Total No. of Questions: 6 *Total No. of Printed Pages:3*

Enrollment No.....

Faculty of Science

End Sem (Even) Examination May-2019

BC3EM03 Computer oriented Numerical Method

Programme: B.Sc. (CS) Branch/Specialisation: Computer Science

Duration: 3 Hrs. **Maximum Marks: 60**

Note: All questions are compulsory. Internal choices, if any, are indicated. Answers of Q.1 (MCQs) should be written in full instead of only a, b, c or d.

Q.1 i. The convergence of which of the following method is sensitive to starting value? **1**

(a) False position (b) Gauss seidal method

(c) Newton-Raphson method (d) All of these

ii. In the Gauss elimination method for solving a system of linear algebraic equations, triangularization leads to: **1**

(a) Diagonal matrix (b) Lower triangular matrix

(c) Upper triangular matrix (d) Singular matrix

iii. Lagrange's interpolation is used to interpolate with ____ interval. **1**

(a) Equal (b) Unequal

(c) Both (a) and (b) (d) None of these

iv. Newton forward interpolation formula, the first and second terms will give the _____interpolation. **1**

(a) Linear (b) Parabolic (c) Hyperbolic (d) Non-linear

v. In Simpson's 3/8th rule which is applicable only when_____. **1**

(a) n is multiple of 3 (b) n is multiple of 6

(c) n is multiple of 8 (d) n is multiple of 24

vi. The modified Euler method is based on the average of ____. **1**

(a) Straight line (b) Ellipse

(c) Slope (d) Points

vii. The partial differential equation $\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial y^2} = 0$ is classified as **1**

(a) Elliptic (b) Parabolic (c) Hyperbolic (d) None of these

P.T.O.

viii. The equation $u_{xx} + u_{yy} = f(xy)$ is known as **1**

(a) Laplace (b) Poisson (c) Heat (d) Wave equation

ix. The principal of least squares state that **1**

(a) The sum of square of all points from curve is minimum

(b) The sum of square of all points from curve is maximum

(c) The sum of square root of all points from curve is minimum

(d) The sum of square root of all points from curve is maximum

x. The line obtained by the method of least square is known as the line of _____ **1**

(a) Straight line (b) Interpolation

(c) Best fit (d) Polynomial equation

Q.2 Attempt any two:

i. Find the positive root of $f(x) = 2x^3 - 3x - 6 = 0$ by Newton-Raphson method correct to 4 decimal places. **5**

ii. Solve by Gauss-Seidel method, the following system of equations: **5**

$28x + 4y - z = 32$; $x + 3y + 10z = 24$; $2x + 17y + 4z = 35$

iii. Prove that order of convergence of Newton's method is 2 or quadratic. **5**

Q.3 Attempt any two:

i. By mean of Newton's divided difference formula find the value of $f(8)$ from the following table: **5**

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

ii. Using Newton's forward interpolation formula, estimate the number of students, who obtained marks between 40 and 45 from the table: **5**

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

iii. Using Stirling's formula, to compute y_{35} given that: $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$ where y_x represent the number of persons of age x year in a life table. **5**

Q.4 Attempt any two:

i. Derive Simpson's 1/3 rd rule. Calculate the value of $\int_0^{\pi/2} \sin x \, dx$ by Simpsons 1/3 rule using 11 ordinates. **5**

ii. Find the value of $y(1.2)$ by Runge Kutta formula of fourth order solving the equation $\frac{dy}{dx} = y^2 + xy$, with $y(1) = 1$. **5**

iii. Apply Modified Euler's method to find the solution of equation $y' = x - y^2$ in the range $0 \le x \le 1$ for the boundary condition $y = 0$ at $x = 0$. **5**

Q.5 Attempt any two:

i. Solve the equation $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides $x = 0, y = 0, x = 3$ and $y = 3$ with $u = 0$ on the boundary and mesh length = 1 **5**

ii. Solve $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial t^2}$ with the condition $u(0,t) = 0$, $u(4,t) = 0, u(x,0) = x(4-x)$ taking $h = 1, k = 1$ **5**

iii. Solve $u_{tt} = 4u_{xx}$ with the boundary condition $u(0,t) = 0$, $u(4,t) = 0$, $u(x,0) = x(4-x)$ and $u_t(x,0) = 0$ taking $h = 1$, upto $t = 5$ **5**

Q.6 Attempt any two:

i. Fit a second-degree parabola to the following data using method of least squares- **5**

