Enrollment No......................................

Faculty of Engineering

End Sem (Odd) Examination Dec-2019

EN3BS06 Discrete Mathematics

Programme: B.Tech. Branch/Specialisation: CSBS

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. For which one of these mappings, the function $f(x) = x^2$ will be one-one? **1**

(a) $N \rightarrow N$ (b) $Z \rightarrow Z$ (c) $R \rightarrow R$ (d) None of these.

ii. Let R be a binary relation on the set of all positive integers such that R={(a,b) : a-b is an odd positive integer}. Is R? **1**

(a) reflexive (b) Symmetric

(c) transitive (d) None of these

iii. A cyclic group of order 8 have____ generators: **1**

(a) 2 (b) 3 (c) 4 (d) 5

iv. Set G={1,2,3,4,5} is a group under **1**

(a) Addition modulo 6 (b) Multiplication modulo 6

(c) Both (a) and (b) (d) None of these

v. If ${}^{n}C_{15} = {}^{n}C_{8}$, then value of ${}^{n}C_{21}$ is: **1**

(a) 251 (b) 252 (c)253 (d) 254

vi. How many persons must be chosen in order that at least 7 of them will have birthdays in same calendar month. **1**

(a) 72 (b) 71 (c) 73 (d) 77

vii. A connected graph G is an eulerian graph if the degree of every vertex in G is **1**

(a) Odd (b) More than 2

(c) Even (d) None of these

viii. Every bipartite graph is ____ chromatic. **1**

(a) 3 (b) 2 (c) 4 (d) None of these

ix. P and Q are two propositions. Which of the following logical expressions are equivalent? **1**

I. P∨¬Q

II. ¬(¬P∧Q)

III. (P∧Q)∨(P∧¬Q)∨(¬P∧¬Q)

IV. (P∧Q)∨(P∧¬Q)∨(¬P∧Q)

(a) Only I and II (b) Only I, II and III

(c) Only I, II and IV (d) All of I, II, III and IV

x. Which of the following statement is a proposition? **1**

(a) Get me a glass of milkshake

(b) God bless you!

(c) What is the time now?

(d) The only odd prime number is 2

Q.2 Attempt any two:

i. Define function. If $f : X \to Y$ is one-one and onto, then prove that $f^{-1} : Y \to X$ is also one-one and onto. **5**

ii. Prove that intersection of any two partial order relations is also a partial order relation. **5**

iii. Among 100 students, 32 study mathematics, 20 study physics, 45 study biology, 15 study mathematics and biology, 7 study mathematics and physics, 10 study physics and biology, and 30 do not study any of the three subjects. **5**

(a) Find the number of students studying all three subjects.

(b) Find the number of students studying exactly one of the three subjects.

Q.3 Attempt any two:

i. Prove that a ring R is commutative, if $(a+b)^2 = a^2 + 2ab + b^2$, $\forall\, a, b \in R$. **5**

ii. Prove that every subgroup of a cyclic group is also cyclic. **5**

iii. Show that the set of numbers of the form $a + b\sqrt{2}$, with a and b as rational numbers is a field. **5**

Q.4 Attempt any two:

i. If n is a fixed positive integer, then prove that **5**

$C(n,1) + 2C(n,2) + 3C(n,3) + \cdots \ldots \ldots \ldots \ldots + rC(n,r) + \cdots \ldots \ldots + nC(n,n) = n.\, 2^{n-1}$

ii. Solve the recurrence relation $a_r - 4a_{r-1} + 4a_{r-2} = 2^r r^2, r \geq 2$ **5**

iii. Find the numeric function corresponding to the generating function given by $A(z) = \frac{(1+z)^2}{(1-z)^4}$ **5**

Q.5 Attempt any two:

i. Define bipartite graph. Is the graph $K_{3,4}$ planar or not? **5**

ii. Define chromatic polynomial. Show that a graph with n vertices is a tree if and only if $P_n(\lambda) = x(\lambda - 1)^{n-1}$ **5**

iii. Prove that in a simple graph with ‘n’ vertices and ‘k’ components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges. **5**

