Total No. of Questions: 6 Total No. of Printed Pages:3

Enrollment No......................................

Faculty of Engineering / Science

End Sem Examination May-2024

EN3BS11 / BC3BS01 Engineering Mathematics -I

Programme: B.Tech. / B.Sc. Branch/Specialisation: All

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d. Assume suitable data if necessary. Notations and symbols have their usual meaning.

Q.1 i. Condition for unique solution of system of non-homogeneous linear equation (where A is coefficient Matrix, C is augmented matrix and 'n' is number of unknowns) is ______. **1**

(a) $Rank(A) = Rank(C) < n$

(b) $Rank(A) = Rank(C) = n$

(c) $Rank(A) \neq Rank(C)$

(d) None of these

ii. If A be an $n \times n$ identity matrix, then rank of A is given by- **1**

(a) n^n (b) n^2 (c) n (d) None of these

iii. The function $f(x,y) = \log_e\left(\frac{x^4+y^4}{x+y}\right)$ is- **1**

(a) Not homogeneous (b) Homogeneous

(c) can't say (d) None of these

iv. $if\ f(x,y) = 3x^3y + y^3x$ then the value of f_{xy} is- **1**

(a) $9x^2 + y$ (b) $9x^2y$

(c) $9x^2y + 3x$ (d) None of these

v. The general form of the series $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n}\right]$ is- **1**

(a) $\frac{r}{n}$ (b) $\frac{r}{n+r}$

(c) $\sum_{r=0}^{r=2n} \frac{1}{n+r}$ (d) None of these

vi. The value of $\beta(3,4)$ is- **1**

(a) 1/12 (b) 60 (c) 1/30 (d) 1/60

vii. Order and Degree of differential equation $(\frac{d^2y}{dx^2})^4 + 6\frac{dy}{dx} = e^x$ is- **1**

(a) 2, 2 (b) 2, 4 (c) 1, 1 (d) None of these

viii. Differential equation $Mdx + Ndy = 0$ where M and N are function of x and y is said to be exact differential equation if- **1**

(a) $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$ (b) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(c) $x\frac{\partial M}{\partial x} = y\frac{\partial N}{\partial y}$ (d) None of these

ix. Cauchy's Riemann equations for the function- **1**
$\omega = f(z) = u(x,y) + i\,v(x,y)$ is-

(a) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ (b) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(c) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -x\frac{\partial v}{\partial x}$ (d) None of these

x. $\omega = \frac{1}{z}$ is analytic everywhere except at- **1**

(a) $z = 1$ (b) $z = 0$ (c) $z = -1$ (d) None of these

Q.2 i. Why a system of homogeneous equation is always consistent? Explain it. **2**

ii. Reduce the given matrix A into its Echelon form and hence find its rank- **3**

$$A = \begin{bmatrix} 3 & -4 & -1 & 2 \\ 1 & 7 & 3 & 1 \\ 5 & -2 & 5 & 4 \end{bmatrix}$$

iii. Check whether the system of linear equations are consistent or not- **5**

$$a - 2b + c - d + 1 = 0$$
$$3a - 2c - 3d + 4 = 0$$
$$5a - 4b + d + 3 = 0$$

OR iv. Find the Eigen values and corresponding Eigen vectors of given matrix- **5**

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Q.3 Attempt any two:

i. Find the maxima and minima of the function- **5**

$$f(x) = x^3 + y^3 - 3x - 12y + 20$$

ii. Expand $\tan\left(x + \frac{\pi}{4}\right)$ as far as the term x^4 using Taylor series and evaluate $tan46.5^{\circ}$ upto three significant digits. **5**

iii. If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$ then by Euler's theorem prove that- **5**

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

Q.4 Attempt any two:

i. Evaluate the following- **5**

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\ldots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$$

ii. Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\,dxdydz.$ **5**

iii. Change the order of the integration- **5**

$$\int_0^4 \int_x^{2\sqrt{x}} f(x,y)dxdy$$

Q.5 i. Solve the following differential equation- **4**

$$ydx - xdy + xy\ \log x\,dx = 0$$

ii. Solve the following differential equation- **6**

$$(D^2 - 4D + 4)y = x^2 + e^x + \cos 2x$$

OR iii. Solve the differential equation- **6**

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$$

Q.6 i. Define analytic function and harmonic function for complex function. **4**

ii. Evaluate $\int_{1-i}^{2+i}(2x + iy + 1)\,dz$ along the two paths: $x = t + 1$, $y = 2t^2 - 1$. **6**

