Enrollment No......................................

Faculty of Science

End Sem (Even) Examination May-2019

BC3CO15 Mathematics-IV

Programme: B.Sc. (CS) Branch/Specialisation: Computer Science

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. The inverse of element i of group $(G,.)$ where $G=\{1,-1,i,-i\}$ is **1**

(a) -1 (b) i (c) $-i$ (d) None of these

ii. Every abelian group is cyclic **1**

(a) True (b) False (c) Can't say (d) None of these

iii. Every permutation can be expressed as product of **1**

(a) Transpositions (b) Disjoint cycles

(c) Can't say (d) None of these

iv. A subgroup H of a group (G, .) is a normal subgroup of G, if $xHx^{-1}=H,\ \forall x\in G$. **1**

(a) True (b) False

(c) Both (a) and (b) (d) None of these

v. If the system $(R, +, .)$ is a ring then which one is true **1**

(a) $a.0=0=0.a, \forall a\in R$ (b) $a.(-b)=(-a).b, \forall a,b\in R$

(c) $(-a).(-b)=a.b, \forall a,b\in R$ (d) All of these

vi. If $f(x)$ and $g(x)$ are two non-zero polynomials of degree n in $R[x]$ where $R[x]$ is the set of all polynomials over a ring R then **1**

(a) $\deg f(x)+\deg g(x)\le \deg[f(x)+g(x)]$

(b) $\deg f(x)+\deg g(x)\le \deg[f(x).g(x)]$

(c) Both (a) and (b)

(d) None of these

vii. The number of elements in any basis of a finite dimensional vector space is called ....of the vector space. **1**

(a) Basis (b) Linear sum

(c) Dimension (d) None of these

viii. A set of vectors which contains at least one zero vector is........ **1**

(a) Linearly dependent (b) Linearly independent

(c) Can't say (d) None of these

ix. Let T be a linear transformation from a finite dimensional vector space U to the vector space V over the same field F then $rank(T)+nullity(T)=$. **1**

(a) $\dim V$ (b) $\dim U$

(c) Both (a) and (b) (d) None of these

x. The kernel of linear transformation T from vector space U to vector space V over the field F is a subspace of $U(F)$ **1**

(a) True (b) False (c) Can't say (d) None of these

Q.2 Attempt any two:

i. Prove that every subgroup of a cyclic group is cyclic. **5**

ii. Show that the set of all integers I forms a group with respect to the binary operation $*$ defined by the rule $a*b=a+b+1, \forall a,b\in I$. **5**

iii. State and prove Lagrange's Theorem. **5**

Q.3 Attempt any two:

i. Prove that the set of all cosets of a normal subgroup is a group with respect to multiplication of complexes as the composition. **5**

ii. If f is a homomorphism of a group $(G,.)$ on to a group $(G',.)$, then prove that kernel K of f is normal subgroup of G. **5**

iii. State and prove Cayley's theorem. **5**

Q.4 Attempt any two:

i. Prove that every finite integral domain is a field. **5**

ii. Prove that under addition and multiplication the set $S=\{0,2,4,6,8\}(\text{mod}10)$ is a ring. **5**

iii. Prove that a subset S of a ring $(R,+,.)$ is a subring if and only if **5**

(a) $a\in S, b\in S \Rightarrow a-b\in S\ \forall a,b\in S$

(b) $a\in S, b\in S \Rightarrow a.b\in S\ \forall a,b\in S$

Q.5 Attempt any two:

i. Show that the vectors (2,1,4), (1, -1,2), (3,1, -2) forms a basis for R^3. **5**

ii. Prove that intersection of any two subspaces of a vector space V over the field F is also a subspace of $V(F)$. **5**

iii. If W_1, W_2 are two subspaces of a finite dimensional vector space over the field F then **5**

$\dim(W_1+W_2)=\dim W_1+\dim W_2-\dim(W_1\cap W_2)$

Q.6 Attempt any two:

i. Show that the mapping $T:R^2\rightarrow R^3$ defined by $T(a,b)=(a-b,b-a,-a)\forall a,b\in R$ is a linear transformation from R^2 into R^3. Also find the nullity of T. **5**

ii. Find the matrix representation of linear transformation T on vector space V_3 over the field R defined as $T(a,b,c)=(2b+c,a-4b,3a)$ corresponding to the basis $B=\{(1,1,1),(1,1,0),(1,0,0)\}$. **5**

iii. Define: **5**

(a) Diagonalisation of linear operator

(b) Quotient space

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Q.1

i. (c) $-i$ 1

ii. (b) False 1

iii. (a) Transpositions 1

iv. (a) True 1

v. (d) All of these. 1

vi. (d) None of these 1

vii. (c) Dimension 1

viii. (a) Linearly dependent 1

ix. (b) Dim U 1

x. (a) True. 1

Q.2

i) Let G be the cyclic group generated by a. If subgroup $H = G$ or $\{e\}$ then H is cyclic. +1

Let H be proper subgroup of G, then H contains integral power of a.

