Total No. of Questions: 6 *Total No. of Printed Pages:3*

Enrollment No......................................

Faculty of Engineering

End Sem (Odd) Examination Dec-2017

CS3BS03 / IT3BS03 Discrete Mathematics / Structure

Programme: B.Tech. Branch/Specialisation: CS/IT

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. If A={a,b,c} and B={1,2,3} then which of the following is function from A to B: **1**

(a) {(a,1), (a,2), (b,3)} (b) {(a,1), (b,2), (c,3)}

(c) {(b,1), (a,2), (b,2)} (d) None of these

ii. The relation "is divisor of" on the set of positive integers is **1**

(a) Reflexive (b) Symmetric

(c) Transitive (d) Both (a) and (c)

iii. Let D_{30} = {1, 2, 3, 5, 6, 10, 15, 30} be the set of all positive integral divisors of 30 and relation I be a partial ordering on D_{30} The least upper bound of 10 and 15 respectively is **1**

(a) 30 (b) 15 (c) 10 (d) 6

iv. In a Boolean Algebra, the property $x + xy = x$ is known as **1**

(a) Distributive (b) Absorption

(c) Associative (d) Commutative

v. The inverse of - i in the multiplicative group, $\{1, -1, i, -i\}$ is **1**

(a) 1 (b) -1 (c) i (d) $-i$

vi. A monoid is always a **1**

(a) Group (b) Abelian group

(c) Non-abelian group (d) Groupoid

vii. A graph in which all the vertices are of equal degree is known as **1**

(a) Tree (b) Complete (c) Regular (d) Simple

viii. The number of colors required to properly color the vertices of a bipartite graph is **1**

(a) 2 (b) 3 (c) 4 (d) 5

P.T.O.

ix. How many numbers can be formed taking only 3 out of the digits 3, 4, 5, 6, 7? **1**

(a) 3! (b) $P(5,3)$ (c) $C(5,3)$ (d) 5!

x. A sequence is defined by the recurrence relation $u_{n+1} = \frac{1}{4}u_n + 8$ with $u_0 = 32.$ The value of u_2 is? **1**

(a) 4 (b) 8 (c) 12 (d) 16

Q.2 i. If $f: X \to Y$ be one-one onto mapping then prove that $f^{-1}of = I_X$ and $fof^{-1} = I_Y$ where I_X and I_Y are the identity mappings in set X and Y respectively. **4**

ii. Determine the number of integers between 1 and 250 which are divisible by any of the number 2 , 3 , 5 and 7. **6**

OR iii. If R_1 and R_2 are equivalence relations in a set A, show that $R_1 \cap R_2$ is an equivalence relation in A. **6**

Q.3 i. Prove that in a distributive lattice; if an element has a complement then this complement is unique. **4**

ii. Use Karnaugh maps to simplify the sum of product expansion **6**

$xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}yz + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$

OR iii. Let X be the set of all complex numbers $z = x + iy$. For, $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$, let the meaning of $z_1 \leq z_2$ be : $x_1 \leq x_2$ and $y_1 \leq y_2$ where " $\leq$ " has usual meaning for the real numbers. **6**

Prove that " $\leq$ " is a partial order relation on X?

Q.4 i. Define Cyclic group. Show that the multiplicative group of three cube roots of unity is cyclic. How many generators are there in this group, write them. **4**

ii. Show that (I, +, .) where I is a set of all integers is a Ring. **6**

OR iii. Prove that the necessary and sufficient condition for a non-empty subset H of a group G to be a subgroup is that: **6**

$a \in H, b \in H \Rightarrow ab^{-1} \in H$ where b^{-1} is the inverse of b in G.

Q.5 i. Define the following: **4**

(a) Isomorphic graphs (b) Homeomorphic graphs

(c) Bipartite graph (d) Eulerian path

ii. Prove that a connected planar graph with n vertices and e edges has r regions given by r = e - n +2 **6**

OR iii. Define Chromatic number. Prove that every tree with two or more vertices is 2-chromatic but the converse is not true. **6**

Q.6 i. Determine the generating function for the following numeric function $a_r = \frac{r+1}{4^r}; r = 0,1,2 ....$ **4**

ii. Solve the following recurrence relation : **6**

$a_r - 5a_{r-1} + 6a_{r-2} = 2 + r\,; r \geq 2$ with boundary conditions $a_0 = 1$ and $a_1 = 1$

OR iii. (a) Prove that $C(n,r) = C(n-1, r) + C(n-1, r-1)$ **6**

(b) In how many ways can 8 Englishmen and 8 Americans can sit down at a round table, no two Americans being in consecutive positions?

