Enrollment No......................................

Faculty of Engineering / Science

End Sem (Odd) Examination Dec-2022

CA3CO15 Algebra

Programme: BCA /BCA-MCA (Integrated) Branch/Specialisation: Computer Science

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. For each pair of elements a and b in a Boolean Algebra B then $\big(a + (ab)\big)$ is equal to- **1**

(a) 0 (b) 1 (c) a (d) b

ii. If P and Q are two propositions, then $(P \wedge Q) \vee P$ is equivalent to- **1**

(a) P (b) Q (c) π (d) None of these

iii. If G is a group and $a \in G$, then- **1**

(a) $a^{o(G)} = e$ (b) $a^{G} = e$ (c) $a^2 = e$ (d) None of these

iv. The set of all natural numbers- **1**

(a) Is an additive group (b) Is a multiplicative group

(c) Is a cyclic group (d) Is not a group

v. Every quotient group of a cyclic group is cyclic- **1**

(a) Yes (b) No (c) Indefinite (d) None of these

vi. Let G be a group. If $f: G \to G'$ is an isomorphism, then- **1**

(a) $K = \{e\}$ (b) $K = G$ (c) $K = \phi$ (d) None of these

vii. If $v(F)$ be a vector space and W_1, W_2 be its two subspaces, then W_1 and W_2 are said to be \disjoint if their intersection will be- **1**

(a) $W_1 \cap W_2 = 0$ (b) $W_1 \cap W_2 = \emptyset$

(c) $W_1 \cap W_2 = \{\emptyset\}$ (d) $W_1 \cap W_2 = \{0\}$

viii. Even $V(F)$ is a finite dimensional vector space, then the number of elements in any two bases for- **1**

(a) Even in numbers (b) odd in numbers

(c) Equal in numbers (d) None of these

ix. Let $T: U \rightarrow V$ be a linear transformation and if U be a finite dimensional vector space, then- **1**

(a) dim $R(T) <$ dim U (b) dim $R(T) \leq$ dim U

(c) dim $R(T) \geq$ dim U (d) dim $R(T) >$ dim U

x. If a matrix be orthogonal similar to a diagonal matrix, then it must be- **1**

(a) Symmetric (b) Skew-symmetric

(c) Hermitian (d) Skew-hermitian

Q.2 i. If a and b are arbitrary elements of a boolean algebra, then Show that- **3**

(a) $(a+b)' = a'b'$ (b) $(ab)' = a' + b'$

ii. Describe conjunctive normal form. Write the following functions into conjunctive normal form in which maximum number of variables are used. $f(x, y, z) = xy' + xz + xy$. **7**

OR iii. Define proposition with example and prove that- **7**

(a) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(b) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

Q.3 i. Show that intersection of two subgroups of a group G, is a subgroup G. **3**

ii. Show that the set of all integers I forms a group with respect to the binary operation $' * '$ defined by rule **7**

$a * b = a + b + 1 \quad \forall\, a, b \in I.$

OR iii. Define cyclic Group with example and prove that the order of a cyclic group is same as the order of its generator. **7**

Q.4 i. What do you understand by homomorphism and isomorphism of group. Also discuss about kernel of homomorphism. **3**

ii. State and prove first theorem of homomorphism of group. **7**

OR iii. Describe ring and rings without zero divisors. Also describe field in detail. **7**

Q.5 i. Define vector space and the basis for the vector space. **3**

ii. Define linear independent vector and show that if two vectors are linearly dependent, one of them is a scalar multiple of other. **7**

OR iii. Show that the following set S of R^3 from a basis of R^3: **7**

$S = \{(1,2,3), (2,1,0), (1,-1,2)\}$.