OR iii. Evaluate using Cauchy's integral formula- **6**

$$\oint_C \frac{z^2}{(z-1)(z-2)(z-3)}\ C: |z| = 2.5$$

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Q.1 MCQ

(i) (b) Rank $(A) =$ Rank $(C) = n$ +1
(ii) (c) n +1
(iii) (a) Not Homogeneous +1
(iv) (d) None of these +1
(v) (c) $\sum_{r=0}^{r=2n} \frac{1}{n+r}$ +1
(vi) (d) $\frac{1}{60}$ +1
(vii) (b) 2, 4 +1
(viii) (b) $\partial M/\partial y = \partial N/\partial x$ +1
(ix) (b) $\partial u/\partial x = \partial v/\partial y$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ +1
(x) (b) $z = \infty$ +1

Q2 (i) System of Homogeneous eq^n is given as $AX = 0$
here always
$\rho(A) = \rho(A:B)$
therefore system of homogeneous equation is always consistent. +2

Q.2. (ii)

$$A = \begin{bmatrix} 3 & -4 & -1 & 2 \\ 1 & 7 & 3 & 1 \\ 5 & -2 & 5 & 4 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 7 & 3 & 1 \\ 3 & -4 & -1 & 2 \\ 5 & -2 & 5 & 4 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$

$R_3 \rightarrow R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & 7 & 3 & 1 \\ 0 & -25 & -10 & -1 \\ 0 & -37 & -10 & -1 \end{bmatrix}$$ [1]

$R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 7 & 3 & 1 \\ 0 & -25 & -10 & -1 \\ 0 & -12 & 0 & 0 \end{bmatrix}$$ [2]

$R_3 \rightarrow R_3/-12$

$$\sim \begin{bmatrix} 1 & 7 & 3 & 1 \\ 0 & -25 & -10 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$ which is echelon form

$\rho(A)$ = no. of non zero rows [3]

$$\boxed{\rho(A) = 3}$$

Q. 2 (iii)

$$a-2b+c-d+1=0$$
$$3a-2c-3d+4=0$$
$$5a-4b+d+3=0$$

$$\Rightarrow \quad a-2b+c-d=-1$$
$$3a-2c-3d=-4$$
$$5a-4b+d=-3$$

the matrix eqⁿ is given as $Ax=B$ — (1) [1]

Now augmented matrix —

$$[A:B] = \left[\begin{array}{cccc:c} 1 & -2 & 1 & -1 & -1 \\ 3 & 0 & -2 & -3 & -4 \\ 5 & -4 & 0 & 1 & -3 \end{array}\right]$$ [2]

operating $R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 5R_1$

$$\sim \left[\begin{array}{cccc:c} 1 & -2 & 1 & -1 & -1 \\ 0 & 6 & -5 & 0 & -1 \\ 0 & 6 & -5 & 6 & 2 \end{array}\right]$$ [3]

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{cccc:c} 1 & -2 & 1 & -1 & -1 \\ 0 & 6 & -5 & 0 & -1 \\ 0 & 0 & 0 & 6 & 3 \end{array}\right] \quad — (2)$$ [4]

$\rho(A:B) = \rho(A) = 3$ (~~no of~~ < 4 (no. of unknowns.)

$\therefore$ the system of eqⁿ is consistent & have infinite many solution. [5]

Q.2 (iv)

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

its characteristic eq^n is given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0 \qquad (1)$$

$$(3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$$\lambda = 2, 3, 5 \qquad (2)$$

Eigen values of A are 2, 3, 5.