Q.6 Attempt any two:

i. Show that the proposition $(p \Leftrightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$ is a tautology. **5**

ii. Write short note on: **5**

(a) Tautology (b) Contradiction

iii. Construct truth table for following formulas: **5**

(a) (P ∨Q ∨¬R) ∧R (b) (P∧Q)∨(P∧¬Q)∨(¬P∧Q)

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Marking Scheme

End Sem (Odd) Exam. Dec - 2019

EN3BS06 (Discrete Mathematics)

Q.1. (i) (a) $N \to N$ [1 Marks]

(ii) (d) None of these. [1 Marks]

(iii) (c) 4 [1 Marks]

(iv) (d) None of these. [1 Marks]

(v) (c) 253 [1 Marks]

(vi) (c) 73 [1 Marks]

(vii) (c) Even [1 Marks]

(viii) (b) 2 [1 Marks]

(ix) (b) Only I, II and III [1 Mark]

(x) (d) The only odd prime number is 2 [1 Mark]

Q. 2 (i) To show f^{-1} is one - one [2.5 marks]

Let $y_1, y_2 \in Y$ be such that

$$f^{-1}(y_1) = x_1 \text{ and } f^{-1}(y_2) = x_2 \quad [x_1, x_2 \in X]$$

$\therefore$ By definition $f(x_1) = y_1$ and $f(x_2) = y_2$

Then $f^{-1}(y_1) = f^{-1}(y_2) \Rightarrow x_1 = x_2$

$\Rightarrow f(x_1) = f(x_2)$ $\quad \left\{ \begin{array}{l} \because f \text{ is one-one} \\ \therefore x_1 = x_2 \Rightarrow f(x_1) = f(x_2) \end{array} \right.$

$\therefore f^{-1}$ is one-one with $\Rightarrow y_1 = y_2$

To show f^{-1} onto [2.5 marks]

Let $x \in X$ be any element.

Since f is a mapping from X to Y, therefore there definitely exists an element $y \in Y$ such that

$y = f(x)$ or $f^{-1}(y) = x$

It shows that the f^{-1} image of $y \in Y$ is $x \in X$

i.e. every element of X is f^{-1} image of some element in Y.

$\therefore$ Hence f^{-1} is one-one onto.

2. (ii) Since R and R' are relation (partial order) say A

so $R \subseteq A \times A$ and $R' \subseteq A \times A$

Hence $R \cap R' \subseteq A \times A$ and thus $R \cap R'$ is a relation in A

→ $R \cap R'$ is reflexive [1.5 marks]

Since R is reflexive $\therefore (a,a) \in R \ \forall a \in A$

Also R' is reflexive $\therefore (a,a) \in R' \ \forall a \in A$

$\therefore \forall a \in A$, we have

$(a,a) \in R \cap R'$

Therefore $R \cap R'$ is reflexive.

$R \cap R'$ is transitive [1.5 marks]

$(a,b) \in R \cap R'$ and $(b,c) \in R \cap R'$

$\Rightarrow [(a,b) \in R$ and $(a,b) \in R']$ and

$[(b,c) \in R$ and $(b,c) \in R']$

$\Rightarrow [(a,b) \in R$ and $(b,c) \in R]$ and

$[(a,b) \in R'$ and $(b,c) \in R']$

$\Rightarrow (a,c) \in R$ and $(a,c) \in R'$

[Since R and R' are transitive]

$\Rightarrow (a,c) \in R \cap R'$

$\therefore$ $R \cap R'$ is transitive.

$R \cap R'$ is Anti Symmetric [2 Marks]

Let $(a,b) \in R \cap R'$ and $(b,a) \in R \cap R'$

$\Rightarrow [(a,b) \in R$ and $(b,a) \in R']$ and

$[(a,b) \in R$ and $(b,a) \in R']$

$\Rightarrow [(a,b) \in R$ and $(b,a) \in R' \Rightarrow a = b]$

and $[(a,b) \in R$ and $(b,a) \in R' \Rightarrow a = b]$

Since R and R' are antisymmetric

$\Rightarrow a = b$ Thus $R \cap R'$ is also Antisymmetric

Hence $R \cap R'$ is Partial order Relation.

