Total No. of Questions: 6 *Total No. of Printed Pages:3*

Enrollment No......................................

Faculty of Engineering / Science

End Sem Examination Dec-2023

EN3BS11 / BC3BS01 Engineering Mathematics -I

Programme: B.Tech. / B. Sc. Branch/Specialisation: All

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d. Assume suitable data if necessary. Notations and symbols have their usual meaning.

Q.1 i. If 0, 1 and 2 are the eigen values of a matrix A, then the determinant of A is equal to ______. **1**

(a) 0 (b) 1 (c) 2 (d) None of these

ii. If $AX = B$ be a system of linear equations and if $\rho(A) = \rho[A{:}B] = r < n\ (no.of\ unknown\ variables)$, then the system has- **1**

(a) No solution (b) Unique Solution

(c) Infinite many solutions (d) None of these

iii. If $f(x) = 2x^2 - 10x + 29$ be a function defined in the interval $[2,7]$ and if $c \in (2,7)$, then by Lagrange' s Mean Value theorem the value of c is equal to- **1**

(a) 1.54 (b) 2.67 (c) 6.44 (d) None of these

iv. If $z = x^3 + y^3 - 3xy$, then $\frac{\partial^2 z}{\partial y \partial x} =$ _____. **1**

(a) $6x - 3$ (b) $6x$ (c) -3 (d) None of these

v. $1^2 + 2^2 + 3^2 + \cdots + n^2 =$ ____ . **1**

(a) $\frac{n(n+1)(2n+1)}{6}$ (b) $\frac{n(n+1)}{6}$

(c) $\frac{n(n+1)}{2}$ (d) None of these

vi. The value of $\left\lceil \frac{1}{2} \right.$ is equal to- **1**

(a) $\sqrt{\pi}$ (b) π (c) 2π (d) None of these

vii. The integrating factor for the given linear differential equation $\frac{dy}{dx} + \frac{y}{x} = x^2$ is __________. **1**

(a) e^x (b) x (c) $\log x$ (d) None of these

viii. The necessary and sufficient condition to check the exactness of a differential equation $Mdx + Ndy = 0$ is ______. (where M and N are functions of x and y). **1**

(a) $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ (b) $\frac{\partial N}{\partial y} = \frac{\partial M}{\partial x}$

(c) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (d) None of these

ix. The Cauchy Riemann equations for the analytic function $f(z) = u(x,y) + v(x,y)$ are given by _________. **1**

(a) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ *and* $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

(b) $\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$ *and* $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

(c) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ *and* $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(d) None of these

x. If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a simple closed curve C , then $\int_c f(z)\, dz$ is equal to __________. **1**

(a) 0 (b) $2\pi i$ (c) 1 (d) None of these

Q.2 Attempt any two:

i. Reduce the given matrix into its normal form and also find the rank and nullity of the matrix. **5**

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

ii. Find for what values of k the system of equations: **5**

$$2x - 3y + 6z - 5t = 3$$
$$y - 4z + t = 1$$
$$4x - 5y + 8z - 9t = k$$

has (a) no solution (b) infinite number of solutions

iii. Find the eigen values and eigen vectors of the matrix: **5**

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Q.3 Attempt any two:

i. Expand $log_e x$ in power of $(x-1)$ by Taylor's series and hence find the value of $\log_e(1.1)$. **5**

ii. If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then prove the following by Euler's theorem. **5**

(a) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

(b) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \,.\, \cos 2u}{4\cos^3 u}$

iii. Find the maximum and minimum values for the function: **5**

$u = \sin x + \sin y + \sin(x+y)$.

Q.4 Attempt any two:

i. Evaluate the following limit: $\lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$ **5**

ii. Evaluate $\iint_R y\, dxdy$, where R is the region bounded by the parabola $y^2 = 4x$ and $x^2 = 4y$. **5**

iii. Evaluate $\int_1^3 \int_{1/x}^1 \int_0^{\sqrt{xy}} xyz\ \ dx\, dy\, dz.$ **5**

