Total No. of Questions: 6 *Total No. of Printed Pages:3*

Enrollment No.....

Faculty of Engineering

End Sem (Odd) Examination Dec-2022

EN3BS06 Discrete Mathematics

Programme: B.Tech. Branch/Specialisation: CSBS

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. Let A and B be two disjoint sets then $|A \cup B| =$ **1**

(a) $|A| + |B|$ (b) $|A| - |B|$

(c) $|A||B|$ (d) None of these

ii. The set of all subsets of a set is **1**

(a) Proper Subset (b) Improper Subset

(c) Power Set (d) None of these

iii. In group $G = \{1, -1, i, -i\}$ inverse of element i with respect to multiplication is- **1**

(a) 1 (b) -1 (c) $-i$ (d) None of these

iv. Let B ={1, 2, 3, 5, 15} is a set of positive divisors of 15 and, for any two elements a, b in B defines $a \wedge b = gcd\ (a, b)$, $a \vee b = lcm\ (a, b)$ then complements of element 3 is- **1**

(a) 1 (b) 5 (c) 10 (d) None of these

v. The number of integers in $A = \{1, 2, 3, ...1000\}$, are divisible by 3- **1**

(a) 300 (b) 333 (c) *289* (d) None of these

vi. If $|A| > |B|$ then any function $f : A \rightarrow B$ can never be- **1**

(a) One One (b) On to (c) Many one (d) None of these

vii. A graph $G = (V, E)$ is called Hamiltonian if it contains- **1**

(a) Boolean circuit (b) Hamiltonian circuit

(c) Euler circuit (d) None of these

viii. The number of parallel edges or loop in a tree: **1**

(a) 1 (b) 2 (c) 0 (d) None of these

P.T.O.

ix. Consider the statement: **1**
$1. A \Leftrightarrow A \vee \sim A$, $2. A \Leftrightarrow A \vee A$, which of the above is tautology
(a) 1 (b) 1 and 2 (c) 2 (d) None of these

x. A preposition which is true for all the truth values of its sub preposition or component is called _____. **1**
(a) Contradiction (b) Tautology
(c) Both (a) and (b) (d) None of these

Q.2 Attempt any two:

i. State and prove De morgan's law for two subsets of A and B of universal set U. **5**

ii. Define poset and partial order relation. **5**

iii. If R is an equivalence relation, then prove that R^{-1} is also an equivalence relation in the set A. **5**

Q.3 Attempt any two:

i. Define group and abelian group for a binary operation addition. **5**

ii. In a lattice $(L, \wedge, \vee)$ for each a in L prove that: **5**
(a) $a \wedge a = a$ (b) $a \vee a = a$

iii. Show that the set of numbers of the form $a + b\sqrt{2}$ where a and b are real numbers is a field. **5**

Q.4 Attempt any two:

i. Show that if 5 points are selected in a square whose sides have length 1 inch, at least two of the points must be no more than $\sqrt{2}$ inches apart. **5**

ii. Apply the generating function technique to solve the initial value problem $y_{k+2} - 2\, y_{k+1} + y_k = 2^k$ with $y_0 = 2$ and $y_1 = 1$. **5**

iii. How many different four-digit numbers can be formed from the digits 1,2,3,5 assuming that the digits can be repeated any number of times. **5**

Q.5 Attempt any two:

i. Define chromatic number in graph and statement of four colour problem. **5**

ii. Define planar graph. Show that a connected planar graph with n vertices and e edges has r regions given by $r = e - n + 2$. **5**

iii. Prove that a tree with n vertices has exactly $n-1$ edges. **5**

Q.6 Attempt any two:

i. Explain tautology and contradiction with using two statements example. **5**

ii. Show that $p \Longrightarrow (q \Longrightarrow r) \equiv (p \wedge q) \Longrightarrow r$. **5**

iii. Express the following formula into conjunctive normal form $\sim(p \vee q) \leftrightarrow (p \wedge q)$. **5**

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No. Marks

Medicaps University
End Sem Exam Dec 2022
B.Tech (CSBS)
Discrete Mathematics (EN3BS06)

Q1 MCQ:

i) a) $|A \cup B| = |A| + |B|$
ii) c) Power set
iii) c) $-i$
iv) b) 5
v) b) 333
vi) a) ~~Many~~ one-one
vii) b) Hamiltonian circuit
viii) c) 0
ix) a) 1
x) b) Tautology

