3 + \lambda) = 0$

$(4-\lambda)(\lambda^2 - 2\lambda + 5) - 12 + 6\lambda + 3 + \lambda = 0$

$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0 \quad -(1)$ 1

For Cayley Hamilton theorem Putting $\lambda = A$

$$A^3 - 6A^2 + 6A - 11I = 0 \quad -(2)$$

Now $A^2 = A \cdot A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 125 & 84 & -12 \\ 36 & 23 & 0 \\ 48 & 30 & -7 \end{bmatrix}$$

Now $A^3 - 6A^2 + 6A - 11I$

$$\cancel{= \begin{bmatrix} 125 & 84 & -12 \\ 36 & 23 & 0 \\ 48 & 30 & -7 \end{bmatrix} - 6 \begin{bmatrix} 23 & 17 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}}$$ 2

$$= \begin{bmatrix} 125 & 84 & -12 \\ 36 & 23 & 0 \\ 48 & 30 & -7 \end{bmatrix} - 6 \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} + 6 \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} - 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

This verifies Cayley Hamilton theorem.

To find A^{-1}: Multiplying both sides of (2) by A^{-1} we get

$$A^2 - 6A + 6I - 11A^{-1} = 0$$

$$A^{-1} = \frac{1}{11}(A^2 - 6A + 6I)$$ 2

$$A^{-1} = \frac{1}{11}\left\{ \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} - 6 \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

$$= \frac{1}{11} \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$$

Q.3

Ans i) Let $f(x) = \tan x$, $f(\pi/4) = \tan \pi/4 = 1$

Now Differentiating w.r. to x we get

$$f'(x) = \sec^2 x = 1 + \tan^2 x$$

$$= 1 + [f(x)]^2 , \quad f'(\pi/4) = 1+1 = 2$$

$$f''(x) = 2f(x)f'(x) , \quad f''(\pi/4) = 2f(\pi/4)f'(\pi/4)$$

$$f''(\pi/4) = 4$$

$$f'''(x) = 2f(x)f''(x) + 2[f'(x)]^2$$ 2

$$\therefore f'''(\pi/4) = 2f(\pi/4)f''(\pi/4) + 2[f'(\pi/4)]^2$$

$$= 2 \times 1 \times 4 + 2 \times 4 = 16$$

$$f^{iv}(x) = 2f(x)f'''(x) + 6f'(x)f''(x)$$

$$\therefore f^{iv}(x) = 2f(\pi/4)f'''(\pi/4) + 6f'(\pi/4)f''(\pi/4)$$

$$= 2 \times 1 \times 16 + 6 \times 2 \times 4 = 80$$

By Taylor's theorem

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots \quad \text{—(1)}$$ 1

Putting $h = x$, $a = \pi/4$, we get

$$f(x+\pi/4) = f(\pi/4) + xf'(\pi/4) + \frac{x^2}{2!}f''(\pi/4) + \frac{x^3}{3!}f'''(\pi/4) + \frac{x^4}{4!}f^{iv}(\pi/4)$$

$$\tan(x+\pi/4) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \cdots \quad \text{—(2)}$$ 2

Now Putting $x = 1.5° = \frac{3}{2} \times \frac{\pi}{180}$ rad $= 0.02618$

$$\tan(1.5° + 45°) = 1 + 2\times(0.02618) + 2\times(0.02618)^2 + \frac{8}{3}(0.02618)^3 + \cdots$$

$$\tan(46.5°) = 1.05378$$

ii) Given

$$u = \tan^{-1}(y^2/x)$$

$$\Rightarrow \tan u = y^2/x$$

Let $Z = \tan u$

$$\Rightarrow Z = y^2/x = x\, y^2/x^2 = x f(y/x)$$

Thus Z is a homogeneous function of x and y of degree 1. Hence by Euler's theorem

$$x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} = 1.Z$$ 1

$$\Rightarrow x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = \tan u$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = \tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u} = \frac{\sin u}{\cos u}\times \cos^2 u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin u\cos u$$ 2

