Total No. of Questions: 6 *Total No. of Printed Pages:3*

Enrollment No......................................

Faculty of Science

End Sem (Even) Examination May-2022

CA3CO08 Mathematics -II

Programme: BCA, BCA+MCA (Integrated) Branch/Specialisation: Computer Application

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. The n^{th} derivative of $y = e^{ax}$ is- **1**

(a) e^{ax} (b) $a^{n-1}e^{ax}$ (c) $a^n e^{ax}$ (d) None of these

ii. Which of the theorem is used to evaluate n^{th} derivate of product of two functions? **1**

(a) Rolle's Theorem (b) Leibnitz Theorem

(c) Cauchy's Theorem (d) None of these

iii. A partial derivative requires- **1**

(a) Exactly one independent variable

(b) Two or more independent variable

(c) Equal number of dependent and independent variable

(d) None of these

iv. If u is a homogeneous function of x and y of degree n, then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} =$____. **1**

(a) 0 (b) u (c) nu (d) None of these

v. The value of an integral $\int_0^\infty x^9 e^{-x} dx$ is equal to- **1**

(a) $\Gamma(10)$ (b) $\Gamma(9)$ (c) $\Gamma(8)$ (d) None of these

vi. The value of $\beta(2,3)$ is- **1**

(a) 1/12 (b) 1 (c) 2 (d) None of these

P.T.O.

vii. The value of the integral $\int_0^1\int_0^1 x\,dx\,dy$ is- **1**

(a) 1 (b) $1/2$ (c) $1/4$ (d) None of these

viii. The triple integral $\iiint_D dxdydz$, where D is the bounded surface in 3-dimensional space represents. **1**

(a) Volume of D (b) Length of D
(c) Area of D (d) None of these

ix. The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^4+\left(\frac{dy}{dx}\right)^3+2y=\sin x$ is- **1**

(a) 3 (b) 4 (c) 2 (d) None of these

x. The solution of linear differential equation $xdy-ydx=0$. **1**

(a) $\ln(y/x)=c$ (b) $x+y=c$
(c) $y-x=c$ (d) None of these

Q.2 Attempt any two:

i. State and Verify Rolle's theorem for the function: **5**
$f(x)=1-(x-4)^{2/3}$.

ii. Find the Taylor's series expansion of the function $f(x)=\log(\cos x)$ about the point $\frac{\pi}{3}$. **5**

iii. Expand the function $\log(1+e^x)$, using Maclaurin's series. **5**

Q.3 Attempt any two:

i. If $u=e^{xyz}$ then show that- $\frac{\partial^3 u}{\partial x\partial y\partial z}=e^{xyz}(1+3xyz+x^2y^2z^2)$. **5**

ii. If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then show that- **5**

(a) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

(b) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2\cos 3u\sin u$.

iii. Locate the stationary points of $x^4+y^4-2x^2+4xy-2y^2$ and determine their nature. **5**

Q.4 i. Evaluate the integral $\int_a^b x^2\,dx$ using definite integral as a limit of sum. **3**

ii. Evaluate $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)........\left(1+\frac{n}{n}\right)\right]^{1/n}$. **7**

OR iii. Prove that $\Gamma m\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma 2m$. **7**

Q.5 Attempt any two:

i. Evaluate the integral $\int_0^1\int_0^{1-y}\int_0^{1-x-y} xyz\,dz\,dy\,dx$. **5**

ii. Change the order of integration and hence evaluate the integral $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$. **5**

iii. Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, using double integral. **5**

Q.6 Attempt any two:

i. Form the differential equation from the relation $y=e^x(A\cos x+B\sin x)$. **5**

ii. Solve the differential equation $\left(1+e^{x/y}\right)dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$. **5**

iii. Solve the differential equation $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=e^x\sin x$. **5**