+10

Q2 1) If U is a universal set and its two subsets A and B then De morgan's law

(a) $(A \cup B)' = A' \cap B'$ (b) $(A \cap B)' = A' \cup B'$ +2

Proof: $(A \cup B)' = A' \cap B'$ if

$(A \cup B)' \subseteq A' \cap B'$ and $A' \cap B' \subseteq (A \cup B)'$

To Prove $(A \cup B)' \subseteq A' \cap B'$ +1

Let $x \in (A \cup B)'$ then

$x \in (A \cup B)' \Rightarrow x \notin A \cup B$ [$x \in A' \Rightarrow x \notin A$]

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in A' \cap B'$

$(A \cup B)' \subseteq A' \cap B'$ — (1)

To Prove $A' \cap B' \subseteq (A \cup B)'$ +1

Let $x \in A' \cap B'$ then

$x \in A' \cap B' \Rightarrow x \in A'$ and $x \in B'$

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$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \in (A \cup B)'$

$A' \cap B' \subseteq (A \cup B)'$ — (2)

from (1) & (2) $(A \cup B)' = A' \cap B'$ +1

Similarly Prove

$(A \cap B)' = A' \cup B'$

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ii) POSET: A set P together with a POR(R) is called POSET or ordered set. denoted by (P, R). usually denoted by $(P, \leq)$ +2

A relation R on a set P is called Partial order relation if

i) R is reflexive: ie $aRa \ \forall a \in P$ +1

ii) R is antisymmetry ie aRb and bRa $\Rightarrow a = b$ for $a, b \in P$ +1

iii) R is ~~symmetry~~ transitive: +1

ie aRb & $bRc \Rightarrow aRc$ for $a, b, c \in P$

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iii) Given R is an equivalence relation in the set A.

To Prove R^{-1} is also an equivalence relation.

1) R^{-1} is reflexive +1

$\because$ R is reflexive $\therefore \forall a \in A \Rightarrow (a, a) \in R$

Now $(a, a) \in R \Rightarrow (a, a) \in R^{-1}$

$\therefore \forall a \in A$, $(a, a) \in R^{-1}$

$\therefore R^{-1}$ is reflexive

2) R^{-1} is symmetric

let $a, b \in A$ and $(a, b) \in R$ then

No. Marks

$(a,b) \in R \Rightarrow (b,a) \in R^{-1}$

Now $(a,b) \in R \Rightarrow (b,a) \in R$

$\Rightarrow (a,b) \in R^{-1}$ +2

$\therefore (b,a) \in R^{-1} \Rightarrow (a,b) \in R^{-1}$

$\therefore R^{-1}$ is symmetric.

iii) R^{-1} is transitive

Let $(a,b) \in R$ and $(b,c) \in R$ then

$(b,a) \in R^{-1}$ and $(c,b) \in R^{-1}$ +2

Now

$(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$

[R is transitive

$\therefore (c,b) \in R^{-1}$ and $(b,a) \in R^{-1} \Rightarrow (c,a) \in R^{-1}$

$\therefore R^{-1}$ is transitive

$\therefore R^{-1}$ is an equivalence relation

Q3 i) An algebraic structure $(G,+)$ where G is non empty set and '+' is binary operation defined on G is called a group if satisfy following postulates:

1) Closure – $a \in G, b \in G \Rightarrow a+b \in G$ $\forall a,b \in G$ +1

2) Associative – $a+(b+c) = (a+b)+c$ $\forall a,b,c \in G$ +1

3) Identity – $a+0 = 0+a = a \ \forall a \in G$ +1

0 is called identity for addition

4) Inverse – for each $a \in G \ \exists -a \in G$ such that $a+(-a) = (-a)+a = 0 \in G$ +1

Abelian group: A group $(G,+)$ is said to be abelian or commutative if in addition to (A) postulates the following is also satisfied +1

$a+b = b+a \ \forall a,b \in G$

No. Marks

ii) In a lattice $(L, \wedge, \vee)$ for each $a \in L$
i) $a \wedge a = a$ ii) $a \vee a = a$ (Idempotent) +1

Proof:
Let a be any arbitrary element of L then

i) $a \wedge a = a \wedge (a \vee (a \wedge b))$ (Absorption law) +2

$= a$ (Absorption law)