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\,2\sin u\cos u = \frac{1}{2}\sin 2u \quad -(1)$$

Differentiating (1) Partially w. r to x

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = \cos 2u\frac{\partial u}{\partial x}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} = (\cos 2u - 1)x\frac{\partial u}{\partial x} \quad —(2)$$ 1

Similarly D. w. r to y, we get

$$xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (\cos 2u - 1)y\frac{\partial u}{\partial y} \quad —(3)$$ 1

Adding (2) and (3) we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (\cos 2u - 1)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= (-2\sin^2 u)\,\frac{1}{2}\sin 2u$$

$$= -\sin^2 u\,\sin 2u$$

Q.3
Ans. iii) Let $u = x^3 + y^3 - 3axy$

Partially Differentiating w.r. to x and y we get

$$\frac{\partial u}{\partial x} = 3x^2 - 3ay, \quad \frac{\partial u}{\partial y} = 3y^2 - 3ax$$ 1/2

Thus for maxima and minima of u.

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0$$ 1/2

$$\Rightarrow \quad 3x^2 - 3ay = 0, \quad 3y^2 - 3ax = 0$$

by $3x^2 - 3ay = 0$, $x^2 = ay$ 1 1/2

$3y^2 - 3ax = 0$, $y^2 = ax$

$\Rightarrow \quad x = y = a$

$\therefore$ The Point (a, a) is a critical Point i.e (a, a) is the extreme Point of the given function u

Now $r = \frac{\partial^2 u}{\partial x^2} = 6x$, $s = \frac{\partial^2 u}{\partial x \partial y} = -3a$, $t = \frac{\partial^2 u}{\partial y^2} = 6y$ 1

At the Point (a, a) we have

$$r = 6a, \quad s = -3a, \quad t = 6a$$

which gives $rt - s^2 = 36a^2 - 9a^2$ 1

$$= 27a^2 > 0$$

Since $rt - s^2 > 0$, $r > 0$ we have Maximum of u and minimum according as a is -ve or +ve at $\boxed{x = y = a}$ 1/2

Q.4

Ans i) Let $P = \lim_{n \to \infty} \left[(1+\frac{1}{n})(1+\frac{2}{n}) \cdots (1+\frac{n}{n}) \right]^{1/n}$

$\Rightarrow \log P = \lim_{n \to \infty} \frac{1}{n} \left[\log(1+\frac{1}{n}) + \log(1+\frac{2}{n}) + \cdots + \log(1+\frac{n}{n}) \right]$

$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \log(1+\frac{r}{n}) = \int_0^1 \log(1+x)\,dx$ 2

$= [x \log(1+x)]_0^1 - \int_0^1 \frac{x}{1+x}\,dx$ 1

$= \log 2 - \int_0^1 (1 - \frac{1}{1+x})\,dx$

$= \log 2 - [x - \log(1+x)]_0^1$ 1

$= \log 2 - [(1-\log 2) - (0 - \log 1)]$

$= \log 2 - 1 + \log 2$ 1

$= 2\log 2 - 1 = \log 2^2 - 1$

$= \log 4 - \log e$

$= \log 4/e$

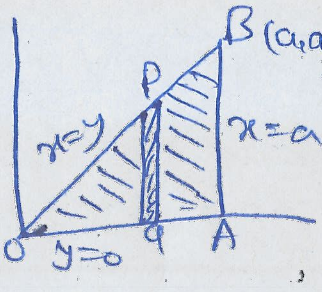
$\boxed{P = 4/e}$

ii) Given Integral

$$I = \int_{y=0}^{a} \int_{x=y}^{a} \frac{x}{x^2+y^2}\,dy\,dx$$

The limit of Integration are 2

$x=y$, $x=a$, $y=0$ and $y=a$



Clearly OABO is the area of Integration. Now to Change the order of Integration it is clear that y varies