Q.6 i. Describe linear transformation and show that, if f is a homomorphism of $U(F)$ into $V(F)$, then $f(0) = 0'$ where 0 and $0'$ are the zero vectors of U and V respectively. **3**

ii. State and prove rank-nullity theorem. **7**

OR iii. Show that transformation $T: V_2(R) \rightarrow V_3(R)$ defined by $T(a, b) = (a+b, a-b, b)$ is a linear transformation from $V_2(R)$ into $V_3(R)$. Find the range, rank, null space and nullity of T. **7**

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Solution and Marking Scheme of
CA3CO15 - Algebra
BCA - V-Sem

Ans: No. 1		Marks
	(i) - (c). a	1
	(ii) (d) - None of these	1
	(iii) - (a) - $a^{o(a)} = e$	1
	(iv) (b) Is a multiplicative group	1
	(v) (a) yes	1
	(vi) (d) None of these	1
	(vii) (c) $W_1 \cap W_2 = \{0\}$	1
	(viii) (d) None of these	1
	(ix) (b) $\dim R(T) \le \dim U$	1
	(x) a Symmetric	1

Ans(2) (a) part

(i) To prove $(a+b)' = a'.b'$

we are to show that the complement of $(a+b)$ is $a'b'$ for which it is a sufficient to prove

$$\left.\begin{array}{l}(a+b) + a'.b' = 1 \\ \text{and } (a+b).(a'.b') = 0\end{array}\right\} \text{ — (A)}$$ 0.5

Now $(a+b) + a'.b' = [(a+b)+a'].[(a+b)+b']$

$= [(a+a')+b].[a+(b+b')]$

$= (1+b).(a+1) = 1.1 = 1$ 0.5

and $(a+b).(a'.b') = (a'.b')(a+b)$

$= (a'.b').a + (a'.b').b$

$= (b'.a').a + a'.(b'.b)$

$= b'.(a'.a) + a'.0$

$= b'.0 + a'.0 = 0+0 = 0$ 0.5

1.5

Hence Result is Proved.

No. 2 (b) Part. To show $(ab)' = a' + b'$, $\forall a, b \in B$. Marks.

here we are to show that the complement of $a.b$ is $(a'+b')$ for which it is sufficient to prove

$$\left.\begin{array}{l}(a.b) + (a'+b') = 1 \\ (a.b) \cdot (a'+b') = 0\end{array}\right\}$$ 0.5

Now $(a.b) + (a'+b') = (a'+b') + (a.b)$

$= [(a'+b') + a].[(a'+b') + b]$

$= [b' + (a'+a)].[a' + (b'+b)]$

$= (b'+1)(a'+1)$

$= 1.1 = 1$ 0.5

and $(a.b).(a'+b') = (a.b).a' + (a.b).b'$

$= (b.a).a' + a.(b.b')$

$= b.(a.a') + a.0$ 0.5

$= b.0 + a.0 = 0.0 = 0$

1.5

Ans: 2.(ii)

Definition: (CNF)

A Boolean polynomial is called in CNF if it is the product of distinct factors where each factor is sum of variables x_1, x_2, x_3 --- and their complements x_1', x_2', x_3' --- and in each factor variables or their complements do not occur more than one. (2)

To write $f(x,y,z) = xy' + xz + xy$ in CNF.

we have $f(x,y,z) = xy' + xz + xy$

$= xy' + xy + xz$

$= x(y'+y) + xz$ 0.5

$= x + xz$ 1.

No. | Marks

$f(x,y,z) = x + x \cdot z$

$= (x+x)(x+z)$ 1·5

$= x(x+z)$

$= (x+yy')(x+z)$ 2·0

$= (x+y)(x+y')(x+z)$ 2·5

$= (x+y+zz')(x+y'+zz')(x+z+yy')$ 3·5

$= \cancel{(x+y+z)(x+y+z')(x+y'+z)}$

$(x+y'+z')(x+z+y)(x+z+y')$ 4·5

$= (x+y+z)(x+y'+z)(x+y+z')(x+y'+z')$ 5·

Ans 2 (iii)

The algebra of proposition is a Boolean Algebra. i.e. ~~let~~ B is the set of propositions (statements) p, q, r, ... Then $(B, \vee, \wedge, \sim)$ is a Boolean algebra. 1

Part (a) and (b) Prove by Truth Table. 3+3

Ans 3 (i) Let H_1 & H_2 be any two subgroup of G (group)

