Enrollment No......................................

Faculty of Engineering

End Sem Examination May-2024

EN3BS08 Linear Algebra

Programme: B.Tech. Branch/Specialisation: CSBS

Duration: 3 Hrs. Maximum Marks: 60

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d. Assume suitable data if necessary. Notations and symbols have their usual meaning.

Q.1 i. If k rows of a determinant A becomes identical when $x = a$, then $(x-a)^{k-1}$ is a ______ of A. **1**

(a) Multiple (b) Factor (c) Reciprocal (d) None of these

ii. Matrix $A = \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$ is a ______ matrix. **1**

(a) Skew symmetric (b) Symmetric

(c) Diagonal (d) None of these

iii. Matrix A is said to be positive definite if all eigen values of A is ______ 0. **1**

(a) $=$ (b) $<$ (c) $>$ (d) None of these

iv. Rank of the matrix $A = \begin{bmatrix} -3 & 2 \\ 1 & 3 \end{bmatrix}$ is______. **1**

(a) 0 (b) 1 (c) 3 (d) 2

v. Which of the following is/are correct? **1**

I. If two vectors are linearly dependent one of them is scalar multiple of the other.

II. A set which contains zero vector only is linearly independent.

III. A set of vectors which contains atleast one zero vector is linearly dependent.

(a) I and II (b) I and III (c) II and III (d) None of these

vi. ______ is a vector space? **1**

(a) $Q(R)$ (b) $C(R)$ (c) $Q(C)$ (d) None of these

vii. If W be any subspace of a finite dimensional inner product space V, then, which one is correct? **1**

(a) $V = W \oplus W^{\perp}$ (b) $W \cap W^{\perp} = \{0\}$

(c) $W^{\perp\perp} = W$ (d) All of these

	viii.	If $f: V_1 \rightarrow V_2$ is a one-one, onto and linear transformation, then $f^{-1}: V_2 \rightarrow V_1$ is ______.	**1**
		(a) On to but not linear (b) One-one but not linear	
		(c) Linear (d) None of these	
	ix.	______ value decomposition of a matrix can be written as multiplication of three simpler matrices.	**1**
		(a) Singular (b) Double (c) Multiple (d) None of these	
	x.	______ is a method used to simplify complex data by reducing its dimension.	**1**
		(a) Temporary component analysis	
		(b) Principal component analysis	
		(c) Both (a) and (b)	
		(d) None of these	
Q.2	i.	Define Hermitian matrix with example.	**4**
	ii.	Solve the system of equation by Cramer's rule: $3x + y + 2z = 3, 2x - 3y - z = -3, x + 2y + z = 4$	**6**
OR	iii.	Solve by inverse matrix method the system of equation: $x + y + z = 4, x - y + z = 0, 2x + y + z = 5$	**6**
Q.3	i.	Using LU decomposition method find the matrix L for the matrix- $A = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$	**4**
	ii.	Find the eigen values and eigen vectors of the matrix- $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.	**6**
OR	iii.	Apply Gauss Elimination method to solve the equations- $x + 4y - z = -5; x + y - 6z = -12; 3x - y - z = 4.$	**6**
Q.4	i.	Define dimension of vector space and state dimension theorem.	**4**
	ii.	Prove that the vectors (2,1,4), (1,-1,2),(3,1,-2) forms a basis for R^3.	**6**
OR	iii.	Check whether $W = \{(x, 2y, 3z): x, y, z \in R\}$ is a subspace of $V_3(R)$ or not where R is the field of real numbers.	**6**
Q.5	i.	Prove that $V_n(R)$ is an inner product space with an inner product defined on $\alpha = (a_1, a_2, \dots, a_n)$.	**4**
	ii.	Show that the function $T: V_2 \rightarrow V_2$ defined by $T(x, y) = (2x + 3y, 3x - 4y)$ is a linear transformation.	**6**

OR	iii.	Orthogonaize the set of linearly independent vectors $B = \{\beta_1, \beta_2, \beta_3\}$ of $V_4(R)$, where $\beta_1 = (1,0,1,1), \beta_2 = (-1,0,-1,1), \beta_3 = (0,-1,1,1)$.	**6**
Q.6	i.	Write application of principle component analysis in image processing.	**4**
	ii.	Explain singular value decomposition in detail with suitable example.	**6**
OR	iii.	Explain principle component analysis in detail with suitable example.	**6**