$\therefore a \wedge a = a \quad \forall a \in L$

ii) $a \vee a = a \vee (a \wedge (a \vee b))$ [Absorption law] +2

$= a$

$\therefore a \vee a = a \quad \forall a \in L$

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iii) Let $Q[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in Q\}$

Let $a_1 + b_1\sqrt{2}$ and $a_2 + b_2\sqrt{2} \in Q[\sqrt{2}]$ +1/2
then $a_1, b_1, a_2, b_2 \in Q$

we have
$$(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = (a_1 + a_2) + (b_1 + b_2)\sqrt{2} \in Q[\sqrt{2}]$$
$[\because a_1 + a_2, b_1 + b_2 \in Q]$ +1/2

Also
$$(a_1 + b_1\sqrt{2}) \cdot (a_2 + b_2\sqrt{2}) = (a_1a_2 + 2b_1b_2) + (a_1b_2 + a_2b_1)\sqrt{2} \in Q[\sqrt{2}]$$

is closed w.r.t to addition and multiplication

All the elements of $Q[\sqrt{2}]$ are real numbers and w.r.t + addition and multiplication are both associative as well as commutative in the set of real numbers. +1

$\therefore 0 + 0\sqrt{2}$ is the additive identity

No. Marks

Again if $a+b\sqrt{2} \in Q[\sqrt{2}]$ then $(-a)+(-b)\sqrt{2} \in Q[\sqrt{2}]$ and we have

$$[(-a)+(-b)\sqrt{2}]+(a+b\sqrt{2})=0+0\sqrt{2}$$

$\therefore$ each element of $Q[\sqrt{2}]$ possesses additive inverse.

Further in the set of real numbers, multiplication is distributive with respect to addition

Again $1+0\sqrt{2} \in Q[\sqrt{2}]$ and +1

$$(1+0\sqrt{2})(a+b\sqrt{2})=(a+b\sqrt{2})(1+0\sqrt{2})$$

$\therefore$ $1+0\sqrt{2}$ is the multiplicative inverse

Thus $(Q[\sqrt{2}], +, \cdot)$ is a commutative ring with unity. $0+0\sqrt{2}$ is zero element and $1+0\sqrt{2}$ is unit element.

Now $(Q[\sqrt{2}], +, \cdot)$ will be field if each non zero element of $Q[\sqrt{2}]$ possesses multiplicative inverse.

Let $a+b\sqrt{2}$ be any non zero element of this ring i.e at least $a \neq 0$ or $b \neq 0$ +1

then $\dfrac{1}{a+b\sqrt{2}} = \dfrac{a-b\sqrt{2}}{a^2-2b^2}$

If at least one of the rational number a and b is non zero then we cannot have $a^2-2b^2=0$ i.e $a^2=2b^2$

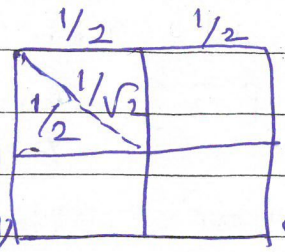
$\therefore$ $\left(\dfrac{a}{a^2-2b^2}\right)+\left(\dfrac{-b}{a^2-2b^2}\right)\sqrt{2}$ is a non zero element of $Q[\sqrt{2}]$ and it is M.I of $a+b\sqrt{2}$ +1

Hence $a+b\sqrt{2} \in Q[\sqrt{2}]$ and $(Q[\sqrt{2}], +, \cdot)$ is a field

No. Marks

Q4

i) We divide the given square into four small regions (square) Now the given five points (Pigeons) and 4 (Pigeonholes) squares. Hence by Pigeonhole Principle some small square must contain at least two points. The distance between these points can not exceed the length of the hypotaneous of the small square which is $1/\sqrt{2}$. Hence there exist 2 points within a distance at most $1/\sqrt{2}$ inches



+2

+2

+1

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ii) Given equation is

$$y_{k+2} - 2y_{k+1} + y_k = 2^k \text{ and } y_0 = 2,\ y_1 = 1 \quad (1)$$

let generating function

$$Y(t) = \sum_{h=0}^{\infty} y_h t^h = y_0 + y_1 t + y_2 t^2 + \cdots \quad (2)$$ +1