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

ii. Using the Chebyshev polynomials obtain the least squares approximations of second degree for $f(x) = x^4$ on $[-1,1]$. **5**

iii. Write short note on Cubic Splines and it's solution methods. **5**

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Que.1(i) (C) Newton Raphson method +1
(ii) (C) Upper triangular matrix +1
(iii) (C) Both (a) and (b) +1
(iv) (a) Linear +1
(v) (a) n is multiple of 3. +1
(vi) (C) slope +1
(vii) (C) Hyperbolic +1
(viii) (b) Poisson +1
(ix) (a) The sum of square of all points from curve is minimum. +1
(x) (C) Best fit. +1

Que.2(i) $f(x) = 2x^3 - 3x - 6 = 0 \longrightarrow$ eqn. (1) (say)

$f(0) = -6$ (−ve)

$f(1) = -7$ (−ve)

$f(2) = +4$ (+ve)

Clearly $f(1)$ and $f(2)$ are of opposite signs.

$\therefore$ The root of eqn. (1) lies between 1 and 2.

Taking initial approximation at $x_0 = \frac{1+2}{2} = 1.5$ +1 (Mark)

Also $f(x_n) = 2x_n^3 - 3x_n - 6$

$\therefore f'(x_n) = 6x_n^2 - 3$

By Newton Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

$$\therefore x_{n+1} = x_n - \frac{(2x_n^3 - 3x_n - 6)}{6x_n^2 - 3}$$

or $$x_{n+1} = \frac{4x_n^3 + 6}{6x_n^2 - 3} \longrightarrow \text{eqn. (2)}$$ (+1 Mark)

putting $n=0$ in equ. (2) we get

First Approximation

$$x_1 = \frac{4x_0^3+6}{6x_0^2-3}$$

$$x_1 = \frac{4\times(1.5)^3+6}{6\times(1.5)^2-3}$$

$$x_1 = \frac{13.5+6}{13.5-3} = \frac{19.5}{10.5}$$

$$x_1 = 1.8571$$ +1 Mark

putting $n=1$ in equ. (2) we get Second Approximation

$$x_2 = \frac{4x_1^3+6}{6x_1^2-3} = \frac{4\times(1.8571)^3+6}{6\times(1.8571)^2-3}$$

$$x_2 = \frac{31.6190}{17.6929} = 1.7871$$

i.e. $x_2 = 1.7871$

putting $n=2$ in equ. (2) we get Third Approximation

$$x_3 = \frac{4\times x_2^3+6}{6\times x_1^2-3} = \frac{4\times(1.7871)^3+6}{6\times(1.7871)^2-3}$$

$$x_3 = \frac{28.8298}{16.1623}$$ +1 Mark

$$x_3 = 1.7837$$

putting $n=3$ in equ. (2) we get Fourth Approximation

$$x_4 = \frac{4x_3^3+6}{6x_3^2-3} = \frac{4\times(1.7837)^3+6}{6\times(1.7837)^2-3}$$

$$x_4 = \frac{28.69997}{16.089} \Rightarrow x_4 = 1.7838$$

putting $n=4$ in equ. (2) we get Fifth Approximation

$$x_5 = \frac{4x_4^3+6}{6x_4^2-3} = \frac{4\times(1.7837)^2+6}{6\times(1.7838)^2-3}$$

$$x_5 = \frac{28.7034}{16.09188} = 1.7838$$

i.e. clearly we see that fourth and fifth approximation are equivalent i.e. $x_4 = x_5 \approx 1.7838$

Thus the root of equ. (1) lies nearer to 1.7838. +1 Mark.

Que.2(ii) Since in each equation one of the coefficient is larger than the other i.e. satisfying the condition for Gauss-Siedel method, we write the given equation in the following form

$$x = \frac{1}{28}(32 - 4y + z)$$

$$y = \frac{1}{17}(35 - 2x - 4z)$$

$$z = \frac{1}{10}(24 - x - 3y)$$ (+1) Mark

Start with, $y=0$, $z=0$ and using the most recent values of x, y, z

First Iteration

$$x^{(1)} = \frac{1}{28}(32 - 4\times 0 + 0) = 1.1429$$

$$y^{(1)} = \frac{1}{17}(35 - 2\times(1.1429) - 4\times 0) = 1.9244$$

$$z^{(1)} = \frac{1}{10}(24 - 1.1429 - 3\times 1.9244) = 1.8084$$ (+1) Mark

Second Iteration

$x^{(2)} = 0.9325$, $y^{(2)} = 1.5230$, $z^{(2)} = 1.8497$

Third Iteration

$x^{(3)} = 0.9913$, $y^{(3)} = 1.5070$, $z^{(3)} = 1.8488$ (+1) Mark

Fourth Iteration

$x^{(4)} = 0.9936$, $y^{(4)} = 1.5069$, $z^{(4)} = 1.8486$ (+1) Mark

Fifth Iteration

$x^{(5)} = 0.9936$ $y^{(5)} = 1.5069$ $z^{(5)} = 1.8486$

Since 4th and 5th iterations are same, we get

$x = 0.9936$, $y = 1.5069$, $z = 1.8486$ (+1) Mark

Que.2(iii) Let α be the exact root of the equ. $f(x)=0$.
Let x_n and x_{n+1} be the approximations for the root after n^{th} and $(n+1)^{th}$ iteration respectively. Let e_n and e_{n+1} be the corresponding errors.