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Ans.-1

(I) (c) $a^n e^{ax}$

(II) (b) Leibnitz Theorem

(III) (b) two or more independent variable

(IV) (c) nu

(V) (a) $\Gamma 10$

(VI) (a) $\frac{1}{12}$

$$\beta(2,3) = \frac{\Gamma 2 \, \Gamma 3}{\Gamma 2+3}$$

$$= \frac{\lfloor 1 \, \lfloor 2}{\lfloor 4} = \frac{\cancel{\lfloor 2}}{4 \times 3 \times \cancel{\lfloor 2}}$$

$$= \frac{1}{12} \text{ Ans}$$

(VII) (b) $\frac{1}{2}$

$$\int_0^1 \int_0^1 x \, dx \, dy = \int_0^1 x (y)_0^1 dx$$

$$= \int_0^1 x \, dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2} \text{ Ans}$$

(VIII) (a) volume of D

(IX) (b) 4

(X) (a) $\log\left(\frac{y}{x}\right) + c$

$$x\,dy - y\,dx = 0$$

$$x\,dy = y\,dx$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\log y - \log x = c$$

$$\log(y/x) = c \text{ Ans}$$

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Ans. 2 (I) Given

$$f(x) = 1-(x-4)^{2/3}$$

Put $f(x)=0$

$$1-(x-4)^{2/3} = 0$$

$$(x-4)^{2/3} = 1$$

$$(x-4)^2 = 1$$

$$x^2-8x+16 = 1$$

$$x^2-8x+15=0$$

$$x^2-3x-5x+15=0$$

$$x(x-3)-5(x-3)=0$$

$$(x-3)(x-5)=0 \quad \text{— (1)}$$

$x = 3,5$

$\therefore$ interval will be $[3,5]$

now

(a) $f(x)$ is continuous in $[3,5]$ because it is a polynomial in x

(b) $f(x) = 1-(x-4)^{2/3}$

$$f'(x) = -\frac{2}{3}(x-4)^{-\frac{1}{3}} \quad \text{— (1)}$$

$\therefore f'(x)$ exist for all values of x in $[3,5]$ $\therefore$ $f(x)$ is differentiable in $[3,5]$ — (2)

(c) $f(3) = 1-(3-4)^{2/3}$

$$= 1-(-1)^{2/3} = 1-\left[(-1)^{\frac{1}{3}}\right]^2 = 0$$

$$f(5) = 1-(5-4)^{2/3} = 1-(+1)^{2/3} = 0$$

$\therefore f(3) = f(5)$

all the conditions of Rolle's theorem is satisfied ∴ by Rolle's theorem in between $x=3$ and $x=5$ their exist $x=c$ at which

$$f'(c)=0$$

now from eq-①

$$f'(x) = -\frac{2}{3}(x-4)^{-\frac{1}{3}}$$

$$f'(c) = -\frac{2}{3}(c-4)^{-\frac{1}{3}}$$

but $f'(c)=0$

$$\Rightarrow -\frac{2}{3}(c-4)^{-\frac{1}{3}} = 0 \quad \text{— 2}$$

$$(c-4)^{-\frac{1}{3}} = 0$$

$$\Rightarrow c-4 = 0$$

$$c = 4 \in (3,5)$$

∴ Rolle's theorem is verified.

Ans. 2(II) ~~Let~~ Given $a = \pi/3$ and

$$f(x) = \log(\cos x) \qquad f(a) = \log\left(\frac{1}{2}\right)$$

$$f'(x) = -\frac{\sin x}{\cos x} = -\tan x \qquad f'(a) = -\sqrt{3} \quad \text{— 2}$$

$$f''(x) = -\sec^2 x \qquad f''(a) = -4$$

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now by Taylor's theorem

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \cdots$$

$$\log(\cos x) = \log\left(\frac{1}{2}\right) + \left(x - \frac{\pi}{3}\right)(-\sqrt{3}) + \frac{\left(x-\frac{\pi}{3}\right)^2}{\lfloor 2}(-4) + \cdots \quad \text{— 3}$$

$$\log(\cos x) = \log\left(\frac{1}{2}\right) - \sqrt{3}\left(x - \frac{\pi}{3}\right) - 2\left(x - \frac{\pi}{3}\right)^2 + \cdots \quad \underline{\underline{\text{Ans}}}$$