Q.5 Attempt any two:

i. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\, e^x \sin x.$ **5**

ii. Solve $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0.$ **5**

iii. Solve **5**

$$\frac{dx}{dt} + 2y = e^t$$
$$\frac{dy}{dt} - 2x = e^{-t}$$

Q.6 Attempt any two:

i. Show that $u = 2x - x^3 + 3xy^2$ is harmonic and determine its conjugate. **5**

ii. Evaluate $\int_0^{2+i} (\bar{z})^2\, dz$, along the real axis from $z = 0$ to $z = 2$ and then along a line parallel to y-axis from $z = 2$ to $z = 2 + i$. **5**

iii. Using Cauchy Integral Formula, evaluate $\int_c \frac{e^{2z}}{(z+1)^4} dz$, where c is the circle $|z| = 3$. **5**

******

End Sem Examination Dec-2023

Faculty of Engineering

Date :

P. No. : 1

EN3BS11 / BC3 BS01 - Engineering Mathematics-I

Program : B.tech / B.SC.

Branch : All

Q.1. MCQ -

i) 0 (a) 1

ii) (c) Infinite many Solution 1

iii) (d) None of these 1

iv) (c) -3 1

v) (a) $\frac{n(n+1)(2n+1)}{6}$ 1

vi) (a) $\sqrt{\pi}$ 1

vii) (b) x 1

viii) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ 1

ix) (d) None of these 1

x) 0 1

Q 2. Reduce matrix in normal form -

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Sol:- $R_1 \longleftrightarrow R_2$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 3R_1$

$R_4 \rightarrow R_4 - R_1$

$$A = \begin{bmatrix} \cancel{0}1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$ +1

$R_3 \rightarrow R_3 - R_2$

$R_4 \rightarrow R_4 - R_2$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ +1

3

Date :

P. No. :

$\cancel{A} =$ $C_3 \to C_3 - C_1$

$C_4 \to C_4 - C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ +1

$C_3 \to C_3 + 3C_2$

$C_4 \to C_4 + C_2$

$$A = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{bmatrix}$$

$$A = \begin{bmatrix} I_{2\times 2} & \vdots & 0 \\ 0 & \vdots & 0 \end{bmatrix}$$

$\rho(A) = 2$ +1

$\text{nullity} = 4 - 2 = n - r$

$\eta(A) = 2$ +1

ii) System of equations:

$$2x - 3y + 6z - 5t = 3$$
$$y - 4z + t = 1$$
$$4x - 5y + 8z - 9t = k$$

Sol:-

$$Ax = B$$

$$\begin{bmatrix} 2 & -3 & 6 & -5 \\ 0 & 1 & -4 & 1 \\ 4 & -5 & 8 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ k \end{bmatrix}$$

$\Rightarrow$ Argumented matrix $C = [A : B]$

$$C = \left[\begin{array}{cccc:c} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 4 & -5 & 8 & -9 & k \end{array}\right]$$ +1

$$R_3 \to R_3 - 2R_1$$

$$C = \left[\begin{array}{cccc:c} 2 & -3 & 6 & -5 & 3 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 1 & -4 & 1 & k-6 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$C = \begin{bmatrix} 2 & -3 & 6 & -5 & : & 3 \\ 0 & 1 & -4 & 1 & : & 1 \\ 0 & 0 & 0 & 0 & : & k-7 \end{bmatrix}$$ +1

(a) If equation has no Solution -

if $\rho(A) \neq \rho(C)$

$k-7 \neq 0$ or $k \neq 7$ +1

then

$\rho(A) = 2$ and $\rho(C) = 3$.

b) Infinite no. of Solutions if

$\rho(A) = \rho(C) = 2$

$k-7 = 0$ or $k = 7$ +1

Solution of eq :-
$$\begin{bmatrix} 2 & -3 & 6 & -5 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$2x - 3y + 6z - 5t = 3$ — (1)

$y - 4z + t = 1$ — (2)

Let $t = k_1$ and $z = k_2$

then

$\boxed{y = 1 + 4k_2 - k_1}$ +1

$\boxed{x = 3 + 3k_2 + k_1}$

iii) Find the eigen values and eigen vectors of matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol→

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$ +1

$\Rightarrow (6-\lambda)(9-6\lambda+\lambda^2-1)+2(-6+2\lambda+2)+$

$2(2-6+2\lambda)=0$

$\Rightarrow \lambda^3+12\lambda^2-36\lambda+32=0$

$\Rightarrow (\lambda-2)(\lambda^2-10\lambda+16)=0$ or

$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-8)=0$

$\boxed{\lambda = 2, 2, 8}$ — eigen values +1.

