Total No. of Questions: 6 *Total No. of Printed Pages:3*

Enrollment No......................................

Faculty of Science

End Sem (Even) Examination May-2022

MA5CO09 Complex Analysis -II

Programme: M.Sc. Branch/Specialisation: Mathematics

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. Which of the following is true about f(z) = z^2? **1**

(a) Continuous and differentiable

(b) Continuous but not differentiable

(c) Neither continuous nor differentiable

(d) Differentiable but not continuous

ii. $\Gamma z\,\Gamma(1-z) =$ ______. **1**

(a) π (b) $\frac{\pi}{\cos\pi}$ (c) $\frac{\pi}{\sin\pi}$ (d) None of these

iii. Taylor series for $f(z)$ with circle c at centre $z = a$ is given by: **1**

(a) $f(z) = \sum_{n=0}^{a} \frac{(z-a)^n}{n!} f^n(a)$

(b) $f(z) = \sum_{n=0}^{a} \frac{(z-a)}{n} f^n(a)$

(c) $f(z) = \sum_{n=0}^{a} \frac{(z+a)}{n} f^n(a)$

(d) $f(z) = \sum_{n=0}^{a} \frac{(z+a)}{n}$

iv. The radius of convergence of power series $\sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}}$ is ______. **1**

(a) ∞ (b) 1 (c) 2 (d) $\frac{1}{2}$

v. Harmonic conjugate of $u(x,y) = e^y \cos x$ is: **1**

(a) $e^x \cos y + c$ (b) $e^x \sin y + c$

(c) $e^y \sin x + c$ (d) $-e^y \sin x + c$

vi. The Minimum principle holds for: **1**

(a) Subharmonic functions only

(b) Superharmonic functions only

(c) Either sub-harmonic or Super-harmonic

(d) Neither sub-harmonic nor super harmonic

P.T.O.

vii. If $f(z)$ is an entire transcendental function with maximum modulus $M(r)$, then $\lim_{\{r\to\infty\}} \log\frac{M(r)}{\log r} =$ ______. **1**

(a) 0 (b) 1 (c) ∞ (d) π

viii. The order of function $\cos z$ is ______. **1**

(a) 0 (b) 1 (c) ½ (d) 2

ix. Let f be an analytic function in a region continuous $B(0,R)$ contains a disc of radius is: **1**

(a) $\frac{1}{72}R\,|f'(0)|$ (b) $\frac{1}{70}R\,|f'(1)|$

(c) $r \le \frac{1}{72}$ (d) $\frac{1}{56}R\,|f'(0)|$

x. By Little Picard theorem if f be an entire function that omits two values, then f is: **1**

(a) Connected (b) Continuous

(c) Constant (d) None of these

Q.2 i. Define Riemann Zeta function with example. **2**

ii. Show that $C(G,\Omega)$ is a complete metric space. **3**

iii. State and prove Weirstrass factorization theorem. **5**

OR iv. State and prove Riemann mapping theorem. **5**

Q.3 i. Prove that there cannot be more than one continuation of an analytic function $f(z)$ into the same domain. **2**

ii. Let $f(z)$ be a function of z analytic in a domain D which contains a segment of x-axis which D is symmetric. Then prove that $\overline{f(\bar{z})} = f(z),\ z \in D$ i. e. $f(z)$ takes conjugate values of z if and only f(x) is real for each point on the segment of x- axis. **8**

OR iii. If the radius of convergence of the power series $f(z) = \sum_{n=0}^{\infty} a_n\, z^n$ is non-zero finite, then prove that f(z) has at least one singularity on the circle of convergence. **8**

Q.4 i. Let $u: G \to \mathbb{R}$ be a continuous function which has the MVP. Then prove that u is harmonic. **3**

ii. Let G be a bounded Drichlet Region, then prove that for each $a \in G$, there is a Green's Function on G with singularity at a. **7**

OR iii. Prove that Poisson Kernel $\Pr(\theta)$ can be expressed as- **7**

$$R_e\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) \text{ for } 0 \le r \le 1 - \infty < \theta < \infty$$

Q.5 i. If $f(z)$ is an entire transcendental function with maximum modulus $M(r)$, then prove that $\lim_{r\to\infty} \frac{\log M(r)}{\log r} = \infty$. **4**

ii. State and prove Hadmard's factorization theorem. **6**

OR iii. Let $f(z)$ be an entire function with $f(0) \neq 0$. Also let $r_1, r_2, \dots, r_n$ be the moduli of zeros $z_1, z_2, \dots, z_n$ of $f(z)$, arranged as nondecreasing sequence, multiple zero being repeated. Then prove that $R^n|f(0)| \le M(R) r_1 r_2 \dots r_n$ if $r_n < R < r_{n+1}$. **6**