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Marking Scheme

Total No. of Printed Pages:1

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Answers of Q.1 (MCQs)

Q.1	i.	(b)	
	ii.	d. Both a and c	1
	iii.	a. 30	1
	iv.	b. Absorption	1
	v.	c. i	1
	vi.	d. groupoid	1
	vii.	c. regular	1
	viii.	a. 2	1
	ix.	b. $P(5,3)$	1
	x.	c. 12	1

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Q1 - Answers of MCQ's are given seperately

Q2. (i) Let x be any arb. elem of X then

$f(x) = y$, where $y \in Y$,

then $f^{-1}(y) = x$ ——— (1)

$\therefore (f^{-1} \circ f)(x) = f^{-1}[f(x)]$ (by defn)

$= f^{-1}(y)$

$= x$ +(1)

It shows that funcn $(f^{-1} \circ f)$ maps every elem. $x \in X$ onto itself.

Hence $(f^{-1} \circ f)$ is an iden. mapping on X

$f^{-1} \circ f = I_X$ ——— +(1)

Similarly, $f \circ f^{-1} = I_Y$ ——— +(2)

(same process).

Q2 (ii) Let A_1 denotes the set of integers bet 1 & 250, divisible by 2

A_2 " " " " " " by 3

A_3 " " " " " " by 5

A_4 " " " " " " by 7

$\therefore |A_1| = \left|\frac{250}{2}\right| = 125$, $|A_2| = \left|\frac{250}{3}\right| = 83$

$|A_3| = \left|\frac{250}{5}\right| = 50$, $|A_4| = \left|\frac{250}{7}\right| = 35$ ——— (2)

$$|A_1 \cap A_2| = \left|\frac{250}{\text{LCM of } 2 \& 3}\right| = 41$$

$$|A_1 \cap A_3| = \left|\frac{250}{\text{LCM of } 2 \& 5}\right| = 25$$

$$|A_1 \cap A_4| = \left|\frac{250}{\text{LCM of } 2 \& 7}\right| = 17$$

Similarly

$$|A_2 \cap A_3| = 16, \quad |A_2 \cap A_4| = 11, \quad |A_3 \cap A_4| = 7$$

Now,

$$|A_1 \cap A_2 \cap A_3| = \left|\frac{250}{\text{LCM of } 2,3,5}\right| = 8$$

$$|A_1 \cap A_2 \cap A_4| = 5, \quad |A_1 \cap A_3 \cap A_4| = 3$$

$$|A_2 \cap A_3 \cap A_4| = 2 \text{ and } |A_1 \cap A_2 \cap A_3 \cap A_4| = 1$$ +(2)

Using Principle of ~~Mathem~~ Ind^n & Excl^n

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = 125 + 83 + 50 + 35 - 41 - 25 - 17 - 16 - 11 - 7 + 8 + 5 + 3 + 2 - 1$$

$$= 193$$ +(2)

R Q2 (iii) Since R_1 & R_2 are rel^{n}s in A

so, $R_1 \subseteq A \times A$ & $R_2 \subseteq A \times A$.

Hence, $R_1 \cap R_2 \subseteq A \times A$ and thus,

$R_1 \cap R_2$ is also a rel^n in A. +(1)

(a) $R_1 \cap R_2$ is reflexive

Since R_1 is reflexive $\Rightarrow (a,a) \in R_1 \ \forall a \in R_1$

" R_2 " " $\Rightarrow (a,a) \in R_2 \ \forall a \in R_2$

$\therefore \forall a \in A \Rightarrow (a,a) \in R_1 \cap R_2$

$R_1 \cap R_2$ is reflexive. +(1)

(ii) $R_1 \cap R_2$ is symmetric :-

let $(a,b) \in R_1 \cap R_2$

Now $(a,b) \in R_1 \cap R_2 \Rightarrow (a,b) \in R_1$ & $(a,b) \in R_2$

$\Rightarrow (b,a) \in R_1$ & $(b,a) \in R_2$

($\because R_1$ & R_2 are sym)

