OR iii. Show that a topological space (X, τ) is T_2-space if and only if every convergent filter on X has unique limit. **6**

Q.6 Attempt any two:

i. Show that a topological space is completely regular if and only if the family of all continuous real-valued function on it distinguishes points from closed sets. **5**

ii. Show that a second countable space is metrisable if and only if it is T_3. **5**

iii. Show that a topological space is a Tychonoff space if and only if it is embeddable into a cube. **5**

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Total No. of Questions: 6 *Total No. of Printed Pages:4*

Enrollment No......................................

Faculty of Science

End Sem (Even) Examination May-2022

MA5CO08 Topology -II

Programme: M.Sc. Branch/Specialisation: Mathematics

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. Every topological space (X, τ) is compact if- **1**

(a) X is not finite
(b) τ is not finite
(c) Every open cover of X is reducible to finite subcover
(d) None of these

ii. Which of the following is not true? **1**

(a) Every sequentially compact metric space is compact.
(b) Every sequentially compact metric space is totally bounded.
(c) Every sequentially compact metric space has the Bolzano Weierstrass property.
(d) Every sequentially compact metric space is not separable.

iii. Which one of the following is not true? **1**

(a) A second countable space is always first countable.
(b) First countable space always implies second countable space.
(c) Every subspace of a second countable space is second countable space.
(d) A second countable space is always separable.

iv. Let (X, τ) be a T_3-space, then which one is true? **1**

(a) X is a T_2-space
(b) X is a T_1-space
(c) X is a T_4-space
(d) None of these

v. Finite product of normal spaces is- **1**

(a) Normal (b) Need not be normal
(c) Hausdorff (d) Need not Hausdorff

vi. Tychonoff theorem states- **1**

(a) Any arbitrary product of compact spaces is compact in the product topology.

(b) A topological space for which every open covering contains a countable subcovering.

(c) Path connected topological space is connected but converse is not true.

(d) None of these.

vii. Which of the following statement is not true? **1**

(a) Let $S: D \to X$ be a net in a topological space (X, τ). Then the family $\{B_m: m \in D\}$ has finite intersection property, where each $m \in D$, $B_m = \overline{A_m}$ and $A_m = \{S(n): n \in D, n \geq m\}$.

(b) Let X be a set and τ_1, τ_2 topologies on X. Then τ_1 is stronger than τ_2 if and only if whenever a net converges to a point w.r.t τ_1, it does so w.r.t τ_2 also.

(c) In a first countable compact topological space (X, τ) every sequence has a convergent subsequence.

(d) None of these.

viii. Which of the following statement is true? **1**

(a) The singleton family $\{X\}$ is not a filter on X, where X is any arbitrary nonempty set.

(b) A topological space (X, τ) is Hausdorff then every ultrafilter converges to all points of X in it.

(c) Let (X, τ) be a topological space and suppose $A \subset X, x \in X$. Then $x \in \overline{A}$ iff there exists a filter $\mathcal{F}$ on X such that $A \in \mathcal{F}$ and $\mathcal{F}$ converges to X.

(d) None of these.

ix. Every metrizable space is- **1**

(a) Hausdorff (b) Disjoint (c) Normal (d) None of these

x. Which of the following is false? **1**

(a) Every para compact space X is normal.

(b) Every closed subspace of a para compact space is para compact.

(c) An arbitrary subspace of a para compact space and product of para compact spaces need not be para compact.

(d) Every metrizable space need not be para compact.

Q.2 i. Show that every compact subset of compact topological space need not be closed. **2**

ii. Define countably compact space and then show that in usual topological space $(\mathbb{R}, \tau)$, the closed interval $I = [a,\ b]$, where $a, b \in \mathbb{R}$ is countable compact space. **3**

iii. Define locally compact space and then show that every compact topological space is locally compact, but converse need not be true. **5**

OR iv. Define sequentially compact space and then show that sequentially compact space is a countably compact. **5**

Q.3 i. Show that regularity is a hereditary property. **2**

ii. Show that every Lindelöf regular space is normal. **8**

OR iii. Let A be a nonempty subset of a normal topological space (X, τ_X) and $f: A \to [-1,\ 1]$ be a continuous map. Then show that there exists a continuous map $g: X \to [-1,\ 1]$ such that- **8**

$g(x) = f(x), \forall\, x \in A$.