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Q.1.
i) b) factor
ii) a) Skew symmetric
iii) c) $>$
iv) d) 2
v) b) I and III
vi) b) $C(\mathbb{R})$
vii) d) All of these
viii) c) Linear
ix) a) Singular
x) b) Principal component analysis

Q2.i)

A square matrix (complex or real) is said to be Hermitian if:

$$A^{\theta} = A$$

where $A^{\theta} = (\overline{A})^{T}$ (transpose of conjugate) 2

Example.

$$A = \begin{bmatrix} 1 & 4+3i \\ 4-3i & 5 \end{bmatrix}$$ is

hermitian matrix as.

$$(\overline{A})^{T} = \begin{bmatrix} 1 & 4+3i \\ 4-3i & 5 \end{bmatrix} = A$$ 2

Q2 ii)

$$3x + y + 2z = 3$$
$$2x - 3y - z = -3$$
$$x + 2y + z = 4$$

A = matrix of coefficient

$$= \begin{bmatrix} 3 & -1 & 2 \\ 2 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3(-1+2) - 1(2+1) + 2(4+1)$$

$$= 3(1) - 3 + 2 \times 5$$

$$= 3 - 3 + 10$$

$$\boxed{|A| = \Delta = 10} \neq 0$$ 1

$$\Delta_1 = \begin{vmatrix} 3 & 1 & 2 \\ -3 & -1 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 3(-1+2) - 1(-3+4) + 2(-6+4)$$

$$= 3(1) - \cancel{3}1 - 2 \times 2$$

$$= 3 - 1 - 4$$

$$= 3 - 5 = -2 \neq 0$$ 1

$$\boxed{\Delta_1 = -2}$$

$$\Delta_2 = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 3(-3+4) - 3(2+1) + 2(8+3)$$

$$= 3 - 9 + 22$$

$$\boxed{\Delta_2 = 16} \neq 0$$ 1

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$$\Delta_3 = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -1 & -3 \\ 1 & 2 & 4 \end{vmatrix} = 3(-4+6) - 1(8+3) + 3(4+1)$$

$= 3(2) - 11 + 3\times5$

$= 6 - 11 + 15$

$= 6 + 4$

$\boxed{\Delta_3 = 10} \neq 0.$ 1

$x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$ 1

$x_1 = \frac{-2}{10}$, $y = \frac{16}{10}$, $z = \frac{10}{10}$

$\Rightarrow \boxed{x = -0.2, \quad y = 1.6, \quad z = 1}$ 1

Q2.
OR (iii)

$x + y + z = 4$
$x - y + z = 0$
$2x + y + z = 5$

Matrix of coefficient, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ 0.5

$X = A^{-1}B$. [inverse matrix method]

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$$\boxed{A^{-1} = \frac{1}{|A|}(\mathrm{Adj}\ A)}$$ 0.5

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= 1(-1-1) - 1(1-2) + 1(1+2)$$

$$= -2 + 1 + 3$$ 1

$$\boxed{|A| = 2}$$

$$\mathrm{Adj}\ A = \begin{bmatrix} -2 & -1 & 3 \\ 0 & -1 & -1 \\ 2 & 0 & -2 \end{bmatrix}^T$$

$$\mathrm{Adj}\ A = \begin{bmatrix} -2 & 0 & 2 \\ -1 & -1 & 0 \\ 3 & -1 & -2 \end{bmatrix}$$ 1

$$\boxed{A^{-1} = \frac{1}{2}\begin{bmatrix} -2 & 0 & 2 \\ -1 & -1 & 0 \\ 3 & -1 & -2 \end{bmatrix}}$$ 1

$$\boxed{X = A^{-1}B}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -1/2 & -1/2 & 0 \\ 3/2 & -1/2 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$ 1

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$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\boxed{x = 1, \quad y = -2, \quad z = 1}$$

Q3 ii)

$$A = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

$A = LU$ $\quad$ L - lower triangular matrix

U - Upper triangular matrix

$R_3 : R_3 - R_1$

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 0 & 2 & -6 \end{bmatrix}$$ 1

$R_2 : R_2 - \frac{2}{3} R_1$

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 2 & -6 \end{bmatrix}$$ 1

$R_3 : R_3 - \frac{6}{5} R_2$

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} = U$$ 1

Pg: 6

Here $L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix}$ 1

Q3 ii)