$\Rightarrow (b,a) \in R_1 \cap R_2$

$\therefore R_1 \cap R_2$ is symmetric ——— +(2)

(c) $R_1 \cap R_2$ is transitive :-

$(a,b) \in R_1 \cap R_2$, $(b,c) \in R_1 \cap R_2$

$\Rightarrow [(a,b) \in R_1, (b,c) \in R_1]$ & $[(a,b) \in R_2$ & $(b,c) \in R_2]$

$\Rightarrow [(a,c) \in R_1$ & $(a,c) \in R_2]$

$\Rightarrow (a,c) \in R_1 \cap R_2$

$\Rightarrow R_1 \cap R_2$ is transitive

Hence $R_1 \cap R_2$ is ~~reflex~~ equival. rel^n. ——— +(2)

Q.3 (i) Def A gp $(G, \cdot)$ is called cyclic if for $a \in G$, every elem. $x \in G$ is of the form a^n, where $n \in I$

The elem. a is called the generator of G ——— (2)

The mul. gp. of three cube roots of unity $\{1, \omega, \omega^2\}$ can be written as $\{\omega, \omega^2, \omega^3\}$

$\therefore$ it is a cyclic gp. of order 3. ~~The gen~~

Similarly, $\{\omega^2, (\omega^2)^2, (\omega^2)^3\}$ ——— +(1)

The generators of this gp. are ω & ω^2 ——— +(1)

Q 3) (ii) To show $(I, +, \cdot)$ is a ring

R_1 : $(I, +)$ is abelian gp.

Closure: $\forall\ a, b \in I \Rightarrow a+b \in I$

Associative: $\forall\ a, b, c \in I \Rightarrow (a+b)+c = a+(b+c)$

Exis. of Inverse: $\forall\ a \in I$, $\exists$ an int. $-a$ s.t.

$$a + (-a) = 0 = (-a) + a$$

Exis of Iden: $\forall\ a \in I$, $\exists$ an int '0' s.t. +(3)

$$a + 0 = 0 + a = a$$

Commutative: $\forall\ a, b \in I$, $a+b = b+a$ (2)

R_2 : $(I, \cdot)$ is a semigp.

Closure: $\forall\ a, b \in I \Rightarrow a \cdot b \in I$ +(2)

Asso. : $\forall a, b, c \in I \Rightarrow a\cdot(b\cdot c) = (a\cdot b)\cdot c$ +(2)

R_3 : Distributive Law

$\because$ mul. of int. is distributive w.r.t. add^n

i.e., $a\cdot(b+c) = a\cdot b + a\cdot c \ \forall\ a, b, c \in I$ +(1)

$(b+c)\cdot a = b\cdot a + c\cdot a \ \forall a, b, c \in I$ +(2)

Hence $(I, +, \cdot)$ is a ring.

OR Q 3. (iii) The cond^n is necessary:

Let H be a subgp. of G. Let $a \in H$, $b \in H$.

Now each elem. of H must possess inverse b/c H itself is a gp.

$\therefore\ b \in H \Rightarrow b^{-1} \in H$ +(1)

Further H is closed w.r.t. mul^n

$\therefore\ a \in H, b \in H \Rightarrow a \in H, b^{-1} \in H$

$\Rightarrow ab^{-1} \in H$ +(1)

Necessary cond^n is satisfied (2)

<u>The cond^n is sufficient</u>:

Let H be non-empty subset of G s.t.

$a \in H, b \in H \Rightarrow ab^{-1} \in H$ ——— (1)

T.P. H is a subgp. of G

Exis. of Identity : we have, $a \in H, a \in H$

by (1) $aa^{-1} \in H$

$\Rightarrow e \in H$ +(1)

$\therefore$ e is the iden. of H

Exis of Inverse : Let $a \in H$, then by (1)

$e \in H, a \in H \Rightarrow ea^{-1} \in H$

$\Rightarrow a^{-1} \in H$ +(1)

$\therefore$ each elem. of H possess inverse.

