Enrollment No......................................

Faculty of Engineering

End Sem (Odd) Examination Dec-2017

EN3BS01 Engineering Mathematics-I

Programme: B. Tech. Branch/Specialisation: All

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. If for the system $AX=B$, we get $\rho(A)=\rho(A:B)=r=n$, the number of unknowns , then system has **1**

(a) Unique solution (b) No solution

(c) Infinite no. of solution (d) None of these

ii. Let A be a matrix such that there exists a square sub matrix of order r which is non-singular and every sub matrix of order $r+1$ or higher is singular, then the rank of A is **1**

(a) $= r+1$ (b) $< r$

(c) $> r$ (d) $= r$

iii. The necessary condition for the existence of a maxima or a minima of $f(x, y)$ *at* $x=a$ *and* $y=b$ are **1**

(a) $\left(\frac{\partial f}{\partial x}\right)_{(a,b)}=0$ and $\left(\frac{\partial f}{\partial y}\right)_{(a,b)}=0$ (b) $\left(\frac{\partial f}{\partial x}\right)_{(a,b)}\neq 0$ and $\left(\frac{\partial f}{\partial y}\right)_{(a,b)}=0$

(c) $\left(\frac{\partial f}{\partial x}\right)_{(a,b)}=0$ and $\left(\frac{\partial f}{\partial y}\right)_{(a,b)}\neq 0$ (d) $\left(\frac{\partial f}{\partial x}\right)_{(a,b)}\neq 0$ and $\left(\frac{\partial f}{\partial y}\right)_{(a,b)}\neq 0$

iv. $Sin\, x =$ **1**

(a) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots$ (b) $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots$

(c) $\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots$ (d) $\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\dots$

v. The value of $\Gamma n\,\Gamma(1-n)$, is **1**

(a) $\beta(n,n)$ (b) $\beta(n,1-n)$

(c) $\beta(n,1+n)$ (d) $\beta(1-n,1-n)$

vi. The value of $\beta(2,3)$ is **1**

(a) 1 (b) 12

(c) $\frac{1}{12}$ (d) 2

vii. The necessary and sufficient condition that the differential equation $M\,dx + N\,dy = 0$ be exact is that **1**

(a) $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ (b) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(c) $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$ (d) $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial y}$

viii. Particular integral of $(D^2 + a^2) = cosax$ is **1**

(a) $\frac{x}{2a}\,sinax$ (b) $\frac{-x}{2a}\,sinax$

(c) $\frac{x}{2a}\,cosax$ (d) $\frac{-x}{2a}\,cosax$

ix. $y = x$ is a part of C.F. of the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ if **1**

(a) $Px + Q = 0$ (b) $Px - Q = 0$

(c) $P + Qx = 0$ (d) $P - Qx = 0$

x. The C.F. of the equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}$ is given by **1**

(a) $y_c = c_1\,e^{3x} + xc_2\,e^{3x}$ (b) $y_c = c_1\,e^{3x} - xc_2\,e^{3x}$

(c) $y_c = c_1\,e^{3x} + c_2\,e^{3x}$ (d) $y_c = c_1\,e^{3x} - c_2\,e^{3x}$

Q.2 i. Reduce the matrix to normal form and find its rank, where **4**

$$A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 3 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$$

ii. Verify Cayley-Hamilton theorem for the matrix **6**

$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Also find A^{-1}.

OR iii. Find the Eigen values and Eigen vectors of matrix **6**

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Q.3 i. Find the first four terms in the expansion of by e^{sinx} McLaurin's theorem. **4**

ii. Find the maximum and minimum values of the function $u = x^3y^2(1 - x - y)$. **6**

OR iii. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then prove that **6**

$(i)\ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$. (ii) $(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z})^2 u = \frac{-9}{(x+y+z)^2}$.

