Total No. of Questions: 6 *Total No. of Printed Pages:3*

Enrollment No......................................

Faculty of Science

End Sem (Odd) Examination Dec-2018

BC3CO11 Mathematics–III

Programme: B.Sc. (CS) Branch/Specialisation: Computer Science

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. The series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+........\infty$ is divergent if, **1**

(a) $p>1$ (b) $p<=1$ (c) $p=1$ (d) None of these

ii. The sequence {0, 1, 0, $\frac{1}{2}$, 0, $\frac{1}{3}$, -----} has the n^{th} term **1**

(a) $\frac{1+(-1)^n}{n}$ (b) $\frac{n-(1)^n}{n}$ (c) $\frac{n+(1)^n}{n}$ (d) None of these

iii. The equation $(1-x^2)\frac{d^2y}{dx^2}$ - $2x\frac{dy}{dx}+n(n+1)y=0$ is called **1**

(a) Legendre's differential equation
(b) Bessel's differential equation
(c) Partial differential equation
(d) Ordinary differential equation of first order

iv. Ordinary differential equation involves- **1**

(a) Only one independent variable
(b) Only two independent variables
(c) Infinite independent variable
(d) No independent variable

v. Equation $p^4+qx^2+z^3=0$ is of degree: **1**

(a) Two (b) Three (c) Four (d) None of these

P.T.O.

vi. The solution of Partial differential equation $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=0$ is: **1**

(a) $z=f_1(y+x)+f_1(y-x)$ (b) $z=f_1(y+x)+f_2(y-x)$

(c) $z=f_2(y+x)+f_2(y-x)$ (d) $z=f(y^2-x^2)$

vii. A vector $\vec{F}(t)$ is called solenoidal if: **1**

(a) $\nabla X\,\vec{F}=0$ (b) $\nabla.\vec{F}=0$

(c) $\nabla.(\nabla X\,\vec{F})=0$ (d) None of these

viii. The relation between the line integral and surface integral is: **1**

(a) Gauss's Theorem (b) Stokes' Theorem

(c) Green's Theorem (d) Bernoulli's Theorem

ix. If a function $f(z)$ is analytic at a point $z=z_0$, then following statement is false- **1**

(a) f is differential at z_0 (b) f is not continuous at z_0

(c) f is defined at z_0 (d) f is continuous at z_0

x. The zero of first order is known as- **1**

(a) Complex Zero (b) Simple Zero

(c) Singularity (d) None of these

Q.2 i. Define monotonic sequence and comparison test. **2**

ii. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{\lfloor n}{n^n}$ **3**

iii. Test the series for convergence **5**

$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+.......$

OR iv. Examine the convergence of $\sum_{n=1}^{\infty} ne^{-n^2}$ **5**

Q.3 i. Prove that $\frac{d}{dx}(x^n J_n)=x^n J_{n-1}$ **4**

ii. Solve $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$ **6**

OR iii. Express $f(x)=4x^3+6x^2+7x+2$ in terms of Legendre Polynomials. **6**

Q.4 i. Solve $yq-xp=z$ **3**

ii. Solve by charpit's method: $z=px+qy+p^2+q^2$ **7**

OR iii. Solve $\frac{\partial^3 z}{\partial x^3}-4\frac{\partial^3 z}{\partial x^2\partial y}+4\frac{\partial^3 z}{\partial x\partial y^2}=2\sin(3x+2y)$ **7**

Q.5 i. If $\vec{V}=xy^2\,\hat{i}+2yx^2z\,\hat{j}-3yz^2\,\hat{k}$, then curl $\vec{V}$ is at the point $(1,-1,1)$. **2**

ii. If $\vec{F}=2z\,\hat{i}-x\,\hat{j}+y\,\hat{k}$, evaluate $\iiint_v \vec{F}dv$ where v is the region bounded by the surface $x=0,\ y=0,\ x=2,\ y=4,\ z=x^2,\ z=2$ **8**