Then $e_n = x_n - \alpha$ and $e_{n+1} = x_{n+1} - \alpha$

i.e. $x_n = e_n + \alpha$ and $x_{n+1} = e_{n+1} + \alpha$ (+1 Mark)

By Newton Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\Rightarrow (e_{n+1} + \alpha) = (e_n + \alpha) - \frac{f(e_n+\alpha)}{f'(e_n+\alpha)}$$

$$\Rightarrow e_{n+1} = e_n - \frac{f(\alpha+e_n)}{f'(\alpha+e_n)} \xrightarrow{\text{equ.}(1)} \text{(say)}$$ (+1 Mark)

(Since expanding $f(\alpha+e_n)$ and $f'(\alpha+e_n)$ by Taylor series we have

$$f(\alpha+e_n) = f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{\lfloor 2} f''(\alpha) + \cdots$$

$$f(\alpha+e_n) = e_n f'(\alpha) + \frac{e_n^2}{\lfloor 2} f''(\alpha) + \cdots \quad (\because f(\alpha)=0$$

and

$$f'(\alpha+e_n) = f'(\alpha) + e_n f''(\alpha) + \cdots$$ (+1 Mark)

putting these values in equ. (1) we get

$$e_{n+1} = e_n - \frac{\left\{e_n f'(\alpha) + \frac{e_n^2}{\lfloor 2} f''(\alpha) + \cdots\right\}}{\left(f'(\alpha) + e_n f''(\alpha) + \cdots\right)}$$

$$\text{or } e_{n+1} = \frac{e_n^2 f''(\alpha)}{2f'(\alpha)}\left[1 + e_n \frac{f''(\alpha)}{f'(\alpha)}\right]^{-1}$$

$$= \frac{e_n^2 f''(\alpha)}{2f'(\alpha)}\left[1 - e_n \frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]$$

$$\text{or } e_{n+1} = \frac{e_n^2 f''(\alpha)}{2f'(\alpha)} \quad (\because (1+x)^{-1} = 1 - x + x^2 - x^3 \cdots$$ (+1 Mark)

(neglecting small quantities)

Comparing by definition of order of convergence $e_{n+1} \le k|e_n|^p$

Hence $p=2$ and $k = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$. This shows that order of convergence of N-R method is 2 i.e. N-R method is quadratic convergent. (+1 Mark)

Que 3 (i) The divided difference table is:

x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4}=52$			
5	100		$\frac{97-52}{7-4}=15$		
		$\frac{294-100}{7-5}=97$		$\frac{21-15}{10-4}=1$	
7	294		$\frac{202-97}{10-5}=21$		$\frac{1-1}{11-4}=0$
		$\frac{900-294}{10-7}=202$		$\frac{27-21}{11-5}=1$	
10	900		$\frac{310-202}{11-7}=27$		$\frac{1-1}{13-5}=0$
		$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	
11	1210		$\frac{409-310}{13-10}=33$		
		$\frac{2028-1210}{13-11}=409$			
13	2028				

(+2 Marks)

We know that Newton's divided difference formula

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + (x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta^4 f(x_0) \longrightarrow (1)$$

(+1 Mark)

put $x=8$ in eqn. (1), we get

$$\therefore f(8) = 48 + (8-4)\times 52 + (8-4)(8-5)\times 15 + (8-4)(8-5)(8-7)\times 1$$

(+1 Mark)

$$\therefore F(8) = 48 + 208 + 180 + 12$$

i.e. $F(8) = 448$. Answer.

