Enrollment No......................................

Faculty of Science

End Sem (Odd) Examination Dec-2019

BC3CO03 Mathematics-I

Programme: B.Sc. (CS) Branch/Specialisation: Computer Science

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. The equation $2x^7 - x^5 + 4x^3 - 5x$ can have: **1**

(a) 2 negative roots (b) More than 2 negative roots
(c) 1 negative roots (d) Only real roots

ii. One root of equation $3x^3 - 4x^2 + x + 88 = 0$ is $2+\sqrt{7}\,i$ then second root is: **1**

(a) 2 (b) $2-\sqrt{7}\,i$ (c) $\sqrt{7}\,i$ (d) None of these

iii. The rank of two equivalent matrices are: **1**

(a) Unequal (b) Equal (c) Less than 1 (d) Zero

iv. If rank of a matrix is equal to the number of unknowns, then the equations AX=0 has: **1**

(a) Trivial solution
(b) Finite non trivial solution
(c) Infinite non trivial solution
(d) None of these

v. Every constant function is: **1**

(a) Continuous somewhere (b) Not continuous
(c) Continuous everywhere (d) None of these

vi. Derivative of $log(1+x^2)$ is: **1**

(a) $\frac{x}{1+2x}$ (b) $\frac{2x}{1+x^2}$ (c) $\frac{x}{1+x^2}$ (d) None of these

vii. If $f(x)$ is continuous function and its derivative upto n^{th} and higher order exists, then in Maclaurin's theorem $f(x)=$ **1**

(a) $f(a)+ hf'(a)+ \frac{h^2}{2!}f''(a)+....$ (b) $f(0)+xf'(0)+ \frac{x^2}{2!}f''(0)+....$
(c) $1+x+ \frac{x^2}{2!}+.....$ (d) $f(0 - xf'(0)+ \frac{x^2}{2!}f''(0) - ....$

viii. The tangent at any point on the curve will be parallel to X axis if and only if: **1**

(a) $f'(x)=0$ (b) $f'(x)=c$ (c) $f''(x)=0$ (d) None

ix. Beta Function is a **1**

(a) Symmetric (b) Skew Symmetric

(c) Non symmetric (d) None of these

x. The value of $\int_0^2 \int_0^1 xydxdy$ **1**

(a) 2 (b) 1 (c) 1/2 (d) 4

Q.2 i. Solve the equation $x^3 - 7x^2+36=0$, given that one root is double of another. **2**

ii. Transform the equation $x^3 - 6x^2+5x+8=0$ into another in which the second term is missing. **3**

iii. Solve the equation by cardan's method $x^3 -3x^2+3=0$ **5**

OR iv. Solve $6x^6 - 25x^5+31x^4 - 31x^2+25x - 6=0$ **5**

Q.3 i. Define rank and nullity of a matrix. Give an example **3**

ii. Find eigen values and eigen vectors of the matrix A=$\begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$ **7**

OR iii. Examine the following equations for consistency and if consistent find the complete solution $x+y+z+3=0,\ 3x+y -2z+2=0,\ 2x+4y+7z - 7=0$ **7**

Q.4 i. Define uniform continuity of the function and prove that $f(x)=x$ is uniform continuous in the interval $(0, \infty)$ **3**

ii. Examine the continuity of the function at x=1 and x=2 **7**

$$f(x) = \begin{cases} x\ , 0 < x < 1 \\ 2 - x\ , \quad 1 \le x \le 2 \\ 3x - 5\ ,\ x > 2 \end{cases}$$

OR iii. State Mean value theorem and Verify Lagrange's mean value theorem for the function $f(x) = 2x^2 - 10x + 29$ in [2,7]. **7**

Q.5 Attempt any two:

i. Expand $\tan^{-1}x$ in powers of $(x- \pi/4)$ up to 4 terms by Taylors Series. **5**

ii. Find all asymptotes of the curve $x^3+3x^2y - 4y^3 - x+y+3=0$ **5**

iii. Find radius of curvature at the point (r, θ) of the curve $r=a\ (1+\cos\theta\ )$ **5**