OR iii. Apply stroke's theorem to evaluate $\int_c[(x+y)dx+(2x-z)dy+(z+y)dz]$ where, C is the boundary of the triangle with vertices (2,0,0), (0,3,0) and (0,0,6). **8**

Q.6 Attempt any two:

i. Show that the function $e^x(\cos y+i\sin y)$ is analytic and find its derivative. **5**

ii. Determine the pole of the function $f(z)=\frac{z^2}{(z-1)^2(z+1)}$ and the residue at each pole. **5**

iii. Evaluate $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}$ **5**

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Solution

End Sem Examination Dec-2018

BC3C011, mathematics III

Q.1

(i) b: $p \leq 1$ 1

(ii) a: $\frac{1+(-1)^n}{n}$ 1

(iii) a; Legendre's differential Equation 1

(iv) a; Only one independent variable. 1

(v) c: Four 1

(vi) b: $z = f_1(y+x) + f_2(y-x)$ 1

(vii) b: $\nabla . \vec{F} = 0$ 1

(viii) b; Stoke's Theorem 1

(ix) b; f is not continuous at z_0 1

x b: Simple zero 1

10

Q.2 (i) Sequence $\{a_n\}$

monotonic increasing if $a_{n+1} > a_n \ \forall n$

monotonic decreasing if $a_{n+1} < a_n \ \forall n$ +1

<u>Comparison test</u>

$\sum u_n$ is positive term series, let $\sum v_n$ be a series such that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \text{finite non zero quantity}$$ +1

$\Rightarrow \sum u_n$ and $\sum v_n$ converges or diverges simultaneously. (2)

(ii) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$u_n = \frac{n!}{n^n} \qquad u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$ +1

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \frac{(n+1)}{(n+1)^{n+1}} n^n = \left(\frac{1}{1+\frac{1}{n}}\right)^n$$ +1

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{e} < 1$$ +1

$\Rightarrow$ series is convergent by Ratio test (3)

(iii) $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} --- \infty$

Ignoring 1st term,

$$u_n = \frac{n^n}{(n+1)^{n+1}}$$

$$u_{n+1} = \frac{(n+1)^{n+1}}{(n+2)^{n+2}}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^{n+1}}{(n+2)^{n+2}} \cdot \frac{(n+1)^{n+1}}{n^n}$$

$$= \frac{(1+1/n)^n \, (1+1/n)^n \, (n+1)^2}{(1+2/n)^n \cdot (n+2)^2}$$ (+1)

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1 \quad \text{Ratio test fails}$$ (+1)

Now $n\left(\frac{u_n}{u_{n+1}} - 1\right)$

$$= n\left(\frac{(n+2)^2}{(n+1)^2} - 1\right)$$

$$= n\left[\frac{(n^2+4n+4) - (n^2+2n+1)}{(n+1)^2}\right]$$ (+1)

$$= n\left(\frac{2n+3}{(n+1)^2}\right)$$

$$= \frac{(2+3/n)}{(1+1/n)}$$ (+1)

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 2 > 1$$

Using Raabe's Test Σu_n is convergent.

(5)

Q2 (iv) $\sum_{n=1}^{\infty} n e^{-n^2}$

$u_n = n e^{-n^2}$

$u_n^{1/n} = n^{1/n} e^{-n}$ +1

$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} n^{1/n} . \lim_{n\to\infty} e^{-n}$ +1

$= 1 . e^{-\infty}$ +1

$= 0 < 1$ As $\lim_{n\to\infty} n^{1/n} = 1$ +1

$e^{-\infty} = 0$

By Root test $\sum n e^{-n^2}$ is convergent. +1

(5)

3 (i) Prove that

$$\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}$$

We know

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$ +1

$$x^n J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, (n+r)!\, 2^{n+2r}} x^{2n+2r}$$ +1

$$\frac{d}{dx}\left(x^n J_n(x)\right) = \sum_{r=0}^{\infty} \frac{(-1)^r \, 2(n+r)\, x^{2n+2r-1}}{r!\, (n+r)!\, 2^{n+2r}}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r \, x^n . x^{(n-1)+2r}}{r!\, (n-1+r)!\, 2^{(n-1)+2r}}$$ +1