Q.6 Attempt any two:

i. Let f be analytic in $D = \{z: |z| < 1\}$ and let $f(0) = 0, f'(0) = 1$, and $|f(z)| \le M$ for all z in D. Then prove that $M \ge 1$ and $f(D) \supset B\left(0; \frac{1}{6M}\right)$. **5**

ii. Let f be an analytic function in a region containing the closure of the $disc\ D = \{z: |z| < 1\}$ and $f(0), f'(0) = 1$. Then prove that $f(D)$ contains a disc of radius 1. **5**

iii. Let f be an analytic function that has an essential singularity at $z = z_0$. Then prove that in each neighbourhood of z_0, f assumes each complex numbers, with one possible exception, an infinite number of times. **5**

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Solution Set of

End Sem (Even) Exam May-2022

MA5C003 Complex Analysis II

Q.1

(i) a. continuous and differentiable 1

(ii) d. None of these. (Correct $\frac{\pi}{\sin \pi z}$) 1

(iii) a. $f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^n(a)$ 1

(iv) c. 2 1

(v) c. $e^y \sin x + c$ 1

(vi) b. Superharmonic functions only 1

(vii) c. ∞ 1

(viii) b. 1 1

(ix) c. $r \leq \frac{1}{72}$ 1

(x) c. Constant 1

Q 2 (i) The Reimann zeta function $\zeta(z)$ is defined for $z > 1$ by

$$\zeta(z) = \sum_{n=}^{\infty} n^{-z}$$

when z is complex number n +ve Integer

$$|n^z| = \exp(z \log n) = \exp(\operatorname{Re} z . \log n)$$

i) To prove $C(G,\Omega)$ is complete metric space

Consider $\{f_n\}$ a Cauchy sequence in $C(G,\Omega)$ s.t.

$$\sup\{d(f_n(z), f_m(z)) : z \in K \subseteq G\} < \delta$$

for $n, m \geq N$

$\{f_n\}$ Cauchy in Ω

$$\lim_{n\to\infty} f_n(z) = f(z)$$

$\Rightarrow \exists f \in G \quad f : G \to \Omega$

to show f is continuous

$\rho(f_n, f) \to 0 \quad n \to \infty$ (1)

Let K is compact $\exists m \geq N$ s.t

$d(f(z), f_m(z)) \leq \delta$

But

$d(f(z), f_n(z)) \leq$

$d(f, f_m) + d(f_m, f_n) \leq 2\delta$

$\forall n \geq N$, N does not depends on z (+1)

$\Rightarrow \sup(d(f(z), f_n(z)) : z \in K) \to 0$
$n \to \infty$

$\Rightarrow \{f_n\}$ cgt uniformly on every compact set in G. (+1)

3 mark

(ii) Weierstrass factorization theorem :-

Let $f(z)$ be an entire fun and let $\{z_n\}$ be sequence of non zeros of f whose limiting point is the point at infinity repeated according to multiplicity. Suppose f has zero at $z=0$, of order $m>0$. then there is there is an entire function $g(z)$ and a sequence of integers $\{P_n\}$ such that (3)

$$f(z) = z^n e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right) \qquad +(2)$$

Pf Assume that $z_1, z_2 -- z_n$ are such that
$$|z_1| \le |z_2| \le --- |z_n|$$

define
$$E_p(z) = (1-z) \exp\left\{z + \frac{z^2}{2} + --- + \frac{z^p}{p}\right\} \quad -(1)$$
$$E_0(z) = (1-z)$$

$$\Rightarrow \log E_p(z) = -\sum_{n=1}^{\infty} \frac{z^{p+n}}{p+n} \qquad +1$$

that
$$|\log E_p(z)| \le \frac{|z|^{p+1}}{1-|z|} \quad \text{if } |z|<1$$
$$\le \frac{k}{k-1} |z|^{p+1} \quad \text{if } |z| \le \frac{1}{k} \quad k>1$$

$r_n = |z_n|$
$$\sum_{n=1}^{\infty} \left(\frac{R}{r_n}\right)^{P_n+1} \text{ converges } \forall R>0 \qquad 2)$$

