Q.5 Attempt any two:

i. Prove that the complete graph of five vertices in non-planar. **5**

ii. State and prove Euler's formula. **5**

iii. Prove that the set of all circuit vectors in the vector space associated with the graph *G* is a subspace. **5**

Q.6 Attempt any two:

i. Define Bipartite graph, complete Bipartite graph, properly colored graph with examples. **5**

ii. Prove that every tree with two or more vertices is 2-chromatic but converse need not be true. **5**

iii. Prove that in a bipartite graph a complete matching of V_1 and V_2 exists if there is a positive integer m for which the following condition is satisfied: degree of every vertex in $V_1 \geq m \geq$ degree of every vertex in V_2. **5**

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Total No. of Questions: 6 *Total No. of Printed Pages:4*

Enrollment No....................................

Faculty of Science

End Sem (Odd) Examination Dec-2019

BC3EM01 Graph Theory

Programme: B.Sc. (CS) Branch/Specialisation: Computer Science

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated.Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. A graph with more than 1 vertex in which all vertices having zero degree is known as a ___________ **1**

(a) Nullgraph (b) Multigraph

(c) Simple graph (d) Complete Graph

ii. The sum of the in-degrees over all vertices in any directed graph is equal to **1**

(a) The number of the edges

(b) The sum of the out-degrees over all vertices

(c) Both (a) and (b)

(d) None of these

iii. If C(*G*) is the cut-set matrix of connected graph *G* with *n* vertices, then rank of C(*G*) = **1**

(a) n (b) $n+1$ (c) $2n$ (d) $n-1$

iv. Two identical columns in cut set matrix represents____ in the graphs **1**

(a) Two components (b) Self loops

(c) Parallel edges (d) None of these

v. The total number of vertices in a binary tree is always **1**

(a) Even (b) Odd (c) Prime (d) 0

vi. If in a graph *G* there is one and only one path between every pair of vertices, then G is a **1**

(a) Tree (b) Connected

(c) Both (a) and (b) (d) None of these

vii. Graph K_4 is a **1**

(a) Complete graph (b) Planar graph
(c) Non-planar graph (d) Both (a) and (b)

viii. The circuit subspace and the cut set subspace in the vector space of a graph are **1**

(a) Orthogonal to vector space
(b) Symmetric
(c) Orthogonal to each other
(d) None of these.

ix. A graph consisting of only n- isolated vertices is **1**

(a) 1-chromatic (b) n-chromatic
(c) $(n-1)$-chromatic (d) None of these.

x. A covering exists for a graph if and only the graph has no ________ vertex **1**

(a) Pendant (b) Isolated
(c) Non-isolated (d) None of these

Q.2 Attempt any two:

i. Define walk, spanning subgraph, isomorphic graph, Euler and Hamiltonian graph with examples. **5**

ii. Prove that a connected graph G is an Euler graph if and only if all vertices of G are of even degree. **5**

iii. Prove that a simple disconnected graph G with vertices $'n'$ and $'k'$ components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges. **5**

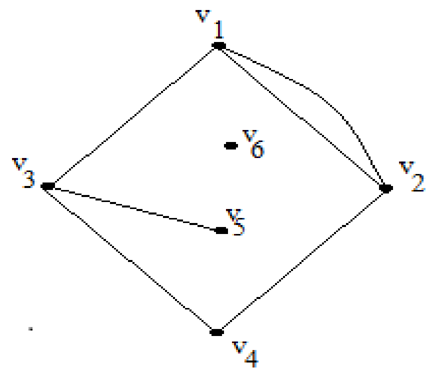
Q.3 Attempt any two:

i. Draw the graph corresponding to following adjacency matrix and verify observations. **5**

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

ii. If B is a circuit matrix of a connected graph G with e edges and n vertices, then show that the rank of B = $e-n+1$. **5**