Closure prop : Let $a, b \in H$ then $b \in H$

$\Rightarrow b^{-1} \in H$

Thus, $a, b^{-1} \in H \Rightarrow a(b^{-1})^{-1} \in H$ (by (1)) +(1)

$\Rightarrow ab \in H$

H is closed w.r.t. comp^n '$\cdot$' in G.

Associativity : $\forall a, b, c \in H$

+(1)

$\Rightarrow (ab)c = a(bc)$ ——— (4)

Asso. law holds.

$\therefore$ H is a subgp of G

Q. ~~4~~3 (i) Let $(L, \leq)$ be a bdd. distributive lattice. Let $a \in L$ be an elem. and let b & c are its complements. Then

$a \vee b = 1$, $a \wedge b = 0$

$a \vee c = 1$, $a \wedge c = 0$ ——— (1.5)

To prove the theorem, we show $b = c$

$b = b \wedge 1$

$= b \wedge (a \vee c)$

$= (b \wedge a) \vee (b \wedge c)$ [$\because \wedge$ is dist. over $\vee$]

$= (a \wedge b) \vee (b \wedge c)$ (comm.)

$= a \vee (b \wedge c)$

$= (a \wedge c) \vee (b \wedge c)$

$= (a \vee b) \wedge c$

$= 1 \wedge c$ ($\because a \vee b = 1$)

$= c$ $+(2 \cdot 5)$

Q 43 (ii)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1	1	1
$\bar{x}$	1		1	1

(4)

Studying the K-map

Minimized Boolean expansion is

$\boxed{x + \bar{y} + z}$ $+(2)$

OR" Q 43 (iii) Let $X = \{ z = x + iy ; x, y \in R \}$

Then,

(a) Reflexivity : For $z = x + iy \in X$,

we have $x = x$, $y = y$

$\Rightarrow x \leq x, \; y \leq y$

$\Rightarrow z \leq z \;\; \forall z \in X$

Hence the relⁿ $\leq$ is reflexive (2)

(b) Anti-symmetric : Let $z_1 = x_1 + iy_1, \in X$
$z_2 = x_2 + iy_2 \in X$

and let $z_1 \le z_2$ & $z_2 \le z_1$

Then, $z_1 \le z_2$, $z_2 \le z_1$

$\Rightarrow x_1 \le x_2$ & $y_1 \le y_2$, $x_2 \le x_1$ & $y_2 \le y_1$

$\Rightarrow x_1 = x_2$ & $y_1 = y_2$

$\Rightarrow x_1 + iy_1 = x_2 + iy_2$

$\Rightarrow z_1 = z_2$

Hence relⁿ $\le$ is anti-symm. on the set X — +(2)

(c) Transitive :

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_3 = x_3 + iy_3 \in X$

& $z_1 \le z_2$, $z_2 \le z_3$

$\Rightarrow x_1 \le x_2$ & $y_1 \le y_2$, $x_2 \le x_3$ & $y_2 \le y_3$

$\Rightarrow x_1 \le x_3$ & $y_1 \le y_3$

$\Rightarrow z_1 \le z_3$

Hence the relⁿ is transitive on X — +(2)

$\therefore$ Partial ordered relⁿ

~~"OR" Q4 (iii)~~

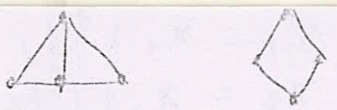
Q5. (i) Isomorphic gph – Two graphs are said to be isomorphic if their graph is a one-one correspondence b/w their vertices & between their edges s.t. their incidence relationship is preserved. Thus,

(i) they have same no. of edges

(ii) " " " " " vertices

(iii) they have equal no. of vertices with a given degree.

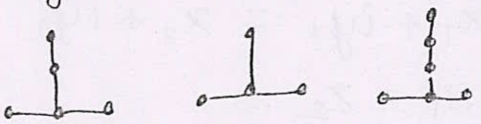
Ex



(a) (b)

Isomorphic graphs ——— (1)

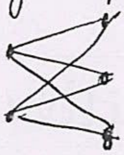
(b) <u>Homeomorphic graphs</u> – Two graphs G & G' are called homeomorphic graphs if they can be obtained from the same graph H by dividing an edge H with additional vertices. <u>Ex</u>



——— +(1)