Q.4 i. If (X, τ_1), (Y, τ_2) and (Z, τ_3) are topological spaces then the function $f: Z \to X \times Y$ is continuous if and only if the function $\pi_1 o f: Z \to X$ and $\pi_2 o f: Z \to Y$ are continuous, where π_1 and π_2 are projection maps. **3**

ii. Define quotient space, and then show that- **7**

If (X, τ) is a topological product of a family of topological spaces $\{(X_i, \tau_i): 1, 2, \dots n\}$, then τ_i is the quotient topology with respect to X and π_i.

OR iii. Show that a product of topological spaces is path-connected if and only if each coordinate space is path-connected. **7**

Q.5 i. Show that a topological space is Hausdorff if and only if limits of all nets in it are unique. **4**

ii. Let $(X,\ \tau)$ be a topological space and $A \subseteq X$. Then show that $x \in \overline{A}$ if and only if there exist a net $S(D)$ in A such that $S(D) \to x$. **6**

Scheme of Marking

MEDI-CAPS UNIVERSITY Knowledge is Power	Faculty of Science End Sem (Even) Examination May-2022 Topology-II(MA5CO08)	
	Programme: M.Sc.	Branch/Specialisation:

Note: The Paper Setter should provide the answer wise splitting of the marks in the scheme below.

Q.1	i)	Every topological space (X,τ) is compact if, c) every open cover of X is reducible to finite subcover.	1
	ii)	Which of the following is not true? d) Every sequentially compact metric space is not separable.	1
	iii)	Which one of the following is not true? b) First countable space always implies second countable space.	1
	iv)	Let (X,τ) be a T_3-space, then which one is true? b) X is a T_1-space.	1
	v)	Product of normal spaces is b) Need not be normal	1
	vi)	Tychonoff theorem states a) Any arbitrary product of compact spaces is compact in the product topology.	1
	vii)	Which of the following statement is not true? d) None of the above.	1
	viii)	Which of the following statement is true? c) Let (X,τ) be a topological space and suppose $A \subset X, x \in X$. Then $x \in \overline{A}$ iff there exists a filter $\mathcal{F}$ on X such that $A \in \mathcal{F}$ and $\mathcal{F}$ converges to X.	1
	ix)	Every metrizable space is c) normal	1
	x)	Which of the following is false? d) Every metrizable space need not be para compact.	1
Q.2	i.	Show that every compact subset of compact topological space need not be closed. **Solution**: Let (X,τ) be indiscrete topological space containing more than one point. Then clearly it is compact. **(1 marks)** Now let Y be any nonempty proper subset of X.Then the only τ_Y-open covering of Y being $\{Y\}$ and hence (Y, τ_Y) is compact. But Y is not clearly closed. **(1marks)**	
	ii.	Define countably compact space and then show that a continuous image of a countably compact space is countably compact. **Solution**: Definition: A topological space is said to be countably compact space if every countable open cover of it has a finite sub-cover. **(1 marks)** Let $G \subset I$ is an infinite set then it is bounded and from Bolzono-Weistrass theorem G has a limit point x and since $G \subset I$ then x is limit point of I and since I is closed set then $x \in I$ which implies that I is countably compact. **(2 marks)**	
	iii.	Define locally compact space and then show that every compact topological space is locally compact, but converse need not be true. **Solution:** Definition: - A topological space X is said to be locally compact at a point $x \in X$ if x has a compact nbd in X. X is locally compact if it is locally compact at every point. **(1marks)** If part: - Let (X,τ) be a compact topological space and $x \in X$. Then X is nbd of X such that $\overline{X} = X$ is compact. Thus, every point in X has a nbd, namely X whose closure is compact and hence it is locally compact. **(2 marks)** Conversely, consider discrete topological space (X,τ), where X is infinite. Then the collection $\{\{x\}: x \in X\}$ is an infinite open covering of X having no finite sub covering, so (X,τ) is not compact. But (X,τ) is locally compact, if for $x \in X$ then $\{x\}$ is a nbd of x and $\overline{\{x\}} = \{x\}$ which being finite is compact i.e., X is locally compact at each x. Hence (X,τ) is locally compact. **(2 marks)**	
OR	iv.	Define sequentially compact space and then show that sequentially compact space is a countably compact. **Solution**: A topological space is said to be sequentially compact if every sequence in it has a convergent subsequence. **(1 marks)** Let $Y \subset X$ is an infinite set. Then there exists an infinite sequence $\{x_n\}$ of distinct points of Y. If (X,τ) be a sequentially compact space, then there exists a subsequence $\{x_{i_n}\}$ of $\{x_n\}$ which converges to a point x in X this means that there is $G \in \tau$ such that $x \in G$ and there exists $n_0 \in N$ such that $x_{i_n} \in G$ whenever $i_n \geq n_0$this means that $G \cap Y$ is an infinte set which implies that	