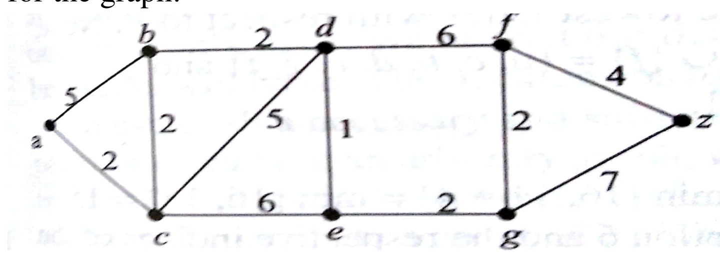
iii. Describe the use of matrix representation of graph and find the incidence matrix representation for the following graph **5**



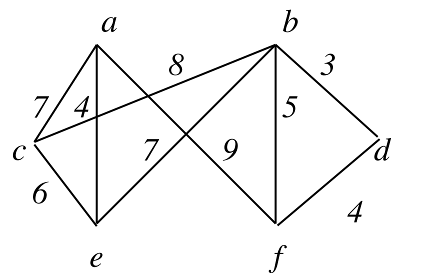
Q.4 Attempt any two:

i. Define tree and prove that a tree with n vertices has $n-1$ edges. **5**

ii. Find shortest path form vertex 'a' to 'z' using Dijkstra's algorithm for the graph: **5**



iii. Use Prim's alogrithm to find the minimum spanning tree for the weighted graph **5**



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P.T.O.

Medi-Caps University, Indore

Faculty of Science

Course Code: BC3EM01

Course Name: Graph Theory

Q.1

(i) (a) Nullgraph +1

(ii) ~~(b)~~ (C) Both (a) and (b) +1

(iii) (d) (n-1) +1

(iv) (C) Parallel edges +1

(v) (b) odd +1

(vi) (C) Both (a) and (b) +1

(vii) (d) Both (a) and (b) +1

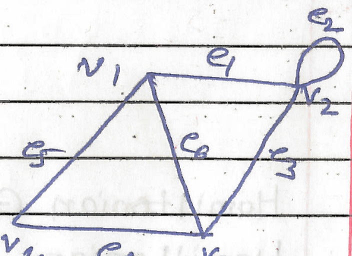
(viii) (c) orthogonal to each other +1

(ix) (a) 1-chromatic +1

(x) (b) Isolated vertex. +1

Q.2 (i) Walk: A walk in a graph G is defined as a finite ~~number~~ alternating sequence of vertices and edges which begins and ends with vertices, such that each edge is incident with the vertices preceding and following it.

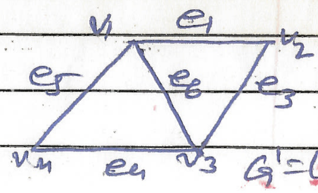
For example: $v_1\ e_1\ v_2\ e_2\ v_2\ e_3\ v_3$ is a walk. +1



Graph G=(V,E)

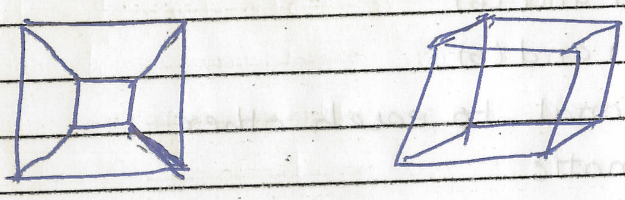
Spanning subgraph: A subgraph G' is said to be a spanning subgraph of a graph G if G' contains all the vertices of the graph G

For example, in above graph, A subgraph $G' = (V', E')$ is a spanning subgraph of ~~G~~ $G=(V,E)$ +1



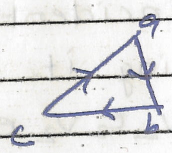
$G' = (V', E')$

Isomorphic graph: Two graphs G and G' are said to be isomorphic (to each other) if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved. For example



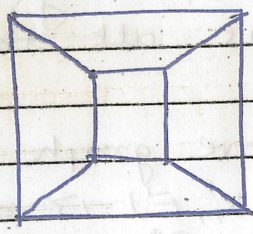
Isomorphic graphs. +1

Euler graph: If some closed walk in a graph contains all the edges of the graph, then the walk is called an Euler line and the graph an Euler graph.
For example:



Euler graph +1

Hamiltonian Graph: A graph G = (V, E) is called Hamiltonian graph if it is contains a Hamiltonian circuit.