From $y=0$ to $y=x$ and x varies from $x=0$ to $x=a$. Hence on changing the order of integration given integral will be 1

$$\int_0^a \int_y^a \frac{x\,dy\,dx}{x^2+y^2} = \int_0^a \int_0^x \frac{x\,dx\,dy}{x^2+y^2}$$

$$= \int_0^a x\,dx \left[\int_0^x \frac{dy}{x^2+y^2}\right]$$

$$= \int_0^a x\,dx \left(\frac{1}{x}\tan^{-1} y/x\right)_0^x$$ 1

$$= \int_0^a \cancel{x}\cdot\frac{1}{\cancel{x}}\tan^{-1} 1\,dx$$

$$= \pi/4 \int_0^a dx$$

$$= \pi/4\,(x)_0^a$$ 1

$$= \frac{\pi a}{4}$$

Ans iii) $\Gamma\frac{1}{2} = \sqrt{\pi}$

We know that,

$$B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n} \quad (m, n > 0)$$ 1/2

Putting $m=n=\frac{1}{2}$

$$B(1/2, 1/2) = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 1} = (\Gamma\tfrac{1}{2})^2$$ 1/2

$$\Rightarrow (\Gamma\tfrac{1}{2})^2 = B(1/2, 1/2) = \int_0^1 x^{-1/2}(1-x)^{-1/2}dx$$ 1

$$= \int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}}dx$$

Putting $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\, d\theta$

$$\therefore \left(\Gamma\tfrac{1}{2}\right)^2 = \int_0^{\pi/2} \frac{1}{\sin\theta\sqrt{1-\sin^2\theta}}\, 2\sin\theta\cos\theta\, d\theta$$ 1

$$= 2\int_0^{\pi/2} \frac{1}{\cancel{\sin\theta}\cdot\cancel{\cos\theta}}\cdot\cancel{\sin\theta}\,\cancel{\cos\theta}\, d\theta$$ ~~1~~

$$= 2\int_0^{\pi/2} d\theta$$ 1

$$= 2\,[\theta]_0^{\pi/2}$$

$$= \cancel{2}\cdot\frac{\pi}{\cancel{2}} = \pi$$

$$\boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$$ 1

5

i Given equation in symbolic form:

$$(D^3+2D^2+D)y = e^{2x}+x^2+x.$$

For C.F. : The A.E. is

$$m^3+2m^2+m=0$$

$$\Rightarrow \quad m(m^2+2m+1)=0 \quad \Longrightarrow \quad m(m+1)^2=0 \qquad 1$$

$$\Rightarrow \quad m=0,-1,-1 \quad \text{(roots)} \qquad 1$$

$$\therefore \quad C.F = C_1+(C_2+C_3x)e^{-x}$$

For P.I.:

$$P.I. = \frac{1}{(D^3+2D^2+D)}(e^{2x}+x^2+x) \qquad 1$$

$$= \frac{1}{D(D+1)^2}e^{2x}+\frac{1}{D(D+1)^2}(x^2+x)$$

$$= \frac{1}{2\cdot(2+1)^2}\cdot e^{2x}+\frac{1}{D}[1+D]^{-2}[x^2+x]$$

$$= \frac{1}{18}\cdot e^{2x}+\frac{1}{D}[1-2D+3D^2-\cdots](x^2+x)$$

$$[\text{Because } (1+x)^{-2}=1-2x+3x^2-\cdots] \qquad 1$$

$$= \frac{1}{18}\cdot e^{2x}+\frac{1}{D}[x^2+x)-2(2x+1)+3(2+0)]$$

$$= \frac{e^{2x}}{18}+\frac{1}{D}[x^2-3x+4]$$

$$= \frac{e^{2x}}{18}+\frac{x^3}{3}-\frac{3}{2}x^2+4x \quad \left[\because \frac{1}{D}=\int\right] \qquad 1$$

