Total No. of Questions: 6 *Total No. of Printed Pages:3*

Enrollment No......................................

Faculty of Science

End Sem (Even) Examination May-2022

MA5CO07 Real Analysis -II

Programme: M.Sc. Branch/Specialisation: Mathematics

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. Let $\{A_n\}$ be a collection of sets of real number and m* is outer measure, then- **1**

(a) $m^*(\cup A_n) \geq \sum m^*(A_n)$

(b) $m^*(\cup A_n) \leq \sum m^*(A_n)$

(c) $m^*(\cup A_n) = \sum m^*(A_n)$

(d) None of these

ii. If A is countable set, then m*(A) is equal to- **1**

(a) 0 (b) 1 (c) $\emptyset$ (d) 2

iii. If c is a constant and f is measurable real valued function then- **1**

(a) f+c is measurable function

(b) f+c is non-measurable function

(c) c.f is not measurable function

(d) None of these

iv. If f ’(x) = max{f(x) , 0 } and f ’(x) = min{-f(x) , 0 } then- **1**

(a) $|f| = f^+ + f^-$

(b) $|f| = f^+ - f^-$

(c) $|f| = f^+$

(d) $|f| = f^-$

v. If f and g is measurable function and f ≤ g almost everywhere then- **1**

(a) $\int_E f = \int_E g$

(b) $\int_E f > \int_E g$

(c) $\int_E f \leq \int_E g$

(d) None of these

P.T.O.

vi. A non-negative measurable function f is integrable over the measurable set E, if- **1**

(a) $\int_E f = \infty$

(b) $\int_E f > \infty$

(c) $\int_E f < \infty$

(d) None of these

vii. Let f be monotonic and bounded on [a, b] and f is of bounded variation on [a, b], then- **1**

(a) V(f) = f(b) – f(a)

(b) V(f) = f(b) + f(a)

(c) V(f) = f(b) /f(a)

(d) None of these

viii. If f and g are square integrable in the lebesgue sense, then- **1**

(a) $\|f+g\|_2 \geq \|f\|_2 + \|g\|_2$

(b) $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$

(c) $\|f+g\|_2 = \|f\|_2 . \|g\|_2$

(d) None of these

ix. If $L^\infty[0,1]$ is the class of all measurable function on [0, 1]. Then $L^\infty[0,1]$ is- **1**

(a) Not complete (b) Unbounded

(c) Complete (d) None of these

x. If f is abounded measurable function on [a, b], then for a given $\in > 0, \exists\ a\ continous\ function\ on\ [a,b]$ such that- **1**

(a) $\|f-g\|_2 < \in$

(b) $\|f-g\|_2 > \in$

(c) $\|f-g\|_2 \geq \in$

(d) None of these

Q.2 Attempt any two:

i. Prove that the outer measure of an interval is its length. **5**

ii. Let A be any set, and $E_1, E_2, \ldots, E_n$ be a finite sequence of disjoint measurable sets. Then show that- **5**

$$m^*(A \cap [\cup_{i=1}^{n} E_i]) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

iii. If E_1 and E_2 are any measurable sets, then prove that- **5**

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$$

Q.3 Attempt any two:

i. Let f_i, i =1,2,3……., n be measurable functions defined on a measurable set E, then prove that $\max\{f_1, f_2, \ldots, f_n\}$ and $\min\{f_1, f_2, \ldots, f_n\}$ are measurable. **5**

ii. If a measurable function defined on a measurable set E if f and g are equivalent (equal a. e.) functions, then g is a measurable function on E. **5**

iii. Prove that a continuous function defined on measurable set is measurable function. **5**

Q.4 Attempt any two:

i. Let f be a bounded function defined on [a, b]. if f is Riemann Integrable on [a, b], then prove that it is Lebesgue Integrable on [a , b] and **5**

$$R\int_a^b f(x)dx = \int_a^b f(x)dx$$

ii. State and prove Fatou's Lemma. **5**

iii. Find the four derivatives of the function $f: R \to R$ defined by $f(x) = |x|\ \forall\ x \in R$. **5**