i) eigen vector for $\lambda = 2$

$$[A-2I]X = 0$$

Date :

P. No. :

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 2R_3 - R_1$

$\Rightarrow$ $R_2 \to 2R_2 + R_1$

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $4x_1 - 2x_2 + 2x_3 = 0$ +1.5

Let $x_1 = K_1$

$x_2 = K_2$

then

$$x_3 = \frac{1}{2}(2K_2 - 4K_1) = K_2 - 2K_1$$

ii) for $\lambda = 8$

$$[A - 8I][X] = 0$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + R_1$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0$$

$$x_1 + x_2 - x_3 = 0 \quad \text{--- } 1$$

$$-3x_2 - 3x_3 = 0$$

$$0x_1 + x_2 + x_3 = 0 \quad \text{--- } 2$$

let ~~$x_1 = K$~~

$$\frac{x_1}{1+1} = \frac{-x_2}{1+0} = \frac{x_3}{1-0} = K$$

$x_1 = 2K$

$x_2 = -K$ +1.5

$x_3 = K$

Hence eigen vector are,

$$X_1 = \begin{bmatrix} k_1 \\ k_2 \\ \frac{1}{2}(k_2 - 2k_1) \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2K \\ -K \\ K \end{bmatrix}$$

Ques-3.) Expand $\log_e x$ in power of $(x-1)$

Sol-) by Taylor series -

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

+1

$$f(x-1+1) = f(1) + \frac{x-1}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1)$$

$$+ \frac{(x-1)^3}{3!} f'''(1) + \cdots \quad (1)$$

+1

For the given problem: $a=1$

$f(x) = \log_e x \quad , \quad f(1) = 0$

$f'(x) = \frac{1}{x} \quad , \quad f'(1) = 1$

+1

$f''(x) = \frac{-1}{x^2} \quad , \quad f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \quad , \quad f'''(1) = 2$

$f^{(iv)}(x) = \frac{-6}{x^4} \quad , \quad f^{iv}(1) = -6$

Substituting in (1), we get

$$\log_e x = 0 + (x-1) \cdot 1 + \frac{(x-1)^2}{2}(-1) + \cdots$$

$$\log_e x = 0 + (x-1)\cdot 1 + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{2\cdot 3}\cdot 2$$

$$+ \frac{(x-1)^4}{2\cdot 3\cdot 4}(-6) + \cdots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots \quad +1$$

Ans.

Put $x = 1.1$

$$\log_e 1.1 = (1.1-1) - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 -$$

$$\frac{1}{4}(1.1-1)^4 + \cdots$$

$$= 0.1 - 0.005 + 0.00033 - 0.000025 + \cdots$$

$$= 0.095305 \quad \text{Ans.} \quad +1$$

ii) If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then by Euler's theorem -

Sol:- we have,

$$u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$z = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left[1+\frac{y}{x}\right]}{\sqrt{x}\left[1+\frac{\sqrt{y}}{\sqrt{x}}\right]} \quad +1$$

$$= x^{1/2} \phi\left(\frac{y}{x}\right)$$

$$z = f(u) = \sin u$$

z is a homogenous function of degree 1/2. +1

i) by euler deduction I

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u}$$

+1.5

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

H.P.

ii) $$g(u) = \frac{1}{2}\tan u.$$

by Euler deduction II

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} =$$

$$g(u)[g'(u)-1]$$

+1.5

$$= \quad = \frac{1}{2}\tan u\left(\frac{1}{2}\sec^2 u - 1\right)$$

$$= \frac{1}{4}\frac{\sin u}{\cos u}\left(\frac{1}{\cos^2 u} - 2\right) = \frac{1}{4}\frac{\sin u}{\cos^3 u}(1 - 2\cos^2 u)$$

$$= \frac{-\sin u \cdot \cos 2u}{4\cos^3 u}.$$ H.P.

iii) $u = \sin x + \sin y + \sin(x+y)$

Sol:- for maxima or minima of u, we have

$$\frac{\partial u}{\partial x} = \cos x + \cos(x+y) = 0 \quad \text{—— (1)}$$

+1

$$\frac{\partial u}{\partial y} = \cos y + \cos(x+y) = 0 \quad \text{—— (2)}$$

(1) and (2)