Q.4 i. Evaluate $\lim_{n\to\infty} \left\{\frac{n!}{n^n}\right\}^{1/n}$. **4**

ii. Express the integral $\int_0^1 x^m(1 - x^n)^p\,dx$ in terms of beta/gamma function and hence evaluate **6**

$$(i)\int_0^1 x^2(1 - x^2)^4\,dx \quad (ii)\int_0^1 x^5(1 - x^3)^{10}\,dx$$

OR iii. Find the area enclosed by the parabolas $y^2 = 4ax$ and$x^2 = 4ay$. **6**

Q.5 i. Solve $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$. **4**

ii. Solve $\frac{dx}{dt} + y = sint$, $\frac{dy}{dt} + x = cost$, given that $x = 2$ and $y = 0, when\ t = 0$. **6**

OR iii. Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = sin(logx)$. **6**

Q.6 i. Solve by the method of variation of parameters **4**

$\frac{d^2y}{dx^2} + 4y = 4tan2x$

ii. Solve $x\frac{d^2y}{dx^2} - (2x - 1)\frac{dy}{dx} + (x - 1)y = e^x$, given that $y = e^x$ is one integral. **6**

OR iii. Solve in series the equation $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$ **6**

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Medi-Caps University, Indore

Engineering Mathematics - I

EN3BS01, DEC-2017

		MARKS
Que. 1 (i)	(a) Unique solution.	1
(ii)	(d) $= 2$	1
(iii)	(a) $\left(\frac{\partial f}{\partial x}\right)_{(a,b)} = 0$ and $\left(\frac{\partial f}{\partial y}\right)_{(a,b)} = 0$	1
(iv)	$\sin x =$ (C) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$	1
(v)	(b) $\beta(n, 1-n)$	1
(vi)	(C) $\frac{1}{12}$	1
(vii)	(b) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$	1
(viii)	(a) $\frac{x}{2a} \sin ax$	1
(ix)	(c) $P + Qx = 0$	1
(x)	(a) $y_c = c_1 e^{3x} + x c_2 e^{3x}$	1

Que. 2 (i) $A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

$C_1\left(\frac{1}{8}\right), C_4\left(\frac{1}{2}\right)$

$\sim \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 3 & 2 & 1 \\ -1 & -1 & -3 & 2 \end{bmatrix}$

$C_{21}(-1), C_{31}(-3)\ C_{41}(-3)$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 \\ -1 & 0 & 0 & 5 \end{bmatrix}$

$R_{31}(1)$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

$C_2\left(\frac{1}{3}\right)\ C_3\left(\frac{1}{2}\right)$

$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

+2

MARKS

C_{34}

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

$C_3\left(\frac{1}{5}\right)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim [I_3, 0]$$ +1

$\therefore \; \rho(A) = 3$

ue.2(ii)

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

we know that the characteristic equation of the matrix A is given by

$$|A - \lambda I| = 0$$

$$\therefore \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 0$$

or $\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \longrightarrow (1)$ +1

To verify Cayly Hamilton theorem, we will show that

$$A^3 - 6A^2 + 9A - 4I = 0 \longrightarrow (2)$$

$$A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix},$$ +1

$$A^3 = A^2 . A = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$ +1

Hence $A^3 - 6A^2 + 9A - 4I = 0$ +1

Now multiplying equ. ② by A^{-1}, we get

$$A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$$ +1

$$\therefore \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$ +1

Que. 2(iii)

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

characteristic equ. of matrix A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$-\lambda^3 + 18\lambda^2 - 45\lambda = 0$$ +1

$\lambda = 0, 3, 15$ are the eigen values of A. +1

If X_1 is an eigen vector corresponding to $\lambda = 0$

then $(A - 0I)X_1 = 0$; $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{x_1}{(\frac{1}{2})} = \frac{x_2}{1} = \frac{x_3}{1} = k$$

$$X_1 = \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$$ +1

The eigen vector X_2 corresponding to $\lambda = 3$

$(A - 3I)X_2 = 0$; $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

on solving

$\Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = k$

$\therefore \quad X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is the eigen vector for $\lambda = 3$. +1.5

The eigen vector X_3 corresponding to $\lambda = 15$

$(A - 15I)X_3 = 0$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

on solving

$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$

$X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is the eigen vector for $\lambda = 15$ +1.5

ue. 3(i) Let $y = e^{\sin x} \Rightarrow (y)_0 = 1$

Differentiating successively, we get

$y_1 = \cos x . e^{\sin x}$

or $y_1 = y \cos x \qquad \Rightarrow (y_1)_0 = 1$ +1

$y_2 = y_1 \cos x - y \sin x \quad \Rightarrow (y_2)_0 = 1$ +1

$y_3 = y_2 \cos x - 2y_1 \sin x - y \cos x \quad \Rightarrow (y_3)_0 = 0$

$y_4 = y_3 \cos x - 3y_2 \sin x - 3y_1 \cos x + y \sin x \Rightarrow (y_4)_0 = -3$ +1

