Enrollment No......................................

Faculty of Science

End Sem (Odd) Examination Dec-2019

CA3CO15 Algebra

Programme: BCA Branch/Specialisation: Computer Application

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. If a' is a compliment of a in a Boolean algebra $(B,+,\cdot,')$, then $a+a'$ is equal to **1**

(a) 1 (b) a' (c) 0 (d) None of these

ii. Two switches x and y are connected in series if and only if **1**

(a) x-y (b) $x.y$ (c) $x + y$ (d) None of these

iii. The inverse of an element in a group is always **1**

(a) Unique (b) 2 (c) 3 (d) No

iv. The generators of a cyclic group $(\{1,-1,i,-i\},\cdot)$ is /are **1**

(a) 1 and -1 (b) -1 (c) $i,-i$ (d) None of these

v. If $Z/3Z$ is a factor group, then the identity element in $Z/3Z$ is **1**

(a) Z (b) $Z/3Z$ (c) 2Z (d) 3Z

vi. If $T:G\to G'$ is the group homomorphism and e is an identity element of G, and $KerT=G$, then which of the following is correct **1**

(a) $T(G)=\{e\}$ (b) $KerT=\{1,-1\}$

(c) $KerT=\{e\}$ (d) None of these

vii. If α,β are two vectors in a vector space such that $\alpha=2\beta$, then the which of the following is correct **1**

(a) α,β are L.I.

(b) α,β are L.D.

(c) $a\alpha+b\beta=0\Rightarrow a=0,b=0$

(d) None of these.

viii. If w_1 and w_2 are two vector subspace of vector space V(F) then $\dim(w_1 + w_2)$ is equal to **1**

(a) $\dim w_1 - \dim w_2 + \dim(w_1 \cap w_2)$

(b) $\dim w_1 + \dim w_2 + \dim(w_1 \cap w_2)$

(c) $\dim w_1 + \dim w_2 - \dim(w_1 \cap w_2)$

(d) $\dim w_1 - \dim w_2 - \dim(w_1 \cap w_2)$

ix. If $T: U(F) \to U(F)$ is a linear transformation and B be the ordered basis for n- dimensional vector space U and c is any scalar, then which of the following is correct **1**

(a) $[cT]_B \neq c[T]_B$ (b) $[cT]_B = cI + [T]_B$

(c) $[cI + T]_B = c[T]_B$ (d) $[cT]_B = c[T]_B$

x. If W is a subspace of a finite dimensional vector space V, then the $\dim(V/W)$ is equal to **1**

(a) $\dim V \cdot \dim W$ (b) $\dim V + \dim W$

(c) $\dim V - \dim W$ (d) None of these

Q.2 i. Consider the function $f(x, y, z) = (x+z)(y+z)$. Then **3**

(a) Simplify f algebraically.

(b) Draw the simplified network of f.

ii. Define Boolean algebra. State and prove the De-Morgan's law in a Boolean algebra. **7**

OR iii. Define the disjunctive normal form and change the following function into disjunctive normal form $f(x, y, z) = (x+y)z'$. **7**

Q.3 i. Show that the intersection of two normal subgroup of a group is normal subgroup. **3**

ii. Let G be the set of all rational numbers other than -1 and let $a * b = a + b + ab,\ \forall a, b \in G$, then show that $(G, *)$ is an abelian group. **7**

OR iii. State and prove Lagranges theorem. **7**

Q.4 i. Define group homomorphism, kernel of group homomorphism with example. **3**

ii. State and prove the fundamental theorem of homomorphism. **7**

OR iii. Prove that $(\{0,1,2,3,4,5,6\}, +_7, \times_7)$ is a field. **7**

Q.5 i. Show that $W = \{(x, y, z) \in R^3 : x, y, z \in R \text{ and } z = 0\}$ is a subspace of $V_3(R)$. **3**

ii. Show that the set of all continuous function $f:[0,1] \to R$ is a vector space over the field of real numbers with respect to addition of functions and scalar multiplication function. **7**