(c) <u>Bipartite graph</u> – Let ~~G~~ G(V, E) be a graph. G is bipartite graph if its set V can be partitioned into two subsets H & S ~~such~~ such that each edge of G connects a vertex of H to a vertex of S. ~~If each vertex of H is connected to each vertex of S, then such a graph is called~~

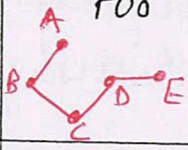
<u>Ex</u>

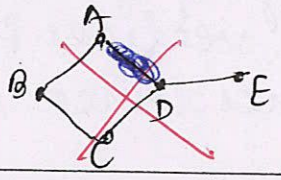


$K_{2,3}$ ——— +(1)

(d) <u>Eulerian Path</u> → G = (V, E) is defined as a path which traverses each edge in the graph G once & only once.

For ex:





Eulerian path A-B-C-D-E ——— +(1)

Q5 ~~(ii) We shall prove the theorem~~

Q5 (ii)

Proof. We shall prove the theorem by induction on the number of edges in G, where G is a connected planar graph.

Suppose $e = 1$ then n may be equal to 1 or 2.

In case $e = 1, n = 2$, see Fig. 4.75 (a), then number of regions $r = 1 - 2 + 2 =$

Clearly graph (a) in Fig. 4.75 has one region (only infinite region)

$e = 1, n = 2$ (a) $\quad$ $e = 1, n = 2$ (b) —— (2)

Fig. 4.75

Again in case $e = 1, n = 1$, then $r = 1 - 1 + 2 = 2$. The graph (b) in Fig. 4.75 has two regions r_1 and r_2.

Hence the result is true for $e = 1$.

Now suppose that the result holds for all graphs with at most $e - 1$ edges. Assume that G is a connected graph with e edges and r regions. In case G is a tree then $e = n - 1$ and number of regions (*i.e.*, faces) is 1 (only infinite region).

In this case by the formula, we have

$$r = e - n + 2 = (n - 1) - n + 2 = 1. \quad —— \quad +(2)$$

Hence the theorm holds in case G is a tree. Now consider the case when G is not a tree, then G has some circuits. Consider an edge 'c' say, in some circuit. By removing this edge 'c' from the plane representation of G, the regions are merged into a new region. Therefore $G - \{c\}$ is a connected graph with n vertices, $e - 1$ edges and $r - 1$ regions (where the number of regions in G is r). Thus by induction hypothesis, we have

$$r - 1 = (e - 1) - n + 2 \quad \text{or} \quad r = e - n + 2. \quad —— \quad +(2)$$

This proves the theorem.

OR Q5 (iii) chromatic no. – is the minimum no. of colours required for proper coloring the vertices of G is denoted by $\chi(G)$.
Thus if graph G requires n diff. colors (but not less than n) for proper coloring then $\chi(G) = n$ & is n-chromatic graph —— (2)

5 (iii) -biparti

roof. Consider a tree T with v as one of its vertex.