Q. 2. (iii) (a) Find the number of students studying all three subjects. [2.5 Marks]

Let X denote the set of all students. Let M, P and B denote the set of students studying Mathematics, Physics and Biology respectively. Then we have.

$$|X| = 100, \quad |M| = 32, \quad |P| = 20, \quad |B| = 45$$

$$|M \cap B| = 15, \quad |M \cap P| = 7, \quad |P \cap B| = 10,$$

$$|(M \cup P \cup B)'| = 30$$

$$\therefore |M \cup P \cup B| = |X| - |(M \cup P \cup B)'|$$

$$= 100 - 30 = 70$$

By the principle of inclusion and exclusion, we have

$$|M \cap P \cap B| = |M \cup P \cup B| - |M| - |P| - |B| + |M \cap P| + |M \cap B| + |P \cap B|$$

$$= 70 - 32 - 20 - 45 + 7 + 15 + 10.$$

$$= 5$$

Hence the number of students studying all the three subject is 5.

(ii) Find the number at Students Studying exactly one at the three Subject [2.5 Marks]

Let M_1, P_1 and B_1 be the Set at Students who Study only Mathematics, Physics and Biology respecsively.

Then

~~[illegible]~~

The number at Students who Study only Mathematics

$= |M| - |M \cap P| - |M \cap B| + |M \cap P \cap B|$

$= 32 - 7 - 15 + 5 = 15$

The number at Students who Study only Physics.

$= |P| - |P \cap M| - |P \cap B| + |P \cap M \cap B|$

$= 20 - 7 - 10 + 5 = 8$

The number at Students who Study only Biology

$= |B| - |B \cap M| - |B \cap P| + |B \cap M \cap P|$

$= 45 - 15 - 10 + 5 = 25$

Hence the number at Students Studying exactly One at the three Subject

$= |M_1| + |P_1| + |B_1| = 15 + 8 + 25 = \underline{48}.$

Q.3 (i) Prove that a ring R is commutative, if $(a+b)^2 = a^2+2ab+b^2$, $\forall a, b \in R$ [5 Marks]

Sol^n Let R is Commutative i.e.

$$ab = ba \quad \forall a, b \in R$$

The we have to prove $(a+b)^2 = a^2+2ab+b^2$, $\forall a, b \in R$

$$(a+b)^2 = (a+b)(a+b)$$
$$= a^2+ab+ba+b^2$$
$$= a^2+ab+ab+b^2$$
$$= a^2+2ab+b^2$$

Conversely suppose that $(a+b)^2 = a^2+2ab+b^2$, $\forall a, b \in R$

Then we are to prove that R is commutative.

We have $(a+b)^2 = a^2+2ab+b^2$

$$\Rightarrow (a+b)(a+b) = a^2+2ab+b^2$$

$$\Rightarrow a^2+ab+ba+b^2 = a^2+2ab+b^2$$

$$\Rightarrow ab+ba = ab+ab$$

$$\Rightarrow ba = ab$$

$\Rightarrow$ R is Commutative ring.

Q.3. (ii) Prove that every subgroup of a cyclic group is also cyclic. [5 Marks]

Let $G = \{a\}$ be a cyclic group generated by 'a'. If $H = G$ or $\{e\}$, then H is cyclic.

Let H be a proper subgroup of G, then H contains integral power of 'a'.

Let $a^s \in H$ then

$$(a^s)^{-1} \in H$$

$\Rightarrow$ H has the elements of positive and negative integral power of 'a'.

Let m be the least positive integer, such that $a^m \in H$. Then we shall prove that

$H = \{a^m\}$ i.e. H is cyclic group generated by a^m.

Suppose $a^t \in H$ arbitrarily, By division algorithm $\exists$ integers q and r such that

$$t = mq + r \quad , \quad 0 \leq r < m$$

Now $a^m \in H \Rightarrow (a^m)^q \in H$ (by closure property)

$$\Rightarrow a^{mq} \in H$$

$$\Rightarrow (a^{mq})^{-1} \in H$$

$$\Rightarrow a^{-mq} \in H$$

Also $a^t \in H$, $a^{-mq} \in H \Rightarrow a^t a^{-mq} \in H$

$$\Rightarrow a^{t-mq} \in H$$

$$\Rightarrow a^r \in H \quad [\because r = t - mq]$$

Now, m be the least positive integer such that

$a^m \in H$ and $0 \leq r < m$

Then $r = 0$, therefore $t = mq$

$$a = a^{mq} = (a^m)^q$$

Thus every element $a^t \in H$ is of the form of $(a^m)^q$. So H is cyclic having a^m as its generator.