+1 Mark

Que. 3(ii)

Marks less than (x)	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

+2 Marks

Now at $x=45$, then $u = \frac{x-x_0}{h} = \frac{45-40}{10} = 0.5$

Using Newton's forward interpolation formula

$$f(x) = y_0 + \frac{u}{\lfloor 1}\Delta y_0 + \frac{u(u-1)}{\lfloor 2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{\lfloor 3}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{\lfloor 4}\Delta^4 y_0 + \cdots$$ (+1 Mark)

$$\Rightarrow f(45) = 31 + (0.5)\times 42 + \frac{(0.5)(-0.5)}{2}\times 9 + \frac{(0.5)(-0.5)(-1.5)}{6}\times(-25) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}\times 37$$ (+1 Mark)

i.e. $f(45) = 47.87 \approx 48$

Hence the no. of students getting marks between 40 and 45

$= f(45) - f(40)$

$= 48 - 31 = 17$. Answer. (+1 Mark)

Que. 3(iii) Given $y_{20} = 512$, $y_{30} = 439$; $y_{40} = 346$, $y_{50} = 243$

$u = \frac{x-a}{h}$; Taking origin at $a = 30 = x_0$

We know that by Stirling's formula

$$y_u = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{\lfloor 3}\times\left\{\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right\}$$ (+ Marks)

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_{-1} 20	y_{-1} 512	$\Delta y_{-1} = -73$		
x_0 30	y_0 439	$\Delta y_0 = -93$	$\Delta^2 y_{-1} = -20$	$\Delta^3 y_{-1} = 10$
x_1 40	y_1 346	$\Delta y_1 = -103$	$\Delta^2 y_0 = -10$	
x_2 50	y_2 243			

(+2 Marks)

Here $u = \frac{35-30}{10} = 0.5$

$$\therefore y_{35} = 439 + (0.5)\left(\frac{-93-73}{2}\right) + \frac{(0.5)^2}{2}(-20)$$ (+1 Mark)

$$+ \frac{(0.5)((0.5)^2-1)^2)}{6} \times \frac{10}{2}$$

$$y_{35} = 394.6875 \approx 395$$ Answer (+1 Mark)

Que.4(i) Here given 11 ordinates i.e. no. of sub-intervals is $n = 10$

Taking $h = \frac{b-a}{n} \Rightarrow h = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$ (+1 Mark)

x	$y = \sin x$
$x_0 = 0$	$y_0 = 0$
$x_1 = \pi/20$	$y_1 = 0.1564$
$x_2 = \pi/10$	$y_2 = 0.3090$
$x_3 = 3\pi/20$	$y_3 = 0.4540$
$x_4 = \pi/5$	$y_4 = 0.5878$
$x_5 = \pi/4$	$y_5 = 0.7071$
$x_6 = 3\pi/10$	$y_6 = 0.8090$
$x_7 = 7\pi/20$	$y_7 = 0.8910$
$x_8 = 2\pi/5$	$y_8 = 0.9511$
$x_9 = 9\pi/20$	$y_9 = 0.9877$
$x_{10} = \pi/2$	$y_{10} = 1.0000$

(+2 Marks)

Using Simpson's 1/3 rule

$$\int_0^{\pi/2} \sin x\,dx = \frac{h}{3}\left[(y_0+y_{10}) + 4(y_1+y_3+y_5+y_7+y_9) + 2(y_2+y_4+y_6+y_8)\right]$$ (+1 Mark)

$$= \frac{\pi}{20\times 3}\left[(0+1.0) + 4(0.1564+0.4540+0.7071+0.8910+0.9877) + 2(0.3090+0.5878+0.8090+0.9511)\right]$$

i.e. $\int_0^{\pi/2} \sin x\,dx = 1.0006$. Answer. (+1 Mark)

Que.4(ii) Given $\frac{dy}{dx} = y^2 + xy$ with $x_0 = 1$, $y_0 = 1$
$= f(x,y)$ (say) → (1)

To find $y(1.2)$ Take $h = \frac{x - x_0}{n}$, $n = 1$ one step size

$$h = \frac{1.2 - 1}{1} = 0.2$$

By Runge-kutta Method of fourth order

$k_1 = h\, f(x_0, y_0)$

$k_2 = h\, f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$k_3 = h\, f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

$k_4 = h\, f(x_0 + h, y_0 + k_2)$

and $K = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ (+1 Mark)

and

$y_1 = y_0 + K$

$\therefore k_1 = (0.2) f(1, 1)$

$= (0.2)[1^2 + (1 \times 1)]$ ($\because$ By defn. (1))

$k_1 = 0.4$ (+1 Mark)

Simi. $k_2 = (0.2) f(1 + \frac{0.2}{2}, 1 + \frac{0.4}{2})$

$\therefore k_2 = (0.2) f[1.1, 1.2]$

$= (0.2)[(1.2)^2 + (1.1) \times (1.2)]$

$k_2 = 0.552$ (+1 Mark)

and $k_3 = (0.2) f(1.1, 1.276)$

$k_3 = 0.60635$

$k_4 = (0.2) f(1.2, 1.6063)$

$k_4 = 0.90159$

Thus $K = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$K = 0.60304$ (+1 Mark)

Hence

$y_1 = y_0 + K$

$y_1 = 1 + 0.60304$

i.e. $y_1 = 1.60304$ Answer. (+1 Mark)