Ans 2 (III) (a) Given

$f(x) = \log(1+e^x)$ $\qquad f(0) = \cancel{\log 1 + e^0}$ $= \log 2$

$$f'(x) = \frac{e^x}{1+e^x} \qquad f'(0) = \frac{1}{2} = \frac{1}{2}$$

$$f''(x) = \frac{(1+e^x)e^x - e^x \cdot e^x}{(1+e^x)^2} \quad \text{— 2}$$

$$f''(x) = \frac{e^x + \cancel{e^{2x}} - e^{2x}}{(1+e^x)^2}$$

$$f''(x) = \frac{e^x}{(1+x)^2} \qquad f''(0) = 1$$

now by Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\underline{|2}} f''(0) + \cdots$$

$$\log(1+e^x) = \log 2 + x \cdot \frac{1}{2} + \frac{x^2}{2} \cdot 1 + \cdots$$ 3

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{2} + \cdots \quad \text{Ans}$$

ns. 3(I) Given

$$u = e^{xyz} \quad \text{—— (1)}$$

on parr. diff. eq. (1) wrt. z

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot xy$$ 1

on parr. diff wrt. y

$$\frac{\partial^2 u}{\partial y \partial z} = x \cdot \frac{\partial}{\partial y}\left[\underset{I}{e^{xyz}} \cdot \underset{II}{y}\right]$$ 1

$$= x\left[e^{xyz} \cdot 1 + y \cdot e^{xyz} \cdot xz\right]$$

$$= (x + x^2yz)\, e^{xyz}$$

on again parr. diff. wrt. x

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (x + x^2yz)\, e^{xyz} \cdot yz + e^{xyz}(1 + 2xyz)$$

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$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1+3xyz+x^2y^2z^2)e^{xyz}$$ Ans — 3

Ans. 3 (II)(a) Given

$$u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$$

1 — let

$$f = \frac{x^3+y^3}{x-y} \quad \text{— (1) then}$$

$$u = \tan^{-1}(f)$$

$$\Rightarrow f = \tan u \quad \text{— (2)}$$ — 1

now from eq. (1) f is a homogeneous function in x & y of degree 3-1 = 2 ∴ by Euler's theorem

$$x \cdot \frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f \quad \text{— (3)}$$

from eq. (2)

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2\tan u$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\frac{\sin u}{\cos u}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\sin u}{\cos u}\times\frac{1}{\sec^2 u}$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\sin u}{\cancel{\cos u}}\times\cos^{\cancel{2}} u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\sin u\cdot\cos u$$

—(1)

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u \quad \text{—(3)} \quad \text{Ans}$$

Ans. 3 (II) (b)

From eq. (3) of part a.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u \quad \text{—(4)}$$

on parr. diff. eq. (4) wrt x

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\cdot 1+y\cdot\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\cdot\frac{\partial u}{\partial x}$$

on multipling both sides by x

—(3)

$$x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+xy\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\cdot x\frac{\partial u}{\partial x}$$

—(5)

on parr. diff eq. (4) wrt y

$$x\frac{\partial^2 u}{\partial x\partial y}+y\cdot\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}\cdot 1=2\cos 2u\frac{\partial u}{\partial y}$$

on multipling both sides by y

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$$xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=2\cos 2u\cdot y\frac{\partial u}{\partial y} \quad \text{---} (6)$$

adding eqs (5) and (6)

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=2\cos 2u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

putting the value from eq. (4)

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+\sin 2u=$$
$$2\cos 2u\cdot\sin 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\sin 2u=\sin 4u$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 4u-\sin 2u$$

now

$$\sin C-\sin D=2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y}=2\cos\left(\frac{4u+2u}{2}\right)\cdot$$
$$\sin\left(\frac{4u-2u}{2}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2\cos 3u\cdot\sin u$$