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\, \Gamma(n+r)} \frac{x^{(n-1)+2r}}{2^{(n-1)+2r}}$$ +1

(4)

$x^n J_n(x)$

6

1 . Solve $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$

$x = 0$ is an ordinary point. +1

Let the solution be $y = \sum_{n=0}^{\infty} a_n x^n$ +1

Substituting the values of y, y_1, and y_2 in the given equation, we get

$$(1+x^2)\sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} + x\sum_{n=0}^{\infty} a_n n x^{n-1}$$

$$\cancel{+} \sum_{n=0}^{\infty} a_n x^n = 0$$

Arranging the terms of same powers of x, we get

$$\sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n [n(n-1) + n - 1] x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n (n^2-1) x^n = 0$$ +1

comparing coeff of x^n to 0

$$a_{n+2}(n+2)(n+1) + a_n(n^2-1) = 0$$

$$a_{n+2} = \frac{(1-n)}{(n+2)} a_n$$

$$\Rightarrow a_2 = \frac{1}{2} a_0$$ +1

$$a_3 = 0$$

$$a_4 = -\frac{1}{4} a_2 = -\frac{1}{8} a_0$$ +1

$$a_5 = 0$$ +1

$$\Rightarrow y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 - \frac{1}{8} a_0 x^4 \dots$$

… x^4 …) $+ a_1 x$ 5

Q.3 (iii) $f(x) = 4x^3 + 6x^2 + 7x + 2$

To express in term of Legendre polynomial

we know

$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$ +1

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

Let $f(x) = A P_0(x) + B P_1(x) + C P_2(x) + D P_3(x)$

$\Rightarrow 2 + 7x + 6x^2 + 4x^3 = A + Bx + \frac{C}{2}(3x^2 - 1)$ +1

$+ \frac{D}{2}(5x^3 - 3x)$

$= \left(A - \frac{C}{2}\right) + \left(B - \frac{3D}{2}\right)x + \frac{3C}{2}x^2 + \frac{5D}{2}x^3$ +1

$\Rightarrow \quad A - \frac{C}{2} = 2$

$B - \frac{3D}{2} = 7$

$\frac{3}{2}C = 6 \quad \Rightarrow \quad \boxed{C = 4}$ +1

$\frac{5D}{2} = 4 \quad \Rightarrow \quad \boxed{D = \frac{8}{5}}$

$\Rightarrow \quad A = 4$

$B - \frac{3 \times \cancel{8}^{4}}{\cancel{2} \times 5} = 7$ +1

$B = 7 + \frac{12}{5}$

$\boxed{B = \frac{47}{5}}$

$\Rightarrow \quad f(x) = 4 P_0(x) + \frac{47}{5} P_1(x) + 4 P_2(x) + \frac{8}{5} P_3(x)$ +1

Ans

6

8

Q4. (i) Solve $yq - xp = z$.

Comparing with Lagrange's eq $Pp + Qq = R$

$$P = -x \quad Q = y \quad R = z$$

Auxiliary Eq

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{z}$$ +1

Taking $\frac{dx}{-x} = \frac{dy}{y}$

$$-\log x = \log y + \log c_1$$

$$\Rightarrow xy = c_1$$

Taking $\frac{dy}{y} = \frac{dz}{z}$

$$\log y = \log z + \log c_2$$ +1

$$\Rightarrow y/z = c_2$$

3

Solu $f(xy, y/z) = 0$

(ii) Solve by Charpit's method

$$z = px + qy + p^2 + q^2$$

$$f = z - px - qy - p^2 - q^2$$

$f_x = -p \quad f_y = -q \quad f_z = 1 \quad f_p = -x - 2p$ +2

$f_q = -y - 2q$

Charpit's Auxiliary equation :-

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$ +2

Or $$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-p(x+2p)-q(y+2q)} = \frac{dx}{-(x+2p)} = \frac{dy}{-(y+2q)} = \frac{dF}{0}$$