Que 4 (iii) Given $y' = \frac{dy}{dx} = x - y^2 = f(x,y)$ (say) $\rightarrow$ ① in the range $0 \le x \le 1$

Also given boundary condition $y = 0$ at $x = 0$

i.e. at $x_0 = 0$; $y_0 = 0$

To find the value of y at $x = 1$ i.e. $y(1)$

We take here $h = 0.5$ ($n = 2$ step size)

then $x_1 = x_0 + h = 0 + 0.5 = 0.5$

By Modified Euler's method

$$y_{n+1} = y_n + h f\{x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\} ; n = 0, 1, 2, \ldots \rightarrow ②$$

(+1) (Mark)

put $n = 0$ in equ. ② ; we get

$$y_1 = y_0 + h f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))$$

$$\Rightarrow y_1 = 0 + (0.5) f(0 + \frac{0.5}{2}, 0 + (\frac{0.5}{2}) f(0,0))$$

$$y_1 = 0 + (0.5) f(0.25, 0.25(0 - 0^2)) \quad (\because \text{By ①})$$

$$y_1 = (0.5) f(0.25, 0)$$

$$y_1 = (0.5)(0.25 - 0^2) \quad (\because \text{By ①})$$

i.e. $y_1 = (0.5) \times (0.25) = 0.125$

i.e. at $x_1 = 0.5$ $y_1 = 0.125$ (+2 Marks)

Again put $n = 1$ in equ. ② we get

$$y_2 = y_1 + h f\{x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\}$$

$$y_2 = (0.125) + (0.5) f\{(0.5 + \frac{0.5}{2}), (0.125) + \frac{0.5}{2} f(0.5, 0.125)$$

$$= (0.125) + (0.5) f(0.75, (0.125) + (0.25)(0.5 - (0.125)^2))$$

$$= (0.125) + (0.5) f(0.75, (0.125) + (0.25) \times (0.34375))$$

$$= (0.125) + (0.5) f(0.75, (0.125) + 0.0859375))$$

$$= (0.125) + (0.5) f(0.75, 0.2109375)$$

$$= (0.125) + (0.5)\{0.75 - (0.2109375)^2\}$$

$$= (0.125) + (0.5)\{0.75 - 0.04449462\}$$

$$= (0.125) + (0.5)(0.70550537)$$

$$= (0.125) + 0.35275$$

$$y_2 = 0.477752$$ (+2 Marks)

i.e. at $x_2 = x_0 + 2h = 0 + 2 \times 0.5 = 1$ $y_2 = 0.47775$

Answer.

Q.5(i) Draw the figure with sides $x=0$, $y=0$ and $x=y=3$ and $h=k=1$. Let u_1, u_2, u_3, u_4 be the values of u at the internal mesh-points.

We know that $\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$

and $\frac{\partial^2 u}{\partial y^2} = \frac{1}{k^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})$

Hence given eqn. becomes

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10(i^2+j^2+10) \quad \text{eqn. (1)}$$

(+1 Mark)

For finding u_1, u_2, u_3 and u_4

putting $i=1$, $j=2$ then eqn. (1) becomes

$$0 + u_2 + u_3 + 0 - 4u_1 = -10\{(1)^2 + 2^2 + 10\}$$

$$\Rightarrow u_1 = \frac{1}{4}(u_2 + u_3 + 150) \longrightarrow \text{eqn. (2)}$$

putting $i=2$, $j=2$ then eqn. (1) becomes

$$u_1 + u_4 - 4u_2 = -10(2^2 + 2^2 + 10)$$

$$u_2 = \frac{1}{4}(u_2 + u_3 + 150) \longrightarrow \text{eqn. (3)}$$

putting $i=1$, $j=1$ in (1) becomes.

$$0 + u_4 + 0 + u_1 - 4u_3 = -10(1+1+10)$$

$$u_3 = \frac{1}{4}(u_1 + u_4 + 120) \longrightarrow \text{eqn. (4)}$$

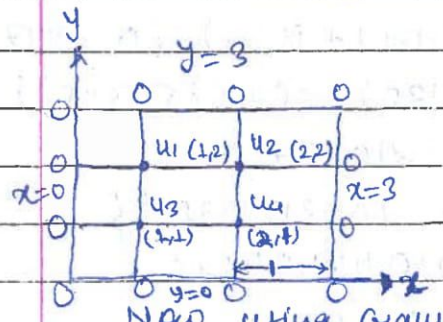
putting $i=2$, $j=1$, in eqn. (1) becomes

$$u_3 + 0 + u_2 - 4u_4 = -10(4+1+10)$$

$$u_4 = \frac{1}{4}(u_2 + u_3 + 150) \longrightarrow \text{eqn. (5)}$$

From eqn. (2) and (5) $\Rightarrow u_4 = u_1$

Hence $u_1 = \frac{1}{4}(u_2 + u_3 + 150)$



$$\left.\begin{aligned} u_2 &= \frac{1}{4}(2u_1 + 180) \\ u_3 &= \frac{1}{4}(2u_1 + 120) \end{aligned}\right\} \longrightarrow \text{eqn. (6)}$$