Q.6 Attempt any two:

i. Solve $\lim_{n\to\infty} \left\{\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{1}{2n}\right\}$ **5**

ii. If $Un = \int_0^{\pi/2} \theta \sin^n\theta d\theta$, Prove that $Un = \frac{n-1}{n} U_{n-2} + \frac{1}{n^2}$ Hence Evaluate U_3 and U_5 **5**

iii. Express $\int_0^1 x^m(1-x^n)^p dx$ in terms of beta/ gamma function and hence evaluate $\int_0^1 x^5(1-x^3)^{10} dx$ **5**

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Faculty of Science
Course Name:- Mathematics
course code:- BC3C003
Branch: BSc-Ist year

Q. 1

(i) (b) more than 2 negative roots.

(ii) (b) $2-\sqrt{7}i$

(iii) (b) Equal

(iv) (a) Trivial solution

(v) (a) Continuous everywhere

(vi) (b) $2x/1+x^2$

(vii) (b) $f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots$

(viii) (a) $f'(x)=0$

(ix) (a) Symmetric

(x) (b) 1.

Q.2(i) Let the roots be α, β, γ such that $\beta = 2\alpha$

Also $\alpha+\beta+\gamma=7$, $\alpha\beta+\beta\gamma+\gamma\alpha=0$, $\alpha\beta\gamma=-36$

$\therefore$ $3\alpha+\gamma=7$, $2\alpha^2+3\alpha\gamma=0$, $2\alpha^2\gamma=-36$ +1

Solving $\alpha=3$, $\gamma=-2$, $\beta=6$ +1

Q.2(ii) Sum of the roots of the given eq. $=6$

put $y = x-(6/3) = x-2$ +1

$\Rightarrow x = y+2$ in given eq., we have

y^3-7y+2 +2

Q.2(iii) Given equation $x^3-3x^2+3=0$ ---(i)

To remove the x^2 term put $y = x-(\frac{+3}{3})$

$\Rightarrow y = x-1 \Rightarrow x = y+1$

so that (i) becomes y^3-3y+1 --- (ii) +1

To solve it put $y = u+v$

so that $y^3 = u^3+v^3+3uv(u+v)$

$\Rightarrow y^3-3uvy-(u^3+v^3)=0$ --- (iii) +1

Comparing (ii) and (iii), we get

$uv = 1, \quad u^3 + v^3 = -1$

$\therefore$ u^3, v^3 are the roots of the equation

$t^2 + t + 1 = 0$.

Hence $u^3 = \frac{-1 + i\sqrt{3}}{2}$, and $v^3 = \frac{-1 - i\sqrt{3}}{2}$

$\therefore u = \left(\frac{-1 + i\sqrt{3}}{2}\right)^{1/3}$

Put $-\frac{1}{2} = r\cos\theta, \ \frac{\sqrt{3}}{2} = r\sin\theta$.

$\Rightarrow r = 1, \ \theta = 2\pi/3$

$\therefore u = [r\cos\theta + i\, r\sin\theta]^{1/3} = [\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]^{1/3}$

$n = 0, 1, 2$

$u = \cos\left(\frac{\theta + 2n\pi}{3}\right) + i\sin\left(\frac{\theta + 2n\pi}{3}\right)$

Put $n = 0, 1, 2$,

We get $\cos\frac{2\pi}{9} + i\sin\left(\frac{2\pi}{9}\right)$, $\cos\left(\frac{8\pi}{9}\right) + i\sin\frac{8\pi}{9}$, +1

$\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}$,

The corresponding value of v

$\cos\frac{2\pi}{9} - i\sin\frac{2\pi}{9}$, $\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}$, $\cos\frac{14\pi}{9} - i\sin\frac{14\pi}{9}$ +1

$\therefore y = u + v$

$y = 2\cos\frac{2\pi}{9}, \ 2\cos\frac{8\pi}{9}, \ 2\cos\frac{14\pi}{9}$.