Q.5 Attempt any two:

i. Prove that a bounded monotone function is a function of bounded variation; that is, a monotone function on [a, b] has finite variation. **5**

ii. State and prove Jensen's Inequality. **5**

iii. State and prove Minkowski's Inequality. **5**

Q.6 Attempt any two:

i. Prove that the L^p spaces are complete. **5**

ii. Let $\emptyset$ have a second derivative at each point of (a, b), then prove that $\emptyset$ is convex on (a, b) if and only if $\emptyset''(x) \geq 0\ for\ each\ x \in (a,b)$. **5**

iii. If f is a bounded measurable function on [a, b] then prove that for a given $\in > 0, then\ \exists\ a\ continouos\ function\ g\ on\ [a,b]$ such that- **5**

$$\|f-g\|_2 < \in$$

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Medicaps University Indore
End sem Exam 2022 (May)

10/6/2022 exam date

Date :
P. No. : 1

M.Sc. Real Analysis - II (Mathematics)

(MASC0 07) Solution by scheme

		Marks
Q1	MCQs	
1	Ans (b) $m^*(\bigcup A_n) \le \sum m^*(A)$	1
2	Ans (a) 0	1
3	Ans (a) $f+c$	1
4	Ans (a) $\lvert f\rvert = f^+ + f^-$	1
5	Ans (c) $\int_E f \le \int_E g$	1
6	Ans. (c) (c) $\int_E f < \infty$	1
7	Ans. (a) (a) $V(f) = f(b) - f(a)$	1
8	Ans. (b) $\lVert f+g\rVert_2 \le \lVert f\rVert_2 + \lVert g\rVert_2$	1
9	Ans. (C) Complete	1
10	Ans. (a) $\lVert f-g\rVert_2 < \varepsilon$	1

Q 2(i)

2 (i) Case I:-

Suppose I is a closed finite interval say $[a,b]$.

For given $\varepsilon>0$, we see that open interval $\left(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2}\right)$ contains $[a,b]$. 1

So we have $m^*(I) \le l\left(\left(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2}\right)\right)$

$= b-a+\varepsilon$

Since this being true for each $\varepsilon>0$ so on letting $\varepsilon \to 0$ we must have $m^*(I) \le b-a = l(I)$

Thus In order to complete the proof

we need to show that
$$m^*(I) \geq b-a \quad \text{——— (1)}$$
By definition of outer measure we have $m^*(I) = \inf \sum_i l(I_i)$

where the infimum is taken over all countable collections $\{I_i\}$ of open intervals such that $I \subset \bigcup_i I_i$

Let $\varepsilon > 0$ be given
$$m^*(I) \geq \sum_{n=1}^{\infty} l(I_n) - \varepsilon \quad \text{——— (2)}$$

1

$$\sum_n l(I_n) \geq \sum_{i=1}^{k} l(a_i, b_i)$$
$$= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \cdots \cdots + \cdots \cdots + (b_1 - a_1)$$
$$\geq b_k - a_1 \qquad \left\{ \because a_i < b_{i-1} \ \forall i \right.$$
$$> b - a \qquad \left. \because b_k > b \text{ and } a_1 < a \right\}$$

So $m^*(I) > b - a - \varepsilon$

so as letting $\varepsilon \to 0$ we must have
$$m^*(I) \geq b-a \quad \text{——— (3)}$$
Hence from (1) and (2), we get
$$m^*(I) = b-a$$

Case 2:- Suppose I is any finite interval:-

1

Then given $\varepsilon > 0$ there is a closed interval $J \subset I$ such that

$l(J) > l(\bar{I}) - \varepsilon$

Therefore $l(\bar{I}) - \varepsilon < l(J)$

$= m^*(J)$

$\le m^*(I) \le m^*(\bar{I})$

$= l(\bar{I}) = l(I)$.

$\Rightarrow l(\bar{I}) - \varepsilon < m^*(I) \le l(I)$

This is true for each $\varepsilon > 0$ Hence $m^*(I) = l(I)$ 2

Case 3:- Suppose I is an infinite interval

Then given any real number $K > 0$

there exists a closed finite interval 2

$J \subset I$ such that $l(J) = K$

Thus, $m^*(I) \ge m^*(J) = l(J) = K$;

$m^*(I) \ge K$ for any arbitrary real

number $K > 0$ Hence

$m^*(I) = \infty = l(I)$ Q2(i)

Sol: (ii) We prove the theorem by induction on n

Sol (ii) For $n=1$

the result is trivially true

For $n=2$

we have $m^*\left(A \cap \left[\bigcup_{i=1}^{2} E_i\right]\right) = m^*(A \cap [E_1 \cap E_2])$