From (1) 2 last

$\Rightarrow dp=0$

$\Rightarrow \boxed{p=a}$ +2

From (2) 2 last

$dq=0$

$\Rightarrow \boxed{q=b}$

Complete equation +4

$dz = p\,dx + q\,dy$

$= a\,dx + b\,dy$ (7)

$\Rightarrow z = ax+by+c$ Ans

Q11) $$\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 2\sin(3x+2y)$$

$\frac{\partial}{\partial x} \equiv D \quad \frac{\partial}{\partial y} = D'$

$(D^3 - 4D^2D' + 4DD'^2)z = 2\sin(3x+2y)$

Auxiliary equation :- $D \to m \quad D' \Rightarrow 1$

$m^3 - 4m^2 + 4m = 0$

$\Rightarrow m = 0, 2, 2$ (+2)

C.F. $\phi_1(y) + \phi_2(y+2x) + x\,\phi_3(y+2x)$

P.I $= \frac{1}{D^3 - 4D^2D' + 4DD'^2} 2\sin(3x+2y)$ (+)

$= \frac{2}{D(D^2 - 4DD' + 4D'^2)} \sin(3x+2y)$

$$= \frac{-6\cos(3x+2y)}{D^2 - 4DD' + 4D'^2}$$ As $\frac{1}{D} = \int dx$ (10)

Now $\frac{1}{f(D^2, DD', D'^2)} \cos(ax+by)$

$$= \frac{\cos(ax+by)}{f(-a^2, -ab, -b^2)}$$ +2

$D^2 \to -a^2$, $DD' \to -ab$

$D'^2 \to -b^2$

+4

$\Rightarrow$ P.I $\Rightarrow 6\cos(3x+2y)$

+1

Ans $z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) + 6\cos(3x+2y)$ (7)

Q.5 (i) $\vec{v} = xy^2\hat{i} + 2yx^2z\hat{j} - 3yz^2\hat{k}$

$\text{Curl}\,\vec{v} = \nabla \times \vec{v}$ +1

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2yx^2z & -3yz^2 \end{vmatrix}$$

+1

$$= \hat{i}\left(-3z^2 - 2yx^2\right) + \hat{k}\left(4xyz - 2xy\right)$$ Ans (2)

(ii) $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$

$$\iiint_V \vec{F}\,dv = \int_0^2\int_0^4\int_{x^2}^{2} (2z\hat{i} - x\hat{j} + y\hat{k})\,dz\,dy\,dx$$ +2

$$= \int_0^2\int_0^4 \left[(4-x^4)\hat{i} - x(2-x^2)\hat{j} + y(2-x^2)\hat{k}\right] dy\,dx$$ +1

+1

$$= \int_0^2 4(4-x^2)\hat{i} - 4x(2-x^2)\hat{j} + 8(2-x^2)\,dx$$ +1

$$= \left(16\times 2 - 4\times\frac{8}{3}\right)\hat{i} - \left(4\times 4 - 4\times\frac{8}{3}\right)\hat{j} + \left(32 - \frac{8\times 8}{3}\right)\hat{k}$$

$= \frac{6y}{3}\hat{i} - \frac{16}{3}\hat{j} + \frac{32}{3}\hat{k}$ Ans 11 +2

8

Q 5 (iii)

By Stoke's theorem :-

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

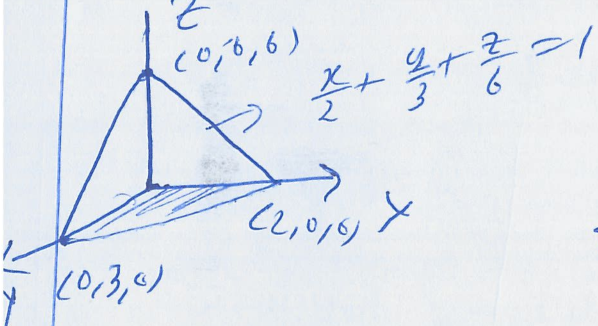
+1

$$\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (z+y)\hat{k}$$ +2

$$\text{curl}\,\vec{F} = 2\hat{i} + \hat{k}$$

$$S = 3x+2y+z = 6$$

$$\phi = 3x+2y+z-6$$



$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{1}{\sqrt{14}}(3\hat{i}+2\hat{j}+\hat{k})$$ +1