Ans

ns. 3 (III) Given that

$$f(x) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$p = \frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$$

$$q = \frac{\partial f}{\partial y} = 4y^3 - \cancel{4y}\, 4y + 4x$$

$$r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 4$$ 1

$$t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

Put $p = 0$ and $q = 0$

$$4x^3 - 4x + 4y = 0 \quad \text{--- (1)}$$

$$4y^3 + 4x - 4y = 0 \quad \text{--- (2)}$$

eq. (1) + eq. (2)

$$4(x^3 + y^3) + \cancel{2x} = 0$$

$$x^3 + y^3 = 0$$

$$(x+y)(x^2 - xy + y^2) = 0$$

take $x + y = 0 \Rightarrow x = -y$ --- (3)

put $y = -x$ in eq. (1)

$$4x^3 - 4x - 4x = 0$$

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$4x^3 - 8x = 0$

$4x(x^2 - 2) = 0$

$4x = 0$ or $x^2 - 2 = 0$

$\Rightarrow x = 0$ or $x = \pm\sqrt{2}$

from eq. 3 — 2

$y = 0$ or $y = \mp\sqrt{2}$

$\therefore$ stationary points are

$(0,0)$, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

Now at point $(0,0)$

$r = -4$, $s = 4$, $t = -4$

$rt - s^2 = (-4)(-4) - 16 = 0$

Case is doubtful.

at Point $(\sqrt{2}, -\sqrt{2})$

$r = 24 - 4 = 20 > 0$, $s = 4$

$t = 24 - 4 = 20$

$rt - s^2 = 20 \times 20 - 16 > 0$

$r = 20 > 0$

$\therefore$ at point $(\sqrt{2}, -\sqrt{2})$ function has minima.

(10)

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at point $(-\sqrt{2}, \sqrt{2})$

$r = 24-4 = 20, \quad s = 4, \quad t = 20$

$rt - s^2 = 20 \times 20 - 4 > 0$

$r = 20 > 0$ — (2)

$\therefore$ at point $(-\sqrt{2}, \sqrt{2})$ function has ~~max~~ minima.

ns. 4 (I) $\int_a^b x^2 dx$

we know that by the formula of limit of sum

$$\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + f(a+2h) + \cdots + f\{a+(n-1)h\}]$$ — (1½)

where at $h \to 0$ $nh = b-a$

now

$f(x) = x^2$

$f(a) = a^2, \; f(a+h) = (a+h)^2, \; ---$

$$\int_a^b f(x)dx = \lim_{h\to 0} h[a^2 + (a+h)^2 + (a+2h)^2 + \cdots + \{a+(n-1)h\}^2]$$ (½)

$$= \lim_{h\to 0} h[a^2 + a^2 + 2ha + h^2 + a^2 + 4ha + h^2 + \cdots + a^2 + 2(n-1)ha + \cdots + (n^2-2n+1)h^2]$$

$$= \lim_{h\to 0} h\left[na^2 + 2ha\{1+2+3+\dots+(n-1)\} + \dots + h^2\{1^2+2^2+3^2+\dots+(n-1)^2\}\right] \quad \text{--- (1)}$$

now we know that

$$1+2+3+\dots n^{th} = \frac{n(n+1)}{2}$$

$$1+2+3+\dots+(n-1)^{th} = \frac{(n-1)n}{2} \quad \text{--- (2)}$$

$$1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^2+2^2+3^2+\dots+(n-1)^2 = \frac{(n-1)n(2n-1)}{6} \quad \text{--- (3)}$$

putting the values from eqs (2) & (3) in eq. (1) we get

$$\Rightarrow \lim_{h\to 0} h\left[na^2 + \cancel{2}ha\frac{(n-1)n}{\cancel{2}} + \dots + \frac{h^2(n-1)n(2n-1)}{6}\right] \quad \text{--- (1)}$$

$$\Rightarrow \lim_{h\to 0}\left[(nh)a^2 + a(nh)(nh-h) + \dots + \frac{h(nh-h)(nh)(2nh-h)}{6}\right]$$

$$\Rightarrow \lim_{h\to 0}\left[(b-a)a^2 + (b-a)^2a + \frac{(b-a)(b-a)\cdot 2(b-a)}{6}\right]$$

$$\Rightarrow \left[(b-a)a^2 + (b-a)^2a + \frac{(b-a)^3}{3}\right]$$

$$\Rightarrow (b-a)\left[a^2 + (b-a)a + \frac{(b-a)^2}{3}\right]$$

(12)