$r_n \to \infty$ $n \to \infty$ $P_n = n$ (1)

since $\left(\frac{R}{r_n}\right)^n < \frac{1}{2^n}$

for $\frac{R}{r_n} < \frac{1}{2}$ series is conv

now let $f(z) = z^n e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$

$\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$ cgt uniformly $|z| < R$

Introduce factor z^m in above

$z^m \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$ entire fun. whose

zeros are $0, z_1, z_2 \ldots z_n$

$f(z)$ is arbitrary entire function the

$$f(z) \Big/ \left(z^m \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)\right) = e^{g(z)} \text{ is}$$

entire

$$\Rightarrow f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

+1

+5

(iv) Riemann Mapping Theorem :- Let Ω be simply connected region in z plane which is neither z plane itself nor the extended z plane and let $z_0 \in \Omega$ then there is unique analytic function $f: \Omega \to \mathbb{C}$ having

(a) $f(z_0) = 0$ and $f'(z_0) > 0$

(b) f is one - one

(c) $w = f(z)$ maps Ω onto the disc $|w| < 1$

+2

Pf

+3

for

Proof

Q.3

(i) Let $(f_1(z), D_1)$ and $(f_2(z), D_2)$ are function element in analytic continuous of each other

$$D_1 \cap D_2 \neq \phi$$

$$f_1(z) = f_2(z) \quad \forall \; z \in D_1 \cap D_2 \qquad (+1)$$

it required to prove $f_2(z)$ is unique in D_2

Let $g_2(z)$ be another function in analytic continuation of $f_1(z)$ the

$$f_1(z) = g_2(z) \quad \forall \; z \in D_1 \cap D_2$$

$$\Rightarrow \quad f_2(z) = g_2(z) \quad \forall \; z \in D_1 \cap D_2$$

ie $f_2(z)$ & $g_2(z)$ coincide in D_2

i.e. the coinc in $D_1 \cap D_2$

$$\Rightarrow \quad f_2(z) = g_2(z) \quad \forall \quad z \in D_2 \qquad (+1)$$

2 marks

(ii) Observe that $\overline{\overline{f(z)}} = f(z)$ the condition $f(\bar{z}) = \overline{f(z)}$ is equivalent to the condition

$$\overline{f(\bar{z})} = f(z) \qquad -(1)$$

Suppose the domain D have the segment AB of the real axis in its interior and let D be symmetrical about segment AB

Let $f(x)$ be real for each point x on AB of x axis contained in D. Then we have to show eq (1) hold. For this we show that $\overline{f(\bar{z})}$ is analytic throught D. Let $g(z) = \overline{f(\bar{z})}$

$$g(z) = \overline{f(\bar{z})} \qquad -(2)$$

and write

$$f(z) = u(x,y) + iv(x,y) \qquad -(3)$$

$$g(z) = p(x,y) + iq(x,y)$$

$$f(\bar{z}) = u(x,-y) + iv(x,-y)$$

$$\overline{f(\bar{z})} = u(x,-y) - iv(x,-y) \qquad -(4)$$

$$g(z) = p(x,y) + iq(x,y)$$

$$= u(x,-y) - iv(x,-y)$$

$$\Rightarrow p(x,y) = u(x,\eta) \qquad q(x,y) = -v(x,\eta)$$

$$\eta = -y$$

using C.R.-eq $p_x = q_y \quad p_y = -q_x$

$f(x)$ is real on segment of real axis

$$f(x) = u(x,0) + iv(x,0)$$

$$\Rightarrow v(x,0) = 0$$

$$g(x) = p(x,0) + iq(x,0) = u(x,0) - iv(x,0)$$

$$= u(x,0) = f(x)$$

$g(z) = f(z)$ on real axis

$$g(z) - f(z) = 0 \quad \forall z \in D$$

$$\Rightarrow \overline{f(\bar{z})} = f(z) \quad \forall z \in D$$

Similarly prove contradiction.

Pf. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ Let R be the radius of convergence Γ_0 defined by with z_0 of power series (1) then $f(z) = \sum a_n (z-z_0)^n$ is analytic within Γ_0 defined by $|z-z_0| = R$ (+1)

Construct concentric circle Γ of radius $r < R$

Let ρ be radius of convergence of at any point z of Γ. δ be the least value of ρ'. (+1)

$\delta \geq R - r$ Since $\rho \geq R - r$

if $\delta > R - r$ $R < r + \delta$ (+1)

$|z - z_0| < r + \delta$.

