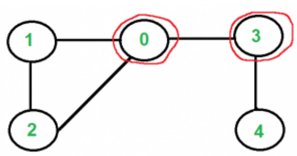
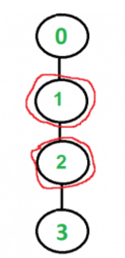
Articulation Points (or Cut Vertices) in a Graph

A vertex in an undirected connected graph is an articulation point (or cut vertex) iff removing it (and edges through it) disconnects the graph. Articulation points represent vulnerabilities in a connected network – single points whose failure would split the network into 2 or more disconnected components. They are useful for designing reliable networks.
For a disconnected undirected graph, an articulation point is a vertex removing which increases number of connected components.

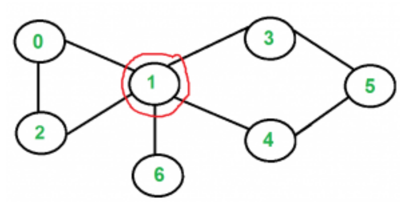
Following are some example graphs with articulation points encircled with red color.



Articulation points are 0 and 3



Artiulation Points are 1 & 2



Articulation Point is 1

How to find all articulation points in a given graph?
A simple approach is to one by one remove all vertices and see if removal of a vertex causes disconnected graph. Following are steps of simple approach for connected graph.

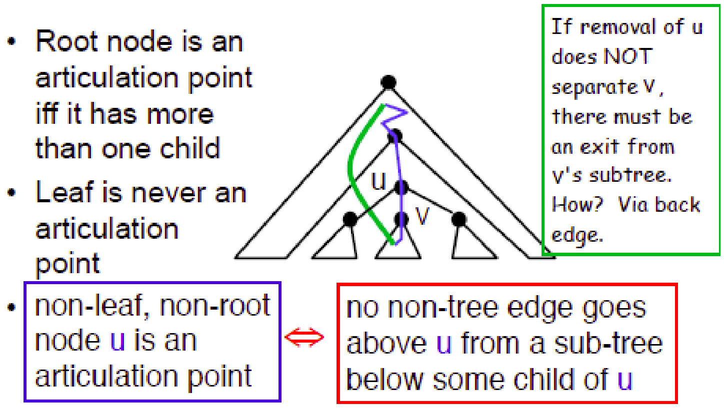
1) For every vertex v, do following
…..a) Remove v from graph
…..b) See if the graph remains connected (We can either use BFS or DFS)
…..c) Add v back to the graph

Time complexity of above method is O(V*(V+E)) for a graph represented using adjacency list. Can we do better?

A O(V+E) algorithm to find all Articulation Points (APs)
The idea is to use DFS (Depth First Search). In DFS, we follow vertices in tree form called DFS tree. In DFS tree, a vertex u is parent of another vertex v, if v is discovered by u (obviously v is an adjacent of u in graph). In DFS tree, a vertex u is articulation point if one of the following two conditions is true.

1) u is root of DFS tree and it has at least two children.
2) u is not root of DFS tree and it has a child v such that no vertex in subtree rooted with v has a back edge to one of the ancestors (in DFS tree) of u.

Following figure shows same points as above with one additional point that a leaf in DFS Tree can never be an articulation point. (Source Ref 2)



We do DFS traversal of given graph with additional code to find out Articulation Points (APs). In DFS traversal, we maintain a parent[] array where parent[u] stores parent of vertex u. Among the above mentioned two cases, the first case is simple to detect. For every vertex, count children. If currently visited vertex u is root (parent[u] is NIL) and has more than two children, print it.
How to handle second case? The second case is trickier. We maintain an array disc[] to store discovery time of vertices. For every node u, we need to find out the earliest visited vertex (the vertex with minimum discovery time) that can be reached from subtree rooted with u. So we maintain an additional array low[] which is defined as follows.

```
low[u] = min(disc[u], disc[w])
where w is an ancestor of u and there is a back edge from
some descendant of u to w.
```

Following are C++, Java and Python implementation of Tarjan's algorithm for finding articulation points.