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$$= \frac{(b-a)}{3}\left\{3a^2+3(b-a)a+b^2+a^2-2ba\right]$$

$$= \frac{(b-a)}{3}\left\{\cancel{3a^2}+3ba-\cancel{3a^2}+b^2+a^2-2ba\right]$$

$$\frac{(b-a)}{3}\left(b^2+ab+a^2\right)$$

$$\Rightarrow \frac{(b-a)(b^2+ab+a^2)}{3}$$

$$\Rightarrow \frac{b^3-a^3}{3} \quad \underline{\text{Ans}}$$

ns. 4 (II) Let

$$f(n) = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)+\cdots+\left(1+\frac{m}{n}\right)\right]^{\frac{1}{n}}$$

$$\log f(n) = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+ \cdots + \log\left(1+\frac{n}{n}\right)\right]$$

$$\Rightarrow \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \int_a^b \log(1+x)\,dx$$

where $\frac{1}{n} = dx,\ \frac{r}{n} = x$

$$a = \lim_{n\to\infty}\frac{r}{n} \Rightarrow \lim_{n\to\infty}\frac{1}{n} = 0$$

$$b = \lim_{n\to\infty}\frac{r}{n} \Rightarrow \lim_{n\to\infty}\frac{n}{n} = 1$$

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$$= \int_0^1 \log(1+x)\, dx$$

$$\Rightarrow \int_0^1 \underset{I}{\log(1+x)} \times \underset{II}{1}\, dx$$

$$\Rightarrow \left[\log(1+x) \cdot x\right]_0^1 - \int_0^1 \frac{1}{1+x} \times x\, dx \qquad \text{— } 1$$

$$\Rightarrow \log 2 - \int_0^1 \left[\frac{x+1-1}{1+x}\right] dx$$

$$\Rightarrow \log 2 - \int_0^1 \left[1 - \frac{1}{1+x}\right] dx$$

$$\Rightarrow \log 2 - \left[x - \log(1+x)\right]_0^1$$

$$\Rightarrow \log 2 - \left[1 - \log 2\right]$$

$$\Rightarrow 2\log 2 - 1$$

$$\Rightarrow \log 4 - 1 \qquad \text{— } 3$$

$$\log f(n) = \log 4 - 1$$

$$\log f(n) = \log 4 - \log 10$$

$$\log f(n) = \log\left(\frac{4}{10}\right)$$

$$f(n) = \frac{4}{10} \quad \text{Ans}$$

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Ans. 4 (III) we know that by the definition of beta function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$\beta(m,m) = \int_0^1 x^{m-1}(1-x)^{m-1}\,dx \quad (n=m)$$

$$\beta(m,m) = \int_0^1 \{x(1-x)\}^{m-1}\,dx$$

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

at $x=0 \quad \theta = 0$

$x = 1 \quad \theta = \pi/2$ — (2)

$$\beta(m,m) = \int_0^{\pi/2} \{\sin^2\theta(1-\sin^2\theta)\}^{m-1}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$\beta(m,m) = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-2}\cdot(\sin\theta\cos\theta)\,d\theta$$

$$\beta(m,m) = 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\,d\theta$$

$$\beta(m,m) = 2\int_0^{\pi/2} \left(\frac{2\sin\theta\cos\theta}{2}\right)^{2m-1} d\theta$$