Let $a^s \in H$ then $a^{-s} \in H$ +1

$\Rightarrow$ H has +ve an -ve integral powers of a.

Let m be the least +ve integer such that $a^m \in H$. Now we are going to prove that a^m is generator of H +1

Let $a^t \in H$ arbitrarily

so $t = mq + r$; $0 \le r < m$ by division algorithm. where $q, r \in I$

So, $a^m \in H \Rightarrow a^{mq} \in H$

$\Rightarrow a^{mq} \in H$

$\Rightarrow (a^{mq})^{-1} \in H$

$\Rightarrow a^{-mq} \in H$

Also, $a^t \in H, a^{-mq} \in H \Rightarrow a^{t-mq} \in H$
$\Rightarrow a^r \in H$

As m is the least +ve integer such that $a^m \in H$ +1

$\Rightarrow r = 0$

$\therefore \quad t = mq$

$\Rightarrow a^t = a^{mq} = (a^m)^q$

So, every element a^t is of the form $(a^m)^q$.

So, H is cyclic generated by a^m. +1

ii) G_1: closure law:

$\forall a, b \in I \Rightarrow a+b+1 \in I$

$\Rightarrow a*b \in I$

$\therefore$ I is closed wrt $*$. +1.25

G_2: Associative law:

$\forall a, b, c \in I$

$(a*b)*c = (a+b+1)*c$

$= a+b+1+c+1$

$= a+1+b+c+1$

$= a+1+(b*c)$

$= a*(b*c)$

Hence associative +1

G_3: Identity law:

$\forall a \in I \Rightarrow \exists e \in I$

s.t $a*e = a$

$\Rightarrow a+e+1 = a$

$\Rightarrow e = -1$

as $-1 \in I$

So, Identity law is satisfied. +1

G_4: Inverse law:

$\forall a \in I \Rightarrow \exists a^{-1} \in I$

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$\Rightarrow n = mk$

$\Rightarrow k = \frac{n}{m}$

$\Rightarrow k = \frac{o(G)}{o(H)}$

as $k \in I \Rightarrow o(H)$ is divisor of $o(G)$. +1

Hence proved.

Q.3

i) Let N be a normal subgroup of any group G. So each right coset of N in G is also a left coset of N in G. Let G/N be the set of all cosets of N in G.

ie. $G/N = \{Na : a \in G\} = \{Ne, Na_1, Na_2, \ldots\}$ +1

G_1: Closure property:

$\forall\ Na_i, Na_j \in G/N$

$(Na_i)(Na_j) = N(a_i N a_j) = NN a_i a_j = N a_i a_j$

as $a_i, a_j \in G \Rightarrow a_i a_j \in G$. [closure law in G]

$\therefore N a_i a_j \in G/N$.

Hence closure law is satisfied. +1

G_2: Associative law:

~~Let~~ $\forall Na_i, Na_j, Na_k \in G/N$

$\therefore Na_i[(Na_j)(Na_k)] = Na_i[Na_j a_k]$ [closure in G/N]

$= Na_i(a_j a_k)$ ["]

$= N(a_i a_j)a_k$. [associativity in G]

$= (Na_i a_j) Na_k$

$= [(Na_i)(Na_j)] Na_k$

s.t $a * a^{-1} = -1$

$\Rightarrow a + a^{-1} + 1 = -1$

$\Rightarrow a^{-1} = -2 - a$

as $-2-a \in I \ \forall a \in I$

$\Rightarrow$ Inverse Law is satisfied

Hence $(I, *)$ is a group +1.25

iii) Lagranges Theorem:

The order of each subgroup of a finite group is divisor of the order of the group. +1

Proof: Let G be the group and H be the subgroup of the group G.

Let $o(G) = n$ and $o(H) = m$

Let $a \in G$, So aH is left coset of H in G. +1

Now we shall prove that $o(H) = o(aH)$

Let $h_1, h_2, ---, h_m$ be the m distinct members of H.