Hence, the complete solution is $y = C.F. + P.I.$

$$\Longrightarrow \quad y = C_1+(C_2+C_3x)e^{-x}+\frac{1}{18}\cdot e^{2x}+\frac{x^3}{3}-\frac{3}{2}x^2+4x.$$

Ans

ii Given $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0.$

We observe that equation (1) is not exact, because 1

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Here $\quad M = 3x^2y^4+2xy$ and $N=2x^3y^3-x^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 12x^2y^3+2x \text{ and } \frac{\partial N}{\partial x} = 6x^2y^3-2x.$$

We have $\displaystyle \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = \frac{6x^2y^3-2x-12x^2y^3-2x}{3x^2y^4+2xy}$ 1

$$= \frac{-6x^2y^3-4x}{xy(3xy^3+2)} = \frac{-2x(3xy^3+2)}{xy(3xy^3+2)} = \frac{-2}{y} = f(y), \text{ say}$$

$\therefore$ I.F. $= e^{\int f(y)dy} = e^{\int -\frac{2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$. 1

Now multiplying equation (1) by $\frac{1}{y^2}$, we get

$$\frac{1}{y^2}(3x^2y^4+2xy)\,dx + \frac{1}{y^2}(2x^3y^3-x^2)\,dy = 0$$

$$\Rightarrow \left(3x^2y^2+\frac{2x}{y}\right)dx + \left(2x^3y-\frac{x^2}{y^2}\right)dy = 0$$

where $M = 3x^2y^2+\frac{2x}{y}$ and $N = 2x^3y-\frac{x^2}{y^2}$

$\therefore \frac{\partial M}{\partial y} = 6x^2y-\frac{2x}{y^2}$ and $\frac{\partial N}{\partial x} = 6x^2y-\frac{2x}{y^2}$. 2

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Longrightarrow$ equation (2) is an exact.

Therefore, solution of (2) is

$$\int\left(3x^2y^2+\frac{2x}{y}\right)dx + \int_{(\text{free from } x)} 0\,dy = c$$

$$\Rightarrow x^3y^2+\frac{x^2}{y} = c$$

$$\Rightarrow x^3y^3+x^2 = cy.$$

(ii) Given equations are

$$\frac{dx}{dt}+5x+y = e^t \quad — (1)$$

$$\frac{dy}{dt}-x+3y = e^{2t} \quad — (2)$$

Equation (1) & (2) can be written as

$$(D+5)x + y = e^t \quad — (3)$$

$$-x + (D+3)y = e^{2t} \quad — (4)$$

For eliminating y Multiplying by $(D+3)$ in (3) and by 1 in (4) we get

$$(D+5)(D+3)x + \cancel{(D+3)y} = (D+3)e^t$$

$$\mp x \quad \pm \cancel{(D+3)y} = \mp e^{2t}$$

$$[(D+5)(D+3)+1]x = (D+3)e^t - e^{2t}$$

$$(D^2 + 8D + 16)x = De^t + 3e^t - e^{2t}$$

$$(D^2 + 8D + 16)x = 4e^t - e^{2t} \quad -(5)$$ $1\frac{1}{2}$

The A.E of (5) $m^2 + 8m + 16 = 0$

$$\Rightarrow (m+4)^2 = 0$$

$$\Rightarrow m = -4, -4$$

$$\therefore \quad CF = (C_1 + C_2 t)\, e^{-4t}$$

For P.I:

$$P.I = \frac{1}{(D+4)^2} 4e^t - \frac{1}{(D+4)^2} e^{2t}$$ $1\frac{1}{2}$

$$= \frac{4}{(1+4)^2} e^t - \frac{1}{(2+4)^2} e^{2t}$$

$$= \frac{4}{25} e^t - \frac{1}{36} e^{2t}$$

Hence $\quad x = CF + P.I$

$$x = (C_1 + C_2 t)\, e^{-4t} + \frac{4}{25} e^t - \frac{1}{36} e^{2t} \quad -(6)$$