OR iii. Show that $(2, 1, 4)$, $(1, -1, 2)$, $(3, 1, -2)$ from a basis for R^3. **7**

Q.6 i. Define the linear transformation. **3**

ii. State and prove the rank-nullity theorem. **7**

OR iii. Let $V(R)$ be the vector space of all polynomials in indeterminate x with coefficients on R of the form $f(x) = a_0x^0 + a_1x^1 + a_2x^2$. The differential operator D is a linear transformation on $V(R)$ and the set $B = \{x^0, x^1, x^2\}$ is an ordered basis of $V(R)$. Find the matrix of D relative to ordered basis and also find the eigen values of the transformation. **7**

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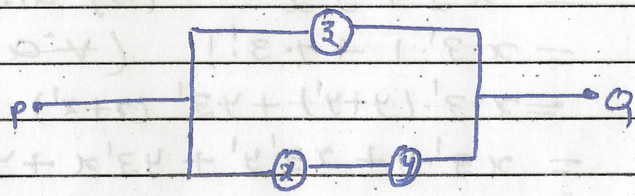
Faculty of Science
Course Name: Algebra
Course code: BCA3C015
Programme - BCA

Q.1
(i) (a) 1
(ii) (b) $x \cdot y$
(iii) (a) Unique
(iv) (c) $+i, -i$
(v) (d) $3\mathbb{Z}$
(vi) (a) $T(G) = \{e\}$
(vii) (b) α, β are L.D.
(viii) (c) $\dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$
(ix) (d) $[cT]_B = c[T]_B$
(x) (c) $\dim V - \dim W$

Q.2(i) (a) $f(x,y,z) = (x+z)(y+z)$
$= (xy) + z$ +1

(b)



+2

(ii) let B be a non-empty set of elements a, b, c, ..., with two binary operation V (join) and ∧ (meet) and a unary operation ', then the algebraic structure (B, V, ∧, ') is called a Boolean Algebra, if the elements of B obey the following axioms.
B1. Closure laws. B2. Commutative laws.
B3. Distributive laws B4. Existence of identity elem. (+2)
B5. Laws of inverse.

De-Morgan's laws:
For any two elements 'a' and 'b' of Boolean algebra B, we have
(i) $(a+b)' = a'.b'$ (ii) $(a.b)' = a'+b'$ +2
proof: (i) It remains to prove
$(a+b) + a'.b' = 1$ +1.5
$(a+b).a'.b' = 0$ +.5

2 (iii) Definition: A Boolean polynomial which can be written as sum of the minimal Boolean functions, is called disjunctive normal form of the Boolean function.
Ex. $f(x,y) = x.y + x'.y$ [+2]

Given $f(x,y,z) = (x+y)z'$
$= xz' + y.z'$ (By distributive law)
$= x.z'.1 + y.z'.1$ ($\forall a \in B$, $a.1 = a$)
$= x.z'.(y+y') + y.z'(x+x')$ ($\forall a \in B$, $a+a'=1$)
$= xz'.y + xz'y' + yz'x + yz'x'$
$= xyz' + xy'z' + xyz' + x'yz'$
$f(x,y,z) = xyz' + xy'z' + x'yz'$ ($a+a=a$ [+5]

3 (i) let H_1 and H_2 be two normal subgroup of a group G.
To prove that $H_1 \cap H_2$ is a normal subgroup.
$\because$ H_1 and H_2 are normal subgroup of a group G
$\Rightarrow e \in H_1$ and $e \in H_2$ where e is an identity element of a group G [+1]

$\Rightarrow e \in H_1 \cap H_2 \Rightarrow H_1 \cap H_2 \neq \phi$

If $h \in H_1 \cap H_2$, $x \in G$ (+1)

$\Rightarrow h \in H_1$ and $h \in H_2$, $x \in G$

$\Rightarrow x h x^{-1} \in H_1$, $x h x^{-1} \in H_2$ [$\because H_1$ & H_2 are normal sub. of G]