So, $aH = \{ah_1, ah_2, ---, ah_m\}$

if $ah_i = ah_j$; $1 \le i, j \le m$ & $i \neq j$

$\Rightarrow h_i = h_j$ +1

which is not possible $\Rightarrow ah_i \neq ah_j$

$\Rightarrow o(aH) = m$

Let the number of dist left cosets of H in G be equal to k (say). +1

then $G = a_1H \cup a_2H \cup --- \cup a_kH$

$\therefore$ total number of elements in G is mk

$\therefore o(G) = mk$

Hence associative law is satisfied. +1

G_3: Identity law:

$$\forall Na_i \in G/N \Rightarrow \exists Ne \in G/N$$

$$\text{s.t } (Na_i)(Ne) = Na_ie = Na_i$$
$$= Nea_i$$
$$= (Ne)(Na_i)$$

Hence identity law is satisfied. +1

G_4: Inverse law:

$$\forall Na_i \in G/N \Rightarrow \exists Na_i^{-1} \in G/N$$

$$\text{s.t. } Na_i Na_i^{-1} = Na_ia_i^{-1}$$
$$= Ne$$
$$= Na_i^{-1}a_i$$
$$= (Na_i^{-1})(Na)$$

Hence inverse law is satisfied.

Hence $(G/N, \cdot)$ is a group. +1

(ii) Let f be a homomorphism of a group G in to a group G' and e, e' be the identities of G and G' respectively. Let K be the kernal of f i.e. +1

$$K = \{x \in G ; f(x) = e'\}$$

$$\text{as } f(e) = e', e \in G$$

$$\Rightarrow e \in K \Rightarrow K \neq \phi$$ +1

$$\forall a, b \in K \Rightarrow f(a) = e' = f(b)$$

Now, $f(ab^{-1}) = f(a) f(b^{-1})$

$$= f(a)(f(b))^{-1}$$
$$= e'(e')^{-1}$$

$= e' \cdot e' = e'$

$\therefore f(ab^{-1}) = e' \Rightarrow ab^{-1} \in K$

$\therefore K$ is subgroup of G +1

Let x be any arbitrary element of G,

then $f(xax^{-1}) = f(x) f(a) f(x^{-1})$

$= f(x)\, e' [f(x)]^{-1}$

$= f(x) [f(x)]^{-1}$

$= e'$ +1

$\Rightarrow xax^{-1} \in K$

as $x \in G, a \in G \Rightarrow xax^{-1} \in G$

Hence ~~xax^{-1}~~ K is normal subgroup of G +1

ii) Cayleys Theorem:

Every finite group G is isomorphic to a permutation group.

Proof:

Let G be a finite group of order n where $G = \{a_1, a_2, \dots a_n\}$

Let $a \in G$ be arbitrary. Consider a mapping $f_a : G \to G$ defined by the rule

$f_a(x) = ax \quad \forall x \in G$ +1

This mapping is well defined since

$a \in G, x \in G \Rightarrow ax \in G$

Clearly f_a is one one as well as onto

So f_a is a permutation of degree n of G.

we have $f_a = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ aa_1 & aa_2 & \dots & aa_n \end{pmatrix}$

Let G' be the set of all such permutations defined on G corresponding to every element of G.

$G' = \{ f_a : a \in G \}$ +1

First of all prove that G' is a group wrt composition of two functions. +1

Since $o(G) = n$, $G' = \{f_a : a \in G\} \Rightarrow o(G') = n$
Thus G' is a finite group of order n.

Now we shall prove that group G is isomorphic to the group G'. +1
Let us define a function
$\phi : G \longrightarrow G'$
defined by $\phi(a) = f_a \ \forall a \in G$
Now, prove the ϕ is one-one, onto homomorphic.
Hence $G \cong G'$. +1

Q 4

(i) Let D be a finite commutative ring without zero divisors with n elements $a_1, a_2, \ldots a_n$ i.e. $D = \{a_1, a_2, \ldots, a_n\}$. +1
So in order to prove that D is a field it is sufficient to prove that
a) $\exists$ unity element $1 \in D$
b) Each non zero element posses multiplicative inverse. +1
Let $0 \neq a \in D$. Consider the n product $aa_1, aa_2, \ldots aa_n$. All of these are elements of D written in some what different order.