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$$\beta(m,m) = \frac{2}{2^{2m-1}} \int_0^{\pi/2} (\sin 2\theta)^{2m-1} \, d\theta$$

put $2\theta = t$

$\theta = t/2$

$d\theta = dt/2$

at $\theta = 0 \quad t = 0$

$\theta = \frac{\pi}{2} \quad t = \pi$

$$\beta(m,m) = \frac{2}{2^{2m-1}} \int_0^{\pi} \sin^{2m-1} t \, \frac{dt}{2}$$

by the property of definite - integral

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \quad \text{if}$$

$$f(2a-x) = f(x)$$

$$\beta(m,m) = \frac{2}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1} t \, dt$$

$$\beta(m,m) = \frac{2}{2^{2m-1}} \int_0^{\pi/2} \sin^{2m-1} t \cdot \cos^{0} t \, dt$$

on appling gamma function

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$$\beta(m,m) = \frac{\cancel{2}}{2^{2m-1}} \times \frac{\Gamma\frac{2m-1+1}{2}\,\Gamma\frac{0+1}{2}}{\cancel{2}\,\Gamma\frac{2m-1+0+2}{2}}$$

we know that

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n} \quad \text{--- (3)}$$

$$\frac{\Gamma m\,\cancel{\Gamma m}}{\Gamma m+m} = \frac{1}{2^{2m-1}}\,\frac{\cancel{\Gamma m}\cdot\sqrt{\pi}}{\Gamma m+\frac{1}{2}}$$

$$\Gamma m\,\Gamma m+\tfrac{1}{2} = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma 2m \qquad \underline{\text{Ans}}$$

Ans. 5 (I) Let

$$I=\int_0^1\int_0^{1-y}\int_0^{1-x-y} xyz\,dz\,dy\,dx$$

$$I=\int_0^1\int_0^{1-y}\int_0^{1-x-y} xyz\,dy\,dx\,dz$$

$$I=\int_0^1\int_0^{1-y} xy\left[\frac{z^2}{2}\right]_0^{1-x-y} dy\,dx$$

$$I=\frac{1}{2}\int_0^1\int_0^{1-y} xy(1-x-y)^2\,dy\,dx \quad (2)$$

$$I=\frac{1}{2}\int_0^1\int_0^{1-y} xy\left(1+x^2+y^2-2x+2xy-2y\right)dy\,dx$$

$$I=\frac{1}{2}\int_0^1\int_0^{1-y} y\left(x+x^3+xy^2-2x^2+2x^2y-2xy\right)dy\,dx$$

$$I=\frac{1}{2}\int_0^1\int_0^{1-y} y\left[(1-2y+y^2)x-2(1-y)x^2+x^3\right]dy\,dx$$

$$I=\int_0^1 y\left[(1-y)^2\frac{x^2}{2}-2(1-y)\left(\frac{x^3}{3}\right)+\frac{x^4}{4}\right]_0^{1-y} dy$$

$$I=\int_0^1 y\left[\frac{(1-y)^4}{2}-\frac{2(1-y)^4}{3}+\frac{(1-y)^4}{4}\right]dy$$

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$$I = \frac{1}{2}\int_0^1 y\left[\frac{6(1-y)^4 - 8(1-y)^4 + 3(1-y)^4}{12}\right] dy$$

$$I = \frac{1}{24}\int_0^1 y(1-y)^4\, dy \quad \leftarrow (1)$$

$$I = \frac{1}{24}\int_0^1 y\left[{}^4C_0\, 1 + {}^4C_1(-y) + {}^4C_2(-y)^2 + {}^4C_3(-y)^3 + {}^4C_4(-y)^4\right] dy$$

$$I = \frac{1}{24}\int_0^1 y\left[1 - 4y + 6y^2 - 4y^3 + y^4\right] dy$$

$$I = \frac{1}{24}\int_0^1 \left[y - 4y^2 + 6y^3 - 4y^4 + y^5\right] dy$$

$$I = \frac{1}{24}\left[\frac{y^2}{2} - \frac{4y^3}{3} + \frac{6y^4}{4} - \frac{4y^5}{5} + \frac{y^6}{6}\right]_0^1$$

$$I = \frac{1}{24}\left[\frac{1}{2} - \frac{4}{3} + \frac{\cancel{6}^{3}}{\cancel{4}_{2}} - \frac{4}{5} + \frac{1}{6}\right] \quad \leftarrow (2)$$