Let $F(z)$ be the complete analytic function in the $|z - z_0| < r + \delta$ (+1)

$F(z) = f(z)$ $|z - z_0| < R$

Since the radius of convergence of $F(z)$ and $f(z)$ are the same $\Rightarrow$ radius of convergence of $f(z)$ must be $r + \delta$. Which contradicts the fact that radius of convergence of $f(z)$ is R, $R < r + \delta$ (+1) (+1)

$\Rightarrow \delta > R - r$ so $\delta = R - r$ & $\rho = R - r$

at point on circle Γ let pt $re^{i\alpha}$ is a pt $Re^{i\alpha}$ on circle Γ_0 must be singularity of $f(z)$. (+1)

taking $z_0 = 0$ we get the result for the given (+1)

8 marks

(8)

4 (i) If $u: G \to R$ be continuous function which has MVP then for $\bar{B}(a;r) \subseteq G$

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta$$

(+1)

holds for harmonic function.

Let D is disc $\overline{B(a,r)} \subseteq D \subseteq G$

$$\gamma : \{z : |z-a| = r\}$$

using cauchy integral th.

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)}\, dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+e^{i\theta}r)}{re^{i\theta}}\, re^{i\theta} i\, d\theta$$

$z - a = re^{i\theta}$

$$= \frac{1}{2\pi i} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta$$

(+1)

$u = \operatorname{Re} f \quad v = \operatorname{img} f$

$$u + iv = \frac{1}{2\pi} \int_0^{2\pi} [u(a + re^{i\theta}) + iv(a + re^{i\theta})]\, d\theta$$

Eq real part

$$\Rightarrow u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta \quad \text{proves}$$

(+1)

3 marks

(ii) Define $f: \delta G \to R$ by $f(z) = \log|z-a|$ Let $u: \bar{G} \to R$ continuous function which is harmonic in G such that $u(z) = f(z)$ for $z \in \delta G$

$$g_a(z) = u(z) - \log|z-a| \text{ is a function}$$

(+1)

having (a) harmonic in G with singularities at 'a'.

(+1)

(b) $G(z) = g_a(z) + \log|z-a| = u(z)$ is harmonic in G.

(+1)

$$g_a(z) = \lim_{z \to w} [u(z) - \log|z-a|]$$

(+2)

$$= \lim_{z \to w} [f(z) - \log|z-a|] \text{ for } z \in \delta G$$

hence we conclude that $g_a(z)$ is required Green function on G with singularity at a. (8)

+2

(7)

Q4

(iii) Poisson kernel $P_r(\theta)$

$$= \sum r^{|n|} e^{in\theta} \quad \text{for } 0 \le r < 1, \quad -\infty < \theta < \infty$$ +1

Suppose $z = re^{i\theta}$, $0 \le r < 1$ +1

$$\frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{1+z}{1-z} = (1+z)(1-z)^{-1}$$

$$= (1+z)(1+z+z^2+\cdots)$$

$$= 1+2\sum z^n$$

$$= 1+2\sum r^n e^{in\theta}$$ +2

$$= 1+2\sum r^n (\cos n\theta + i\sin n\theta)$$

hence $\mathrm{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right)$

$$= 1+2\sum_{n=1}^{\infty} r^n \cos n\theta$$ +3

$$= 1+\sum r^n (e^{in\theta}+e^{-in\theta})$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = P_r(\theta)$$

7 marks

Q.5

(i) If $f(z)$ is an entire trascedental function with max modulus $M(r)$ assume the contradiction

$$\liminf_{r \to \infty} \frac{\log M(r)}{\log r} = \mu < \infty$$

then given $\epsilon > 0$, there is increasing sequence $\{r_n\}$ cgt to ∞ such that ~~so~~

$$\frac{\log M(r)}{\log r_n} < \mu + \epsilon \qquad +2$$

Or

$$\log M(r) < \mu + \epsilon (\log r_n)$$

Or $$M(r) < r_n^{\mu+\epsilon} \quad \forall r_n$$

using cauchy inequality, we obtain

$$|a_n| < \frac{M(r_n)}{r_n^{\lambda}} < r_n^{\mu+\epsilon-\lambda} \qquad +1$$

Taking r_n sufficiently large $\Rightarrow a_\lambda = 0$ $\forall \lambda > \mu + \epsilon$

hence $f(z)$ is polynomial of degree not greater than (μ)

$\Rightarrow$ proposition. $+1$

4 marks

(16)

(i) Hadamard's Factorization theorem

Let $f(z)$ be an entire function of order λ then

$$f(z) = z^m e^{g(z)} p(z)$$

Where m is the order of the possible zeros of $f(z)$ at $z=0$, g is poly of degree not exceeding λ and $p(z)$ is canonical product associated with sequence of non zero zeros of $f(z)$

+2

Proof

+4

6 mark

(iii) Proof of Jensen's inequality +1

Consider

$$F(z) = f(z) \prod_{i=1}^{n} \left(\frac{(R^2 - z\bar{z}_i)}{R(z - z_i)} \right) \quad (1)$$