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$= (1-\lambda)\begin{vmatrix} 5-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$-1\begin{vmatrix} 1 & 1 \\ 3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} 1 & 5-\lambda \\ 3 & 1 \end{vmatrix} = 0$$

$$0 = [(1-\lambda)((5-\lambda)(1-\lambda)-1)] - (1-\lambda-3)$$
$$+3(1-15+3\lambda)$$

$$0 = [(1-\lambda)(5-5\lambda-\lambda+\lambda^2-1)] - (-\lambda-2)$$
$$+3(-14+3\lambda)$$

$$= 5-5\lambda-\lambda+\lambda^2-1-5\lambda+5\lambda^2+\lambda^2-\lambda^3+\lambda$$
$$+\lambda+2-42+9\lambda$$

$$0 = -\lambda^3+7\lambda^2+0\lambda-36$$

at $\lambda=-2$, $-\lambda^3+7\lambda^2-36=0$

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$$\lambda+2 \overline{) \lambda^3 - 7\lambda^2 + 36} \left(\lambda^2 - 9\lambda + 18 \right.$$

$$\lambda^3 + 2\lambda^2$$

$$- \quad -$$

$$-9\lambda^2 + 0\lambda + 36$$

$$-9\lambda^2 - 18\lambda$$

$$+ \quad +$$

$$18\lambda + 36$$

$$18\lambda + 36$$

$$\times$$

$$(\lambda+2)\left[\lambda^2 - 9\lambda + 18\right] = 0$$

$$(\lambda+2)(\lambda-6)(\lambda-3) = 0$$

$\Rightarrow \lambda = -2, 6, 3$ are eigen values. 2

Eigen vector corresponding to $\lambda = -2$

$$[A + 2I] = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

$R_3 : R_3 - R_1 \quad R_2 : 3R_2 - R_1$

$$\simeq \begin{bmatrix} 3 & 1 & 3 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A + 2I] X = 0$$

$$\begin{bmatrix} 3 & 1 & 3 \\ 0 & 20 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$20x_2 = 0 \Rightarrow x_2 = 0$$

$$3x_1 + x_2 + 3x_3 = 0 \Rightarrow x_1 = -x_3 = k$$

Eigen vector corresponding to $\lambda = -2$ is $\begin{bmatrix} k \\ 0 \\ -k \end{bmatrix}$

1.5

$\boxed{\lambda = 6}$

$$[A - 6I] = \begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}$$

$R_2 : 5R_2 + R_1$
$R_3 : 5R_3 + 3R_1$

$$\sim \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 8 & -16 \end{bmatrix}$$

$R_3 : R_3 + 2R_2$

$$\sim \begin{bmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A - 6I]\, Y = 0$$

$$\begin{bmatrix} -5 & 1 & 3 \\ 0 & -4 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$-4y_2 + 8y_3 = 0 \Rightarrow y_2 = 2y_3$$

$$-5y_1 + y_2 + 3y_3 = 0 \Rightarrow -5y_1 + 2y_3 + 2y_3 = 0$$

$$\Rightarrow 5y_1 = 4y_3$$

$$\Rightarrow 5y_1 = 4y_3 = 2y_2 = k$$

$\Rightarrow$ Eigen vector corresponding to $\lambda = 6$

is $\begin{bmatrix} k/5 \\ k/4 \\ k/2 \end{bmatrix} \simeq \begin{bmatrix} 4k \\ 5k \\ 10k \end{bmatrix}$ 1

$\boxed{\lambda = 3}$

$$[A - 3I] = \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}$$

$R_3 : R_3 + R_1$

$$\sim \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$R_3 : R_3 - R_2$

$$\sim \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A - 3I]\,Z = 0$$

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$z_1 + 2z_2 + z_3 = 0$$
$$-2z_1 + z_2 + 3z_3 = 0$$

let $z_3 = k$.

$$z_1 + 2z_2 + k = 0 \quad -(1)$$
$$-2z_1 + z_2 + 3k = 0 \quad -(2)$$

(1) × 2

$$2z_1 + 4z_2 + 2k = 0$$
$$-2z_1 + z_2 + 3k = 0$$
$$+$$
$$5z_2 + 5k = 0$$
$$\boxed{z_2 = -k}$$

(2) × 2

$$z_1 + 2z_2 + k = 0$$
$$-4z_1 + 2z_2 + 6k = 0$$
$$+ \quad - \quad -$$
$$5z_1 - 5k = 0$$
$$\boxed{z_1 = k}$$