(+2 Marks)

Now using Gauss Siedel iterative method:

Let us assume initial value $u_2 = u_3 = 0$

First iteration

$$u_1^{(1)} = \frac{1}{4}(0+0+150) = 37.5$$

$$u_2^{(1)} = \frac{1}{4}(2\times 37.5+180) = 63.75$$

$$u_3^{(1)} = \frac{1}{4}(2\times 37.5+120) = 48.75$$

Second iteration

$$u_1^{(2)} = \frac{1}{4}(63.75+48.75+150) = 65.63$$

$$u_2^{(2)} = \frac{1}{4}(2\times 65.63+180) = 77.82$$

$$u_3^{(2)} = \frac{1}{4}(2\times 65.63+120) = 62.82$$

Similarly,

Third iteration $u_1^{(3)} = 72.66$, $u_2^{(3)} = 81.33$, $u_3^{(3)} = 66.33$

Fourth iteration $u_1^{(4)} = 74.42$, $u_2^{(4)} = 82.21$, $u_3^{(4)} = 62.71$ (+1 Mark)

Fifth iteration $u_1^{(5)} = 74.86$, $u_2^{(5)} = 82.43$, $u_3^{(5)} = 67.43$

Sixth iteration $u_1^{(6)} = 74.97$, $u_2^{(6)} = 82.48$, $u_3^{(6)} = 67.49$

Seventh iteration $u_1^{(7)} = 74.99$, $u_2^{(7)} = 82.5$ $u_3^{(7)} = 67.5$

Eight iteration $u_1^{(8)} = 74.99$, $u_2^{(8)} = 82.5$, $u_3^{(8)} = 67.5$

Clearly seventh and eight iteration gives the same values

Hence

$u_1 = 74.99 \approx 75$

$u_2 = 82.5$

$u_3 = 67.5$

$u_4 = 74.99 \approx 75$

(+1 Mark)

Que.5(ii) Given $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$ or $2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$\therefore \frac{\partial^2 u}{\partial x^2} = a \cdot \frac{\partial u}{\partial t}$; Here $a = 2$.

To use Bender-Schmidt's equation $k = \frac{a}{2}h^2 = 1$
step size in time $= k = 1$. The values of $u_{i,j}$ are tabulated below (+1 Mark)

x direction $\longrightarrow$

	j \ i	0	1	2	3	4
t direction	0	0	3	4	3	0
	1	0	2	3	2	0
	2	0	1.5	2	1.5	0
	3	0	1	1.5	1	0
	4	0	0.75	1	0.75	0
	5	0	0.5	0.75	0.5	0

(+2 Marks)

Analysis.

Range for x : $(0,4)$ for t : $(0,5)$

$u(x,0) = x(4-x)$ This gives $u(0,0)=0$, $u(1,0)=3$
$u(2,0)=4$ $u(3,0)=3$ $u(4,0)=0$ (+1 Mark)

For all t, at $x=0$ $u=0$ and for all t at $x=4$, $u=0$.
using these values we fill up column under $x=0, x=4$ and row against $t=0$.

$a \quad b$
$\searrow c \swarrow$ This means $c = \frac{a+b}{2}$

The value of u at $t=1$ are written by seeing the values of u at $t=0$ and using the average formula. (+1 Mark)

Que-5(iii) To solve $u_{tt} = 4u_{xx}$

Here $a^2 = 4$ i.e. $a = 2$, taking $h = 1$ we get $k = \frac{h}{a} = \frac{1}{2} = 0.5$

From the initial conditions

$u(0,t) = 0 \Rightarrow u = 0$ along entire line $x = 0$

$u(4,t) = 0 \Rightarrow u = 0$ along entire line $x = 4$ (+1 Mark)

In difference form, these are

$u_{0,j} = 0$ and $u_{4,j} = 0$ for all j.