The eigen vector corresponding to $\lambda = 2$ is the non-zero solution of the following eq^n

$$\begin{bmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 - 2R_3$

$$\sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow x_1 + x_2 + 4x_3 = 0$

$\boxed{x_3 = 0}$

$\Rightarrow \boxed{x_1 = -x_2}$

$\therefore$ Eigen Vector $X = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ for $\lambda = 2$. (3)

To find Eigen vector for $\lambda = 3$

$$\begin{bmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

or $$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_2 + 4x_3 = 0$

$-x_2 + 6x_3 = 0$

$2x_3 = 0$

$\Rightarrow \boxed{x_3 = 0}$

$\boxed{x_2 = 0}$

let $\boxed{x_1 = 1}$ (say)

$\therefore X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ eigen vector for $\lambda = 3$ (4)

Q2 (iv)

To find eigen vector for $\lambda=5$

$$\begin{bmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$-2x_1 + x_2 + 4x_3 = 0$$

$$-3x_2 + 6x_3 = 0$$

$$-3x_2 = -6x_3$$

$$x_2 = 2x_3$$

ie $\frac{x_2}{2} = \frac{x_3}{1} = k$ (say)

$$x_2 = 2k$$

$$x_3 = k$$

& $2x_1 = x_2 + 4x_3$

$$= 2k + 4(k)$$

$$2x_1 = 6k$$

$$\boxed{x_1 = 3k}$$

$\therefore \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$

$\therefore X = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ are the eigen vector for ~~$\lambda=3$~~ $\lambda=5$. [5]

Que 3 (i)

given that
$$f(x) = x^3 + y^3 - 3x - 12y + 20$$

For maxima or minima we have

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0 \qquad [1]$$

$\Rightarrow$ $3x^2 - 3 = 0$ $\qquad$ $3y^2 - 12 = 0$

$\Rightarrow$ $3x^2 = 3$ $\qquad$ $3y^2 = 12$

$x^2 = 1$ $\qquad$ $y^2 = 4$

$x = \pm 1$ $\qquad$ $y = \pm 2$

$\Rightarrow$ $(1, 2), (-1, 2), (-1, -2), (1, -2)$ are critical points. [2]

Now $r = \frac{\partial^2 f}{\partial x^2} = 6x$ $\qquad$ $t = \frac{\partial^2 f}{\partial y^2} = 6y$

$s = \frac{\partial^2 f}{\partial x \partial y} = 0$ [3]

Thus at (1, 2) : $r = 6$, $t = 6(2) = 12$, $s = 0$

$rt - s^2 = 6(12) - 0 = 24 > 0$ & $r = 6 > 0$

Hence fun f has minimum at (1, 2) [~~4~~]

$f_{min} = 1 + 8 - 3 - 24 + 20$
$= 2$

at (−1, 2) : $r = -6$ $\quad$ $t = 12$

then
$rt - s^2 = -6(12) - 0 = -72 < 0$.

Hence f(x) is neither maximum nor minimum at (−1, 2) [4] at $(-1,-2)$: $r = -6$, $t = 6(-2) = -12$

$rt - s^2 = (-6)(-12) - 0 = 72 > 0$

and $r < 0$

Hence fun $f(x)$ has maximum at $(-1,-2)$. [5]

$f_{max} = 38$

at $(1,-2)$: $r = 6$, $t = 6(-2) = \cancel{12}\ -12$

$rt - s^2 = 6(-12) - 0 = -72 < 0$

and $r = 6 > 0$

Hence fun $f(x)$ has minimum at $(1,-2)$ [5]

$f_{min} = 34$

Que 3 (iii)

let $f(x+h) = \tan\left(x+\frac{\pi}{4}\right)$

Now $f(x) = \tan x$

$\therefore f(\pi/4) = \tan\left(\frac{\pi}{4}\right) = 1$ [1]

By successive differentiating w.r.to 'x' & putting $x=\pi/4$ we get,

$f'(x) = \sec^2 x = 1+\tan^2 x$ $\quad f'(\pi/4) = 1+1 = 2$ [1.5]

$f''(x) = 2\tan x \cdot \sec^2 x$
$= 2\tan x(1+\tan^2 x)$
$= 2\tan x + 2\tan^3 x$ $\quad f''(\pi/4) = 2(1)+2(1) = 4$ [2]

$f'''(x) = 2\sec^2 x + 6\tan^2 x \cdot \sec^2 x$
$= 2(1+\tan^2 x) + 6\tan^2 x(1+\tan^2 x)$
$= 2+8\tan^2 x + 6\tan^4 x$
$f'''(\pi/4) = 16$ [2.5]

$f^{iv}(x) = 0 + 16\tan x \cdot \sec^2 x + 24\tan^3 x \sec^2 x$
$= 16\tan x(1+\tan^2 x) + 24\tan^3 x(1+\tan^2 x)$
$= 16\tan x + 16\tan^3 x + 24\tan^3 x + 24\tan^5 x$
$f^{iv}(x) = 16\tan x + 40\tan^3 x + 24\tan^5 x$