∴ eigen vector corresponding to eigen value $\lambda = 3$ is

$$\begin{bmatrix} k \\ -k \\ k \end{bmatrix}$$

1.5

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Q3 OK
iii)

$$x + 4y - z = -5$$
$$x + y - 6z = -12$$
$$3x - y - z = 4$$

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$$

1

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[A \vdots B] = \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 1 & 1 & -6 & -12 \\ 3 & -1 & -1 & 4 \end{array}\right]$$

$R_3 : R_3 - 3R_1$
$R_2 : R_2 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & -13 & 2 & 19 \end{array}\right]$$

$R_3 : 3R_3 - 13R_2$

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 0 & -3 & -5 & -7 \\ 0 & 0 & 71 & 148 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148 \end{bmatrix}$$

2

12

$$\boxed{z = 148/71}$$ 1

$-3y - 5z = -7$

$-3y = -7 + 1240/71$

$$\boxed{y = -81/71}$$ 1

$x + 4y - z = -5$

$x = -5 - 4 \times \left[\frac{-81}{71}\right] + \frac{148}{71}$

$$\boxed{x = \frac{117}{71}}$$ 1

Q4 i)

Let V(F) be the finite dimensional vector space. The number of elements in any basis of V is called the dimension of V and denoted by dim V. 2

Dimension Theorem:

If V(F) is a finite dimensional vector space, then any two bases of V have the same number of elements. 2

Q4. ii) Let $a_1, a_2, a_3 \in \mathbb{R}$ are scalars such that

$a_1(2,1,4) + a_2(1,-1,2) + a_3(3,1,-2) = \vec{0}$ 1.5

$(2a_1 + a_2 + 3a_3, a_1 - a_2 + a_3, 4a_1 + 2a_1 - 2a_3) = \vec{0}$

Matrix of coefficient = A

$$= \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{vmatrix} = 2(2-2)-1(-2-4)+3(2+4)$$
$$= 0+6+18$$
$$= 24 \neq 0$$ 1.5

Hence solution is trivial
$\therefore a_1 = a_2 = a_3 = 0$ 1

And dim $V_3(\mathbb{R}) = 3$
no. of linearly independent elements is a given set $= 3$.
By theorem set containing same no. of LI elements then the ~~basis~~ dimension of the vector space forms basis.

$\therefore$ The given set forms basis. 2

Q4. OR iii)

$W = \{(x, 2y, 3z) \mid x, y, z \in \mathbb{R}\} (\neq \phi)$
is a subset of $V_3(\mathbb{R}) = \{(a, b, c) \mid a, b, c \in \mathbb{R}\}$

$\because$ we know that $W (\neq \phi) \subseteq V$ is a subspace of V
$\Leftrightarrow \forall \alpha, \beta \in W \Rightarrow \alpha + \beta \in W$
$\& \forall \alpha, \beta \in W, a \in F \Rightarrow a\alpha \in W$ 2

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let $\alpha = (x_1, 2y_1, 3z_1) \in W$ $x_1, y_1, z_1 \in \mathbb{R}$
$\beta = (x_2, 2y_2, 3z_2) \in W$ $x_2, y_2, z_2 \in \mathbb{R}$ (1)

$a \in \mathbb{R}$

(1) ~~LHS~~ $\alpha + \beta = (x_1, 2y_1, 3z_1) + (x_2, 2y_2, 3z_2)$
$= (x_1 + x_2, 2(y_1 + y_2), 3(z_1 + z_2))$
$\in W$.

$\because x_1 + x_2, y_1 + y_2, z_1 + z_2 \in \mathbb{R}$ (1.5)

(2) $a \in \mathbb{R}$.
$a\alpha = a(x_1, 2y_1, 3z_1)$
$= (ax_1, 2ay_1, 3az_1)$
$\in W$

$\because ax_1, ay_1, az_1 \in \mathbb{R}$.

$\Rightarrow W$ is a subspace of $V_3(\mathbb{R})$. (1.5)

Q5 i) To prove $V_n(\mathbb{R})$ is an inner product space.
we define a mapping.
$V_n \times V_n \longrightarrow \mathbb{R}$
defined by.

$\langle u, v \rangle = \langle (u_1 \dots u_n), (v_1 \dots v_n) \rangle$
$= u_1v_1 + u_2v_2 + \dots + u_nv_n$

$\forall u, v \in V_n(\mathbb{R})$ & $u = (u_1 \dots u_n)$
$v = (v_1 \dots v_n)$ (0.5)

Now to show.