Now $u(x,0) = x(4-x)$

$\Rightarrow u(0,0) = 0, \quad u(1,0) = 3, \quad u(2,0) = 4$

$u(3,0) = 3, \quad u(4,0) = 0$

In difference notations

$u_{i,0} = u(i,0) = i(4-i)$ for different i

putting $i = 0, 1, 2, 3, 4$, we get

$u_{0,0} = 0, \quad u_{1,0} = 3, \quad u_{2,0} = 4, \quad u_{3,0} = 3, \quad u_{4,0} = 0$ (+1 Mark)

Now the condition $u_t(x,0) = 0$

$\Rightarrow \frac{1}{k}[u_{i,j+1} - u_{i,j}] = 0$ when $j = 0$

$\Rightarrow u_{i,1} = u_{i,0}$ for all i.

i.e. u on the first two rows are equal

Now consider the recurrence relation

$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1}$ (+1 Mark)

If we put $j = 1$, we get

$u_{i,2} = u_{i+1,1} + u_{i-1,1} - u_{i,0}$

putting $i = 1, 2, 3, \ldots$ successively, we get

$u_{1,2} = u_{2,1} + u_{0,1} - u_{1,0} = 4 + 0 - 3 = 1$

$u_{2,2} = u_{3,1} + u_{1,1} - u_{2,0} = 3 + 3 - 4 = 2$

$u_{3,2} = u_{4,1} + u_{2,1} - u_{3,0} = 0 + 4 - 3 = 1$

i.e. the third row is filled in. We can fill the remaining rows as shown below:

(+2 Mark)

j \ i	0	1	2	3	4
0	0	3	4	3	0
1	0	3	4	3	0
2	0	1	2	1	0
3	0	−1	−2	−1	0
4	0	−3	−4	−3	0

Que-6(i) Here $m=5$, which is odd and the values of x are equidistant. Therefore we take the origin for x-series at the middle value 2.

Now suppose $X = x-2$, $Y = y$, then equation of parabola of 2nd degree to be fitted is of the form given by

$$Y = a + bX + cX^2 \longrightarrow \text{eqn. } (1)$$

Thus its normal equ. are;

$$\Sigma Y = ma + b\Sigma X + c\Sigma X^2 \longrightarrow \text{eqn. } (2)$$

$$\Sigma XY = a\Sigma X + b\Sigma X^2 + c\Sigma X^3 \longrightarrow \text{eqn. } (3)$$

(+1 Mark)

$$\Sigma X^2 Y = a\Sigma X^2 + b\Sigma X^3 + c\Sigma X^4 \longrightarrow \text{eqn. } (4)$$

x	y	X	Y	XY	X^2	X^2Y	X^3	X^4
0	1	−2	1	−2	4	1	−8	16
1	1.8	−1	1.8	−1.8	1	1.8	−1	1
2	1.3	0	1.3	0	0	0	0	0
3	2.5	1	2.5	2.5	1	2.5	1	1
4	6.3	2	6.3	12.6	4	25.2	8	16
Total	12.9	0	12.9	11.3	10	33.5	0	34

(+2) Marks

putting the above values in normal equ. (2) (3) and (4) we get

$$5a + 10c = 12.9 \longrightarrow (5)$$

$$10b = 11.3 \longrightarrow (6)$$

$$10a + 34c = 33.5 \longrightarrow (7)$$

Solving equ. (5) (6) and (7) we get

$$a = 1.48, \quad b = 1.13, \quad c = 0.55$$

(+1) Mark

∴ The required equ. is

$$Y = 1.48 + 1.13X + 0.55X^2 \longrightarrow (8)$$

Now changing the origin i.e. $X = x-2$ and $Y = y$ in (8)

$$y = 1.48 + 1.13(x-2) + (0.55)(x-2)^2$$

or

$$y = 1.42 + (-1.07)x + 0.55x^2$$

i.e. $$y = 1.42 - 1.07x + 0.55x^2$$

(+1) Mark

Q.6(ii) We write

$$f(x) = c_0 T_0(x) + c_1 T_1(x) + c_2 T_2(x),$$

where chebyshev polynomials T_0, T_1, T_2 are given by

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1$$

we have

$$I(c_0, c_1, c_2) = \int_{-1}^{+1} \frac{1}{\sqrt{1-x^2}} \left(x^4 - \{c_0 T_0 + c_1 T_1 + c_2 T_2\}\right)^2 dx$$

$$= \text{minimum.}$$ (+1 Mark)

$$\frac{\partial I}{\partial c_0} = -2\int_{-1}^{1} [x^4 - (c_0 T_0 + c_1 T_1 + c_2 T_2)] \cdot \frac{T_0\, dx}{\sqrt{1-x^2}} = 0$$

$$\frac{\partial I}{\partial c_1} = -2\int_{-1}^{+} [x^4 - (c_0 T_0 + c_1 T_1 + c_2 T_2) \cdot \frac{T_1\, dx}{\sqrt{1-x^2}} = 0$$

$$\frac{\partial I}{\partial c_2} = -2\int_{-1}^{+1} [x^4 - (c_0 T_0 + c_1 T_1 + c_2 T_2)] \frac{T_2\, dx}{\sqrt{1-x^2}} = 0$$ (+1 Mark)