Now differentiating (6) we get

$$\frac{dx}{dt} = -4(C_1 + C_2 t)\, e^{-4t} + C_2 e^{-4t} + \frac{4}{25} e^t - \frac{1}{18} e^{2t}$$ 2

Putting the value of x and $\frac{dx}{dt}$ in (1)

$$y = e^t - \frac{dx}{dt} - 5x$$

$$y = -(C_1 + C_2 + C_2 t)\, e^{-4t} + \frac{7}{36} e^{2t} + \frac{e^t}{25} \quad -(7)$$

Thus equations (6) & (7) are reevired solution

6:
ns 1) Given equation can be written as

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x$$

or $$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=e^x$$

To find C.F. i.e solution of

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=0 \quad -(1)$$

Here $P=\frac{1}{x}$, $Q=-\frac{1}{x^2}$

Hence $P+Qx=0$ by inspection

$\therefore$ $y=x$ is a Part of C.F. $\frac{1}{2}$

Let $y=vx$ be solution of (1) So that

$$\frac{dy}{dx}=\frac{dv}{dx}x+v$$

$$\frac{d^2y}{dx^2}=\frac{d^2v}{dx^2}x+2\frac{dv}{dx}$$

Substituting in equation (1) we get

$$\frac{d^2v}{dx^2}+\frac{3}{x}\frac{dv}{dx}=0$$

Let $t=\frac{dv}{dx}$, $\frac{dt}{dx}=\frac{d^2v}{dx^2}$

$$\frac{dt}{dx}+\frac{3}{x}t=0$$

$$\Rightarrow \frac{dt}{t}=-\frac{3}{x}dx$$

$$\Rightarrow \log t=-3\log x+\log c_1$$

$$\Rightarrow \log t=\log c_1/x^3$$

$$\Rightarrow t=c_1/x^3$$

$$\Rightarrow \frac{dv}{dx}=c_1/x^3$$

Integrating we get

$$v = -\frac{c_1}{2x^2} + c_2$$

$\therefore$ C.F of the given equation is

$$y = vx$$

$$y = c_2 x - \frac{c_1}{2x}$$

or $$y = c_1' x + \frac{c_2'}{x}$$

Now let $y = Ax + \frac{B}{x}$ —— (2)

be the Complete solution of the given equation where A and B are function of x. 1½

$$\therefore \frac{dy}{dx} = A - \frac{B}{x^2} + \frac{dA}{dx}x + \frac{1}{x}\frac{dB}{dx} \quad \text{—(3)}$$

Now taking A and B are such that

$$x\frac{dA}{dx} + \frac{1}{x}\frac{dB}{dx} = 0 \quad \text{—— (4)}$$

$$\therefore \frac{dy}{dx} = A - \frac{B}{x^2}$$

Differentiating w. r. to x

$$\frac{d^2y}{dx^2} = \frac{dA}{dx} - \frac{1}{x^2}\frac{dB}{dx} + \frac{2}{x^3}B$$

Putting values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in given equation we get 2

$$\frac{dA}{dx} - \frac{1}{x^2}\frac{dB}{dx} + \frac{2}{x^3}B + \frac{1}{x}\left(A - \frac{B}{x^2}\right) - \frac{1}{x^2}\left(Ax + \frac{B}{x}\right) = e^x$$

or $$\frac{dA}{dx} - \frac{1}{x^2}\frac{dB}{dx} = e^x \quad \text{—— (5)}$$

Solving (4) and (5)

$$\frac{dA}{dx} = \frac{1}{2}e^x \Rightarrow A = \frac{1}{2}e^x + c_1$$

$$\frac{dB}{dx} = -\frac{1}{2}x^2 e^x \Rightarrow B = -\frac{1}{2}\int x^2 e^x dx$$