(1) $\langle u, v \rangle = \langle v, u \rangle$ (symmetry property)

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$$LHS = \langle u, v \rangle = \langle (u_1 \dots u_n), (v_1 \dots v_n) \rangle$$
$$= u_1 v_1 + \dots + u_n v_n$$
$$= v_1 u_1 + \dots + v_n u_n \quad \left[\because \mathbb{R} \text{ is commutative w.r.t '}\cdot\text{'} \right]$$
$$= \langle v, u \rangle$$

0.5

(2) $\langle u, u \rangle > 0, \; u \neq 0$ (positivity property)

$$LHS \quad \langle u, u \rangle = \langle (u_1 \dots u_n), (u_1 \dots u_n) \rangle$$
$$= u_1 u_1 + \dots + u_n u_n$$
$$= u_1^2 + \dots + u_n^2$$
$$> 0 \quad RHS.$$
$$\because u_i^2 > 0 \quad i = 1 \dots n.$$

1

(3) $\langle u+v, z \rangle = \langle u, z \rangle + \langle v, z \rangle$

$$z = (z_1 \dots z_n) \in V_n(\mathbb{R})$$

$$LHS = \langle u+v, z \rangle = \langle (u_1 \dots u_n) + (v_1 \dots v_n), z_1 \dots z_n \rangle$$
$$= \langle u_1 + v_1 + \dots + u_n + v_n, z_1 \dots z_n \rangle$$
$$= (u_1 + v_1) z_1 + \dots + (u_n + v_n) z_n$$

$$RHS = \langle u, z \rangle + \langle v, z \rangle$$
$$= \langle (u_1 \dots u_n), (z_1 \dots z_n) \rangle + \langle (v_1 \dots v_n), (z_1 \dots z_n) \rangle$$
$$= u_1 z_1 + \dots u_n z_n + v_1 z_1 + \dots + v_n z_n$$
$$= (u_1 + v_1) z_1 + \dots + (u_n + v_n) z_n$$

$\because$ LHS = RHS. Hence showed.

1

(4) $\langle cu, v \rangle = c \langle u, v \rangle, \quad c \in \mathbb{R}$

$$LHS \; \langle cu, v \rangle = \langle cu_1 \dots cu_n, v_1 \dots v_n \rangle$$
$$= cu_1 v_1 + \dots + cu_n v_n$$

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RHS: $c\langle u, v\rangle$

$= c\langle (u_1 \ldots u_n), (v_1 \ldots v_n)\rangle$

$= c(u_1v_1 + \ldots + u_nv_n)$

$= cu_1v_1 + \ldots + cu_nv_n$

1

$\because$ LHS = RHS Hence showed,

$\because$ all the condition of Inner product space is satisfied.

Hence $V_n(\mathbb{R})$ is an inner product space.

Q5 ii) Given $T: V_2 \longrightarrow V_2$

df. by $T(x, y) = (2x + 3y, 3x - 4y)$

To show T is linear transformation we need to show

1) $T(\alpha + \beta) = T(\alpha) + T(\beta), \forall \alpha, \beta \in V$

2) $T(a\alpha) = aT(\alpha)\ \forall a \in F\ \alpha \in V$

1

let $\alpha = (x_1, y_1) \in V_2(\mathbb{R})$

$\beta = (x_2, y_2) \in V_2(\mathbb{R})$

$a \in \mathbb{R}$.

1

1) LHS.

$T(\alpha + \beta) = T((x_1, y_1) + (x_2, y_2))$

$= T(x_1 + x_2, y_1 + y_2)$

$= (2(x_1 + x_2) + 3(y_1 + y_2), 3(x_1 + x_2) - 4(y_1 + y_2))$

$= (2x_1 + 3y_1, 3x_1 - 4y_1) + (2x_2 + 3y_2, 3x_2 - 4y_2)$

2

$= T(\alpha) + T(\beta)$

$=$ RHS.

(2) $LHS = T(a\alpha) = T(a(x_1, y_1))$

$= T(ax_1, ay_1)$

$= (2ax_1 + 3ay_1, 3ax_1 - 4ay_1)$

$= a(2x_1 + 3y_1, 3x_1 - 4y_1)$

$= a\,T(\alpha) = RHS.$ (2)

$\because$ LHS = RHS in (1) & (2).

Hence $\therefore T$ is a linear Transformation

Q5 OR iii)

$B = \{(1,0,1,1), (-1,0,-1,1), (0,-1,1,1)\}$

Applying Gram-Schmidt orthogonalizing process.