We find

$$c_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{x^4 T_0}{\sqrt{1-x^2}} dx = \frac{1}{\pi}\int_{-1}^{1} \frac{x^4 \cdot 1}{\sqrt{1-x^2}} dx$$

$$= \frac{2}{\pi}\int_{0}^{1} \frac{x^4\, dx}{\sqrt{1-x^2}}$$

$$= \frac{2}{\pi}\int_{0}^{\pi/2} \frac{\sin^4\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$ (putting $x = \sin\theta$, $dx = \cos\theta\, d\theta$) (+1 Mark)

$$= \frac{2}{\pi}\int_{0}^{\pi/2} \sin^4\theta\, d\theta = \frac{2}{\pi} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3}{8}$$

$$c_1 = \frac{2}{\pi}\int_{-1}^{1} \frac{x^4 T_1}{\sqrt{1-x^2}} dx = \frac{2}{\pi}\int_{-1}^{1} \frac{x^4 \cdot x}{\sqrt{1-x^2}} dx = 0$$ (+1 Mark)

$$c_2 = \frac{2}{\pi}\int_{-1}^{+1} \frac{x^4 \cdot T_2}{\sqrt{1-x^2}} dx = \frac{2}{\pi}\int_{-1}^{1} \frac{x^4 (2x^2 - 1)}{\sqrt{1-x^2}} = \frac{1}{2}$$

Thus the required approximation is

$$f(x) = \frac{3}{8} T_0 + \frac{1}{2} T_2.$$ Answer. (+1 Mark)

Que.6(iii) We are given the $(n+1)$ data points (x_i, y_i) $i=0,1,2,\dots n$
Our aim is to find the ~~very~~ value of y corresponding to x where $x_i < x < x_{i+1}$; $i=0,1,2\dots(n-1)$, by using a smooth polynomial curve. We have already seen many methods. A recent one which is becoming important is the Spline-fitting. The name was derived from a draftsman's device to draw a curve through these $(n+1)$ points in such a way that not only the collocation polynomial but also its slope and the curvature as well are continuous functions.

We define a cubic-spline, $S(x)$ as follows:

(i) $S(x)$ is a polynomial of degree one for $x<x_0$ and $x>x_n$

(ii) $S(x)$ is at most a cubic polynomial in each interval (x_{i-1}, x_i), $i=1,2\dots n$.

(iii) $S(x)$, $S'(x)$ and $S''(x)$ are continuous at each points (x_i, y_i) $i=0,1,2\dots n$.

(iv) $S(x_i) = y_i$; $i=0,1,2,\dots n$. (+2 Marks)

<u>Method</u> On (x_{i-1}, x_i) let $S(x)$ be such that

$$S(x) = P_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i \; ; \; i=1,2\dots n \longrightarrow (1)$$

Now we have four unknowns a_i, b_i, c_i, d_i ; $i=1,2\dots n$.

Using continuity of $S(x)$, $S'(x)$ and $S''(x)$, we have

$$\left.\begin{aligned} P_i(x_i) &= y_i = a_i x_i^3 + b_i x_i^2 + c_i x_i + d_i \\ P_{i+1}(x_i) &= y_i = a_{i+1} x_i^3 + b_{i+1} x_i^2 + c_{i+1} x_i + d_{i+1} \end{aligned}\right\} \longrightarrow (2) \quad (+1 \text{ Marks})$$

for $i = 1,2,\dots(n-1)$

equation (2) give $2(n-1)$ conditions.

$$\left.\begin{aligned} 3a_i x_i^2 + 2b_i x_i + c_i &= 3a_{i+1} x_i^2 + 2b_{i+1} x_i + c_{i+1} \\ 6a_i x_i + 2b_i &= 6a_{i+1} x_i + 2b_{i+1} \end{aligned}\right\} \longrightarrow (3)$$

for $i=1,2,\dots,(n-1)$.

equation (3) gives $2(n-1)$ conditions.

Further $y_0 = a_1 x_0^3 + b_1 x_0^2 + c_1 x_0 + d_1$

and $y_n = a_n x_n^3 + b_n x_n^2 + c_n x_n + d_n$ (+1 Mark)

Hence totally we have $4n-2$ conditions.

Further $S''(x_0) = M_0$, $S''(x_n) = M_n$

Now, we have $4n$ conditions to solve for the $4n$ unknowns. This will give the cubic spline in each subinterval.

If $M_0 = 0$ and $M_n = 0$, we call this cubic spline as natural spline. (+1 Mark)