$$I = \frac{1}{24} \times \frac{11}{30}$$

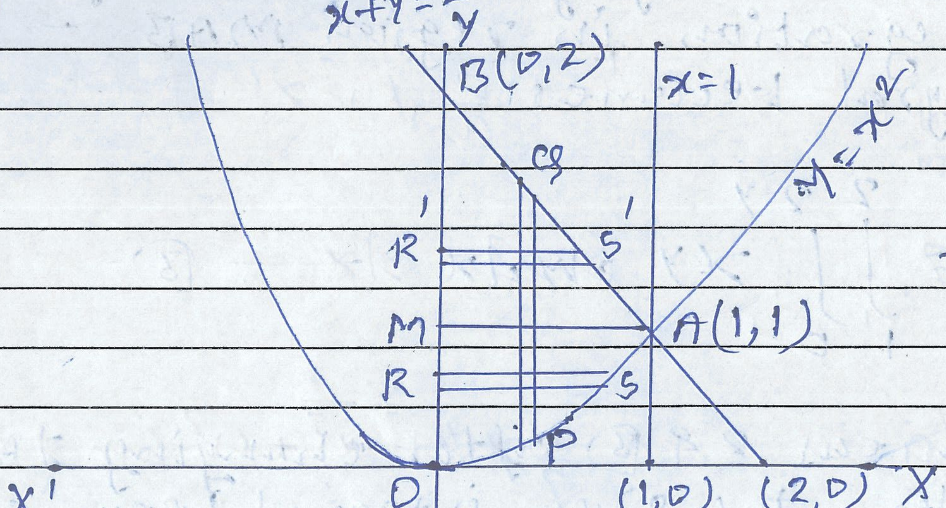
$$I = \frac{11}{720}$$

Ans

Ans 5 (II) Given

$$I = \int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy \quad \text{---} (1)$$

First we determine region of integration as the integration is first perform w.r.t. y ∴ we consider a thin strip parallel to y axis let it be PQ.
Lower limit of y is $y = x^2$ ∴ lower end P lies on parabola $y = x^2$.
Upper limit of y is $y = 2-x$ which is the equation of line ∴ upper end Q lies on line $y = 2-x$ or $x+y = 2$.
This line cuts x-axis at (2,0) and y-axis at (0,2). Lower lim. of x is $x = 0$ which means origin and upper limit of x is $x = 1$ which is the line parallel to y-axis. ∴ region of integration is shown in fig.



— (1)

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from the fig. the region of integration is OABO it is divided into two parts OAM and MAB.

In OAM consider another strip RS in this region the limits of integration wrt x are $x=0$ to $x=\sqrt{y}$ and limits of integration wrt. y are $y=0$ to $y=1$ on changing the order in region OAM integral becomes

$$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy \, dy \, dx \quad \text{—— (2)}$$

now in region MAB consider another strip R'S' the limits of integration in this region wrt. x are $x=0$ to $x=2-y$ and limits of integration wrt y are $y=1$ to $y=2$

∴ after changing the order of integration in region MAB Integral becomes —— (2)

$$I_2 = \int_1^2 \int_0^{2-y} xy \, dy \, dx \quad \text{—— (3)}$$

from eqs 2 & (3) after changing the order the given integral becomes

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$$\int_0^1\int_{x^2}^{2-y} xy\,dx\,dy = \int_0^1\int_0^{\sqrt{y}} xy\,dy\,dx + \int_1^2\int_0^{2-y} xy\,dy\,dx$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \int_0^1 \frac{y}{2}\cdot y\,dy + \int_1^2 \frac{y}{2}(2-y)^2\,dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 y(4-4y+y^2)\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2\left[4y-4y^2+y^3\right]dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left\{8 - \frac{32}{3} + \frac{16}{4} - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right\}$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{3}{8} \quad \text{Ans}$$

2

Ans 5 (III) eq. of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=\frac{b^2}{a^2}(a^2-x^2)$$

$$y=\pm\frac{b}{a}\sqrt{(a^2-x^2)} \quad \text{—— (1)}$$

limits wrt x are from $x=-a$ to $x=a$

$$\text{Area}=\int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dx\,dy$$

$$=\int_{-a}^{a}[y]_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}}\,dx$$

$$=\int_{-a}^{a}\left[\frac{b}{a}\sqrt{a^2-x^2}+\frac{b}{a}\sqrt{a^2-x^2}\right]dx$$

$$=\frac{2b}{a}\int_{-a}^{a}\sqrt{a^2-x^2}\,dx \quad \text{—— (2)}$$

$$\int\sqrt{a^2-x^2}\,dx=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \frac{4b}{a}\int_{0}^{a}\sqrt{a^2-x^2}\,dx$$