$\boxed{W_1 = \beta_1 = (1,0,1,1)}$ (1)

$$W_2 = \beta_2 - \frac{\langle \beta_2, W_1\rangle}{\langle W_1, W_1\rangle} W_1$$ (1)

$$= (-1,0,-1,1) - \frac{\langle(-1,0,-1,1),(1,0,1,1)\rangle}{\langle(1,0,1,1),(1,0,1,1)\rangle} W_1$$

$$= (-1,0,-1,1) - \frac{(-1)}{3}(1,0,1,1)$$

$\boxed{W_2 = \left(\frac{-2}{3}, 0, -2/3, 4/3\right)}$ (1)

$$W_3 = \beta_3 - \frac{\langle \beta_3, W_1\rangle}{\langle W_1, W_1\rangle} W_1 - \frac{\langle \beta_3, W_2\rangle}{\langle W_2, W_2\rangle} W_2$$ (1)

$$= (0,-1,1,1) - \frac{\langle(0,-1,1,1),(1,0,1,1)\rangle}{\langle(1,0,1,1),(1,0,1,1)\rangle} W_1 - \frac{\langle(0,-1,1,1),(\frac{-2}{3},0,\frac{-2}{3},\frac{4}{3})\rangle}{\|(-2/3,0,-2/3,4/3)\|^2} W_2$$

$$= (0,-1,1,1) - (2/3, 0, 2/3, 2/3) + (2/12, 0, 2/12, -4/12)$$

$$\boxed{w_3 = \left(-\tfrac{1}{2}, -1, \tfrac{1}{2}, 0\right)}$$ 2

Orthogonalize set of Linearly independent set B is $\{w_1, w_2, w_3\}$

$$= \left\{(1,0,1,1), \left(-\tfrac{2}{3}, 0, -\tfrac{2}{3}, \tfrac{4}{3}\right), \left(-\tfrac{1}{2}, -1, \tfrac{1}{2}, 0\right)\right\}$$

Q6 i) Application of Principle component analysis in image processing.

1. PCA is used to visualise multidimensional data. 1
2. It is used to reduce the number of dimension in healthcare data. 1
3. It can be used to compress the image. 1
4. It can also be used for finding hidden patterns if data has high dimensions. 1

Q6 ii) Singular ~~Dean~~ Value Decomposition is a way to factor a matrix A into three matrices. as follow:

$$A = U \Sigma V^T$$

where U, V are orthogonal matrices

Σ is diagonal matrix containing singular value of A. 2

Similarly $\lambda = 9$ eigenvector is

$$v_2 = \begin{bmatrix} 1/\sqrt{18} \\ -1/\sqrt{18} \\ 4/\sqrt{18} \end{bmatrix}$$

$\lambda = 0$ $\quad v_1^T v_3 = 0$

$\qquad v_2^T v_3 = 0$

$$v_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ -a \\ -a/2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix}$$

Now $25\, u_1 = \sigma_1^2 v_1$

$\quad 9\, u_2 = \sigma_2^2 v_2$

gives

$$u = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Hence SVD eq becomes.

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$ 4

Q6 OR iii) Principal component Analysis is a powerful technique, used in data analysis, particularly for reducing the dimensionality of dataset while preserving crucial information. 2

Date :

P. No. :

Example:

Find the SVD of the matrix A.

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

(1) compute singular value of A. by finding eigen values of AA^T.

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

eigen value of AA^T are 25, 9.

$\therefore$ singular values are $\sigma_1 = \sqrt{25} = 5$

$\sigma_2 = \sqrt{9} = 3$

Hence $\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$

(2) we find right singular vectors i.e. orthonormal set of eigen vectors of AA^T

eigenvalues of A^TA = 25, 9, 0

eigen vector corresponding to $\lambda = 25$

$$A^TA - 25I = \begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix}$$

row reduce to:

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

unit vector in the direction of it is

$$v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

PCA with Example.

Let's say we have a data set of dimension 300(n) × 50(p), n represent the number of observations & p represents the number of predictors. Since we have a large p = 50, there can be n(p-1)/2 scatterplots, i.e., more than 1000 plots possible to analyse the variable relationship. So it would be a tedious job to perform exploratory analysis on this data.

In this case it would be a lucid approach to select a subset of p ≤< 50 predictor ~~wit~~ which captures so much information, followed by plotting the observation in the resultant low dimensional space.

The image below shows the transformation of high dimensional data (3-D) to low dimensional dat (2-D) using PCA.

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