Let $aa_i = aa_j$; $i \neq j$
$\Rightarrow a(a_i - a_j) = 0$
$\Rightarrow a_i - a_j = 0$
$\Rightarrow a_i = a_j$

which is a contradiction +1

$\therefore aa_i \neq aa_j \quad \forall i,j$

Now,

$a \neq 0,\ a \in D \Rightarrow \exists\ aa_k \in D$ such that $aa_k = a$; $1 \le k \le n$

$\Rightarrow aa_k = a.1,\ a_k \in D.$

$\Rightarrow a_k = 1$

$\Rightarrow 1 \in D.$

such $1.y = y = y.1 \quad \forall\ y \in D.$ +1

Now $1 \in D \Rightarrow \exists\ aa_j \in D$ such that $aa_j = 1$; $1 \le j \le n.$

$\Rightarrow aa_j = a_j a = 1.$

$\Rightarrow a_j \in D.$ is multiplicative inverse of $0 \neq a \in D$ +1

$\therefore D$ is a field.

ii) The composition table for $+_{10}, \times_{10}$.

$+_{10}$	0	2	4	6	8
0	0	2	4	6	8
2	2	4	6	8	0
4	4	6	8	0	2
6	6	8	0	2	4
8	8	0	2	4	6

$\times_{10}$	0	2	4	6	8
0	0	0	0	0	0
2	0	4	8	2	6
4	0	8	6	4	2
6	0	2	4	6	8
8	0	6	2	8	4

+2

Now, prove that

$(R, +_{10})$ is abelian group +1

$(R, \times_{10})$ is a semi group +1

left and right distributive laws. +1

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is satisfied by given cond. 2 +0.5
Associative law wrt multiplication holds by associativity in G.
The operation of multiplication is ~~associative~~ distributive in R hence in S. +1

Hence proved.

Q5

i) Let $S = \{(2,1,4), (1,-1,2), (3,1,-2)\}$
Again, let $a_1, a_2, a_3 \in R$ be such that
$a_1(2,1,4) + a_2(1,-1,2) + a_3(3,1,-2) = 0$ +1

$\therefore$ The coefficient matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$$ +1

here $|A| \neq 0$ +1

$\therefore \rho(A) = 3$

$\therefore$ The set is LI +1

Also dimension of vector space R^3 is 3
Hence any set of 3 LI vectors is a basis of R^3. +1

ii) Let W_1 and W_2 be any two subspaces of a vector space $V(F)$. Since $0 \in V$
$\Rightarrow 0 \in W_1$ and $0 \in W_2$
$\Rightarrow$ ~~and~~ $0 \in W_1 \cap W_2$ +1
$\Rightarrow W_1 \cap W_2 \neq \phi$.

Let $\alpha, \beta \in W_1 \cap W_2$ arbitrarily
Also let $a, b \in F$
Now; $\beta, \alpha \in W_1 \cap W_2 \Rightarrow \alpha, \beta \in W_1$ and $\alpha, \beta \in W_2$ +1

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(iii)

Let $(S,+,\cdot)$ be a subring of the ring $(R,+,\cdot)$

$\therefore a,b \in S \Rightarrow a,-b \in S$ [as S is a subring]
$\Rightarrow a-b \in S$ ["]

Also $a,b \in S \Rightarrow ab \in S$ ["]
Hence both the conditions are satisfied +1.5

Converse:
Let S be non empty subset of the ring $(R,+,\cdot)$
Now to prove that S is subring of the ring R.

i) Identity law:
$\forall a,a \in S \Rightarrow a-a \in S$ [by cond: 1]
$\Rightarrow 0 \in S$
Identity law is satisfied

ii) Inverse law:
$\forall 0,a \in S \Rightarrow 0-a \in S$
$\Rightarrow -a \in S$
Inverse law is satisfied. +1

iii) Closure law:
$\forall a,b \in S \Rightarrow a,-b \in S$ [Inverse law]
$\Rightarrow a+b \in S$ [by cond. 1]
Closure law is satisfied

iv) Associative law holds by associativity in G.