$f^{iv}(\pi/4) = 16+40+24 = 80$ [3.5]

Hence by taylor's theorem

$f(x+h) = f(h) + \frac{x}{\lfloor 1} f'(h) + \frac{x^2}{\lfloor 2} f''(h) + \frac{x^3}{\lfloor 3} f'''(h) + \cdots$ [4]

$\therefore \tan(x+\pi/4) = 1 + x(2) + \frac{x^2}{\lfloor 2}(4) + \frac{x^3}{\lfloor 3}(16) + \frac{x^4}{\lfloor 4}(80) + \cdots$

$\tan(x+\pi/4) = 1+2x+2x^2+\frac{8}{3}x^3+\frac{10}{3}x^4+\cdots$ — (1)

now put $x = 1.5^\circ$

$$= \left(1\tfrac{1}{2}\right)^\circ = \left(\frac{3}{2}\right)^\circ = \frac{3}{2} \times \frac{\pi}{180} = \frac{\pi}{120}$$

$$= 0.0261799$$

$$\simeq \cancel{0.0261 8}\ .02618$$

$$\frac{\pi}{4} = \left(\frac{\pi}{4} \times \frac{180}{\pi}\right)^\circ = 45^\circ$$

$$\therefore x + \frac{\pi}{4} = (1.5)^\circ + (45)^\circ = (46.5)^\circ$$

putting in eqⁿ ①, we get

$$\tan(46.5)^\circ = 1 + 2(0.02618) + 2(0.02618)^2 + \frac{8}{3}(0.02618)^3 + \frac{10}{3}(0.02618)^4 + \ldots$$

$$\boxed{(\tan 46.5^\circ) = 1.053751}$$ [5]

Ques 3 (iii) given $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

clearly u is not homogeneous.

$\therefore$ $\tan u = \frac{x^3+y^3}{x-y} = f(u)$ (say) [1]

$$\therefore f(u) = \frac{x^3+y^3}{x-y} = \frac{x^3\left[1+\frac{y^3}{x^3}\right]}{x\left[1-y/x\right]}$$

$$= x^2 F(y/x)$$ [2]

Thus $f(u) = \tan u$ is a homogeneous funⁿ of x & y of degree 2. ie $n=2$ [3]

Hence by using deduction of Euler's theorem, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$ [4]

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2 \cdot \tan u}{\sec^2 u} = \frac{2\sin u}{\cos u}\cos^2 u = \sin 2u$$

[5]

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Que 4 (ii)

let $P = \lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)*\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

taking $\log_e$ both side

$\log P = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\cdots+\log\left(1+\frac{n^2}{n^2}\right)\right]$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r^2}{n^2}\right)$ [1]

$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left[1+\left(\frac{r}{n}\right)^2\right]$

then by summation of series

$= \int_0^1 \log(1+x^2)\,dx$ $\quad \because \frac{r}{n}\to x,\ \& \ \frac{1}{n}\to dx$ [2]

$= \int_0^1 \log(1+x^2).1\,dx$

$= \left[\log(1+x^2)\cdot x\right]_0^1 - \int \frac{1}{(x^2+1)}\,2x\cdot x\,dx$

$= (\log 2 - 0) - 2\int_0^1\left(\frac{x^2+1-1}{x^2+1}\right)dx$ [3]

$= \log 2 - 2\int_0^1\frac{(x^2+1)}{x^2+1}dx + 2\int_0^1\frac{1}{x^2+1}dx$

$= \log 2 - 2\left[x\right]_0^1 + 2\left[\tan^{-1}x\right]_0^1$

$= \log 2 - 2(1-0) + 2(\tan^{-1}(1) - \tan 0)$

$= \log 2 - 2 + 2\left(\frac{\pi}{4}\right)$ [4]

$\log P = \log 2 - 2 + \frac{\pi}{2}$

$P = e^{\log 2 - 2 + \pi/2} = 2e^{\pi/2 - 2}$ [5]