Multiplying by t^k and summing from $h=0$ to $h=\infty$ we have

$$\sum_{h=0}^{\infty} y_{k+2} t^k - 2\sum_{h=0}^{\infty} y_{k+1} t^k + \sum_{h=0}^{\infty} y_k t^k = \sum_{h=0}^{\infty} 2^k t^k$$ +1

$$(y_2 + y_3 t + y_4 t^2 + \cdots) - 2(y_1 + y_2 t + \cdots) + Y(t) = 1 + 2t + (2t)^2 + \cdots$$

$$\frac{Y(t) - y_0 - y_1 t}{t^2} - 2\,\frac{Y(t) - y_0}{t} + Y(t) = \frac{1}{1-2t}$$ +1

$$Y(t) - 2 - t - 2t\,[Y(t) - 2] + t^2 Y(t) = \frac{t^2}{1-2t}$$

$$(1 - 2t + t^2)\,Y(t) = 2 - 3t + \frac{t^2}{1-2t}$$

$$Y(t) = \frac{2 - 7t + 7t^2}{(1-t)^2(1-2t)}$$

No: Marks

Now solving RHS by Partial fraction

$$\frac{2-7t+7t^2}{(1-t)^2(1-2t)} = \frac{A}{1-t} + \frac{B}{(1-t)^2} + \frac{C}{1-2t}$$ +1

on solving $\boxed{A=3}$ $\boxed{B=-2}$ $\boxed{C=1}$

$$Y(t) = \frac{3}{1-t} - \frac{2}{(1-t)^2} + \frac{1}{1-2t}$$

$$= 3(1-t)^{-1} - 2(1-t)^{-2} + (1-2t)^{-1}$$ +1

$$= 3(1+t+t^2+\dots) - 2(1+2t+3t^2+\dots) + (1+2t+(2t)^2+\dots)$$

$$y_k = 3t^k - 2(k+1) + 2^k$$

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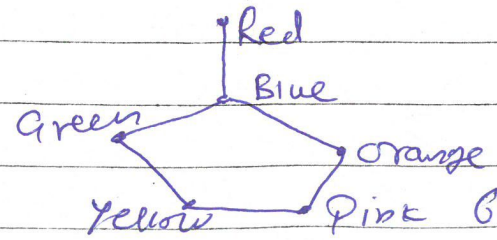
iii) In four digit numbers we can put any of the given four digits at the first place in 4 ways. Since number can be repeated therefore second place, third place, fourth place can be placed in 4 ways +3

Hence number of different four digit numbers $= 4 \times 4 \times 4 \times 4 = 256$ +2

Q5

1) Chromatic number: The chromatic number of a graph G is the minimum number of colours required for proper colouring the vertices of G and denoted by $\chi(G)$. Thus if a graph G requires n different colours but not less than n for its proper colouring then $\chi(G) = n$ and G is called n-chromatic graph. +4



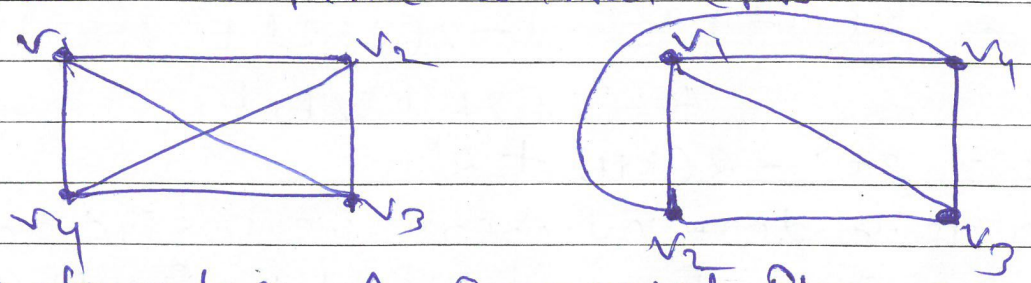
6-colours are used

No. Four Colour Problem:

Every Planar graph is 4 colourable. +1

Marks

11) Planar graph: A graph G is called Planar if it can be drawn on a plane without intersecting edges. The point of intersection is called Crossover. Thus a graph is said to be Planar if it can be drawn on a plane without crossover. +1

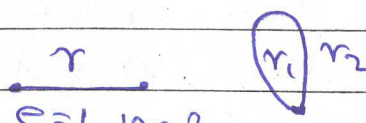


Euler's formula: A connected Planar graph with n vertices e edges has r regions given by $r = e - n + 2$ +1