$\Rightarrow \cos x = \cos y \Rightarrow x = y \Rightarrow \cos x + \cos(x+x) = 0$

$\Rightarrow \cos x + 2\cos^2 x - 1 = 0$

$\Rightarrow 2\cos^2 x + 2\cos x - \cos x - 1 = 0$

$\Rightarrow 2\cos x(\cos x + 1) - 1(\cos x + 1) = 0$

$\Rightarrow (2\cos x - 1)(\cos x + 1) = 0$

+1

$\cos x = -1 = \cos \pi$

$\Rightarrow x = \pi/3$ or $x = \pi$.

$$r = \frac{\partial^2 u}{\partial x^2} = -\sin x - \sin(x+y)$$ +1

$$s = \frac{\partial^2 u}{\partial x \cdot \partial y} = -\sin(x+y)$$

and $$t = \frac{\partial^2 u}{\partial y^2} = -\sin y - \sin(x+y)$$

when

$x = y = \pi/3$, we have

$$r = -\sin\pi/3 - \sin\left(\frac{2\pi}{3}\right) = -\sqrt{3}$$

$$s = -\sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right) = \frac{-\sqrt{3}}{2}$$

$$t = -\sin\frac{\pi}{3} - \sin\left(\frac{2\pi}{3}\right) = -\sqrt{3}$$ +1.5

$$\Rightarrow rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{9}{4} \quad (+ve)$$

Since $rt - s^2$ is +ve and r is −ve, hence we have maxima at $x = y = \pi/3$.

when $x = y = \pi$, we have

$$r = 0, \; s = 0, \; t = 0$$ +0.5

$\therefore$ $rt - s^2 = 0$

This case is doubtful.

Q4. (i) $\lim\limits_{n\to\infty} \left[\dfrac{n!}{n^n}\right]^{1/n}$

Sol:- Let $P = \lim\limits_{n\to\infty} \left\{\dfrac{n!}{n^n}\right\}^{1/n}$

$$P = \lim_{n\to\infty} \left(\frac{1.2.3.\ldots n}{n^n}\right)^{1/n} \quad +1$$

$$P = \lim_{n\to\infty} \left(\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\ldots\frac{n}{n}\right)^{1/n}$$

$$\log P = \lim_{n\to\infty} \frac{1}{n}\left[\log\left(\frac{1}{n}\right) + \log\left(\frac{2}{n}\right) + \log\left(\frac{3}{n}\right) + \ldots \log\left(\frac{n}{n}\right)\right] \quad +1$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n}\log\left(\frac{r}{n}\right) = \int_0^1 \log x\, dx \quad +1$$

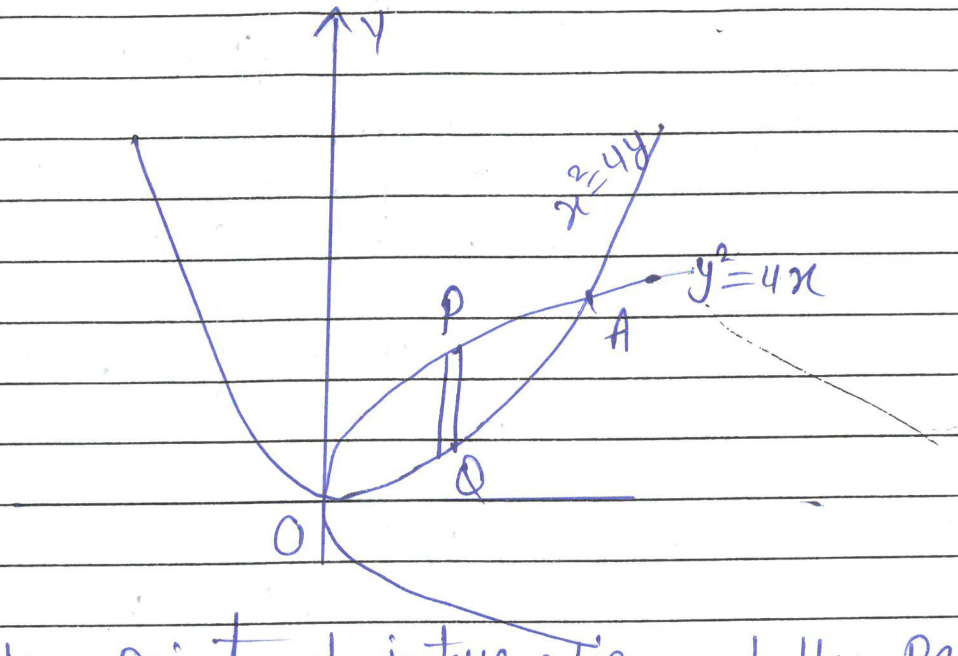
$$= \left[(\log x).x\right]_0^1 - \int_0^1 \frac{1}{x}.x\, dx \quad +1$$