Q. 3. (iii) Show that the set of number of the form $a+b\sqrt{2}$, with a and b as rational numbers is a field. [5 Marks]

Let $Q[\sqrt{2}] = \{a+b\sqrt{2} : a, b \in Q\}$

Let $a_1 + b_1\sqrt{2}$ and $a_2 + b_2\sqrt{2} \in Q[\sqrt{2}]$.

Then $a_1, b_1, a_2, b_2 \in Q$

We have $(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) =$

$$(a_1+a_2) + (b_1+b_2)\sqrt{2} \in Q[\sqrt{2}]$$

$$[\because a_1+a_2, b_1+b_2 \in Q]$$

Also $(a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2}) = (a_1a_2 + 2b_1b_2) +$

$$(a_1b_2 + a_2b_1)\sqrt{2} \in Q[\sqrt{2}]$$

is closed with respect to addition and multiplication.

All the elements of $Q[\sqrt{2}]$ are real numbers and we know that addition and multiplication are both associative as well as commutative compositions in the set of real numbers.

$\therefore$ $0+0\sqrt{2}$ is the additive identity.

Again if $a+b\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$ then

$(-a)+(-b)\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$ and we have

$[(-a)+(-b)\sqrt{2}]+(a+b\sqrt{2}) = 0+0\sqrt{2}$

$\therefore$ Each element of $\mathbb{Q}\sqrt{2}$ possesses additive inverse.

Further in the set of real number, multiplication is distributive with respect to addition.

Again $1+0\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$ and

$$(1+0\sqrt{2})(a+b\sqrt{2}) = a+b\sqrt{2}$$

$$= (a+b\sqrt{2})(1+0\sqrt{2})$$

$\therefore$ $1+0\sqrt{2}$ is the multiplicative identity.

Thus $(\mathbb{Q}[\sqrt{2}], +, \cdot)$ is a commutative ring with unity. The zero element of the ring is $0+0\sqrt{2}$ and the unit element is $1+0\sqrt{2}$

Now $(\mathbb{Q}[\sqrt{2}], +, \cdot)$ will be a field if each non-zero element of $\mathbb{Q}[\sqrt{2}]$ possesses multiplicative inverse.

Let $a+b\sqrt{2}$ be any non-zero element of this ring, i.e. at least $a \neq 0$ or $b \neq 0$

then $$\frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{(a+b\sqrt{2})(a-b\sqrt{2})} = \frac{a-b\sqrt{2}}{a^2-2b^2}$$

Now if at least one of the rational numbers a and b is non-zero, then we cannot have

$a^2-2b^2=0$ i.e. $a^2=2b^2$

$\therefore$ $\frac{a}{a^2-2b^2}$ and $\frac{-b}{a^2-2b^2}$ are both rational numbers

and at least one of them is not zero.

$\therefore$ $\left(\frac{a}{a^2-2b^2}\right)+\left(\frac{-b}{a^2-2b^2}\right)\sqrt{2}$ is a non-zero element of $Q[\sqrt{2}]$ and is the multiplicative inverse of $a+b\sqrt{2}$.

Hence $(Q[\sqrt{2}], +, \cdot)$ is a field.