By McLaurin's theorem $y = y_0 + \frac{x}{\lfloor 1}(y_1)_0 + \frac{x^2}{\lfloor 2}(y_2)_0 + \cdots$

we get $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \cdots$ +1

Que. 3(ii) $u = x^3y^2(1-x-y)$

$\frac{\partial u}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3 \rightarrow (a)$

$\frac{\partial u}{\partial y} = 2x^3y - 2x^4y - 3x^3y^2 \rightarrow (b)$ +1

For max. or min of u, we have put

$\frac{\partial u}{\partial x} = 0$; and $\frac{\partial u}{\partial y} = 0$

On solving eqn. (a) & (b), we get

$x = \frac{1}{2}$; $y = \frac{1}{3}$ +1

$r = \frac{\partial^2 u}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$

$r_{(\frac{1}{2}, \frac{1}{3})} = -\frac{1}{9}$ +1

$s = \frac{\partial^2 u}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$

$s = -\frac{1}{12}$ at $(\frac{1}{2}, \frac{1}{3})$ +1

$t = \frac{\partial^2 u}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$

$t = -\frac{1}{8}$ at $(\frac{1}{2}, \frac{1}{3})$ +1

Finally,

$rt - s^2 = \frac{1}{72} > 0$ (+ve) and $r < 0$

Therefore u has a max. at $(\frac{1}{2}, \frac{1}{3})$

$u_{max} = \frac{1}{432}$ +1

Que. 3(iii) $u = \log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x} = (3x^2-3yz)/(x^3+y^3+z^3-3xyz) \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = (3y^2-3xz)/(x^3+y^3+z^3-3xyz) \rightarrow (2)$$ +1.5

$$\frac{\partial u}{\partial z} = (3z^2-3xy)/(x^3+y^3+z^3-3xyz) \rightarrow (3)$$

adding (1) (2) and (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z} \rightarrow (4)$$ +1.5

for (ii) part

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$ +1.5

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$ +1.5

Que. 4(i) let $I = \lim_{n\to\infty} \left\{\frac{\lfloor n}{n^n}\right\}^{1/n}$

or $I = \lim_{n\to\infty}\left\{\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n}{n}\right\}^{1/n}$

$$\Rightarrow \log I = \lim_{n\to\infty} \frac{1}{n}\left[\log\frac{1}{n}+\log\frac{2}{n}+\cdots+\log\frac{n}{n}\right]$$ +1

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\left(\frac{r}{n}\right) \rightarrow (1)$$

put $\frac{r}{n} = x$, $\frac{1}{n} dx$, Σ by $\int$ in (1) | upper limit at $r=n$ $\lim_{n\to\infty}\frac{r}{n} = 1$ | lower limit at $r=0$ $\lim_{n\to\infty}\frac{r}{n} = 0$

$$= \int_0^1 (\log x)\,dx$$ +1

$$= \left[x\log x\right]_0^1 - \int_0^1 x\cdot\frac{1}{x}\,dx = -1$$ +1

i.e. $\log I = -1$

$\therefore I = e^{-1} = \frac{1}{e}$ +1

Que. 4 (ii) let $I = \int_0^1 x^m (1-x^n)^p dx \longrightarrow (1)$

put $x^n = y$ i.e. $x = y^{1/n}$ in (1)

$\therefore dx = \frac{1}{n} y^{(1/n)-1}$ +1

$\therefore$ By (1) we have

$I = \frac{1}{n}\int_0^1 y^{m/n} (1-y)^p \cdot y^{(1/n)-1} dy$

$= \frac{1}{n}\int_0^1 y^{\left(\frac{m+1}{n}\right)-1} \cdot (1-y)^{(p+1)-1} dy$ +1

$I = \frac{1}{n} \beta\left(\frac{m+1}{n}, p+1\right) \longrightarrow (2)$ +1

putting $m=5$, $n=3$, $p=10$, we get from (2)