$$= 0 - [x]_0^1 = -1 \quad +1$$

$$P = e^{-1} = 1/e$$

ii) Evaluate $\iint_R y \, dx \, dy$, Region R is bounded by $y^2 = 4x$ and $x^2 = 4y$.

Sol→



At the points of intersection of the Parabola

$$y^2 = 4x \quad — (1)$$

$$x^2 = 4y \quad — (2)$$

and

we have $(x^2/4)^2 = 4x \Rightarrow x(x^3 - 64) = 0$

$\Rightarrow \quad x = 0, 4$

The curves (1) and (2) intersect at points O(0,0) and A(4,4).

therefore, the region R is the area OQAPO, where $0 \leq x \leq 4$

and

$$(x^2/4) \leq y \leq 2\sqrt{x},$$

Thus,

$$\iint_R y\,dx\,dy = \int_{x=0}^{4}\int_{y=x^2/4}^{2\sqrt{x}} y\,dx\,dy$$ +1

$$= \int_0^4 \left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx$$ +1

$$= \frac{1}{2}\int_0^4 \left[4x - \frac{x^4}{16}\right] dx$$

$$= \frac{1}{2}\left[2x^2 - \frac{x^5}{80}\right]_0^4$$

$$= \frac{1}{2}\left[2(4)^2 - \frac{1}{80}(4)^5\right]$$

$$= \boxed{\frac{48}{5}}$$ Ans. +1

iii) Evaluate $\int_1^3 \int_{1/x}^{1} \int_0^{\sqrt{xy}} xyz\,dx\,dy\,dz.$

Sol→ $\int_1^3 \int_{1/x}^{1} x.y \left(\int_0^{\sqrt{xy}} z.\,dz\right).\,dy\,dx$

$$\Rightarrow \int_1^3 \int_{1/x}^{1} x.y \left[\frac{z^2}{2}\right]_0^{\sqrt{xy}} dy\,dx$$ +1

$$\Rightarrow \int_1^3 \int_{1/x}^1 x.y \left(\frac{xy}{2} - 0\right) dy\, dx$$

$$\Rightarrow \int_1^3 \int_{1/x}^1 \frac{x^2}{2} y^2\, dy\, dx$$ +1

$$\Rightarrow \frac{1}{2}\int_1^3 x^2 \left[\frac{y^3}{3}\right]_{1/x}^1 dx$$

$$\Rightarrow \frac{1}{2}\int_1^3 x^2 \cdot \left[\frac{1}{3} - \frac{1}{3x^3}\right] \cdot dx$$ +1

$$\Rightarrow \frac{1}{2}\int_1^3 \frac{x^2}{3} - \frac{1}{3x}\, dx$$

$$= \frac{1}{6}\int_1^3 \left(x^2 - \frac{1}{x}\right) dx$$ +1

$$\Rightarrow \frac{1}{6}\left[\frac{x^3}{3} - \log x\right]_1^3$$

$$\Rightarrow \frac{1}{6}\left[\frac{27}{3} - \log 3 - \frac{1}{3} + \log 1\right]$$

$$\Rightarrow \frac{1}{6}\left[9 - \frac{1}{3} - \log 3\right] = \frac{1}{6}\left[\frac{26}{3} - \log 3\right]$$ +1

Q. 5 i) Solve : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

Sol:- $(D^2 - 2D + 1)y = xe^x \sin x$

$\rightarrow$ A. E. is $m^2 - 2m + 1 = 0$ +1

$$(m-1)^2 = 0$$

$$m = 1, 1$$

+1

C. F. $= (C_1 + C_2 x) \cdot e^x$

P. I $= \frac{1}{(D-1)^2} x . e^x \sin x$

$= e^x \frac{1}{(D+1-1)^2} . x \sin x$ +1

$= e^x \frac{1}{D^2} x \sin x$

$= e^x \frac{1}{D} (-x \cos x + \sin x)$ +1

$= e^x [-x \sin x - \cos x - \cos x]$

$= -e^x [x \sin x + 2 \cos x]$

$y = (c_1 + c_2 x)e^x - e^x [x \sin x + 2 \cos x]$ +1

Ans.