		$(G-\{x\}) \cap Y \neq \emptyset$. Hence x is a limit point of Y belon to X which implies that (X,τ) is countably compact. **(4 marks)**	
Q.3	i.	Show that regularity is a hereditary property. **Solution**: - Let X is a regular space and Y is a subspace of X. Let $y \in Y$ and D be a closed subset of Y not containing y. Then D is of the form $C \cap Y$ where C is a closed subset of X. Note that $y \notin C$ for otherwise $y \in D$. Hence by regularity of X, there exist open sets U, V in X such that $y \in U, C \subset V$ and $U \cap V = \phi$. **(1 marks)** Let $G = U \cap Y, H = V \cap Y$. Then G, H are open in the relative topology on Y. Also, $y \in G, D \subset H \, and \, G \cap H = \phi$. Thus, the space Y with the relative topology is regular. **(1 marks)**	
	ii.	Show that every Lindelöf regular space is normal. **Solution**: - Let (X,τ) be a Lindelöf regular space and E and F are two disjoint closed subsets of X. Since Lindelöf is closed hereditary, therefore E and F are also Lindelöf. **(1 marks)** Now since X is regular let $x \in F$ and corresponding to the closed set $E, \exists$ an open set G_x such that $x \in G_x \subseteq \overline{G_x} \subseteq E^c$. Moreover, the family $\{G_x : x \in F\}$ is an open cover of Lindelöf set F, so it must have countable subcover say $\{G_{x_i} : i \in N\}$. **(2 marks)** Again, by regularity of X, let $y \in E$ and corresponding closed set $F, \exists$ open set H_y such that y$\in H_y \subseteq \overline{H_y} \subseteq F^c$. Similarly, $\{H_{y_i} : i \in N\}$ is also countable subcover of E. **(2 marks)** Let $V_n = H_{y_n} - \cup_{i \in N} \{\overline{H_{y_i}} : i \leq n\}$ $= H_{y_n} \cap [X - \cup_{i \in N} \{\overline{H_{y_i}} : i \leq n]$ and $W_n = G_{y_n} - \cup_{i \in N} \{\overline{G_{y_i}} : i \leq n\}$ $= G_{y_n} \cap [X - \cup_{i \in N} \{\overline{G_{y_i}} : i \leq n]$ Then clearly V_n and W_n are open set and therefore $G = \cup \{V_n : n \in N\}$ and H$= \cup \{W_n : n \in N\}$ are open. Now, $E \sqsubseteq \{H_{y_i} : i \in N\}$ and $\overline{H_{y_i}} \cap E = \phi$, so it follows that $\{V_n : n \in N\}$ is open covering of E, $\therefore E \sqsubseteq \cup \{V_n : n \in N\} = G$. Similarly, $F \subseteq H$. Also $= V_n \cap W_n = \phi$, for all $n \Rightarrow G \cap H = \phi$. Hence (X,τ) is normal. **(3 marks)**	
OR	iii.	Let A be a nonempty subset of a normal topological space (X, τ_X) and $f: A \to [-1, 1]$ be a continuous map. Then show that there exists a continuous map $g: X \to [-1, 1]$ such that $g(x) = f(x), \forall x \in A$. **Solution**: The sets $\left[-1, -\frac{1}{3}\right], \left[\frac{1}{3}, 1\right]$ are closed subset of $[-1. 1]$ and $f: A \to [-1, 1]$ is a continuous map implies $A_1 =$	