Hamiltonian graph +1

2.2(ii) Suppose that G is an Euler graph. It therefore contains an Euler line. In tracing this walk we observe that every time the walk meets a vertex v it goes through two "new" edges incident on v - with one we "entered" v and with the other "exited". This is true not only of all intermediate vertices of the walk but also of the terminal vertex, because we "exited" and "entered" the same vertex at the beginning and end of the walk, respectively. Thus if G is an Euler graph, the degree of every vertex is even. +2.5

To prove the sufficiency of the condition, assume that all vertices of G are of even degree. Now we construct a walk starting at an arbitrary vertex v and going through the edges of G such that no edge is traced more than once. We continue tracing as far as possible. Since every vertex is of even degree, we can exit from every vertex we enter; the tracing cannot stop at any vertex but v. And since v is also of even degree, we shall eventually reach v when the tracing comes to an end. If this closed walk h we just traced includes all the edges of G, G is an Euler graph. If not, we remove from G all the edges in h and obtain a subgraph h' of G formed by the remaining edges. Since both G and h have all their vertices of even degree, the degrees of the vertices of h' are also even. Moreover, h' must touch h at least at one vertex a, because G is connected. Starting from a, we can again construct a new walk in graph h'. Since all the vertices of h' are of even degree, this walk in h' must terminate at vertex a; but this walk in h' can be combined with h to form a new walk, which starts and ends at vertex v and has more edges than h. This process can be repeated until we obtain a closed walk that traverses all the edges of G. Thus G is an Euler graph. +2.5

(iii) Let the number of vertices in each of k- component of a graph G be $n_1, n_2, n_3, \ldots, n_k$. Thus we have $\sum_{i=1}^{k} n_i = n$, $n_i \geq 1$. +1

We have

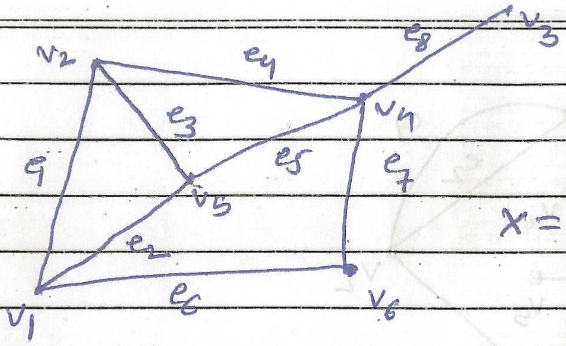
$$\sum_{i=1}^{k} n_i^2 \leq n^2 + k^2 - 2nk + 2n - k \quad \text{--- (1)}$$ +2

Now the maximum no. of edges in the ith component of G (which is a simple connected graph) is $\frac{n_i(n_i-1)}{2}$. Therefore, the maximum number of edges in G is equal to

$$= \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{k} n_i^2 - \frac{n}{2}$$

$$\leq \frac{1}{2}\left[n^2+k^2-2nk+2n-k-n\right] = \frac{(n-k)(n-k+1)}{2}$$ +2

Q.3 (i)



G

$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$ +2.5

Observations: 1. The principal diagonal of X has all zero's if and only if G has no self loop

2. If The sum of any row (or any column) is equal to the degree of the corresponding vertex, since the matrix is symmetric. +2.5