$= m^*((A \cap E_1) \cup (A \cap E_2))$ 1

$= m^*(A \cap E_1) + m(A \cap E_2)$

$= \sum_{i=1}^{2} m^*(A \cap E_i)$

Hence theorem is true for $n=2$

Assume that the result is true for $n-1$

sets E_i, we would have

$$m^*\left(A\cap\left[\bigcup_{i=1}^{n-1}E_i\right]\right)=\sum_{\substack{n=1\\i=1}}^{n-1}m^*(A\cap E_i)$$

Adding $m^*(A\cap E_n)$ on both sides we get

$$m^*\left[A\cap\left[\bigcup_{i=1}^{n-1}E_i\right]\right)\cup m^*(A\cap E_n)=$$

$$\sum_{i=1}^{n}m^*(A\cap E_i) \quad \text{———} \quad (1)$$

Since E_i are disjoint sets, we have 2

$$A\cap\left[\bigcup_{i=1}^{n}E_i\right)\cap E_n = A\cap E_n \quad \text{———} \quad (2)$$

and

$$A\cap\left[\bigcup_{i=1}^{n}E_i\right]\cap E_n^c=\left(A\cap\left[\bigcup_{i=1}^{n}E_i\right]\cap E_n^c\right)$$

$$\cup\,(A\cap E_n\cap E_n^c)$$

$$=\left(A\cap\left(\bigcup_{i=1}^{n-1}E_i\right]\right)\cup(A\cap\phi)$$

$$=\left(A\cap\left[\bigcup_{i=1}^{n-1}E_i\right]\right)\cup\phi$$

2

$$=A\cap\left[\bigcup_{i=1}^{n-1}E_i\right] \quad \text{———} \quad (3)$$

from (1), (2) and (3) eq. gives

$$m^*\left(A\cap\left[\bigcup_{i=1}^{n}E_i\right]\cap E_n^c\right)+m^*\left(A\cap\left[\bigcup_{i=1}^{n}E_i\right]\cap E_n\right.$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i) \text{ --- } (4)$$

Hence

$$m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(A \cap E_i)$$ Ans 2(ii)

2(iii)

Ans 2(iii) Let A be any set. Since E_1 is a measurable set,

we have $m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$ --- (1)

Clearly, the sets $E_1 \cup E_2$, $[E_1 \cup E_2] \cap E_1$ and $[E_1 \cup E_2] \cap E_1^c$ are measurable. 1

So on taking $A = E_1 \cup E_2$ in eq (1) we obtain

$$m(E_1 \cup E_2) = m([E_1 \cup E_2] \cap E_1) + m([E_1 \cup E_2] \cap E_1^c)$$ 2

$\Rightarrow m(E_1 \cup E_2) + m(E_1 \cap E_2) =$

$m([E_1 \cup E_2] \cup E_1) + m((E_1 \cup E_2) \cap E_1^c) + m(E_1 \cap E_2)$ 1

on adding $m(E_1 \cap E_2)$ both sides

$\Rightarrow m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m([E_1 \cap E_2] \cap E_1^c) + m(E_1 \cap E_2)$ --- (2) 1

$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

Q3
(i)
3(i) Proof :-

For each $x \in E$, let us define
$f^*(x) = \max\{f_1, f_2, f_3, \dots f_n\}$

$= \max\{f_1(x), f_2(x), f_3(x), \dots f_n(x)\}$

and $f_*(x) = \min\{f_1(x), f_2(x), \dots f_n(x)\}$ 1

Then $f_*(x)$ can be defined as follows

$f^*(x) = -\max\{-f_1(x), -f_2(x), \dots -f_n(x)\}$

Thus if we show that $f^*(x)$ is measurable on E, the result will follow

To prove this, it is enough to prove that $E(f^* > a)$ is a measurable set for every real number a

f_i is a measurable function on E 2

$\Rightarrow E(f_i > a)$ is a measurable set $\forall i = 1, 2, \dots n$

Evidently $E(f^* > a) = \bigcup_{i=1}^{n} E(f_i > a)$

= a finite union of measurable

sets

= a measurable set

$E(f^* > a)$ is a measurable set by definition. 2

This proves that $E(f^* > a)$ is measurable

Hence

$\max\{f_1, f_2, \dots f_n\}$ and

$\min\{f_1, f_2, \dots f_n\}$ are measurable.