		$f^{-1}\left(\left[-1, -\frac{1}{3}\right]\right), B_1 = f^{-1}\left(\left[\frac{1}{3}, 1\right]\right)$ are closed subset of A. Now $x \in A_1 \cap B_1$ implies $f(x) \in \left[-1, -\frac{1}{3}\right] \cap \left[\frac{1}{3}, 1\right]$ a contradiction. Hence $A_1 \cap B_1 = \phi$. **(2 marks)** Now, A_1 and $B_1$are closed subset of a normal space X. Hence by Urysohn's lemma $\exists$ a continuous map $f_1: X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ such that $f_1(A_1) = \frac{1}{3}$ and $f_1(B_1) = -\frac{1}{3}$ then $\lvert f(x) - f_1(x) \rvert \leq \frac{2}{3}$ for all $x \in A$. Now consider the function $f - f_1: A \to \left[-\frac{2}{3}, \frac{2}{3}\right]$ then $A_2 = (f - f_1)^{-1}([\frac{2}{9}, \frac{2}{3}])$ and $B_2 = (f - f_1)^{-1}([-\frac{2}{3}, -\frac{2}{9}])$ are disjoint closed subset of X. **(2 marks)** Again by using Urysohn's lemma $\exists$ a continuous map $f_2: X \to \left[-\frac{2}{9}, \frac{2}{9}\right]$ such that $f_2(A_2) = \frac{2}{9}$ and $f_2(B_2) = -\frac{2}{9}$ then $\lvert f(x) - (f_1(x) + f_2(x)) \rvert \leq \frac{4}{9}$ for all $x \in A$. By proceeding as above by induction for each $nn \in N$ there exists a continuous function $f_n: X \to \left[-\frac{2^{n-1}}{3^n}, \frac{2^{n-1}}{3^n}\right]$ such that $\lvert f(x) - \sum_{i=1}^{n} f_i(x) \rvert \leq \left(\frac{2}{3}\right)^n$ for all $x \in A$. That is $f_n: X \to [-1, 1]$ is a sequence of continuous function such that $\lvert f_n(x) \rvert \leq \frac{2^{n-1}}{3^n} = M_n$ and $\sum M_n < \infty$. **(2 marks)** By Weierstrass M-test the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on X. Also, each $s_n(x) = \sum_{i=1}^{n} f_i(x), x \in X$ then $s_n(x)$ converges uniformly on X. Hence let s_n converges uniformly to a continuous function g defined by $g(x) = \sum_{n=1}^{\infty} f_n(x)$. Now for each $x \in A, \lvert g(x) - f(x) \rvert = \left\lvert \lim_{n \to \infty} \sum_{i=1}^{n} f_i(x) - f(x) \right\rvert \leq \lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0$ this implies that $f(x) = g(x), x \in A$. **(2 marks)**	
Q.4	i.	If (X, τ_1), (Y, τ_2) and (Z, τ_3) are topological spaces then the function $f: Z \to X \times Y$ is continuous if and only if the function $\pi_1 of: Z \to X$ and $\pi_2 of: Z \to Y$ are continuous, where π_1 and π_2 are projection maps. **Solution**: If $\pi_1 of$ and $\pi_2 of$ are continuous then, $f^{-1}(\pi_1^{-1}(U) = (\pi_1 of)^{-1}(U) \in \tau_3$ for each $U \in \tau_1$ and $f^{-1}(\pi_2^{-1}(V) = (\pi_2 of)^{-1}(V) \in \tau_3$ for each $V \in \tau_2$. This means that inverse of any elements of the sub basis σ of $\tau_1 \times \tau_2$ by a function f is an element of τ_3 and so f is continuous. **(2 marks)** Conversely, if f is continuous and since π_1 and π_2 are continuous then $\pi_1 of$ and $\pi_2 of$ are continuous. **(1 marks)**	

ii. Define quotient space, and then show that

a) if (X, τ) is a topological product of a family of topological spaces $\{(X_i, \tau_i): 1, 2, \dots n\}$, then τ_i is the quotient topology with respect to X and π_i.

b) every quotient space of a discrete space is discrete.