+1

$f(z)$ entire $\Rightarrow$ $F(z)$ is also entire

On the circle $|z| = R$, for $\forall i$ $1 \le i \le n$ +1

$$R^2 (z - z_i)(\bar{z} - \bar{z}_i)$$

$$= (R^2 - \bar{z}_i z)(R^2 - z_i \bar{z})$$

$$\Rightarrow \left| \frac{R^2 - \bar{z}_i z}{R(z - z_i)} \right| = 1 \quad \text{on } |z| = R$$

+1

$$\Rightarrow |F(z)| = |f(z)| \text{ on } |z| = R$$

By max. modulus principle +1

$$|F(z)| \le \max_{|z|=R} |F(z)| = \max |f(z)|$$

$\Rightarrow |F(z)| \le M(R)$

$z=0$ in (1)

$$\left| f(0) \prod_{i=1}^{n} \frac{R^2}{R(-z_i)} \right| = |F(0)| \le M(R)$$

or $$|f(0)| \prod_{i=1}^{n} \frac{R}{|z_i|} \le M(R)$$

or $$\frac{R^n |f(0)|}{r_1 r_2 r_3 \cdots r_n} \le M(R)$$ (+1)

or $$R^n |f(0)| \le M(R) |z_1 z_2 \cdots z_n|$$

6 marks

Q. 6

(i) Let $0<r<1$, f is analytic in D

Maclaurin series

$$\Rightarrow f(z) = f(0) + f'(0) z + \sum_{n=2}^{\infty} \frac{f^n(0)}{n!} z^n$$

$f(0)=0$, $f'(0)=1$

$$\Rightarrow f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad a_n = \frac{f_0^n(0)}{n!}$$ (+1)

$f(z) \le M$, we have (+1)

$$|a_n| \le \frac{M}{r^n}$$

for $n \ge 1$ $1 = |a_1| \le M$ If $|z| = (4M)^{-1}$ (+1)

$$|f(z)| \ge \frac{1}{4M} - \sum_{n=2}^{\infty} \frac{1}{4^n M^{n-1}}$$

$$= \frac{1}{6M} \qquad \text{Re}\ |f(z)| \ge \frac{1}{6M}$$ (+1)

Suppose $|w| < (6m)^{-1}$ $g(z) = f(z) - w$ (B)

$g(z)$ has a zero

for $|z| = (4m)^{-1}$

$|f(z) - g(z)| = |w| \le (6m)^{-1} \le f(z)$ (+1)

$\Rightarrow$ Rouche's the f & $f + (g - f) = g$ have

same number of zeros in $B(0, 1/4m)$

since $f(0) = 0$ $g(z_0) = 0$ for some $z_0 \in B(0, 1/4m)$

$g(z_0) = 0$ $f(z_0) = w$ $|f(z_0)| = w < \frac{1}{6m}$ (+1)

$\Rightarrow f(D) \supset B(0, 1/6m)$

6 marks

6.6.(1)

$f \in F$ $\lambda(f) = \sup \{ r_1 : f(D)$ contains disc of radius r_1

Landau's const L.

$L = \inf \{ \lambda(f) : f \in F \}$ (+1)

It to prove $f(D)$ contains a disc of

radius $\lambda = \lambda(f)$

for each n, let α_n in $f(D)$ (+1)

$B(\alpha_n ; \lambda - 1/n) \subseteq f(D)$

Now $\alpha_n \in f(D) \subseteq f(\bar{D})$ (+1)

$\exists$ $\alpha \in f(\bar{D})$ and subsequence $\{\alpha_{n_k}\}_{k=1}^{\infty}$

for $\{\alpha_n\}_{n=1}^{\infty}$ $\alpha_{n_k} \to \alpha$ $k \to \infty$ (+1)

Let $\alpha = \lim_{n \to \infty} \alpha_n$

If $|w-\alpha| < \alpha$ ~~la~~ n_0 such that

$$|w-\alpha| < \lambda - \frac{1}{n_0}$$

+1

$\exists n_1 \geq n_0$

$$|\alpha_n - \alpha| < \lambda - \frac{1}{n_0} - |w-\alpha|$$

$\forall n \geq n_1$

$$|w - \alpha_n| < \lambda - \frac{1}{n} \quad \forall n \geq n_1$$

$$\Rightarrow w \in B\left(\alpha_{n_1}, \lambda - \frac{1}{n}\right) \subset f(D)$$

w is arbitrary

$$\Rightarrow B(\alpha, \lambda) \subseteq f(D)$$

~~thence~~ hence prove

5 marks

(iii) proof of Great picards theorem.

+5