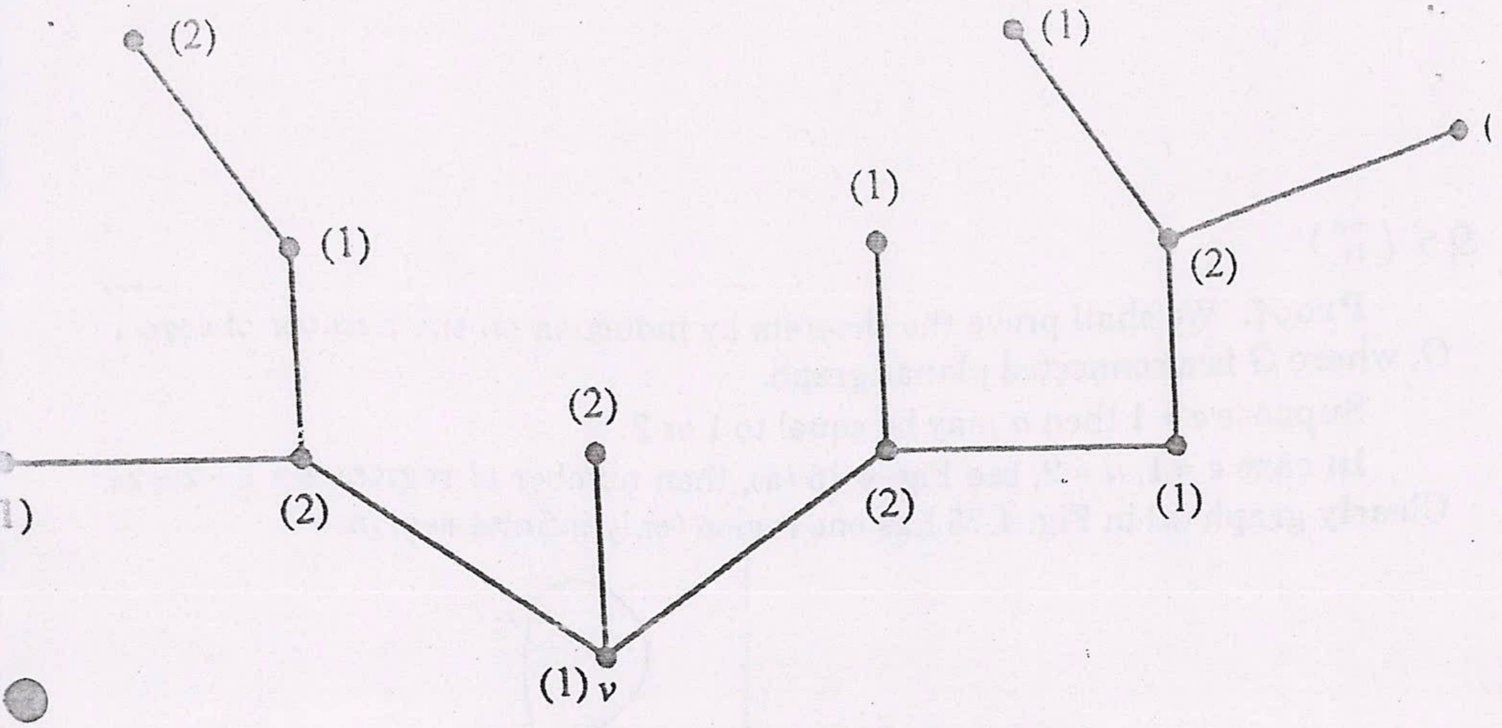


Fig. 4.82 (Proper coloring of tree)

To start proper coloring of T, we paint vertex v with color 1. Now we p ces adjacent to v with color 2. Next we paint all the vertcies adjacent t ces which have been colored with color 2 using color 1. We contin ess till every vertex in the tree T has been painted as shown in Fig. 4.8

We know that in a tree there is one and only one path between a ices, therefore, no two adjacent vertices will have the same color. Thus t s been properly colored with two colors. Again, a tree has atleast two at least one edge and so one color is not enough. + (2)

Hence a tree T is 2-chromatic *i.e.*, $\chi(T) = 2$.

Conversely. Now we shall prove that every 2-chromatic graph is no shall prove it by an example : See the graph in Fig. 4.83 which is 2-ch is not a tree.

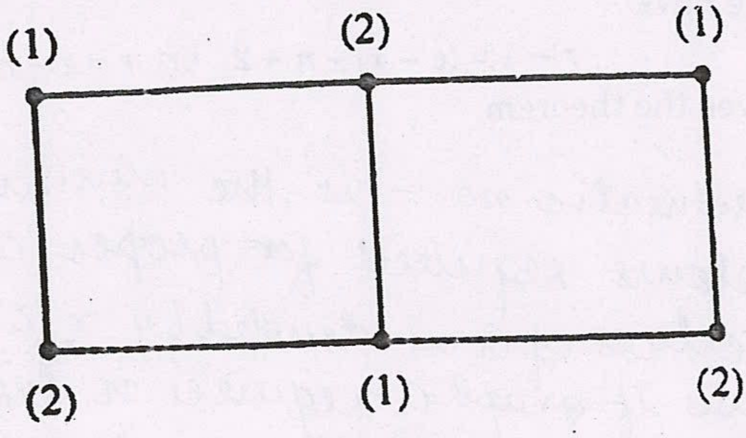


Fig. 4.83 + (2)