ii) Solve: $(1+e^{x/y})dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

Sol:- Solution by Exact differential eq^n:

$$M\,dx + N\,dy = 0$$ +1

$\Rightarrow$ $M = 1+e^{x/y}$

$$\frac{\partial M}{\partial y} = 0 + e^{x/y}\left(\frac{-x}{y^2}\right) = \frac{-x}{y^2}\cdot e^{x/y} \quad \rightarrow (1)$$ +1

$\Rightarrow$ $N = e^{x/y}\left(1-\frac{x}{y}\right)$

$$\frac{\partial N}{\partial x} = \frac{1}{y}\cdot e^{x/y} - \frac{1}{y}\left[\frac{\partial}{\partial x}\, x.e^{x/y}\right]$$

$$= \frac{1}{y}\left[e^{x/y} - \left(x.e^{x/y}.\frac{1}{y} + 1.e^{x/y}\right)\right]$$

$$= \frac{1}{y}\left[e^{x/y} - e^{x/y} - e^{x/y}.\frac{x}{y}\right]$$

$$= \frac{-x}{y^2}\,e^{x/y} \quad \rightarrow (2)$$ +1

So

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (\text{by } (1) \,\&\, (2))$$

Hence eq^n is exact.

Solution of eq^n is

$$\int M\,dx + \int N_{(\text{term not containing } x)}\,dy = C$$ +1

$$\int (1+e^{x/y})\,dx + \int 0\,dy = C$$

$$\Rightarrow x + \frac{e^{x/y}}{1/y} = C$$

$$\Rightarrow \boxed{x + y e^{x/y} = C}$$ Ans. +1

iii) solve : $\frac{dx}{dt} + 2y = e^{t}$

$$\frac{dy}{dt} - 2x = e^{-t}$$

[Let $D = \frac{d}{dt}$]

Sol:-

$$Dx + 2y = e^{t} \quad \times 2 \quad \text{--- (1)}$$

$$-2x + Dy = e^{-t} \quad \times D \quad \text{--- (2)}$$

$$\Rightarrow \cancel{2Dx} + 4y = 2e^{t}$$

$$\cancel{-2Dx} + D^2 y = De^{-t}$$ +1

$$(D^2+4)y = 2e^{t} - e^{-t} \quad \text{--- (3)}$$

solve eqⁿ (3) for y

A.E · $m^2+4=0$

$$m=\pm 2j \quad +1$$

$$C.f=(C_1\cos 2t+C_2\sin 2t)$$

$$P.I=\frac{1}{(D^2+4)}\cdot(2e^t-e^{-t})$$

$$=\frac{1}{(D^2+4)}2\cdot e^t-\frac{1}{(D^2+4)}e^{-t}$$

Put $D=1$ Put $D=-1$

$$=\frac{2\cdot e^t}{5}-\frac{e^{-t}}{5} \quad +1$$

Complete Solution for $y=C.f+P.I.$

$$y=(C_1\cos 2t+C_2\sin 2t)+\frac{2}{5}e^t-\frac{e^{-t}}{5} \quad +1$$

put value in (2)

$$-2x+D\left\{\left[C_1\cos 2t+C_2\sin 2t\right]+\frac{2}{5}e^t-\frac{e^{-t}}{5}\right\}=e^{-t}$$

$$x=\frac{1}{2}\left[-C_1\sin 2t\cdot 2+C_2\cos 2t\cdot 2\right]+\frac{2}{5}e^t+\left.\frac{e^{-t}}{5}\right]-\frac{e^{-t}}{2} \quad +1$$

Ans.

22

Date :

P. No. :

Q. 6.) $u = 2x - x^3 + 3xy^2$ is harmonic.

to prove

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ +1

$$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = -6x.$$

$$\frac{\partial^2 u}{\partial y^2} = 6x$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$$ +1

Hence function is harmonic.

to find conjugate of u.

$$du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$$

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$ +1

$$dv = -6xy\, dx + (2 - 3x^2 + 3y^2)\, dy$$

eq^n is exact.