$\Rightarrow x h x^{-1} \in H_1 \cap H_2$

$\Rightarrow H_1 \cap H_2$ is a normal subgroup of a group G. +[1]

(ii) Given $a * b = a + b + ab$, $\forall a, b \in G$, where $G = \mathbb{Q} - \{-1\}$

G_1 Closure property: $\forall a, b \in G$ we have $a * b \in G$ [+2]

G_2 Associative property: $\forall a, b, c \in G$ we have $a * (b * c) = (a * b) * c$ [+1]

G_3 $\exists$ of an identity element: $\forall a \in G$ $\exists e \in G$ s.t $a * e = a = e * a$ [1.5]

$\Rightarrow e = 0$, $a \neq -1$

G_4 Existence of an inverse of an element: If $a \in G$ $\exists a' \in G$ s.t $a * a' = e = a' * a$ [1.5]

$\Rightarrow a' = -\frac{a}{a+1}$

G_5 Commutative property: $\forall a, b \in G$ we have $a * b = b * a$ [+1]

$\Rightarrow (G, *)$ is a group

(iii) Statement: The order of each subgroup of a finite group is a divisor of the order of the group [+2]

proof: Let H be any subgroup of order m of a finite group G of order n

Let $a \in G$, then aH is the left coset of H in G [+1]

suppose $H = \{h_1, h_2, h_3, \dots h_m\}$

then $aH = \{ah_1, ah_2, ah_3, \dots, ah_m\}$

Since, for the two distinct elements of h_i and h_j of H, $ah_i = ah_j \Rightarrow h_i = h_j$ [+1]

$\therefore$, each left coset of H in G has m distinct elements.

Let the no. of left cosets of H in G be equal to k (say). [+1]

Then $G = a_1H \cup a_2H \cup a_3H \cup \dots \cup a_kH$

$$\Rightarrow O(G) = \underbrace{m + m + \dots + m}_{k\text{-times}}$$

$$\Rightarrow O(G) = km$$

$$\Rightarrow \boxed{\frac{O(G)}{O(H)} = k}$$ [+2]

(i) Group Homomorphism: Let (G, o) and (G', o') be two groups. A mapping f from a group G to a group G' is said to be a homomorphism if $f(a o b) = f(a) o' f(b), \forall a, b \in G$ [+1]

$\ker f = \{a \in G \mid f(a) = e'$, where e' is an identity element in $G'\}$ is called kernel of group homomorphism f: [+1]

Ex: let G be the group of all real no. under addition and let G' be the group of non zero real no. under multiplication and $f: G \to G'$ defined by

$f(a) = 2^a, \forall a \in G$, then it is a group homomorphism.

$$\ker f = \{a \in G \mid f(a) = 1\}$$

$$= \{a \in G \mid 2^a = 1\}$$ [+1]

$$= \{0\}$$

(ii) Fundamental theorem of Homomorphism:

If $f: G \xrightarrow{\text{into}} G'$ is a group Homomorphism with kernel $\ker f$. Then $\frac{G}{\ker f} \cong f(G)$

proof: defined $\phi : \frac{G}{\ker f} \to f(G)$ as

$$\phi(a \ker f) = f(a) \longrightarrow (1)$$ [+1]

To prove

(i) ϕ is well defined [+1]

(ii) ϕ is group homomorphism [+2]

(iii) ϕ is 1-1 [+2]

(iv) ϕ is onto [+1]

$\Rightarrow \phi$ is an isomorphism

$F = \{0, 1, 2, 3, 4, 5, 6\}$

1(iii) Composition Table

$+_7$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

Table-I

$\times_7$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Table-II

[+2]

F1. $(F, +_7)$ is an abelian group. [+2]
F2. $(F, \times_7)$ is a semi-group [+1]
F3. Identity element w.r.t $\times_7$
F4. Inverse of non-zero element w.r.t $\times_7$ [+1]
F5. Commutative laws w.r.t $\times_7$
F6. Distributive laws [+1]

5(i) Since $0 = (0, 0, 0) \in W$.
$\implies W \neq \phi$. [+1]
Let $\alpha, \beta \in W$, where $\alpha = (x_1, y_1, 0)$,
$\beta = (x_2, y_2, 0)$, and $a, b \in \mathbb{R}$.
Then $a\alpha + b\beta \in W$.
$\implies W$ is a sub-space of $V_3(\mathbb{R})$ [+2]

(ii) $V = \{ f \mid f : [0,1] \to \mathbb{R}$ is a continuous fun$^n \}$.
defined addition of function
$(f+g)(x) = f(x) + g(x) \quad \forall x \in [0,1]$ —(i).