v) Commutative law holds by commutativity in G.
$(S,+)$ is abelian group +1

i) Closure law wrt multiplication

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(i) Let $\alpha = (a_1, b_1)$, $\beta = (a_2, b_2)$ be two arbitrary elements of R^2 and $x, y \in R$. Then

$$T(x\alpha + y\beta) = T[x(a_1, b_1) + y(a_2, b_2)]$$
$$= T[xa_1 + ya_2,\ xb_1 + yb_2]$$
$$= (xa_1 - xb_1,\ xb_1 - xa_1,\ -xa_1) + (ya_2 - yb_2,\ yb_2 - ya_2,\ -ya_2)$$
$$= x(a_1 - b_1,\ b_1 - a_1,\ -a_1) + y(a_2 - b_2,\ b_2 - a_2,\ -a_2)$$
$$= xT(\alpha) + yT(\beta)$$

$\therefore$ T is a L.T. +2.5

Nullity of T

$$N(T) = \{(a,b) \in R^2 ;\ T(a,b) = 0\} \text{ where } 0 \text{ is } 0 \text{ vector of } R^3$$
$$= \{(a,b) \in R^2 ;\ (a-b,\ b-a,\ -a) = (0,0,0)\}$$
$$= \{(a,b) \in R^2 ;\ a = 0,\ b = 0\} = \{(0,0)\}$$

$\therefore$ nullity of $T = \dim(N(T)) = 0$. +2.5

ii) $T: V_3(R) \to V_3(R)$ be a linear transformation defined by

$$T(a,b,c) = (2b + c,\ a - 4b,\ 3a)$$

Basis set of $V_3(R)$ is

$$B = \{(1,1,1), (1,1,0), (1,0,0)\}$$ +1

We have $T(1,1,1) = (3, -3, 3)$
$$= 3(1,1,1) - 6(1,1,0) + 6(1,0,0)$$ +1

Again $T(1,1,0) = (2, -3, 3)$

Since W_1 and W_2 are subspaces, so

$a, b \in F$ and $\alpha, \beta \in W_1 \Rightarrow a\alpha + b\beta \in W_1$ +1

& $a, b \in F$ and $\alpha, \beta \in W_2 \Rightarrow a\alpha + b\beta \in W_2$ +1

$a\alpha + b\beta \in W_1, W_2$

$\Rightarrow a\alpha + b\beta \in W_1 \cap W_2$

Hence $W_1 \cap W_2$ is a subspace of $V(F)$. +1

iii) Let the set $S = \{r_1, r_2, \ldots, r_k\}$ be a basis of $W_1 \cap W_2$, so that $\dim(W_1 \cap W_2) = k$

Then $S \subseteq W_1$ and $S \subseteq W_2$

Since S is linearly independent and $S \subseteq W_1$ and $\therefore$ S can be extended to form a basis of W_1. +1

Let $\{r_1, r_2, \ldots, r_k, \alpha_1, \alpha_2, \ldots, \alpha_l\}$ be a basis of W_1. $\therefore$ $\dim W_1 = k + l$

Similarly let $\{r_1, r_2, \ldots r_k, \beta_1, \beta_2, \ldots \beta_m\}$ be a basis of W_2. Then $\dim W_2 = k + m$

$\therefore$ $\dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

$= k + l + k + m - k = k + l + m$ +1

We shall prove that the set

$S_1 = \{r_1, r_2, \ldots, r_k, \alpha_1, \alpha_2, \ldots, \alpha_l, \beta_1, \beta_2, \ldots \beta_m\}$

is a basis of $W_1 + W_2$.

First of all show that S_1 is L.I +1

Then show that $L(S_1) = W_1 + W_2$ +1

$\therefore$ S_1 is basis of $W_1 + W_2$

consequently

$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$ +1

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$= 3(1,1,1) - 6(1,1,0) + 5(1,0,0)$ +1

Finally $T(1,0,0) = (0,1,3)$

$= 3(1,1,1) - 2(1,1,0) - 1(1,0,0)$ +1

$$\therefore [T;B] = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$ +1

iii) a) Diagonalisation of linear operator.

If V is a finite dimensional vector space, then a linear map $T: V \rightarrow V$ is called diagonalizable if there exist an ordered basis of V wrt which T is represented by a diagonal matrix. +2.5

b) Quotient space:

If V(F) is a vector space and W is a subspace of V. If $\alpha \in V$ then the set $W+\alpha$, $\alpha+W$ are right and left cosets respectively. Since $(V,+)$ is commutative +2.5 $\therefore$ $W+\alpha = \alpha+W$. Then the set of all cosets of W in V denoted as

$$\frac{V}{W} = \{W+\alpha \mid \alpha \in V\}$$ is a vector

space wrt vector addition and scalar multiplication known as Quotient space.