$\int_0^1 x^5 (1-x^3)^{10} dx = \frac{1}{3} \beta(2, 11)$

i.e. $\int_0^1 x^5 (1-x^3)^{10} dx = \frac{1}{396}$ +1.5

Again putting $m=2$, $n=2$, $p=4$ in (2), we get

$\int_0^1 x^2 (1-x^2)^4 dx = \frac{1}{2} \frac{\sqrt{\frac{3}{2}}\sqrt{5}}{\sqrt{\frac{3}{2}+5}}$

i.e. $\int_0^1 x^2 (1-x^2)^4 dx = \frac{128}{3465}$ +1.5

Que. 4 (iii) The given parabolas are

$y^2 = 4ax \longrightarrow (i)$

$x^2 = 4ay \longrightarrow (ii)$

The point of intersection of parabolas are obtained by solving (i) and (ii)

i.e. $y^2 = 4a\sqrt{4ay} \Rightarrow y^4 = 64a^3 y$

$\Rightarrow y = 0$ or $y = 4a$

hence from (i) when $y=0$, $x=0$

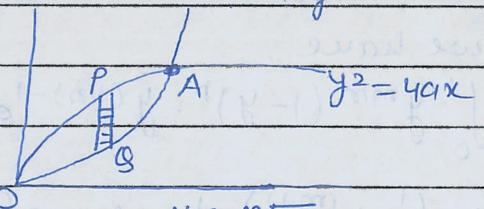
when $y=4a$ $x=4a$

Thus the points of intersection are $O(0,0)$ and $A(4a, 4a)$ +1.5

The required area is OQAPO

The region of integration can be expressed as

$0 \le x \le 4a$; $\frac{x^2}{4a} \le y \le \sqrt{4ax}$



+1.5

$\therefore$ The required area $I = \int_0^{4a}\int_{x^2/4a}^{\sqrt{4ax}} dx\,dy$

$$= \int_0^{4a}\left[\sqrt{4ax} - \frac{x^2}{4a}\right]dx$$ +1.5

$$= 2a^{1/2}\left[\frac{2}{3}x^{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$I = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$ +1.5

Answer

Que.5(i) Given diff. equ. $(1+e^{x/y})dx + e^{x/y}(1-\frac{x}{y})dy = 0 \longrightarrow (1)$

Here $M = 1 + e^{x/y}$; $N = e^{x/y}(1 - x/y)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{-x}{y^2}e^{x/y}$; given equ. is exact. +1.5

$\int_{y\text{-const.}} M dx = \int (1+e^{x/y})dx = x + ye^{x/y}$

$\int_{x\text{-const.}} N dy = \int e^{x/y}(1-\frac{x}{y})dy = \int e^{xt}(1-xt).\left(\frac{-dt}{t^2}\right)$

($\because$ put $y = 1/t$)

$= -\int e^{+xt}\frac{dt}{t^2} + \int xe^{tx}\frac{1}{t}dt$

$= \frac{e^{xt}}{t}$

i.e. $\int_{x\text{-const.}} N dy = ye^{x/y}$ ($\because$ putting $t = \frac{1}{y}$) +1.5

Clearly the term $ye^{x/y}$ is already in integration of M

Hence the required sol. $\int_{y\text{-const.}} M dx + \int N dy = C$ (free from x) $\Rightarrow x + ye^{x/y} = C$ +1

Answer

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Que. 5(ii) Given diff. equ. is

$$\frac{dx}{dt} + y = \sin t \rightarrow (I) \quad \text{and} \quad \frac{dy}{dt} + x = \cos t \rightarrow (II)$$

or $Dx + y = \sin t \rightarrow (iii)$ $\quad x + Dy = \cos t \rightarrow (iv)$

solving equ. (iii) and (iv)

$$D^2x + Dy = D\sin t$$
$$x + Dy = \cos t$$
$$\overline{(D^2-1)x = 0} \rightarrow (V)$$

which is a Linear diff. equ. with constant coefficient. +1.5

The A.E. of (V) is $m^2 - 1 = 0$
$m = \pm 1$

$\therefore$ sol. of (V) is given by $x = c_1e^t + c_2e^{-t}$ +1.5
$\Rightarrow \frac{dx}{dt} = c_1e^t - c_2e^{-t} \rightarrow (VI)$

using (VI) in (I)
$y = \sin t - c_1e^t + c_2e^{-t} \rightarrow (VII)$ +1.5

using boundary conditions i.e put $x = 2, y = 0$ at $t = 0$ in equ. (VI) and (VII), we get

$c_1 + c_2 = 2$; $c_1 - c_2 = 0$

Thus $c_1 = 1, c_2 = 1$

Hence the required solution is

$x = e^t + e^{-t}$ $\quad y = \sin t - e^t + e^{-t}$ Answer +1.5

Que. 5(iii) The given diff. equ. is

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = \sin(\log x) \rightarrow (1)$$ which is homogeneous diff. equ.