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$$= \frac{4b}{a}\left[\frac{1}{2}x\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{2b}{a}\left\{0+a^2\sin^{-1}\left(\frac{a}{a}\right)\right]$$

$$= \frac{2b}{a}\times a^2\cdot\frac{\pi}{2}$$ — (2)

$$= \pi ab$$ **Ans**

Ans. 6 (I) Given

$$y = e^x(A\cos x + B\sin x) \quad \text{— (1)}$$

on diff. wrt x

$$\frac{dy}{dx} = e^x(-A\sin x + B\cos x) + (A\cos x + B\sin x)e^x$$

$$\frac{dy}{dx} = e^x(-A\sin x + B\cos x) + y \quad (2)$$ — (2)

on again diff wrt. x

$$\frac{d^2y}{dx^2} = e^x(-A\cos x - B\sin x) + (-A\sin x + B\cos x)e^x + \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = -e^x(+A\cos x + B\sin x) + \frac{dy}{dx} - y + \frac{dy}{dx}$$

(from eq. (2))

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from eq. (1)

$$\frac{d^2y}{dx^2} = -y + 2\frac{dy}{dx} - y$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0 \quad \underline{\text{Ans}}$$ — (3)

ns 6 (II) Given

$$(1+e^{\frac{x}{y}})dx + e^{\frac{x}{y}}(1-\frac{x}{y})dy = 0 \quad \text{— (1)}$$

$$Mdx + Ndy = 0$$

$$M = 1+e^{x/y}$$

on parr. diff. wrt. y

$$\frac{\partial M}{\partial y} = 0 + e^{x/y}\left(-\frac{x}{y^2}\right)$$

$$= -\frac{xe^{x/y}}{y^2}$$ — (1)

$$N = e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$$

on parr. diff wrt. x

$$\frac{\partial N}{\partial x} = e^{x/y}\cdot\left(-\frac{1}{y}\right) + \left(1-\frac{x}{y}\right)e^{x/y}\cdot\frac{1}{y}$$

$$= -e^{x/y}\cancel{\left(-\frac{1}{y}\right)} + \cancel{e^{x/y}\cdot\frac{1}{y}} - \frac{xe^{x/y}}{y^2}$$

$$\frac{\partial N}{\partial x} = -\frac{xe^{x/y}}{y^2}$$ — (1)

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$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ — 3

$\therefore$ eq. ~~be~~ is exact diff^{tl} eq.
and it's solution is

$\int M\,dx$ (taking y const.) $+ \int$ (Terms of N not contained x) $dy = c$ — 3

$\int (1+e^{x/y})\,dx + \int 0\,dy = c$

$x + ye^{x/y} = c$ Ans

Ans. 6(III) Given

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \cdot \sin x$$

put $\frac{d}{dx} = D$

$$(D^2 - 2D + 1)y = e^x \cdot \sin x$$

$$(D-1)^2 y = e^x \cdot \sin x$$ — 1

auxiliary eq. will be

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$CF = (c_1 + c_2 x)e^x$ — 1

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$$P.I. = \frac{1}{f(D)} \cdot e g$$

$$= \frac{1}{(D-1)^2} \cdot e^x \cdot \sin x$$

$$= e^x \cdot \frac{1}{(D+1-1)^2} \sin x$$

$$= e^x \cdot \frac{1}{D^2} \sin x$$

$$= e^x \cdot \frac{1}{-1} \sin x$$

$$P.I. = -e^x \cdot \sin x$$

3

and solution of eq. will be

$$y = CF + P.I$$

$$y = (c_1 + c_2 x) e^x - e^x \sin x \quad \underline{Ans}$$