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Q4 (ii)

let $$I = \int_{x=0}^{a} \int_{y=0}^{x} \int_{z=0}^{x+y} e^{x+y+z} \, dz \, dy \, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} \left\{ \int_{z=0}^{x+y} e^{x+y+z} \, dz \right\} dy \, dx \quad [1]$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} \left[e^{x+y+z} \right]_{0}^{x+y} dy \, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} \left[e^{2x+2y} - e^{x+y} \right] dy \, dx \quad [2]$$

$$= \int_{x=0}^{a} \left\{ \int_{y=0}^{x} \left(e^{2x+2y} - e^{x+y} \right) dy \right\} dx$$

$$= \int_{x=0}^{a} \left[\frac{e^{2x+2y}}{2} - e^{x+y} \right]_{0}^{x} dx \quad [3]$$

$$= \int_{x=0}^{a} \left[\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^{x} \right] dx$$

$$= \int_{x=0}^{a} \left[\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^{x} \right] dx \quad [4]$$

$$= \left[\frac{e^{4x}}{2(4)} - \frac{3}{2} \frac{e^{2x}}{2} + e^{x} \right]_{0}^{a}$$

$$= \left[\frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} - \frac{1}{8} + \frac{3}{4} - 1 \right]$$

$$\boxed{I = \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} - \frac{3}{8}} \quad \text{Ans.} \quad (5)$$

Que 4 (iii) Change the order of $\int_0^4 \int_x^{2\sqrt{x}} f(x,y)\,dx\,dy$

Here the region of integration is bounded by the following curves:

(i) $x=0$ ie y-axis

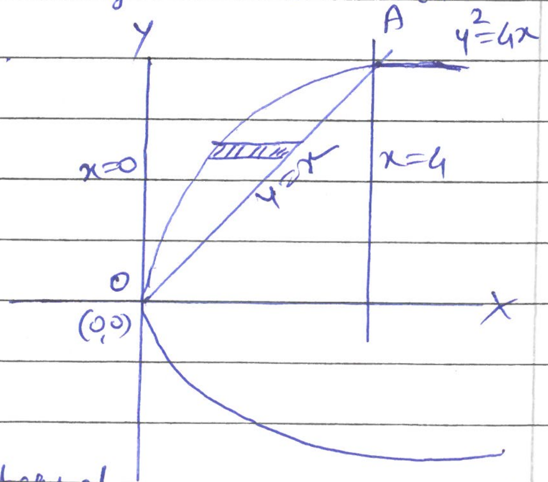
(ii) $x=4$ i.e. straight line parallel to y-axis [1]

(iii) $y=x$ i.e. equation of straight line passing through the origin & makes 45^0 angle with the positive direction of x-axis.

(iv) $y=2\sqrt{x}$ i.e. $y^2=4x$ is a parabola whose vertex is origin and axis is x-axis. [2]

The point of intersection of the parabola $y^2=4x$ & straight line $y=x$ is $(0,0)$ & $(4,4)$. [3]

Hence the region of integration is OAO shown in the figure.



For changing the order of integration draw a strip parallel to x-axis [4]

y varies from $y=0$ to $y=4$

x varies from $x=\frac{y^2}{4}$ to $x=y$

Hence the transformed integral is given by

$$\left[\int_{x=0}^{4} \int_{y=x}^{2\sqrt{x}} f(x,y)\,dx\,dy = \int_{y=0}^{4} \int_{x=\frac{y^2}{4}}^{y} f(x,y)\,dx\,dy\right]$$ [5]

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Que 5 (i)

given diff eq^n $y\,dx - x\,dy + xy \log x\,dx = 0$

divide by xy

$$\frac{1}{x}dx - \frac{1}{y}dy + \log x\,dx = 0$$

$$\left(\frac{1}{x} + \log x\right)dx - \frac{1}{y}dy = 0 \qquad (1)$$

which is ~~an~~ ~~exact~~ of the form

$M\,dx + N\,dy = 0$

here

$M = \frac{1}{x} + \log x \quad , \; N = -\frac{1}{y}$

now $\frac{\partial M}{\partial y} = 0 \quad \& \quad \frac{\partial N}{\partial x} = 0 \qquad (2)$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ therefore the given diff. eq^n is exact diff. eq^n.