A 3(i)

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ii) If f is a measurable function defined on a measurable set E and if f and g are equivalent (equal a.e.) functions, then to show that g is a measurable function on E.

Let a be any real number

Since f and g are equivalent function

i.e. $f \equiv g$, almost everywhere on E

therefore by definition

$$m[E(f \neq g)] = 0$$ 1

Let $E(f \neq g) = A$ and $E(f = g) = B$

$\Rightarrow B = A^c$, where A^c is the complement of A relative to E. Also $E = A \cup B$ and $A \cap B = \phi$

Again $B = A^c$ is a measurable set as 1

A is measurable set.
f is measurable over E ⇒
f is measurable over $B \subset E$
⇒ $B(f>a)$ is a measurable set. 1

But $f=g$ on B, $B(f>a) = B(g>a)$. Thus
⇒ $B(g>a)$ is a measurable set.
⇒ f is a measurable over E
Now $E(g>a) = [B(g>a)] \cup [A(g>a)]$ 1

= Union of two measurable sets;
for $A(g>a) \subset A$ and $m(A)=0$
and also every subset of measure zero is measurable.

Thus $E(g>a)$ is a measurable set 1

Hence function g is measurable on E where f and g are equivalent.

Ans 3(ii)

3(iii)

Ans 3(iii) It is sufficient to show that $E(f \geq a)$ is measurable.
Let f be a continous function defined on a measurable set E.
Let a be any real number.
We now claim that $E(f \geq a)$ is closed
Let $A = E(f \geq a)$ ——— (1) 1
To prove that A is closed, it is just

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sufficient to show that $D(A) \subset A$ — ②
$D(A)$ being derived set of A
Let $\alpha \in D(A)$ be arbitrary. Then
$\alpha \in D(A)$
$\Rightarrow \alpha \in D(A) \Rightarrow \alpha$ is a limit point of A
$\Rightarrow \exists$ a sequence $\{x_n\}$ whose elements
$x_n \in A$ such that $\lim_{n \to \infty} x_n = \alpha$

Moreover, f is continuous at α
it follows by Heine's definition of continuity that $x_n \to \alpha$
$\Rightarrow f(x_n) \to f(\alpha)$ ——— ③ 2
By eq ①, we see that
$x_n \in A \quad \forall n \in N$
$\Rightarrow f(x_n) \geq a \quad \forall n \in N$
$\Rightarrow \lim_{n \to \infty} f(x_n) \geq a$

$\Rightarrow f(\alpha) \geq a$ by eq ③
$\Rightarrow \alpha \in A$, by eq ①
Further, any $\alpha \in D(A) \Rightarrow \alpha \in A$
$\Rightarrow D(A) \subset A$
$\Rightarrow A$ is closed
$\Rightarrow A$ is measurable
$\Rightarrow E(f \geq a)$ is measurable
Hence f is a measurable function on the set E. Ans 3(iii) 2

Q4(i)

Q4(i) Since f is Riemann integrable over $[a,b]$

we have

$$\lim_{\psi_1 \geq f} \int_a^b \psi(x)\,dx = \sup_{\phi_1 \leq f} \int_a^b \phi_1(x)\,dx = R\int_a^b f(x)\,dx$$ 1

where

ϕ are step functions on $[a,b]$

ψ are step functions on $[a,b]$

But every step function is a Simple function

Therefore

$$\sup_{\phi_1 \leq f} \int_a^b \phi_1(x)\,dx \leq \sup_{\phi \leq f} \int_a^b \phi(x)\,dx$$

$$\inf_{\psi_1 \geq f} \int_a^b \psi_1(x)\,dx \geq \inf_{\psi_1 \geq f} \int_a^b \psi(x)\,dx$$ 2

where ϕ and ψ vary over all Simple functions defined on $[a,b]$

Thus, in view of the above relations we have

$$R\int_a^b f(x)\,dx \leq \sup_{\phi \leq f} \int_a^b \phi(x)\,dx \leq$$

$$\inf_{\psi \geq f} \int_a^b \psi(x)\,dx \leq$$

$$R\int_a^b f(x)\,dx$$

B 2

$$\Rightarrow \sup_{\phi \le f} \int_a^b \phi(x)\,dx = \inf_{\psi \ge f} \int_a^b \psi(x)\,dx$$

$$= R\int_a^b f(x)\,dx$$

$$\Rightarrow \int_a^b f(x)\,dx = R\int_a^b f(x)\,dx$$

Hence proved

$$R\int_a^b f(x) = \int_a^b f(x)\,dx \quad [R = R]$$ 2

Ans 4(i). Riemann Integral

4(ii)