So putting $x = e^z \Rightarrow z = \log x$

$x\frac{d}{dx} = D'$; $x^2\frac{d^2}{dx^2} = D'(D'-1)$; where $D' = \frac{d}{dz}$

Then equ. (1) becomes

$(D'(D'-1) - 3D' + 5)y = \sin z$

or $(D'^2 - 4D' + 5)y = \sin z \rightarrow (2)$ +1

The A.E. of (2) is $m^2 - 4m + 5 = 0$ +1

$\therefore m = \dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

$\therefore$ C.F. $= e^{2z}[C_1 \cos z + C_2 \sin z]$

or C.F. $= x^2[C_1 \cos(\log x) + C_2 \sin(\log x)]$ +1

P.I. $= \dfrac{1}{(D'^2 - 4D' + 5)} \sin z$

$= \dfrac{1}{(-1^2 - 4D' + 5)} \sin z = \dfrac{1}{(-4D' + 4)} \sin z$ +1

$= -\dfrac{1}{4}\left[\dfrac{1}{(D'-1)} \times \dfrac{(D'+1)}{(D'+1)} \sin z\right]$

$= -\dfrac{1}{4}\left[\dfrac{(D'+1)}{(D'^2 - 1)} \sin z\right]$

$= -\dfrac{1}{4}\left[\dfrac{(D'+1)}{(-1^2 - 1)} \sin z\right]$ +1

$= \dfrac{1}{8}(D' \sin z + \sin z)$

P.I. $= \dfrac{1}{8}[\cos(\log x) + \sin(\log x)]$

The required general sol. is

$y = $ C.F. + P.I.

$y = x^2[C_1 \cos(\log x) + C_2 \sin(\log x)] + \dfrac{1}{8}[\cos(\log x) + \sin \log x]$ +1

Que. 6 (i) $\frac{d^2y}{dx^2} + 4y = 4\tan 2x \longrightarrow (1)$

On comparing with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, here we have

$P = 0 \quad Q = 4 \quad R = 4\tan 2x$

The A. E. of (1) is $m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$

$\therefore$ C. F. $= y_c = C_1 \cos 2x + C_2 \sin 2x$ +1

Let $u = \cos 2x \quad v = \sin 2x$

$\therefore u' = -2\sin 2x \quad v' = 2\cos 2x$

Now $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$

$w = 2 \neq 0$ +1

Suppose the complete sol. of (1) is

$y = Au + B \cdot v$; where A and B are arbitrary functions of x only; which are obtained by formula

$\frac{dA}{dx} = \frac{-v \cdot R}{w} \Rightarrow \frac{dA}{dx} = -2\left[\frac{1 - \cos^2 2x}{\cos 2x}\right]$

On integrating both sides we get

$A = -2\int(\sec 2x - \cos 2x)\,dx + C_1$

$= -2\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right] + C_1$

$\therefore A = -\log(\sec 2x + \tan 2x) + \sin 2x + C_1$ +1

and $\frac{dB}{dx} = \frac{u \cdot R}{w} \Rightarrow \frac{dB}{dx} = \frac{\cos 2x \cdot 4\tan 2x}{2} = 2\sin 2x$

On integrating both sides

$\therefore B = -\cos 2x + C_2$

Thus the complete sol. is given by +1

$y = C_1 \cos 2x + C_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$

Answer

Que. 6 (ii) The given diff equ. is

$$x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)y=e^x$$

or $$\frac{d^2y}{dx^2}+\left(-2+\frac{1}{x}\right)\frac{dy}{dx}+\left(1-\frac{1}{x}\right)y=\frac{e^x}{x} \longrightarrow (1)$$

On comparing with $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$, we get

$$P=-2+\frac{1}{x}\ ;\quad Q=1-\frac{1}{x}\ ;\quad R=\frac{e^x}{x}$$

But given that $(\underset{say}{u \text{ or } y_1})=e^x$ is a part of C.F. +1

suppose that the complete sol. of (1) is given by

$$y=y_1.v \text{ or } u.v=ve^x \text{ where } v \text{ is a function of } x \text{ only} \quad \to (2)$$

putting $y=v.y_1$ in equ. (1); we get

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{y_1}\frac{dy_1}{dx}\right]\frac{dv}{dx}=\frac{R}{y_1}$$

$$\frac{d^2v}{dx^2}+\left[-2+\frac{1}{x}+\frac{2}{e^x}\cdot e^x\right]\frac{dv}{dx}=\frac{e^x}{x.e^x} \quad \{\because y_1=e^x\} \quad +1$$

$$\frac{d^2v}{dx^2}+\frac{1}{x}\cdot\frac{dv}{dx}=\frac{1}{x} \longrightarrow (3) \quad +1$$

put $\frac{dv}{dx}=t$ and $\frac{dt}{dx}=\frac{d^2v}{dx^2}$

we get

$$\frac{dt}{dx}+\frac{t}{x}=\frac{1}{x} \longrightarrow (4) \quad +1$$

which is L.D.E. of first order.