(12)

Q.4.(i) If n is a fixed positive integer, then prove that

$C(n,1) + 2C(n,2) + 3C(n,3) + \cdots + rC(n,r) + \cdots + nC(n,n) = n \cdot 2^{n-1}$ [5 Marks]

$${}^{n}C_1 + 2 \cdot {}^{n}C_2 + 3\,{}^{n}C_3 + \cdots + n \cdot {}^{n}C_n$$

$$= n + 2 \cdot \frac{n(n-1)}{2!} + 3 \cdot \frac{n(n-1)(n-2)}{3!} + \cdots n \cdot 1$$

$$= n\left[1 + \frac{(n-1)}{1!} + \frac{(n-1)(n-2)}{2!} + \cdots + 1\right]$$

$$= n\left[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \cdots + {}^{n-1}C_{n-1}\right]$$

$$= n \cdot 2^{n-1}$$

Q.4. (iii) Find the numeric function corresponding to the generating function given by $A(z) = \frac{(1+z)^2}{(1-z)^4}$

[5 Marks]

$$A(z) = \frac{(1+z)^2}{(1-z)^4} = (1+2z+z^2)(1-z)^4$$

$$= (1+2z+z^2)\left\{1+4z+\frac{4.5}{2!}z^2+\frac{4.5.6}{3!}z^3+\cdots\right\}$$

[Expanding by binomial theorem]

$$= (1+2z+z^2)\left\{1+4z+10z^2+\cdots\right\} \qquad \text{—— (i)}$$

Now Coefficient at z^r in $A(z)$

$= (\text{Coeff at } z^r + 2 \text{ Coeff at } z^{r-1} + \text{Coeff. at } z^{r-2})$

in the second bracket on R.H.S at (i)

$$= \frac{(r+1)(r+2)(r+3)}{6} + \frac{2r(r+1)(r+2)}{6} + \frac{(r-1)r(r+1)}{6}$$

$$= \frac{(r+1)}{6}\left[r^2+5r+6+2r^2+4r+r^2-r\right]$$

$$= \frac{1}{3}(r+1)(2r^2+4r+3)$$

Hence the numeric function a_r corresponding to $A(z)$ is

$$a_r = \frac{1}{3}(r+1)(2r^2+4r+3), \quad r \geq 0$$

Q.5. (i) Bipartite Graph [2 Marks]

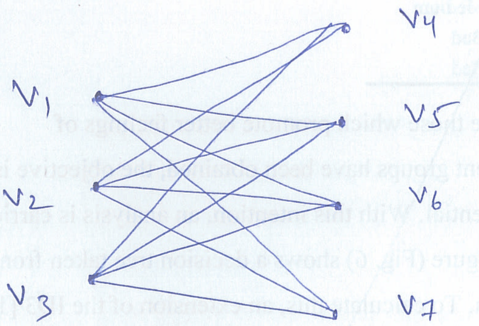
Let G (V, E) be a graph. The graph G is called bipartite graph if its set V of vertices can be partitioned into two subsets H and S such that each edge of G connects a vertex of H to a vertex of S.

If each vertex of H is connected to each vertex of S, then such a graph is called complete bipartite graph and is denoted by $K_{m,n}$

where m = number of vertices in H

n = number of vertices in S

[3 Marks]



$K_{3,4}$

$e \le 2n - 4$

$12 \le 2 \times 7 - 4$

$12 \le 10$

Not correct

So this is not Plannar.

Q.5.(ii) <u>Chromatic Polynomial</u> [2 Marks] 15

Let G be a graph with n vertices, then G can be properly colored in several distinct ways by using different ~~ren~~ colors (say λ). This number can be expressed by a polynomial denoted by $P_n(\lambda)$ is called chromatic polynomial.

Show that a graph with n vertices is a tree if and only if $P_n(\lambda) = \lambda(\lambda-1)^{n-1}$ [3 Marks]

If $n=1$ then $P_n(\lambda) = \lambda$

If $n=2$ then $P_n(\lambda) = \lambda(\lambda-1)$

Thus the result holds for $n=1,2$

Now suppose that the result holds for all trees having $n-1$ vertices. Since G is a tree, therefore it has atleast two pendent vertices, let v be one of the pendent vertex.

Delete v from G, then $G-v = G^*$ say is a tree with $n-1$ vertices. Hence by induction chromatic polynomial of G^* is

$$P_{n-1}(\lambda) = \lambda(\lambda-1)^{n-2}$$

Now suppose a vertex adjacent to v in G is v_1

Add the vertex v to G^* to obtain G

Let the vertex v_1 be colored with one of the λ colored, then v can be properly colored with any of the remaining $\lambda-1$ colors. Hence the total number of ways of properly coloring of tree G i.e. chromatic polynomial of G is given by

$$P_n(\lambda) = \lambda(\lambda-1)^{n-2} \cdot (\lambda-1)$$

$$P_n(\lambda) = \lambda(\lambda-1)^{n-1}$$

Hence by induction hypothesis the result is true.