$\therefore x = 1 + y = 1 + 2\cos\frac{2\pi}{9}, \ 1 + 2\cos\frac{8\pi}{9}$,

$1 + 2\cos\frac{14\pi}{9}$. +1

2(iv) Given equation $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$ ---(i)

This is a reciprocal equation of even degree with opposite signs, $\therefore$ 1, −1 are its roots. +1

Dividing L.H.S. by $x-1$ and $x+1$, the given equation reduces to

$6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0$ +1

Dividing by x^2

$$6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x + \frac{1}{x}\right) + 37 = 0$$

putting $y = x + \frac{1}{x}$, and $x^2 + \frac{1}{x^2} = y^2 - 2$ +1

$\Rightarrow 6(y^2-2) - 25y + 37 = 0$

$\Rightarrow 6y^2 - 12 - 25y + 37 = 0$

$\Rightarrow 6y^2 - 25y + 25 = 0$

$\Rightarrow (2y-5)(3y-5) = 0 \Rightarrow y = 5/2,\ y = 5/3$ +1

$x + \frac{1}{x} = \frac{5}{2},\quad x + \frac{1}{x} = 5/3$

$\Rightarrow 2x^2 - 5x + 2 = 0$ or $3x^2 - 5x + 3 = 0$

$\therefore x = 2, \frac{1}{2}$ or $x = \frac{5 \pm i\sqrt{11}}{6}$ +1

Hence the roots of the given equation are $1, -1, 2, \frac{1}{2}, \frac{5+i\sqrt{11}}{6}, \frac{5-i\sqrt{11}}{6}$.

8 (i) A matrix $A = [a_{ij}]_{m\times n}$ is said to be of rank r, when

(i) it has at least one non-zero minor of order r

(ii) every minor of order higher than r vanishes. +1

Ex: The rank of a matrix $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$ is 2. 0.5

Nullity:- If $A=[a_{ij}]_{n\times n}$ is an square matrix, then the nullity of A = order − rank of A +1

Ex: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$

rank of A = 2

∴ nullity of A = 3−2 = 1. 0.5

(ii) Given $A = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$

The eigen values of A are 3, 4, 5 +3

Eigen vector corresponding to $\lambda=3$:

let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A

corresponding to $\lambda = 3$

$(A-\lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 8 & 1 & 0 \\ 6 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank of the coefficient matrix = 2

no. of unknown variable = 3

∴ 3−2 = 1 variable will be free

say $x_3 = K$,

$8x_1 + x_2 = 0$, $6x_1 + 2x_2 + 2x_3 = 0$

$$\Rightarrow 8x_1 + x_2 = 0$$
$$3x_1 + x_2 = -K$$
$$\Rightarrow 5x_1 = K \Rightarrow x_1 = K/5$$
$$\therefore x_2 = -8x_1 = -8K/5$$

$$\therefore X_1 = \begin{bmatrix} K/5 \\ -8K/5 \\ K \end{bmatrix} = \frac{K}{5}\begin{bmatrix} 1 \\ -8 \\ 5 \end{bmatrix}$$ +2

similarly the eigen vectors corresponding to $\lambda = 4, 5$ are $K\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $K\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ respectively +2

3 (iii) The matrix form of the given system

$$AX = B \quad \text{---} (i)$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}$$ +1

$$\text{Augmented matrix} = [A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{array}\right]$$ +1

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right], \quad R_3 \to R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{array}\right]$$ +3

$\therefore \rho([A|B]) = 3, \quad \text{and } \rho(A) = 2$

$\therefore \rho(A) \neq \rho([A|B])$

$\Rightarrow$ The system will have no solution and system is inconsistent +2

4(i) A function $f: I \to \mathbb{R}$ is said to be uniformly continuous on I if for each $\epsilon > 0 \ \exists \ \delta = \delta(\epsilon)$ such that for any two points $x_1, x_2 \in I$,

$|f(x_1) - f(x_2)| < \epsilon, \quad |x_1 - x_2| < \delta$ +2

If $x_1, x_2 \in (0, \infty)$,

$|f(x_1) - f(x_2)| = |x_1 - x_2| < \epsilon,$

Hence for each $\epsilon > 0 \ \exists \ \delta = \epsilon > 0$ s.t

$|f(x_1) - f(x_2)| < \epsilon,$ whenever $|x_1 - x_2| < \delta$

$\Rightarrow f(x) = x$ is U.C. on $(0, \infty)$ +1

4(ii) Given

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 \leq x \leq 2 \\ 3x-5, & x > 2 \end{cases}$$

At $x = 1$: $f(1) = 1$ +1

$L.H.L. = f(1-) = \lim_{x \to 1^-} f(x)$

Put $x = 1 - h$.