$$\int_{y\,const}^{M} \left(\frac{1}{x} + \log x\right)dx = \log x + x\log x - x \qquad (3)$$

$$\int N\,dy = \int -\frac{1}{y}\,dy = -\log y$$

Hence the required sol^n is

$$\log x + x\log x - x - \log y = c \qquad (4)$$

$$\boxed{\log\frac{x}{y} + x\log x - x = c}$$

Que 5 (ii)

given diff. eqⁿ is

$$(D^2-4D+4)y = x^2+e^x+\cos 2x$$

Here the auxiliary eqⁿ is

$$m^2-4m+4=0$$

$$(m-2)^2=0$$

$$m=2,2 \qquad [1]$$

$$CF=(c_1+c_2x)e^{2x} \qquad [2]$$

$$P.I = \frac{1}{(D^2-4D+4)}(x^2+e^x+\cos 2x) \quad \text{—}④$$

$$= \frac{1}{(D^2-4D+4)}x^2 + \frac{1}{(D^2-4D+4)}e^x + \frac{1}{(D^2-4D+4)}\cos 2x$$

—①

now $\frac{1}{D^2-4D+4}x^2 = \frac{1}{4\left[1+\left(\frac{D^2}{4}-D\right)\right]}x^2$

$$= \frac{1}{4}\left[1+\left(\frac{D^2}{4}-D\right)\right]^{-1}x^2$$

$$= \frac{1}{4}\left[1-\left(\frac{D^2}{4}-D\right)+\left(\frac{D^2}{4}-D\right)^2+\cdots\right]x^2$$

$$= \frac{1}{4}\left[x^2-\left(\frac{2}{4}-2x\right)+2\right]$$

$$= \frac{1}{4}\left[x^2-\frac{2}{4}+2x+2\right]$$

$$= \frac{1}{4}\left[x^2+2x+\frac{3}{2}\right] \qquad [3]$$

& $\frac{1}{D^2-4D+4}e^x = \frac{1}{1-4+4}e^x = e^x$ [4]

Q 5(ii) continue

and $\frac{1}{D^2-4D+4}\cos 2x = \frac{1}{-(2)^2-4D+4}\cos 2x$

$= -\frac{1}{4D}\cos 2x$

$= \frac{-1}{4}\int \cos 2x\,dx$

$= \frac{-1}{4}\cdot\frac{\sin 2x}{2}$

$= \frac{-1}{8}\sin 2x$ (5)

Put these values in ①

$\therefore \quad PI = \frac{1}{4}\left(x^2+2x+\frac{3}{2}\right) + e^x - \frac{1}{8}\sin 2x$ [6]

Que 5 (iii)

given diff. eqⁿ is

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2\log x$$

let $x = e^z$, so that $z = \log x$ (1)

then the given equation becomes

$[D'(D'-1) - D' - 3]y = e^{2z}.z$

$\Rightarrow (D'^2 - 2D' - 3)\,y = z.e^{2z}$ —— ① (2)

$\therefore$ the auxiliary eqⁿ of ① is

$m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

$m = -1, 3$ (3)

$\therefore CF = C_1e^{-z} + C_2e^{3z}$

$CF = C_1\frac{1}{x} + C_2x^3$ [4]

Q 5 (iii) continue

$$P.I = \frac{1}{D'^2 - 2D' - 3} e^{2z} . z$$

$$= e^{2z} \frac{1}{(D'+2)^2 - 2(D'+2) - 3} z$$

$$= e^{2z} \frac{1}{D'^2 + 2D' - 3} . z \qquad (5)$$

$$= e^{2z} \frac{1}{(-3} \left[1 - \left(\frac{2D' + D'^2}{3} \right) \right]^{-1} z$$

$$= \frac{e^{2z}}{-3} \left[1 + \left(\frac{2D' + D'^2}{3} \right) + (\ )^2 - \right] z$$

$$= \frac{e^{2z}}{-3} \left[z + \frac{2}{3} \right]$$

$$= -\frac{e^{2z}}{9} [3z + 2]$$

$$P.I = -\frac{x^2}{9} (3 \log x + 2] \qquad (5)$$

Hence the general solution of the given eq^n is

$$y = c_1 \left(\frac{1}{x} \right) + c_2 x^3 \cancel{+} - \frac{x^2}{9} (3 \log x + 2) \qquad (6)$$

Que 6 (i) ~~If f(z) is a~~ si

Analytic fun :- If $f(z)$ is a single valued function and define in the region R of z-plane and if its derivative ie $f'(z)$ exist at every point in the region R of z-plane then $f(z)$ is said to be analytic fun in R. (2)

* $f(z)$ is said to be analytic at point z_0 if it is diff not only at z_0 but also diff in nbd of z_0.