Solution: Definition: - Let (X, τ_1) and (Y, τ_2) be topological space and $f: X \to Y$ be an onto map. Then f is said to be quotient map of Y or Y is a quotient topology with respect to X and f if τ_2 is the strong topology generated by the singleton family $\{f\}$. In such case we also say that Y is a quotient space of X. **(1 marks)**

a) Let $V \subset X_i$. If V is open in, X_i then $\pi_i^{-1}(V)$ is open in X by continuity of π_i. On the other hand, if $\pi_i^{-1}(V)$ is open in X then $V = \pi_i(\pi_i^{-1}(V))$ is open in X_i because the projection functions are open. Thus V is open in X_i if and only if $\pi_i^{-1}(V)$ is open in X and so τ_i is the quotient space. **(3 marks)**

b) Let $f: X \to Y$ be a quotient map where X is a discrete space. Let , τ be the quotient topology on Y. Recall that τ is the strongest topology on Y which makes f continuous. But since domain X is discrete, any topology on Y renders f continuous. So τ is the strongest topology on Y, that is discrete topology. **(3 marks)**

OR iii. Show that a product of topological spaces is path-connected if and only if each coordinate space is path-connected.

Solution: Let $X = \prod_{i \in I} X_i$ be path-connected. Since each projection map is continuous and onto and X is path-connected therefore each X_i is also path-connected. **(2 marks)**

Conversely, suppose each X_i is path-connected. Let x, y be points in X. For each $i \in I$, there exists a path α_i in X_i from $\pi_i(x)$ *to* $\pi_i(y)$. Define $\alpha: [0, 1] \to X$ by $\alpha(t)(i) = \pi_i(i), \forall\, i \in I$ *and* $t \in [0, 1]$. In other words, $\alpha(t)$ is the point whose i-th coordinate is $\alpha_i(t)$, i. e., $\alpha(t) = \{\alpha_i(t)\}_{i \in I}$. **(2.5 marks)**

We now show that α is continuous which is suffices to show that for each $i \in I$, the composite $\pi_i \circ \alpha$ is continuous. But for $t \in [0, 1]$, $\pi_i(\alpha(t))$ is, by the very definition of α, equal to $\pi_i(t)$. Hence $\pi_i \circ \alpha = \alpha_i$ which is given to be continuous. So α is a path in X. Clearly $\alpha(0) = x$ since $\alpha(0)(i) = \alpha_i(0) = \pi_i(x) = x(i), \forall\, i \in I$. Similarly $\alpha(1) = y$. Thus, any two points of X can be joined by a path and so X is path-connected. **(2.5 marks)**

Q.5 i. Show that a topological space is Hausdorff if and only if limits of all nets in it are unique.

Solution: Suppose X is a Hausdorff space, $S: D \to X$ is a net in X and converges to x and y in X. To show $x = y$.. **(2 marks)**
Conversely suppose that the limits of all nets in X are unique. To show that X is Hausdorff... **(2 marks)**

ii. Let (X, τ) be a topological space and $A \subseteq X$. Then show that $x \in \bar{A}$ if and only if there exist a net $S(D)$ in A such that $S(D) \to x$.

Solution: Let $x \in \bar{A}\ (x \in \bar{A} \Leftrightarrow \forall\, N \in N_x,\ N \cap A \neq \emptyset)$. **(1 marks)**

Take $D = N_x$. In D for n_1, n_2 define $n_1 \geq n_2$ *if* $n_1 \subseteq n_2$. Then D is a directed set. **(1 marks)**

Define the net $S(D)$ in A by for $n \in D$ *as* $N \cap A \neq \emptyset$ TAKE $S(N)$ =any element in $N \cap A$. To show that $S(D)$ in $A \to x$ ... **(2 marks)**

Conversely, let net $S(D)$ in $A \to x$ in X. To show that $x \in \bar{A}$.. **(2 marks)**

OR iii. Show that a topological space (X, τ) is T_2-space if and only if every convergent filter on X has unique limit.