Taking projection on XY Plane

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{1/\sqrt{14}}$$ +1

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \iint_R 7\,dx\,dy$$ +1

$$= 7\int_{x=0}^{2}\int_{y=0}^{\frac{6-3x}{2}} dx\,dy$$ +2

$$= 21$$ Ans

8

Q6. (ii) $f(z) = \dfrac{z^2}{(z-1)^2 (z+1)}$

$z=-1$ simple pole

$z=2$ pole of order 2 (+1)

$\{\text{Res} f(z)\}_{z=-1} = \lim_{z\to -1} \dfrac{z^2}{(z-1)^2} = \underline{\dfrac{1}{4}}$ +1

$\{\text{Res} f(z)\}_{z=1} = \lim_{z\to 1} \dfrac{1}{(2-1)!} \dfrac{d}{dz} \{(z-1)^2 \cdot f(z)\}$ +1

$= \lim_{z\to 1} \dfrac{d}{dz} \left(\dfrac{z^2}{(z+1)}\right)$

$= \lim_{z\to 1} \left(\dfrac{2z}{(z+1)} - \dfrac{z^2}{(z+1)^2}\right)$ +2

$= \left(1 - \dfrac{1}{4}\right) = \underline{\underline{3/4}}$ 5

(iii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$! Consider contour $|z|=1$ +1

$\Rightarrow z = e^{i\theta}$

$\dfrac{dz}{iz} = d\theta$, $\cos\theta = \dfrac{z+\frac{1}{z}}{2}$ +1

$= \dfrac{z^2+1}{2z}$

$\Rightarrow \displaystyle\oint_C \frac{1}{2+\frac{z^2+1}{2z}} \frac{dz}{iz}$

$= \dfrac{2}{i} \displaystyle\oint_C \frac{dz}{z^2+4z+1}$ $C: |z|=1$ +1

$= \dfrac{2}{i}\left[2\pi i \text{ Res}\left\{\dfrac{1}{z^2+4z+1}\right\} \text{ at pole inside } |z|=1\right]$

for $\dfrac{1}{z^2+4z+1}$ =! $z=-2+\sqrt{3}$ is inside $|z|=1$ +1

$z=-2-\sqrt{3}$ is outside $|z|=1$)

$I = \dfrac{2}{i}\left[2\pi i \text{ Res}\left(\dfrac{1}{[z-(-2+\sqrt{3})][z-(-2-\sqrt{3})]}\right)\right]_{\text{at } z=-2+\sqrt{3}}$

$= \dfrac{2}{i} 2\pi i \lim_{z\to -2+\sqrt{3}} \left(\dfrac{1}{z-(-2-\sqrt{3})}\right)$ +1

$= \dfrac{2\pi}{\sqrt{3}} \Rightarrow$ Ans 5

Q6.

(i) $f(z) = e^x \cos y + i\, e^x \sin y$

$u(x,y) = e^x \cos y$ +1

$v(x,y) = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y \quad \frac{\partial u}{\partial y} = -e^x \sin y$ +1

$\frac{\partial v}{\partial x} = e^x \sin y \quad \frac{\partial v}{\partial y} = e^x \cos y$ +1

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous —(1)

Also +1

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ —(2)

$\Rightarrow$ Cauchy Riemann Eq satisfied

(1) & (2) $\Rightarrow$ $f(z)$ is Analytic +1

Also $\frac{d f(z)}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

$= e^x \cos y + i\, e^x \sin y$ 5

$= e^x (\cos y + i \sin y)$

$=$