Proof: Suppose $e = 1$ then n may be equal to 1 or 2

In case $e = 1, n = 2$

$r = 1 - 2 + 2 = 1$



Graph has one region (only infinite region)

Again $e = 1, n = 1, r = 1 - 1 + 2 = 2$ +1/2

Graph has two regions r_1 and r_2

hence result is true for $e = 1$

Now suppose result is true for edges at most $e-1$. In case G is a tree $e = n - 1$ and number of region is 1

$\therefore \quad r = e - n + 2 = (n-1) - n + 2 = 1$ +1/2

hence result holds when G is a tree.

If G is not a tree. G has some circuit

No. Marks

Consider an edge 'e' in some ckt. by removing this edge 'e' from the plane representation of G the regions are merged into a new region therefore G-{e} is a connected graph with n vertices, e-1 edges and r-1 regions thus by induction hypothesis we have +2

$$r-1 = (e-1)-n+2 \quad \boxed{r = e-n+2}$$

Hence Proved

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iii) we shall prove by Induction Method

If $n=1$, T has no edge

$n=2$, T has one edge

Now suppose that the theorem is true for all trees having less than n vertices, +1

let T be a tree with n vertices and there exist only one path between every pair of vertices If we delete one edge from T then graph will be disconnected and will have two components

$$T-e = T_1 + T_2$$ +1

let n_1 and n_2 be the number of vertices of T_1 and T_2 where $n_1 < n$ and $n_2 < n$

Such that $n_1 + n_2 = n$ +1

As no of edges in $T_1 = n_1 - 1$

$T_2 = n_2 - 1$

Number of edges in $T-e = T_1 + T_2$

$= (n_1-1)+(n_2-1)$

$= n_1+n_2-2$

$= n-2$ +2

If we replace edge

then edges in $T = n-2+1$

$= n-1$

No. Marks

Hence the theorem holds for all values of n and hence tree has n-1 edges with n vertices

No. Marks

Q6 1) Tautology: A tautology is a Proposition which is true for all truth values of its Sub propositions or Components.

A tautology is also called logically valid or true. In truth table all entries in the column of tautology are of 'T' only. +1

Example: if p and q are two statements then $(p \wedge q) \Rightarrow p$ is a tautology

p	q	$p \wedge q$	$p \wedge q \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

+1 $\frac{1}{2}$

Contradiction: A contradiction is a Proposition, which is always false for all truth values of its Propositions or Components. +1

A contradiction is also called logically false. Clearly in truth tables, all entries in the column of contradiction are of 'F' only.

Example: $p \wedge (\sim p)$ is a contradiction +1 $\frac{1}{2}$

p	$\sim p$	$p \wedge (\sim p)$
T	F	F
F	T	F

No. Marks

ii) $p \Rightarrow (q \Rightarrow r) \equiv (p \wedge q) \Rightarrow r$

p	q	r	$q \Rightarrow r$	A: $p \Rightarrow (q \Rightarrow r)$	$p \wedge q$	B: $p \wedge q \Rightarrow r$	$A \equiv B$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

+5

iii) Let $P = \sim(p \vee q)$ and $Q = (p \wedge q)$

We know that

$$P \leftrightarrow Q$$

$$\Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P) \quad \text{—(1)}$$ +1

Putting P and Q

$$\sim(p \vee q) \leftrightarrow (p \wedge q)$$

$$\leftrightarrow [\sim(p \vee q) \rightarrow (p \wedge q)]$$

$$\wedge [(p \wedge q) \rightarrow \sim(p \vee q)]$$ +1

$$\leftrightarrow [(p \vee q) \vee (p \wedge q)] \wedge [\sim(p \wedge q) \vee \sim(p \vee q)]$$

$$[\because x \rightarrow y \leftrightarrow [-x \vee y]]$$ +1

$$\leftrightarrow [(p \vee q) \vee (p \wedge q)] \wedge [(\sim p \vee \sim q) \vee (p \wedge \sim q)]$$

[De Morgan law] +1

$$\leftrightarrow (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge (\sim p \vee \sim q \vee \sim p)$$

$$\wedge (\sim p \vee \sim q \vee \sim q)$$

[Distributive law] +1

which is required Conjunctive Normal form