(ii) If A is an incidence matrix of G, we have
$A \cdot B^T = 0 \pmod 2$ +1

Therefore, according to Sylvester's theorem
rank of A + rank of B $\leq$ e

$\Rightarrow$ rank of B $\leq$ e $-$ rank of A. +1

Since rank of A $= n-1$ +1

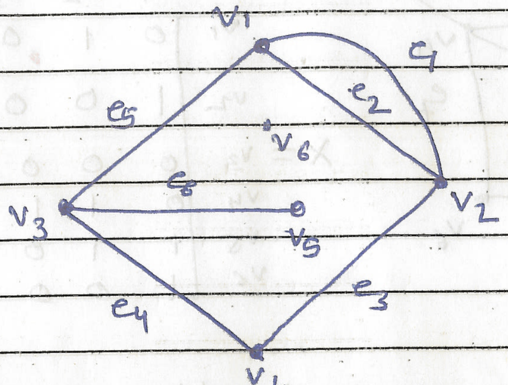
We have rank of B $\leq e-n+1$

But rank of B $\geq e-n+1$. +1

Therefore, we must have

$\boxed{\text{rank of B} = e-n+1}$ +1

(iii)



Incidence matrix

$$
= \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\left[\begin{array}{cccccc}
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\end{array}
$$

+3.0

The Use of matrix representation of graph?

1. The incident ~~matrix~~ of a graph gives the (0, 1) matrix which has a row for each vertex and column for each edge

2. ~~no adja~~ if matrix A contains 1 for edges and 0 otherwise, then A^k contains the number of paths of length k between all vertices. +2

Q.4(ii) Tree: A tree is a connected graph without any circuit.

proof: The result will be proved by induction on the number of vertices.

It is easy to see that the theorem is true for $n=1, 2$, and 3. Assume that the theorem holds for all trees with fewer than n-vertices. +1

Let us now consider a tree T with n-vertices. In T let e_k be an edge with end vertices V_i and V_j, there is no other path between V_i and V_j. ~~Further more, T-e_k consists~~ except e_k. Therefore, deletion of e_k from T will disconnect the graph. ~~Furthermore~~ Furthermore, T-e_k consists of exactly two components, and since there were no circuit in T to begin with, each of these components is a tree. Both these trees t_1 and t_2, have fewer than n vertices each, and therefore, by the induction hypothesis, each contains one less edge than the number of vertices in it. Thus T-e_k consists of n-2 edges. Hence T has exactly (n-1) edges. +3