$= \lim_{h\to 0} f(1-h) = \lim_{h\to 0} (1-h) = 1$ +1

R.H.L. $= f(1+) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} 2-(1+h) = 1$ +1

$\therefore$ L.H.L = R.H.L = $f(1)$

$\Rightarrow \lim_{x\to 1} f(x) = f(1) = 1$

$\Rightarrow f(x)$ is continuous at $x=1$ +1

At $x=2$: $f(2) = 0$ +1

L.H.L. $= f(2-) = \lim_{h\to 0} 2-(2-h) = 0$

R.H.L $= f(2+) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} 3(2+h)-5$ +1
$= 1$

$\Rightarrow$ L.H.L $= 0 = f(2) \neq$ R.H.L

$\rightarrow f(x)$ is discontinuous at $x=2$. +1

P.4(iii) Mean-value theorem: (i) If $f(x)$ is continuous in the closed interval $[a,b]$ and (ii) $f'(x)$ exists in the open interval (a,b), then there is at least one value c of x in (a,b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$ +3

Here $f(x) = 2x^2 - 10x + 29$, $a=2$, $b=7$

We know that polynomial is continuous everywhere and $f'(x) = 4x-10$ exists in $(2,7)$ +1

$f(2) = 2.2^2 - 20 + 29 = 17$

$f(7) = 2\times 49 - 70 + 29 = 98 - 41 = 57$ +1

By Lagrange's mean value theorem $\exists$

$\frac{f(b)-f(a)}{7-2} = f'(c) \Rightarrow \frac{57-17}{5} = 4c-10$

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$8 = 4c - 10 \Rightarrow 4c = 18$

$\Rightarrow c = 4.5 \in [2,7]$ +2

Q.5(i) $f(x) = \tan^{-1}x$, $f'(x) = (1+x^2)^{-1}$

$f''(x) = -2x(1+x^2)^{-2}$, $f'''(x) = -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3} = \frac{-2(1+x^2)+8x^2}{(1+x^2)^3} = \frac{-2+6x^2}{(1+x^2)^3}$

$f^{iv}(x) = 4(1+x^2)^{-3}\cdot(2x) + 8\,2x(1+x^2)^{-3} + 8x^2\cdot(-3)(1+x^2)^{-4}\cdot 2x$

$f^{iv}(x) = \frac{8x}{(1+x^2)^3} + \frac{16x}{(1+x^2)^3} - \frac{48x^3}{(1+x^2)^4}$

$= \frac{24x}{(1+x^2)^3} - \frac{48x^3}{(1+x^2)^4}$ +2

put $x = \pi/4$ in all

$f(\pi/4) = \tan^{-1}(\pi/4)$, $f'(\pi/4) = (1+\frac{\pi^2}{4^2})^{-1}$

$f''(\pi/4) = -2\,\pi/4\,(1+\pi^2/4^2)^{-2}$,

$f'''(\pi/4) = \frac{-2+6(\pi/4)^2}{(1+\pi^2/4^2)^3}$, $f^{iv}(\pi/4) = \frac{6\pi}{(1+\frac{\pi^2}{4^2})^3} - \frac{\frac{48\pi^3}{4^3}}{(1+\frac{\pi^2}{4^2})^3}$ +2

By Taylor's Series.