$$\therefore \quad I.F=e^{\int P_1 dx}=e^{\int \frac{1}{x}dx}=e^{\log x}=x$$

i.e. $I.F.=x$ +1

hence sol. of (4) is $t.x=\int x.\frac{1}{x}dx+C_1$

$(\because y\times I.F=\int(R\times I.F)dx+C)$

$$\therefore \quad tx=C_1+x$$

or $\frac{dv}{dx}=\frac{C_1}{x}+1$ on integrating $v=C_1\log x+x+C_2$

Thus complete sol. (2) is $y=y_1.v$

$$\therefore \quad y=e^x.(C_1\log x+x+C_2)$$ +1

Answer

MARKS

Que. 6 (iii) Given diff. equ. $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0 \longrightarrow (1)$

on comparing with $P_0(x)\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2(x)\frac{dy}{dx} = 0$

Here $P_0(x) = (1-x^2)$

Also $P_0(x) \neq 0$ at $x=0$

Hence $x=0$ is an ordinary point of equ. (1). +1

Let its series sol. is given by

$$y = a_0 + a_1x + a_2x^2 + \dots + a_kx^k + \dots = \sum_{k=0}^{\infty} a_kx^k \longrightarrow (2)$$

Then $\frac{dy}{dx} = a_1 + 2a_2x + \dots + ka_kx^{k-1} + \dots$

and $\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + \dots + k(k-1)a_kx^{k-2} + \dots$ +1

putting the values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in equ. (1) we get

$$(1-x^2)[2a_2 + 6a_3x + \dots + k(k-1)a_kx^{k-2} + \dots]$$
$$-x[a_1 + 2a_2x + \dots + ka_kx^{k-1} + \dots]$$
$$+4[a_0 + a_1x + a_2x^2 + \dots + a_kx^k + \dots] = 0$$

$$\Rightarrow [2a_2 + 6a_3x + \dots + k(k-1)a_kx^{k-2} + \dots]$$
$$-[2a_2x^2 + \dots + k(k-1)a_kx^k + \dots]$$
$$-[a_1x + 2a_2x^2 + \dots k\,a_kx^k + \dots]$$
$$+4[a_0 + a_1x + a_2x^2 + \dots + a_kx^k + \dots] = 0$$ +1

Equating to zero; the coefficient of x^0 i.e. (constant)

$$2a_2 + 4a_0 = 0 \Rightarrow \boxed{a_2 = -2a_0}$$

Equating to zero, the coefficients of x^1, we get

$$6a_3 - a_1 + 4a_1 = 0 \Rightarrow 6a_3 = -3a_1 \Rightarrow \boxed{a_3 = \frac{-a_1}{2}}$$ +1

Equating to zero; the coefficient of x^k, we get

$$(k+2)(k+1)a_{k+2} - k(k-1)a_k - ka_k + 4a_k = 0$$

$$\therefore \boxed{a_{k+2} = \frac{(k-2)}{(k+1)}a_k} \longrightarrow (3)$$ +1

putting $k = 2, 3, 4, 5 \dots$ in equ. (3) we get

$\boxed{a_4 = 0}$; $a_5 = \frac{a_3}{4} = -\frac{1}{8}a_0$; $a_6 = 0$; $a_7 = \frac{-3a_5}{6} \dots$

Finally, putting the values of $a_2, a_3, a_4, a_5, \dots$ in equ. (2)

we get $y = a_0 + a_1x + (-2a_0)x^2 + (-\frac{a_1}{2})x^3 + 0 + (\frac{-a_0}{8})x^2 + \dots$ +1

or $y = a_0(1-2x^2) + a_1\left(x - \frac{x^3}{2} - \frac{x^5}{8} + \frac{x^7}{16} \dots\right) \dots$ Answer.