(iii) Step-I $V = \{a, b, c, d, e, f, g, z\}$

$P_1 = \{a\}$, $T_1 = V - P_1 = \{b, c, d, e, f, g, z\}$

$l(b) = 5$, $l(c) = 2$, $l(d) = \infty$, $l(e) = \infty$, $l(f) = \infty$,

$l(g) = \infty$, $l(z) = \infty$

Here $c \in T_1$ has the minimum index 2 +1

Step-II $P_2 = P_1 \cup \{c\} = \{a, c\}$, $T_2 = \{b, d, e, f, g, z\}$

$\forall x \in T_2$

$l(x) = \min[l(x)\text{ w.r.t } P_1,\ l(c) + w(c, x)]$

$l(b) = \min[5, 2+2] = 4$

$l(e) = 8$, $l(d) = 7$, $l(f) = \infty$, $l(g) = \infty$, $l(z) = \infty$

Here $b \in T_2$ has minimum index 4 +1

Step-III: $P_3 = \{a, c, b\}$, $T_3 = \{d, e, f, g, z\}$

$l(d) = \min(7, 4+2) = 6$, $l(f) = \infty$, $l(g) = \infty$,

$l(e) = \min(8, 4+\infty) = 8$, $l(z) = \infty$ +1

Here $d \in T_3$ has minimum index 6

Step-IV: $P_4 = \{a, c, b, d\}$, $T_4 = \{e, f, g, z\}$

$l(e) = \min(8, 6+1) = 7$, $l(f) = \min(\infty, 6+6) = 12$

$l(g) = \min(\infty, 6+\infty) = \infty$, $l(z) = \infty$

$e \in T_4$ has minimum index 7 +1

Step-V: $P_5 = \{a, c, b, d, e\}$, $T_5 = \{f, g, z\}$

$l(f) = \min(12, 7+\infty) = 12$, $l(g) = 9$, $l(z) = \infty$

$g \in T_5$ has minimum index 9

Step-VI: $P_6 = \{a, c, b, d, e, g\}$, $T_6 = \{f, z\}$

$l(f) = \min(12, 9+2) = 11$, $l(z) = \min(\infty, 9+7) = 16$

$f \in T_6$ has minimum index 9

Step-VII: $P_7 = \{a, c, b, d, g, f\}$, $T_7 = \{z\}$

$l(z) = \min(16, 11+4) = 15$

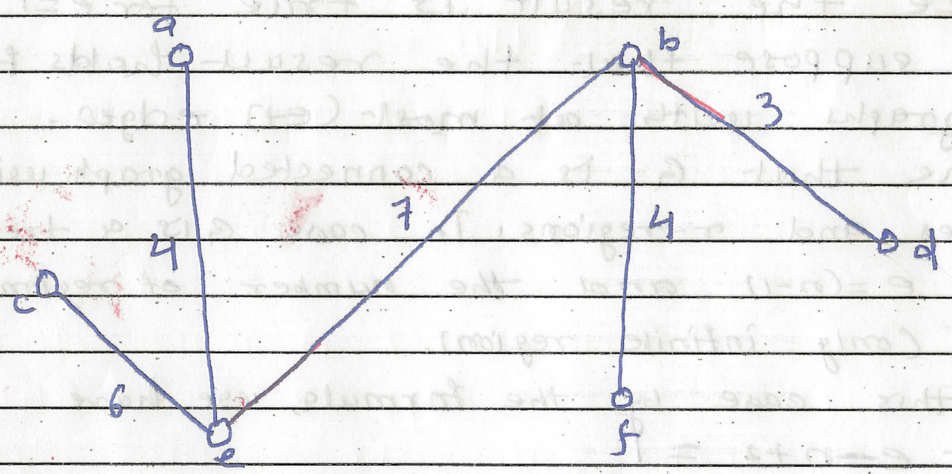
The length of the shortest path is 15

path is $a \to c \to b \to d \to e \to g \to f \to z$ +1

(iii)

	a	b	c	d	e	f
a	–	∞	7	∞	4	9
b	∞	–	8	3	7	5
c	7	8	–	∞	6	∞
d	∞	3	∞	–	∞	4
e	4	7	6	∞	–	∞
f	9	5	∞	4	∞	–

+2



+3

Minimum Spanning tree

(i) We know that the complete graph of 5 vertice K_5 is a simple graph and ~~two~~ $n = 5$. +1.5

$\therefore$ e = number of edges $= \frac{n(n-1)}{2} = 10$ +1.5

Also $3n - 6 = 9 < e (=10)$

Therefore, the inequality $e \leq 3n - 6$ is not satisfied +2

Hence the graph is non-planar

5(ii) A connected planar graph with n-vertices e-edges, r-regions given by $r = e - n + 2$

proof: To prove the formula by induction on the number of edges e of G, where G is a connected planar graph.

Suppose $e = 1$, then n may be equal to 1 and 2.

In case $e = 1$, $n = 2$, then the no. of regions $r = 1 - 2 + 2 = 1 = e - n + 2$ +1

Again in case $e = 1$, $n = 1$, then $r = 1 - 1 + 2 = 2$.

Hence the result is true for $e = 1$ +1

Now suppose that the result holds for all graphs with at most $(e-1)$ edges.