Then $H_1 \cap H_2 \neq \Phi$

To prove $H_1 \cap H_2$ is a subgroup it is sufficient to prove that $a \in H_1 \cap H_2$, $b \in H_1 \cap H_2$

$\Rightarrow a b^{-1} \in H_1 \cap H_2$ 0·5

Now $a \in H_1 \cap H_2 \Rightarrow a \in H_1$ and $a \in H_2$

$b \in H_1 \cap H_2 \Rightarrow b \in H_1$ and $b \in H_2$ 1·0

But H_1 & H_2 are subgroup of group G. So—

$$\left.\begin{array}{l} a \in H_1,\ b \in H_1 \Rightarrow a b^{-1} \in H_1 \\ a \in H_2,\ b \in H_2 \Rightarrow a b^{-1} \in H_2 \end{array}\right\}$$ 1·5

Finally $ab^{-1} \in H_1$ and $ab^{-1} \in H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

Hence $H_1 \cap H_2$ is a subgroup of G. proved 3·0

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No. Ans 3(ii) binary operation $*$ defined by rule

$a*b = a+b+1 \quad \forall\ a,b \in I$

(i) Closure Law: Let $a,b \in I$, then

$a \in I,\ b \in I \Rightarrow a+b+1 = a*b \in I$ | Marks: 1

Associativity: If $a,b,c \in I$, then

$(a*b)*c = (a+b+1)*c = (a+b+1)+c+1$

$= a+b+c+2 \quad \text{— (1)}$

and $a*(b*c) = a*(b+c+1)$

$= a+(b+c+1) = a+b+c+2 \quad \text{—(2)}$. | 2

from (1) & (2) $(a*b)*c = (a*b)*c \quad \forall\ a,b,c \in I$.

Existence of Identity: Let $e \in I$ s.t

$e*a = a \quad \forall\ a \in I$.

$e+a+1 = a$ | 5.

$\Rightarrow e+1 = 0 \Rightarrow \boxed{e=-1}$

Existence of Inverse: If $a \in I$ & $b \in I$ s.t

b is the inverse of b then

$b*a = e$

$b+a+1 = -1$ | 7

$b+a = -2$

$b = (-2-a)$ is the Inverse of a

No. 3(iii) Marks

Def: cyclic group

A group G is called a cyclic group if $\exists$ an element $a \in G$ s.t every element of G can be expressed as a power of a. In that case a is called generator of G. we express this fact by writing $G = \langle a \rangle$ or $G = (a)$.
Thus G is called cyclic group if $\exists$ an element $a \in G$ s.t (2)

$$G = \{a^n : n \in \mathbb{Z}\}.$$

Second Part: Let $G = \langle a \rangle$ be a cyclic group generated by a.

Case (i) $o(a)$ is finite, then n is +ve Integer s.t $a^n = e$
consider. $a^0 = e, a, a^2, \ldots a^{n-1}$
These are all elements of G and are n in number.
Suppose any two element are equal.

$$a^i = a^j \quad \text{with} \quad i > j.$$

Then $\quad a^i a^{-j} = e \Rightarrow a^{i-j} = e$.
But $0 < i-j \leq n-1 < n$, thus $\exists$ a +ve Integer $i-j$ s.t $a^{i-j} = e$ and $i-j < n$,
which is a contradiction to fact $o(a) = n$. Thus no two element are equal. (2)

Now Let $x \in G$, and G is cyclic generated by a, x will be some power of a.

Let $\quad x = a^m$

by division algorithm, we can write

$$m = nq + r, \quad 0 \leq r < n.$$

$$a^m = a^{nq+r} = (a^n)^q \cdot a^r = e^q a^r = a^r$$

$$\Rightarrow x = a^r, \quad 0 \leq r < n.$$

No. | Marks.

i.e x is one of $a^0 = e, a, a^2, -- a^{n-1}$

or G contains precisely n elements.