Q.6 (i) The generating func^n $A(z)$ for the given numeric func^n $a_0=1$, $a_1=\frac{2}{4}$, $a_2=\frac{3}{16}$, $a_3=\frac{4}{64}$ etc. is given by

$$A(z) = a_0 + a_1 z + a_2 z^2 + \text{———}$$

$$= 1 + \frac{2}{4}z + \frac{3}{16}z^2 + \frac{4}{64}z^3 + \cdots + \frac{(r+1)}{4^r}z^r + \text{——} \quad \text{——} (2)$$

$$= 1 + 2\left(\frac{z}{4}\right) + 3\left(\frac{z}{4}\right)^2 + \cdots$$

$$= \left(1 - \frac{z}{4}\right)^{-2}$$

$$= \frac{16}{(4-z)^2} \quad \text{——} +(2)$$

Q 6 (ii) $a_r - 5a_{r-1} + 6a_{r-2} = 2 + r$ ——— (1)

Char. eq^n is $m^2 - 5m + 6 = 0$

$\Rightarrow m = 2, 3$

$a_r^{(h)} = c_1 \cdot 2^r + c_2 \cdot 3^r$ ——— (2) ——— (2)

The particular sol^n (trial sol^n) corresponding to the term $2+r$ on RHS of (1) is $A_0 + A_1 r$

$a_r^{(P)} = A_0 + A_1 r$

substituting (3) in (1) we get

$$(A_0 + A_1 r) - 5\{A_0 + A_1(r-1)\} + 6\{A_0 + A_1(r-2)\} = 2 + r$$

or $(2A_0 - 7A_1) + 2A_1 r = 2 + r$

comparing two sides of (4), we get

$\cancel{(2A_0 - 7A_1) + 2A_1 r = 2 + r}$

$2A_0 - 7A_1 = 2$ & $2A_1 = 1$

Solving $A_0 = \frac{11}{4}$, $A_1 = \frac{1}{2}$

∴ putting A_0 & A_1 in (3), we get

$$a_r^{(p)} = \frac{11}{4} + \frac{1}{2} r \qquad +(2)$$

∴ Total solⁿ of (1) is given by

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = C_1 \cdot 2^r + C_2 \cdot 3^r + \frac{11}{4} + \frac{1}{2} r \quad \text{—(5)}$$

Now put $r = 0, 1$ & using boundary condⁿˢ in (5) we get,

$$C_1 + C_2 + \frac{11}{4} = 1 \Rightarrow C_1 + C_2 = -7/4 \quad \text{—(6)}$$

$$\& \quad 2C_1 + 3C_2 = -9/4 \quad \text{—(7)}$$

Solving (6) & (7)

$$C_1 = -3, \quad C_2 = \frac{5}{4}$$

$$\therefore a_r = -3 \cdot 2^r + \frac{5}{4} \cdot 3^r + \frac{11}{4} + \frac{1}{2} r \qquad +(2)$$

Ans.

Q 6 (ii)(a) L.H.S $= {}^nC_r = \dfrac{n!}{r!(n-r)!} = \dfrac{n!}{r!(n-r)!}\left[\dfrac{n-r}{n} + \dfrac{r}{n}\right]$

$$= \frac{1}{n} \cdot \frac{(n-r) \cdot n!}{r!(n-r)!} + \frac{r}{n} \cdot \frac{n!}{r!(n-r)!} \qquad (1)$$

$$= \frac{1}{n} \cdot \frac{(n-r)\, n\, (n-1)!}{r!\,(n-r)\,(n-r-1)!} + \frac{r}{n} \cdot \frac{n(n-1)!}{r(r-1)!\,(n-r)!}$$

$$= \frac{(n-1)!}{r!\,(n-r-1)!} + \frac{(n-1)!}{(r-1)!\,\{(n-1)-(r-1)\}!}$$

$$= {}^{n-1}C_r + {}^{n-1}C_{r-1} \qquad +(2)$$

$= \text{RHS}.$

Hence proved

(b) Keeping one Englishman in a fixed position & then arranging the remaining 7 in all positions, the total no. of diff. ways in which the Englishmen may be seated $= 7! = 5040$ — 1

Now, the 8 Americans can be seated at 8 positions in between two Englishmen in $8!$ ways — (1.5)

Hence the total no. of required arrangements

$= 7! \times 8! = 5040 \times 40320$

$= 203212800$ Ans — +(1.5)

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