Assume that G is a connected graph with e-edges and r-regions. In case G is a tree then $e = (n-1)$ and the number of regions is 1 (only infinite region). +1

In this case by the formula, we have

$r = e - n + 2 = 1$

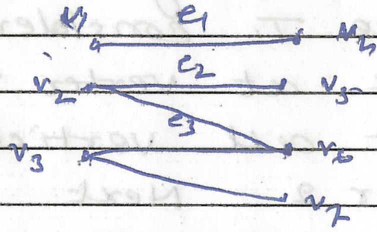
Hence the formula holds in case G is a tree. Now consider the case when G is not a tree, then G has some circuits. Consider an edge 'e' say, in some circuit. By removing this edge from 'e' from the plane representation of G, the regions are merged into a new region. Therefore $G - \{e\}$ is a connected graph with n-vertices, $e-1$ edges and $r-1$ regions. Thus by induction hypothesis, we have +2

$r - 1 = (e-1) - n + 2 \Rightarrow \boxed{r = e - n + 2}$

5(iii) The ring sum of any two circuit vectors is either a third circuit vector or an edge disjoint union of circuit vectors. +2

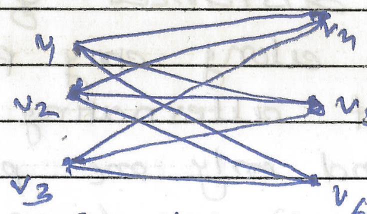
The set of all circuit vectors satisfies the closure axiom, Associativity, and commutativity. And so is the existence of the identity element ϕ and inverse. The set of vectors also satisfies the closure axiom w.r.t scalar multiplication. +3.

6(i) Bipartite graph: A graph G is called bipartite if its vertex set V can be decomposed into two disjoint subset V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2 +2



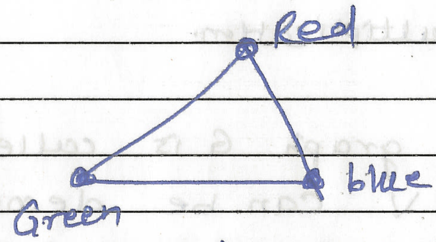
Bi-Partite graph.

Complete bipartite graph: A bipartite graph is said to be a complete bipartite graph if there is one-edge between every vertex of set V_1 to every vertex of set V_2 +1



Complete bipartite graph.

properly colored graph: Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the proper coloring of a graph. A graph in which every vertex has been assigned a color according to a proper coloring is called a properly colored graph.



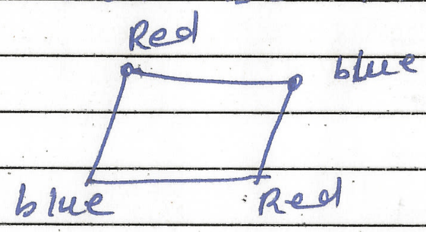
properly colored graph.

+2

5(ii) Select any vertex v in the given tree T. Consider T as a rooted tree at vertex v. Paint v with color 1. paint all vertices adjacent to v with color 2. Next paint the vertices adjacent +1 to these using color 1. continue this process till every vertex in T has been painted. Now in T we find that all +1 vertices at odd distances from v have color 2, while v and vertices at even distances from v have color 1.

Now along any path in T the vertices are of alternating colors. Since there is one and only one path between any two vertices in a tree, no two adjacent +2

vertices have the same color. Thus T has been properly colored with two colors. One color would not have been enough
⇒ every tree with two or more vertices is 2-chromatic but converse need not be true, because given



+1

graph is 2-chromatic but not a tree.

6 (iii) Consider a subset of r vertices in V_1. These r-vertices have at least $m \cdot r$ edges incident on them. Each $m \cdot r$ edge is incident to some vertex in V_2. Since the degree of every vertex in set V_2 is no greater than m, these $m \cdot r$ edges are incident on at least $(m \cdot r)/m = r$ vertices in V_2. +3

Thus any subset of r-vertices in V_1 is collectively adjacent to r or more vertices in V_2. Therefore, according to result " A complete matching of V_1 into V_2 in a bipartite graph exists if and only if every subset of r vertices in V_1 is collectively adjacent to r or more vertices in V_2 for all values of r" there exists a complete matching of V_1 into V_2 +2