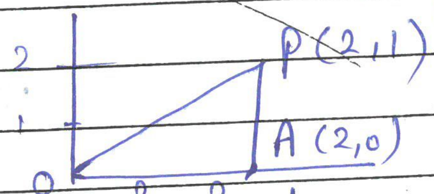
$$v = \int -6xy\,dx + \int 3y^2\,dy + C$$

(exact diffeq:) +1

$$V = -\frac{6x^2}{2}y + \frac{3y^3}{3} + C$$

$$\boxed{V = -3x^2y + \cancel{3}y^3 + C}$$ Harmonic Conjugate +1

ii) $$\int_0^{2+i} (\bar{z})^2\,dz$$



Sol:- $$(\bar{z})^2 = (x-iy)^2 = x^2-y^2-2ixy$$

$$\int_0^{2+i} (\bar{z})^2\,dz = \int_{OA} (x^2-y^2-2ixy)\,dx + \int_{AP} (x^2-y^2-2ixy)\,i\,dy \quad (1)$$ +1

for OA :
$y = 0$
$dz = dx$
limit of $x = 0$ to 2

for AP
$x = 2$
$dz = i\,dy$
Limit of $y = 0$ to 1 +1

~~Put in eq (1)~~

Along OA by eq^n. (1)

$$= \int_0^2 (x^2)\, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$ +1

Along AP, (by eqⁿ. (i)

$$= \int_0^1 (4 - y^2 - 4iy)\, i\, dy$$

$$= \int_0^1 (4i - y^2 i - 4i^2 y)\, dy$$

$$= \left[4iy - \frac{y^3}{3} i + \frac{4y^2}{2}\right]_0^1$$

$$= 4i - \frac{i}{3} + 2 = \frac{11}{3} i + 2.$$ +1

Put both value in eq(i)

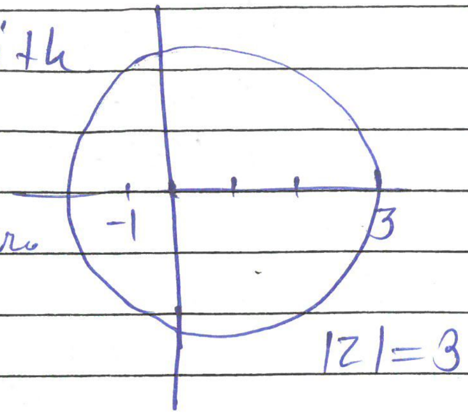
$$\int_0^{2+i} (\bar{z})^2 dz = \frac{8}{3} + \frac{11}{3} i + 2$$

$$= \frac{14}{3} + \frac{11}{3} i \quad \text{Ans.}$$ +1

iii) Cauchy integral formula.

evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$, $|z| = 3$

Sol:- $|z|=3$ is a circle with centre $(0,0)$ and radius 3. as shown in figure +1



we know;

In $\oint_c \frac{f(z)}{(z-a)^n}$

$z=a$ is a singular point.

Here,

$$\oint \frac{e^{2z}}{(z-(-1))^4} dz = \int_c \frac{e^{2z}}{(z+1)^4} dz$$ +1

Clearly $z=-1$ is a singular point which lie inside the circle. by Cauchy Integral formula.

$$f^n(a) = \frac{\lfloor n}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz \quad -(1)$$ +1

Here $f(z) = e^{2z}$

$n = 3$

$a = -1$

from (1), we have

$$f'''(-1) = \frac{3!}{2\pi i} \int_{|z|=3} \frac{e^{2z}}{(z+1)^4} dz$$ +1

$$f'(z) = 2e^{2z}$$

$$f''(z) = 4e^{2z}$$

$$f'''(z) = 8e^{2z} \Rightarrow f'''(-1) = 8e^{-2}$$

Hence,

$$\int_{|z|=3} \frac{e^{2z}}{(z+1)^4}\,dz = \frac{8e^{-2} \times 2\pi i}{6}$$ +1

$$\Rightarrow \int_{|z|=3} \frac{e^{2z}}{(z+1)^4}\,dz = \frac{8}{3} e^{-2} \cdot \pi i$$

Ans.