Sol/4(ii) Statement:-
Let $\{f_n\}$ be a sequence of non negative measurable functions and
$f_n \to f$ a.e. on E 2

Then $\int_E f \le \underline{\lim}_{n\to\infty} \int_E f_n$

Proof:- Without loss of generality we may assume that the sequence $\{f_n\}$ converges to f everywhere on E since the integrals over the sets of measure zero are zero.
Let h be a bounded measurable function such that $h \le f$ and

which vanishes outside a set E' of finite measure, viz

$E' = m(\{x \in E : h(x) \neq 0\}) < \infty$

Define a sequence $\{h\}$ of functions by setting

$$h_n(x) = \min\{h(x), f_n(x)\}$$ 1

Then it is clear that each h_n is bounded by the bounds of h and vanishes outside E'.

Moreover, we note that

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \min\{h(x), f_n(x)\}$$

$$= h(x), \quad x \in E'$$

Thus, $\{h_n\}$ is a uniformly bounded sequence of measurable functions such that $h_n \leq f \Rightarrow h_n \leq h$ on E' 1

Therefore, by the Bounded convergence theorem, by we have

$$\int_{E'} (h) = \int_{E'} (h) \text{ over } E' = \lim_{n\to\infty} \int_{E'} h_n$$

$$\Rightarrow \int_E h = \int_{E'} h = \lim_{n\to\infty} \int_{E'} h_n \leq \lim_{n\to\infty} \int_E f_n$$

Hence taking the sup overall $h \leq f$

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we get

$$\int_E f \leq \underline{\lim}_{n \to \infty} \int_E f_n$$ Hence proved

Ans 4(ii)

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4 (iii)

sol 4(iii) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$ $\forall x \in \mathbb{R}$

where $\mathbb{R}$ is real numbers set

we know that

$$D^+ f(x) = \overline{\lim}_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \quad \text{(upper right d)} \quad \text{---} (1)$$

$$D_+ f(x) = \underline{\lim}_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \quad \text{(lower right d)} \quad \text{---} (2)$$

$$D^- f(x) = \overline{\lim}_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \quad \text{(upper left d)} \quad \text{---} (3)$$

$$D_- f(x) = \underline{\lim}_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \quad \text{(lower left d)} \quad \text{---} (4)$$

Now $\because$ $|x| = +x \quad x > 0$

$= 0 \quad x = 0$

$= -x \quad x < 0.$

$-x \quad 0 \quad +x$

Now from (1) and (2)

$$D^+ f(0) = D_+ f(0) = 1$$

Since $D^+ f(0) = \overline{\lim}_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$

R

$$\Rightarrow D^+f(0) = \overline{\lim_{h\to 0^+}} \frac{f(h)-0}{h}$$

$$= \overline{\lim_{h\to 0^+}} \frac{|h|}{h}$$

$$= \overline{\lim_{h\to 0^+}} \frac{\cancel{h}}{\cancel{h}}$$

$$= \overline{\lim_{h\to 0^+}} (1) = 1$$

$$\Rightarrow \boxed{D^+f(0) = 1} \quad \text{———} \quad (5)$$

$$\text{and } D_+ f(0) = \underline{\lim_{h\to 0^+}} \left(\frac{f(h)}{h}\right) = 1$$

$$\boxed{D_+ f(0) = 1} \quad \text{———} \quad (6)$$

from (5) and (6)

$$D^+f(0) = D_+ f(0) = 1$$

2

Now

$$D^- f(0) = \overline{\lim_{h\to 0^-}} \frac{f(0+h)-f(0)}{h}$$

$$= \overline{\lim_{h\to 0^+}} \left[\frac{f(0-h)-f(0)}{-h}\right]$$

$$= \overline{\lim_{h\to 0^+}} \left[\frac{f(h)-0}{-h}\right]$$

$$= \overline{\lim_{h\to 0}} \left(\frac{|-h|}{-h}\right) = \overline{\lim_{h\to 0}} \left(\frac{|h|}{-h}\right) = \overline{\lim_{h\to 0}} \left(\frac{h}{-h}\right) = -1$$

$$\boxed{D^- f(0) = (-1) = -1}$$

and $D_- f(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0^+} \frac{f(0-h) - f(0)}{-h}$