$\tan^{-1}x = \tan^{-1}(\frac{\pi}{4}) + (x-\frac{\pi}{4})(1+\frac{\pi^2}{4^2})^{-1} +$

$\frac{1}{2!}(x-\frac{\pi}{4})^2\left[\frac{-2+6(\pi^2/4^2)}{(1+\pi^2/4^2)^3}\right] + \cdots$ +2

5(ii) Putting $x=1$, and $y=m$ in the third degree terms

$$\phi_3(m) = 1+3m-4m^3$$ +1

$\therefore \phi_3(m) = 0$ gives $m = 1, -\frac{1}{2}, -\frac{1}{2}$

$\therefore \phi_2(m) = 0$, $\therefore c = -\dfrac{\phi_2(m)}{\phi_3'(m)}$ +1

$= -\dfrac{0}{3-12m^2} = 0$, when $m=1$, $= \frac{0}{0}$ form

when $m = -\frac{1}{2}$

Thus (when $m = -\frac{1}{2}$), c is to be obtained from

$$\frac{c^2}{2!}\phi_3''(m) + c\,\phi_2'(m) + \phi_1(m) = 0$$

$$\frac{c^2}{2!}(-24m) + (-1+m) = 0$$ +2

Putting $m = -\frac{1}{2}$, $c = \pm\frac{1}{2}$

Hence the asymtotes are $y = x$, $y = -\dfrac{x+1}{2}$, +1

$$y = -\frac{x-1}{2}$$

5(iii) $r = a(1+\cos\theta)$

The radius of curvature

$$\rho = \left[\frac{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}\right] = \frac{\{r^2+r'^2\}^{3/2}}{r^2+2r'^2-rr''}$$ +1

$$r' = -a\sin\theta, \quad r'' = -a\cos\theta$$ +1

$$\rho = \frac{2}{3}\sqrt{2ar}$$ +3

5 (I) Let $P = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2}$ +1

$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{1+(\frac{r}{n})^2} \cdot \frac{1}{n}$ +1

$P = \int_0^1 \frac{1}{1+x^2}\, dx = \left[\tan^{-1}x\right]_0^1 = \frac{\pi}{4}$ +3

6 (II) Given

$U_n = \int_0^{\pi/2} \theta \sin^n\theta \, d\theta$

$= \frac{n-1}{n} U_{n-2} + \frac{1}{n^2}$ +2

Put $n=3$

$U_3 = \frac{2}{3} U_1 + \frac{1}{9}$

$U_1 = \int_0^{\pi/2} \theta \cdot \sin\theta \, d\theta = \left[\theta \cdot (-\cos\theta)\right]_0^{\pi/2} + \int_0^{\pi/2} \cos\theta \, d\theta$

$= 1$

$U_3 = \frac{2}{3} + \frac{1}{9} = \frac{6+1}{9} = 7/9$ +2

and $U_5 = \frac{4}{5} U_3 + \frac{1}{25} = \frac{4}{5} \times \frac{7}{9} + \frac{1}{25}$

$= \frac{28}{45} + \frac{1}{25} = \frac{140+9}{225}$ +1

$= \frac{149}{125}$

5(iii) put $x^n = t \Rightarrow x = t^{1/n}$

$$dx = \frac{1}{n} t^{\frac{1}{n}-1} dt = \frac{1}{n} \cdot t^{(1-n)/n} dt$$

$$\int_0^1 x^m (1-x^n)^p dx = \frac{1}{n} \int_0^1 t^{m/n} (1-t)^p \cdot t^{\frac{1}{n}-1} dt$$

$$= \frac{1}{n} \int_0^1 t^{\left(\frac{m+1}{n}\right)-1} (1-t)^{(p+1)-1} dt$$ +3

$$= \frac{1}{n} \cdot \beta\left(\frac{m+1}{n}, p+1\right)$$

putting $m=5$, $n=3$, $p=10$

$$\int_0^1 x^5 (1-x^3)^{10} = \frac{1}{3} B\left(\frac{5+1}{3}, 10+1\right)$$

$$= \frac{1}{3} B(2,11) = \frac{1}{3} \frac{\Gamma(2)\Gamma(11)}{\Gamma(13)}$$

$$= \frac{1}{3} \cdot \frac{1!\,(10)!}{12!} = \frac{1}{396}$$ +2