C++

```
// A C++ program to find articulation points in an undirected graph
#include<iostream>
#include <list>
#define NIL -1
using namespace std;

// A class that represents an undirected graph
class Graph
{
    int V;    // No. of vertices
    list<int> *adj;    // A dynamic array of adjacency lists
    void APUtil(int v, bool visited[], int disc[], int low[],
                int parent[], bool ap[]);
public:
    Graph(int V);   // Constructor
    void addEdge(int v, int w);   // function to add an edge to graph
    void AP();    // prints articulation points
};

Graph::Graph(int V)
```

```
{
    this->V = V;
    adj = new list<int>[V];
}

void Graph::addEdge(int v, int w)
{
    adj[v].push_back(w);
    adj[w].push_back(v);  // Note: the graph is undirected
}

// A recursive function that find articulation points using DFS traversal
// u --> The vertex to be visited next
// visited[] --> keeps tract of visited vertices
// disc[] --> Stores discovery times of visited vertices
// parent[] --> Stores parent vertices in DFS tree
// ap[] --> Store articulation points
void Graph::APUtil(int u, bool visited[], int disc[],
                                      int low[], int parent[], bool ap[])
{
    // A static variable is used for simplicity, we can avoid use of static
    // variable by passing a pointer.
    static int time = 0;

    // Count of children in DFS Tree
    int children = 0;

    // Mark the current node as visited
    visited[u] = true;

    // Initialize discovery time and low value
    disc[u] = low[u] = ++time;

    // Go through all vertices aadjacent to this
    list<int>::iterator i;
    for (i = adj[u].begin(); i != adj[u].end(); ++i)
    {
        int v = *i;  // v is current adjacent of u

        // If v is not visited yet, then make it a child of u
        // in DFS tree and recur for it
        if (!visited[v])
        {
            children++;
            parent[v] = u;
            APUtil(v, visited, disc, low, parent, ap);

            // Check if the subtree rooted with v has a connection to
            // one of the ancestors of u
            low[u]  = min(low[u], low[v]);

            // u is an articulation point in following cases

            // (1) u is root of DFS tree and has two or more chilren.
            if (parent[u] == NIL && children > 1)
               ap[u] = true;

            // (2) If u is not root and low value of one of its child is more
            // than discovery value of u.
            if (parent[u] != NIL && low[v] >= disc[u])
               ap[u] = true;
        }

        // Update low value of u for parent function calls.
        else if (v != parent[u])
            low[u]  = min(low[u], disc[v]);
    }
}
```

```
// The function to do DFS traversal. It uses recursive function APUtil()
void Graph::AP()
{
    // Mark all the vertices as not visited
    bool *visited = new bool[V];
    int *disc = new int[V];
    int *low = new int[V];
    int *parent = new int[V];
    bool *ap = new bool[V]; // To store articulation points

    // Initialize parent and visited, and ap(articulation point) arrays
    for (int i = 0; i < V; i++)
    {
        parent[i] = NIL;
        visited[i] = false;
        ap[i] = false;
    }

    // Call the recursive helper function to find articulation points
    // in DFS tree rooted with vertex 'i'
    for (int i = 0; i < V; i++)
        if (visited[i] == false)
            APUtil(i, visited, disc, low, parent, ap);

    // Now ap[] contains articulation points, print them
    for (int i = 0; i < V; i++)
        if (ap[i] == true)
            cout << i << " ";
}

// Driver program to test above function
int main()
{
    // Create graphs given in above diagrams
    cout << "\nArticulation points in first graph \n";
    Graph g1(5);
    g1.addEdge(1, 0);
    g1.addEdge(0, 2);
    g1.addEdge(2, 1);
    g1.addEdge(0, 3);
    g1.addEdge(3, 4);
    g1.AP();

    cout << "\nArticulation points in second graph \n";
    Graph g2(4);
    g2.addEdge(0, 1);
    g2.addEdge(1, 2);
    g2.addEdge(2, 3);
    g2.AP();

    cout << "\nArticulation points in third graph \n";
    Graph g3(7);
    g3.addEdge(0, 1);
    g3.addEdge(1, 2);
    g3.addEdge(2, 0);
    g3.addEdge(1, 3);
    g3.addEdge(1, 4);
    g3.addEdge(1, 6);
    g3.addEdge(3, 5);
    g3.addEdge(4, 5);
    g3.AP();

    return 0;
}
```