Que 6 (i) continue

Harmonic function :-

A ~~real~~ fun $u(x,y)$ is said to be Harmonic when it is twice continuously differentiable and also satisfies laplace eqⁿ —

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad [4]$$

Que 6 (iii) given $\oint_c \frac{z^2}{(z-1)(z-2)(z-3)} dz$ & $c: |z| = 2.5$

the poles are given by putting denominator equal to zero ie

$$(z-1)(z-2)(z-3) = 0$$

$$z = 1, 2, 3 \qquad [1]$$

only the pole $z = 1, 2$ lies within the circle $c: |z| = 2.5$ [2]

* Partial fraction may be followed

$$\therefore \oint_c \frac{z^2}{(z-1)(z-2)(z-3)} dz = \oint_{c_1} \frac{z^2/(z-2)(z-3)}{(z-1)} dz + \oint_{c_2} \frac{z^2/(z-1)(z-3)}{(z-2)} dz \qquad [3]$$

By cauchy's integral formula

$$= 2\pi i \, f(1) + 2\pi i \, f(2)$$

$$= 2\pi i \left[f(1) + f(2) \right] \longrightarrow (1) \qquad [4]$$

for c_1 $f(z) = \frac{z^2}{(z-2)(z-3)}$

$\Rightarrow f(1) = \frac{1}{(-1)(-2)} = \frac{1}{2}$

for c_2: $f(z) = \frac{z^2}{(z-1)(z-3)}$

$f(2) = \frac{4}{(3)(1)} = 4/3$ [5]

Que 6 (iii) continue

put the value of $f(1)$ & $f(2)$

in ①

$$= 2\pi i\left[\frac{1}{2}+\frac{4}{3}\right]$$

$$= 2\pi i\left[\frac{3+8}{6}\right]$$

$$= 2\pi i\left(\frac{11}{6}\right)$$

$$\oint_C \frac{z^2}{(z-1)(z-2)(z-3)}\,dz = \frac{11\pi i}{3}$$ Ans. [6]

Que 6 (ii)

we have to evaluate $\int_{1-i}^{2+i}(2x+iy+1)\,dz$

along the paths: $x=t+1,\ y=2t^2-1$

$$\int_{1-i}^{2+i}(2x+iy+1)\,dz = \int_{1-i}^{2+i}(2x+iy+1)(dx+i\,dy)$$ [1]

$\because\ x=t+1,\ y=2t^2-1$

$dx=dt,\ dy=4t\,dt$ [2]

& limits:

when $x=1 \Rightarrow t=0$, $y=-1 \Rightarrow t=0$

$x=2 \Rightarrow t=1$, $y=1 \Rightarrow t=1$ [3]

$$= \int_{t=0}^{1}\left[2(t+1)+i(2t^2-1)+1\right].\left[dt+i4t\,dt\right]$$

$$= \int_0^1\{(2t+2)+i2t^2-i+1\}\{1+i4t\}\,dt$$ [4]

$$= \int_0^1\left[(2t+3)+i(2t^2-1).\right]\left[1+4ti\right]dt$$

$$= \int_0^1 \left\{(2t+3)\cdot 1 + 4ti(2t+3) + i\cdot 1(2t^2-1) + 4ti^2(2t^2-1)\right\} dt$$

$$= \int_0^1 \left[(2t+3) + 8t^2 i + 12ti + 2t^2 i - i - 8t^3 + 4t\right] dt$$

$$= \int_0^1 \left[-8t^3 + 6t + 3 + i(10t^2 + 12t - 1)\right] dt \quad [5]$$

$$= \left[-8\frac{t^4}{4} + 6\frac{t^2}{2} + 3t + i\left\{10\frac{t^3}{3} + 12\cdot\frac{t^2}{2} - t\right\}\right]_0^1$$

$$= \left[-2t^4 + 3t^2 + 3t + i\left\{\frac{10}{3}t^3 + 6t^2 - t\right\}\right]_0^1$$

$$= \left[-2 + 3 + 3 + i\left(\frac{10}{3} + 6 - 1\right) - 0\right]$$

$$= 4 + \frac{25}{3} i \qquad \text{Ans.} \qquad [6]$$

—x—