Q.5.(iii) Prove that in a simple graph with n vertices and 'k' components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges.

Let the number of vertices in each of the k component of a graph G be $n_1, n_2, \ldots n_k$

Thus we have

$$n_1 + n_2 + \ldots + n_k = n$$

$$n_i \geq 1$$

The proof of the theorem is based on the inequality

$$\sum_{i=1}^{K} n_i^2 \leq n^2 - (K-1)(2n-K)$$

This inequality can be proved as follows.

$$\sum_{i=1}^{K} (n_i - 1) = n - K$$

Squaring both side

$$\left(\sum_{i=1}^{K} (n_i - 1) \right)^2 = n^2 + K^2 - 2nK$$

or.

$$\sum_{i=1}^{K} (n_i^2 - 2n_i) + K + \text{nonnegative cross term}$$

$$= n^2 + K^2 - 2nK.$$

because $(n_i - 1) \geq 0$ for all i

Therefore $\sum_{i=1}^{K} n_i^2 \leq n^2 + K^2 - 2nK - K + 2n = n^2 - (K-1)(2n-K)$

Thus the required inequality is proved.

Now the maximum number of edges in ith component of G (which is simple connected graph) is

$\frac{1}{2} n_i(n_i - 1)$. Therefore, the maximum number of edges in G is

$$\frac{1}{2}\sum_{i=1}^{k}(n_i - 1)n_i = \frac{1}{2}\left(\sum_{i=1}^{k} n_i^2\right) - \frac{n}{2}$$

$$\leq \frac{1}{2}\left(n^2 - (k-1)(2n-k)\right) - \frac{n}{2}$$

$$= \frac{1}{2}(n-k)(n-k+1)$$

Hence Proved.

Q. 6. (i) Show that the proposition $(P \Leftrightarrow q \wedge r) \Rightarrow (\sim r \Rightarrow \sim p)$ is a tautology [5 Marks]

p	q	r	$q \wedge r$	$p \Leftrightarrow q \wedge r$ (A)	$\sim r$	$\sim p$	$\sim r \Rightarrow \sim p$ (B)	$A \Rightarrow B$
T	T	T	T	T	F	F	T	T
T	T	F	F	F	T	F	F	T
T	F	T	F	F	F	F	T	T
T	F	F	F	F	T	F	F	T
F	T	T	T	F	F	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	F	T	F	T	T	T
F	F	F	F	T	T	T	T	T

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Hence above is tautology.

Q. 6.(ii) (a) Tautology :- A tautology is a proposition which is true for all truth value of its sub-propositions or Components.

A tautology is also called logically valid or logically true.

Clearly in truth table, all entries in the column of tautology are of 'T' only. [2.5 Marks]

(b.) Contradiction :-

A Contradiction is a proposition which is always false for all truth values of its propositions or components.

A Contradiction is also called logically false.

Clearly in truth table, all entries in the column of Contradiction are of F only.

[2.5 marks]

Q.6.(iii) Construct truth table for following

(a) $(P \vee Q \vee \sim R) \wedge R$

P	Q	R	$\sim R$	$(P \vee Q \vee \sim R)$	$(P \vee Q \vee \sim R) \wedge R$
T	T	T	F	T	T
T	T	F	T	T	F
T	F	T	F	T	T
T	F	F	T	T	F
F	T	T	F	T	T
F	T	F	T	T	F
F	F	T	F	F	F
F	F	F	T	T	F

(b) $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$

P	Q	$(P \wedge Q)$ A	$\neg Q$	$(P \wedge \neg Q)$ B	$\neg P$	$(\neg P \wedge Q)$ C	$A \vee B \vee C$
T	T	T	F	F	F	F	T
T	F	F	T	~~F~~ T	F	F	T
F	T	F	F	F	T	~~F~~ T	T
F	F	F	T	F	T	~~T~~ F	F