$\Rightarrow o(G) = n = o(a)$. (1)

~~Case (ii) $o(a)$ is infinite~~

In this case no two powers of a can be equal, as if $a^n = a^m$ $(n > m)$.

Then $a^{n-m} = e$

i.e. it is possible to find a +ve integer $n-m$ s.t $a^{n-m} = e$ it mean a has finite order. (2)

Hence no two power of a can be equal. In other words G would contain infinite number of elements. (5)

Ans: 4 (i).

Let $(G, *)$ and (G', o) be two groups

A mapping $f: G \longrightarrow G'$ is called Homomorphism if $f(a*b) = f(a) \circ f(b)$ $\forall a, b \in G$. (1)

In addition if f is one-one onto, we say f is an Isomorphism, and we write as $G \cong G'$. (1)

The Kernel of Homomorphism ~~a~~ $\operatorname{Ker} f = \{e\}$ (1)

Ans 4 (ii)

Statement: If f is a Homomorphism of a group G into group G', then Kernel K of f is ~~as~~ a normal subgroup of G. (2)

No. | Marks

proof: Let $f: G \to G'$ be a homomorphism.

Therefore $f(e) = e'$, $e \in Ker f$, thus $Ker f \neq \Phi$ (1)

Again Let $x, y \in Ker f \Rightarrow f(x) = e'$, and $f(y) = e'$

Now $f(xy^{-1}) = f(x)\, f(y^{-1})$

$= f(x) [f(y)]^{-1} = e'$, $\because (e')^{-1} = e'$

$\Rightarrow xy^{-1} \in Ker f.$ (2)

Hence its a subgroup of G.

Again for any $g \in G$, $x \in Ker f$

Now $f(g^{-1}xg) = f(g^{-1})\, f(x)\, f(g)$

$= (f(g))^{-1} f(x)\, f(g)$

$= (f(g))^{-1}\, e'\, f(g)$

$= (f(g))^{-1} f(g) = e'$ (2)

$\Rightarrow g^{-1}xg \in Ker f.$

Hence it is Normal Subgroup of G.

(5)

Ans 4(iii) Ring: $(R, +, *) \to$

Note: with r.to + operation Abelian group

$* \to$ oper — semigroup (3)

and two distributive Law should be hold.

Ring with out zero divisore:

Let R be a ring. An element $0 \neq a \in R$ is called a zero-divisor, if $\exists$ an element $0 \neq b \in R$ s.t $ab = 0$ or $ba = 0$.

No. | Marks.

Field: $(F, +, *)$.

$+$ — Abelian group (2)

$*$ → Abelian group

and two distributive Law

(7)

Ans 5(i): Let $\langle V, + \rangle$ be an Abelian group

and

$\langle F, +, \cdot \rangle$ be a field

define function from $F * V \to V$ s.t

$\forall \alpha \in F, v \in V, \Rightarrow \alpha . a \in V$, then V is

said to form a vector space over the F

if for all $x, y \in V$, $\alpha, \beta \in F$. s. t.

(i) $(\alpha+\beta)x = \alpha x + \beta x$

(ii) $\alpha(x+y) = \alpha x + \alpha y$

(iii) $(\alpha\beta)x = \alpha(\beta x)$

(iv) $1 . x = x . 1$, 1 being unity of F. (2)

The member of F are called scalar and V are called vectors.

Basis of Vector space: set of L.I vectors form a basis. (1)

Ans 5(ii) Let $V(F)$ be a vector space. Element $v_1, v_2 -- v_n$ in V are said to be Linearly Independent over F if $\exists$ scalars $\alpha_1, \alpha_2, -- \alpha_n \in F$. all zero such that (2)

$$\alpha_1 v_1 + \alpha_2 v_2 + -- + \alpha_n v_n = 0.$$

IInd Part: Let α, β be two Linearly dependent vectors of V.S $V(F)$. then $\exists$ scalars a, b (not all zero) $\in F$ s.t (1)

No. | Marks

$a\alpha + b\beta = 0 \quad \forall\ a, b \in F,\ \alpha, \beta \in V$

(i) If $\alpha \neq 0$, then

$$a\alpha + b\beta = 0 \Rightarrow a\alpha = -b\beta$$

$$\Rightarrow a^{-1}(a\alpha) = a^{-1}(-b\beta)$$

$$\Rightarrow (a^{-1}a)\alpha = [a^{-1}(-b)]\beta \qquad (2)$$

$$\Rightarrow \alpha = \left(-\frac{b}{a}\right)\beta$$

$\Rightarrow$ α is a scalar multiple of β.