$= \lim_{h \to 0} \left[\frac{|-h| - 0}{-h} \right] = -1$

So $D^- f(0) = D_- f(0) = -1$ 2

So $D^+ f(0) = D_+ f(0) = 1$ —— (7)

$D_- f(0) = D^- f(0) = -1$ —— (8)

from (7) and (8)

Both are different

It is not common to all $f'(0)$

So It is not differentiable at $x = 0$

and

$$\left.\begin{aligned} D^+ f(0) &= 1 \\ D_+ f(0) &= 1 \\ D^- f(0) &= -1 \\ D_- f(0) &= -1 \end{aligned}\right\} \text{Ay } 4(iii)$$

Q5(i)

sol5(i) Let f be monotonic and bounded on $[a, b]$. Then it show that f is of bounded variation on $[a, b]$ and $f(b) - f(a) = V(f)$.

We know that the definition of bounded variation $V_a^b (f, P)$ or $V(f)$ of a function f defined on $[a, b]$ with respect to a partition P of $[a, b]$

is given by
$$V_a^b(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

It is clear that f is bounded variation on $[a,b]$ if and only if $T_a^b(f) < \infty$ —— (1)
$\Rightarrow f \in BV$. 1

Now
Let f be a increasing function defined on the closed interval $[a,b]$ so that
$f(x_r) \leq f(x_{r+1})$
whenever $x_r < x_{r+1}$.
Let $P = \{a = x_0, x_1, x_2, \ldots \ldots x_n = b\}$
be a partition of closed interval $[a,b]$
i.e. $a = x_0 < x_1 < x_2 < x_3 < x_4 < \ldots \ldots < x_n = b$

Now $V_a^b(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$

$$= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$ 2

since $f(x_i) - f(x_{i-1}) \geq 0$

$= f(x_n) - f(x_0)$
$= f(b) - f(a) =$ a finite number
since f is monotone
$\Rightarrow f(b)$ and $f(a)$ are finite number

Thus $T_a^b(f) = \sup_{P \in \mathcal{P}[a,b]} V_a^b(f,P) =$ a finite number $< \infty$

Hence f is of bounded variation proved 2

As 5(i)

4

5(ii)

sol 5(ii) Statement:-

Let ϕ be a convex function on $(-\infty, \infty)$ and f be an integrable function on $[0,1]$. Then

$$\int_0^1 \phi(f(t))\,dt \geq \phi\left[\int_0^1 f(t)\,dt\right]$$ 2

Proof:-

Let $\alpha = \int_0^1 f(t)\,dt$

let $y = m(x-\alpha) + \phi(\alpha)$ be the equation of a supporting line at α. —— (1)

Then by very definition of supporting line, we have for any $t \in [0,1]$

$$\phi(f(t)) \geq m(f(t)-\alpha) + \phi(\alpha). \quad \text{—— (2)}$$ 2

Integrating both sides of eq (2) with respect to t over $[0,1]$,

we get

$$\int_0^1 \phi(f(t))\,dt \geq m\int_0^1 (f(t)-\alpha)\,dt + \phi(\alpha)\int_0^1 dt$$

$$\geq m\left[\int_0^1 f(t)\,dt - \alpha\right] + \phi(\alpha).$$

$$= m[\alpha - \alpha] + \phi(\alpha) = \phi(\alpha)$$

Therefore $\int_0^1 \phi(f(t))\,dt \geq \phi\left[\int_0^1 f(t)\,dt\right]$ 1

Hence proved Ans 5(ii)

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5(ii)

Sol(ii) Statement:-

Let $1 \le p < \infty$ and let $f, g \in L^p(\mu)$.
Then $f+g \in L^p(\mu)$ and
$\|f+g\|_p \le \|f\|_p + \|g\|_p$

Proof:- For $p=1$
The inequality is clearly true.