Scalar multipliplication.

$(af)x = af(x), \forall x \in [0,1], f \in V.$ [+1]

Then To prove that V is a vector space over IR w.r.t + .

V_1. $(V,+)$ is an abelian group [+2]

V_2 V is closed w.r.t scalar multiplication [+1]

V_3: $\forall f,g \in V, a,b \in \mathbb{R}$

V_{31}. $a(f+g) = af+ag$. [+1]

V_{32}. $(a+b)f = af+bf$ [+1]

V_{33}. $a(bf) = (ab)f$

V_{34}. $1.f = f$ [+1]

5(iii) (i) Let $S = \{(2,1,4), (1,-1,2), (3,1,2)\}$

(i) S is L.I.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & -1 & 2 \\ 3 & 1 & 2 \end{bmatrix}$$

$\therefore |A| = 24 \neq 0$

$\Rightarrow$ S is Linearly independent [+3.5]

(ii) $L(S) = \mathbb{R}^3$

~~let~~ Also, the dimension of vector space $\mathbb{R}^3$ is 3. Hence any set of 3 Linearly independent vectors is a basis $\mathbb{R}^3$ [+3.5]

6(I) Let $U(F)$ and $V(F)$ be two vector space over the same field F. A mapping $T: U \to V$ is called a linear transformation, if T is such that

$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$, $\forall \alpha, \beta \in U, a, b \in F$ [+3]

6(II) Let U and V be vector spaces over the field F, and let T be a linear transformation from U into V. Suppose that U(F) is finite dimensional. Then prove that

$\text{rank}(T) + \text{nullity}(T) = \dim U$ [+2]

proof: Let $N(T)$ be the null space of T. Then $N(T)$ is a sub-space of U. Since U is finite dimensional, therefore $N(T)$ is a finite dimensional. Let $\dim N(T) = \text{nullity}(T) = k$; and let $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a [+1] basis of $N(T)$.

Since $\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k\}$ is a L.I. subset of U, therefore by extension theorem we can extend it to form a basis of U.

Let $\dim U = n$ and let $\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k, \alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n\}$ be a basis of U [+1]

Therefore, the vectors $T(\alpha_1), T(\alpha_2), \ldots T(\alpha_k), T(\alpha_{k+1}), \ldots, T(\alpha_n) \in R(T)$

We shall prove that set

(i) $S = \{T(\alpha_{k+1}), T(\alpha_{k+2}), \ldots, T(\alpha_n)\}$ [+1.5] span $R(T)$

(ii) $S = \{T(\alpha_{k+1}), T(\alpha_{k+2}), \ldots, T(\alpha_n)\}$ is a L.I. $\Rightarrow \dim(N(T)) + \dim(R(T)) = \dim U$ [+1.5]

Q.6 (14) given

$$D : V(\mathbb{R}) \to V(\mathbb{R})$$

$$D(P(x)) = \frac{d}{dx} P(x)$$

$$D(x^0) = 0, \quad D(x) = 1, \quad D(x^2) = 2x$$

$$[D]_B^B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$ +4

The characteristic equation is

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 = 0$$

$\Rightarrow \lambda = 0, 0, 0.$

$\Rightarrow$ eigen values of D are $0, 0, 0.$ 1.5

Eigen vectors let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of D corresponding to $\lambda = 0$

$(D - \lambda I) X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow x_2 = 0, \; x_3 = 0, \; x_1 = k$$

$$X = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ 1.5