$= -\frac{1}{2}x^2e^x + xe^x - e^x + c_2$

Substituting in (2) the required solution

$y = c_1x + \frac{c_2}{x} + \frac{1}{2}xe^x - \frac{1}{2}xe^x + e^x - \frac{1}{x}e^x$ 1

or $y = c_1x + \frac{c_2}{x} + e^x - \frac{1}{x}e^x$

Ans ii) Given equation can be written as

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2\sin x^2 \quad \text{--- (1)}$$

Here $P = -\frac{1}{x}$, $Q = -4x^2$, $R = 8x^2\sin x^2$

Choose independent variable z such that

$$\left(\frac{dz}{dx}\right)^2 = |Q| = |-4x^2| = 4x^2$$ 1

$$\Rightarrow \frac{dz}{dx} = 2x \Rightarrow dz = 2x\,dx$$

$$\Rightarrow z = x^2 \quad \text{--- (2)}$$

Now given equation transform to

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \quad \text{--- (3)}$$

Where $P_1 = \dfrac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{(dz/dx)^2} = \dfrac{2 + (-1/x)2x}{4x^2} = 0$ 2

$$Q_1 = \frac{Q}{(dz/dx)^2} = \frac{-4x^2}{4x^2} = -1$$

$$R_1 = \frac{R}{(dz/dx)^2} = \frac{8x^2\sin x^2}{4x^2} = 2\sin x^2 = 2\sin z$$

Hence equation (3) will be

$$\frac{d^2y}{dx^2} - y = 2\sin z$$

$(D^2-1)y = 2\sin z$

A.E will be $m^2-1=0$

$\Rightarrow m^2 = 1$

$\Rightarrow m = \pm 1$

$\therefore \quad C.F = c_1 e^{z} + c_2 e^{-z}$ 1

Now $P.I = \frac{1}{D^2-1} 2\sin z$

$= \frac{1}{-1-1} 2\sin z = \frac{1}{-2} \cdot 2\sin z$

$P.I = -\sin z$

required solution is :

$y = C.F + P.I$

$y = c_1 e^{z} + c_2 e^{-z} - \sin z$

$y = c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2$ 1

$\{\because z = x^2\}$

Ans iii) Given equation

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = e^{x}\sec x$$

Here $P = -2\tan x, \; Q = 5, \; R = e^{x}\sec x$

Now $u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int -2\tan x}$ 1

$= e^{\log \sec x} = \sec x$

Let the general solution be $y = uv$
then v is given by the equation

$$\frac{d^2v}{dx^2} + Iv = R/u$$

where $I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$

$= 5 - \frac{1}{2}\frac{d}{dx}(-2\tan x) - \frac{1}{\cancel{4}}\cancel{4}\tan^2 x$

$= 5 + \sec^2 x - \tan^2 x$

$= 5 + 1 = 6 \Rightarrow \frac{d^2v}{dx^2} + 6v = e^x$ 1

Hence equation becomes:

$(D^2+6)v = e^x \quad , D \equiv \frac{d}{dx}$

A.E will be

$m^2 + 6 = 0$

$\Rightarrow m^2 = -6$ 1

$\Rightarrow m = \pm\sqrt{6}\,i$

$CF = C_1 \cos\sqrt{6}\,x + C_2 \sin\sqrt{6}\,x$

Now

$P.I = \frac{1}{D^2+6} e^x$

$= \frac{1}{1+6} e^x = \frac{1}{7} e^x$ {Putting $D=1$} 2

$\therefore \quad V = CF + P.I$

$V = C_1 \cos\sqrt{6}x + C_2 \sin\sqrt{6}x + \frac{1}{7}e^x$

$\therefore$ Thus Complete solution of the given equation is $y = uv$

or $y = (C_1 \cos\sqrt{6}\,x + C_2 \sin\sqrt{6}\,x + \frac{1}{7}e^x)\sec x$