If $p=\infty$
we note that $|f| \le \|f\|_\infty$ a.e.
$|g| \le \|g\|_\infty$ a.e.
$\Rightarrow |f+g| \le |f| + |g| \le \|f\|_\infty + \|g\|_\infty$ a.e. 1

Hence result follows in this case also.
Thus
assume $1 < p < \infty$
Let $1 < q < \infty$ be such that
$\frac{1}{p} + \frac{1}{q} = 1$

Since $L^p(\mu)$ is a linear space; it follows
that $f, g \in L^p(\mu) \Rightarrow f+g \in L^p(\mu)$
Also we have

$$\int_X |f+g|^p d\mu \le \int_X |f+g|^{p-1} |f| d\mu +$$

$$\int_X |f+g|^{p-1} |g| d\mu$$

Since $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow (p-1)q = p$ 2 we observe that

$$\int_X (|f+g|^{p-1})^q \, d\mu = \int_X |f+g|^p \, d\mu$$

and therefore $|f+g|^{p-1} \in L^q(\mu)$

By theorem

we have $|f+g|^{p-1} \in L^q(\mu)$, $f, g \in L^p(\mu)$

$\Rightarrow |f+g|^{p-1} \in L^q(\mu)$, $|f|, |g| \in L^p(\mu)$

$\Rightarrow |f+g|^{p-1}|f|, |f+g|^{p-1}|g| \in L^1(\mu)$.

By Holder's Inequality, we have

$$\int_X |f+g|^{p-1}|g| \, d\mu \leq \|g\|_p \|(|f+g|)^{p-1}\|_q$$

If $\|f+g\|_p$ is non zero finite

then dividing bothsides by

$(\|f+g\|_p)^{p/q}$ we obtain

$$(\|f+g\|_p)^{p-\frac{p}{q}} \leq \|f\|_p + \|g\|_p$$

i.e. $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ 2

Hence proved Ans 5(iii)

6(i)

6(i) To prove the theorem for the case $p = \infty$

let $\{f_n\}$ be a Cauchy sequence in $L^\infty(\mu)$

Then

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$$

except on a set $A_{m,n} \subset X$ with $\mu(A_{m,n}) = 0$

If $A = \bigcup_{n,m} A_{n,m}$, then it is clear that

$\mu(A) = 0$ and

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$$

for all n, m and for all $x \in X - A$. 1

Consequently, the sequence $\{f_n\}$ converges uniformly to a bounded limit (say) f in $X - A$ and observing the fact that convergence in $L^\infty(\mu)$ is equivalent to uniform convergence outside a set of measure zero

Now we assume that $1 \leq p < \infty$. To prove the result it is enough to show that each absolutely summable sequence in $L^p(\mu)$ is summable in $L^p(\mu)$ to some element in $L^p(\mu)$.

$$\int_X g^p \, d\mu = \lim_{n \to \infty} \int_X (g_n)^p \, d\mu \leq a^p < \infty$$

$\Rightarrow g^p$ is integrable

$\Rightarrow g^p$ is finite a.e. on X

set

$$S_n(x) = \sum_{k=1}^{n} f_k(x)$$ 2

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$h(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x)$, a.e.

$= k \sum_{k=1}^{n} f_n(x)$, a.e.

$\Rightarrow h$ is measurable function

$\int_X |S_n - h^p| \, d\mu \longrightarrow 0$

$\Rightarrow \|S_n - h\|_p \longrightarrow 0$

i.e. $\left\| \sum_{k=1}^{n} f_k(x) - h(x) \right\|_p \longrightarrow 0$ as $n \to \infty$

Hence the sequence $\{f_n\}$ is summable in $L^p(\mu)$ and has the sum h in $L^p(\mu)$.
Hence L^p spaces are complete. proved 2

Ans [6(i)]

6(ii)

6(ii) If Part:-
Let ϕ is convex and
Let $a < u < v < x < y < b$
then we have

$$\frac{\phi(v) - \phi(u)}{v - u} \le \frac{\phi(y) - \phi(x)}{y - x}$$

Taking limit
$v \to u$ and $y \to x$ to left and right hand side

$$\lim_{v\to u} \left(\frac{\phi(v) - \phi(u)}{v - u} \right) < \lim_{y\to x} \frac{\phi(y) - \phi(x)}{y - x}$$ 2