(ii) If $b \neq 0$, then

$$a\alpha + b\beta = 0 \Rightarrow b\beta = -a\alpha$$

$$\Rightarrow \beta = \left(-\frac{a}{b}\right)\alpha \qquad (2)$$

$\Rightarrow$ β is a scalar multiple of α,

Thus one of the vector α and β is a scalar multiple of the other.

(5)

Ans 5(iii) $\quad S = \{(1,2,3), (2,1,0), (1,-1,2)\}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 3 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -5 \\ 0 & -6 & -4 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix} \qquad 3$$

$$\rightarrow \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{-3} & -5 \\ 0 & 0 & \boxed{6} \end{bmatrix} \quad R_3 \to R_3 - 2R_2 \qquad 3$$

Here No of Pivot are 3 = No of vector.
Hence Form a basis

(7)

No. Ans 6 (i) Marks.

Let $U(F)$ & $V(F)$ be two vector space over the same Field F. Then a mapping $T: U \to V$ is called L.T, if T is such that

(i) $T(\alpha+\beta) = T(\alpha) + T(\beta) \quad \forall\, \alpha, \beta \in U$ (2)

(ii) $T(a\alpha) = aT(\alpha) \quad , a \in F, \alpha \in U$.

Second Part

Let $\alpha \in U, \Rightarrow f(\alpha) = V,$

Since $0' \in V$ we have

$f(\alpha) + 0' = f(\alpha) = f(\alpha + 0)$ (1)

$= f(\alpha) + f(0)$

$\Rightarrow \boxed{0' = f(0)}$

(3)

Ans 6 (ii)

Statement:

Let $T: U \to V$, and F be a field Scalars then prove that

$$\text{Rank}(T) + \text{nullity}(T) = \dim U$$ (2)

proof: by Fundamental theorem of vectorspace homomorphism, we have

$U/\text{Ker}(T) \cong \text{Im}(T)$

$\dim U/\text{Ker}(T) = \dim \text{Im}(T)$

$\dim U - \dim \text{Ker}(T) = \dim \text{Im}(T)$ (5)

$\dim \text{Im}(T) + \dim \text{Ker}(T) = \dim U$

$\dim R(T) + \dim N(T) = \dim U$

$\text{rank}(T) + \text{nullity of}(T) = \dim U$

No. Ans 6(iii) | Marks

give $T(a,b) = (a+b, a-b, a)$.

(i) $T(0,0) = (0+0, 0-0, 0) = (0,0,0) = 0$ — 1

0 vector is there

(ii) Let $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$

$T(a+b) = T(a_1+b_1, a_2+b_2)$

$= (a_1+b_1+a_2+b_2, a_1+b_1-a_2-b_2, a_1+b_1)$

$= (a_1+a_2+b_1+b_2, a-a_2+b_1-b_2, a_1+b_1)$

$= (a_1+a_2, a-a_2, a_1) + (b_1+b_2, b_1-b_2, b_1)$ — (1)

$= T(a) + T(b)$

Let $\alpha \in F$, $a = (a_1, a_2) \in V$

~~$T(\alpha a) = T(\alpha a_1, \alpha a_2)$~~

$= T(\alpha a_1 + \alpha a_2, \alpha a_1 - \alpha a_2, \alpha a_1)$

$= \alpha T(a_1+a_2, a_1-a_2, a_1)$ — 4

$= \alpha T(a)$

Hence T is L.T

second Part

$T(1,0) = (1,1,1)$

$T(0,1) = (1,-1,0)$

Now $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$ — (3)

rank(T) = 2, nullity is (0) -