Solution: Let (X, τ) be a T_2-space and let $\mathcal{F}$ be a convergent filter on X. If possible, let $\mathcal{F}$ converges to the distinct points x and y of X. **(1 marks)**

Now, x and y are distinct points of T_2 –space X. Therefore $\exists$ nbd G_x and G_x of x and y resp. such that $G_x \cap G_y = \emptyset$. **(1 marks)**

Also, $\mathcal{F} \to x$ and $x \in G_x \Rightarrow G_x \in \mathcal{F}$. Similarly $\mathcal{F} \to y$ and $y \in G_y \Rightarrow G_y \in \mathcal{F}$. Since $\mathcal{F}$ is a filter then $G_x \cap G_y \in \mathcal{F}$, i.e., $\emptyset \in \mathcal{F}$ which is contradiction hence $x = y$. **(1 marks)**

Conversely, let every convergent filter on X have a unique limit and if possible, let X be not T_2-space. Then $\exists$ distinct points x and y of X having nbd. G and H such that $G \cap H \neq \emptyset$. **(1.5 marks)**
Now $N_x \cap N_y$ is a filter base on X, so it generates a filter $\mathcal{F}$ which is finer than both N_x *and* N_y. Now $N_x \to x$ & $N_y \to y$ and so $\mathcal{F}$ converges to both x and y which is contradiction. Hence X is T_2 –space. **(1.5 marks)**

Q.6

i. Show that a topological space is completely regular if and only if the family of all continuous real-valued function on it distinguishes points from closed sets.

Solution: Let X be a topological space and let C be the family of all continuous real-valued functions on X. Suppose first that X is complete regular. Let a point $x \in X$ and a closed subset C of X,

		not containing x be given. **(1 marks)** Then there exists a continuous function $f: X \to [0,\ 1]$ such that $f(x) = 0$ and $f(c) \subset \{1\}$. Then $f \in C$ and evidently $f(x) \notin \overline{f(c)}$ since $\{1\}$ is closed set in $\mathbb{R}$. So C distinguishes points from closed set in X. **(2 marks)** Conversely, suppose that C distinguishes points from closed set in X. Let a point $x \in X$ and a closed subset C of X not containing x. Then there exists a map $f: X \to \mathbb{R}$ such that $f(x) \notin \overline{f(c)}$. **(1 marks)** Now, $\{f(x)\}$ *and* $\overline{f(c)}$ are disjoint closed subset of $\mathbb{R}$ which is normal space. So there exists a continuous function $f: X \to [0,\ 1]$ which takes values 0 and 1 respectively on them. Let $h: X \to [0,\ 1]$ be composite of $g \circ f$. Then clearly $h(x) = 0$ and $h(y) = 1$ for all $y \in C$.Thus we see that X is a completely regular space. **(2 marks)**	
	ii.	Show that a second countable space is metrisable if and only if it is T_3. **Solution:** Let (X, τ) be a second countable space is metrisable. To show every metrisable space are whether second countable or not. **(3 marks).** Hence X is T_3-space. For the converse, to show that every second countable T_3 –space is homeomorphic to a subspace of the Hilbert cube. **(2 marks).** Since the Hilbert cube is metrisable and metrisability is a hereditary property, hence the result follows. **(1 marks).**	
	iii.	Show that a topological space is a Tychonoff space if and only if it is embeddable into a cube. **Solution:** To show that every cube is a Tychonoff space **(2 marks).** Since Tychonoff property is hereditary, so every subspace of a cube and hence every space homeomorphic to a subspace of a cube is a Tychonoff space. **(1 marks).** Conversely suppose a space X is a Tychonoff space. Let C be a family of all continuous functions from X into $[0, 1]$. Then since X is completely regular, C distinguishes points from closed sets in X. But since X is also T_1, all singleton sets are closed and so it follows that C distinguishes points as well. **(2 marks)** Therefore, the evaluation map $e: X \to [0,\ 1]^C$ is an embedding of X into the cube $[0,\ 1]^C$. **(1 marks)**	

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