Run on IDE

Java

```
// A Java program to find articulation points in an undirected graph
import java.io.*;
import java.util.*;
import java.util.LinkedList;

// This class represents an undirected graph using adjacency list
// representation
class Graph
{
    private int V;   // No. of vertices

    // Array  of lists for Adjacency List Representation
    private LinkedList<Integer> adj[];
    int time = 0;
    static final int NIL = -1;

    // Constructor
    Graph(int v)
    {
        V = v;
        adj = new LinkedList[v];
        for (int i=0; i<v; ++i)
            adj[i] = new LinkedList();
    }

    //Function to add an edge into the graph
    void addEdge(int v, int w)
    {
        adj[v].add(w);  // Add w to v's list.
        adj[w].add(v);  //Add v to w's list
    }

    // A recursive function that find articulation points using DFS
    // u --> The vertex to be visited next
    // visited[] --> keeps tract of visited vertices
    // disc[] --> Stores discovery times of visited vertices
    // parent[] --> Stores parent vertices in DFS tree
    // ap[] --> Store articulation points
    void APUtil(int u, boolean visited[], int disc[],
                int low[], int parent[], boolean ap[])
    {

        // Count of children in DFS Tree
        int children = 0;

        // Mark the current node as visited
        visited[u] = true;

        // Initialize discovery time and low value
        disc[u] = low[u] = ++time;

        // Go through all vertices aadjacent to this
        Iterator<Integer> i = adj[u].iterator();
        while (i.hasNext())
        {
            int v = i.next();  // v is current adjacent of u

            // If v is not visited yet, then make it a child of u
            // in DFS tree and recur for it
            if (!visited[v])
            {
                children++;
                parent[v] = u;
                APUtil(v, visited, disc, low, parent, ap);
```

```
                // Check if the subtree rooted with v has a connection to
                // one of the ancestors of u
                low[u]  = Math.min(low[u], low[v]);

                // u is an articulation point in following cases

                // (1) u is root of DFS tree and has two or more chilren.
                if (parent[u] == NIL && children > 1)
                    ap[u] = true;

                // (2) If u is not root and low value of one of its child
                // is more than discovery value of u.
                if (parent[u] != NIL && low[v] >= disc[u])
                    ap[u] = true;
            }

            // Update low value of u for parent function calls.
            else if (v != parent[u])
                low[u]  = Math.min(low[u], disc[v]);
        }
    }

    // The function to do DFS traversal. It uses recursive function APUtil()
    void AP()
    {
        // Mark all the vertices as not visited
        boolean visited[] = new boolean[V];
        int disc[] = new int[V];
        int low[] = new int[V];
        int parent[] = new int[V];
        boolean ap[] = new boolean[V]; // To store articulation points

        // Initialize parent and visited, and ap(articulation point)
        // arrays
        for (int i = 0; i < V; i++)
        {
            parent[i] = NIL;
            visited[i] = false;
            ap[i] = false;
        }

        // Call the recursive helper function to find articulation
        // points in DFS tree rooted with vertex 'i'
        for (int i = 0; i < V; i++)
            if (visited[i] == false)
                APUtil(i, visited, disc, low, parent, ap);

        // Now ap[] contains articulation points, print them
        for (int i = 0; i < V; i++)
            if (ap[i] == true)
                System.out.print(i+" ");
    }

    // Driver method
    public static void main(String args[])
    {
        // Create graphs given in above diagrams
        System.out.println("Articulation points in first graph ");
        Graph g1 = new Graph(5);
        g1.addEdge(1, 0);
        g1.addEdge(0, 2);
        g1.addEdge(2, 1);
        g1.addEdge(0, 3);
        g1.addEdge(3, 4);
        g1.AP();
        System.out.println();

        System.out.println("Articulation points in Second graph");
        Graph g2 = new Graph(4);
```

```
        g2.addEdge(0, 1);
        g2.addEdge(1, 2);
        g2.addEdge(2, 3);
        g2.AP();
        System.out.println();

        System.out.println("Articulation points in Third graph ");
        Graph g3 = new Graph(7);
        g3.addEdge(0, 1);
        g3.addEdge(1, 2);
        g3.addEdge(2, 0);
        g3.addEdge(1, 3);
        g3.addEdge(1, 4);
        g3.addEdge(1, 6);
        g3.addEdge(3, 5);
        g3.addEdge(4, 5);
        g3.AP();
    }
}
// This code is contributed by Aakash Hasija
```

Run on IDE

Python

```
# Python program to find articulation points in an undirected graph

from collections import defaultdict

#This class represents an undirected graph
#using adjacency list representation
class Graph:

    def __init__(self,vertices):
        self.V= vertices #No. of vertices
        self.graph = defaultdict(list) # default dictionary to store graph
        self.Time = 0

    # function to add an edge to graph
    def addEdge(self,u,v):
        self.graph[u].append(v)
        self.graph[v].append(u)

    '''A recursive function that find articulation points
    using DFS traversal
    u --> The vertex to be visited next
    visited[] --> keeps tract of visited vertices
    disc[] --> Stores discovery times of visited vertices
    parent[] --> Stores parent vertices in DFS tree
    ap[] --> Store articulation points'''
    def APUtil(self,u, visited, ap, parent, low, disc):

        #Count of children in current node
        children =0

        # Mark the current node as visited and print it
        visited[u]= True

        # Initialize discovery time and low value
        disc[u] = self.Time
        low[u] = self.Time
        self.Time += 1

        #Recur for all the vertices adjacent to this vertex
        for v in self.graph[u]:
            # If v is not visited yet, then make it a child of u
```

```
            # in DFS tree and recur for it
            if visited[v] == False :
                parent[v] = u
                children += 1
                self.APUtil(v, visited, ap, parent, low, disc)

                # Check if the subtree rooted with v has a connection to
                # one of the ancestors of u
                low[u] = min(low[u], low[v])

                # u is an articulation point in following cases
                # (1) u is root of DFS tree and has two or more chilren.
                if parent[u] == -1 and children > 1:
                    ap[u] = True

                #(2) If u is not root and low value of one of its child is more
                # than discovery value of u.
                if parent[u] != -1 and low[v] >= disc[u]:
                    ap[u] = True

                # Update low value of u for parent function calls
            elif v != parent[u]:
                low[u] = min(low[u], disc[v])


    #The function to do DFS traversal. It uses recursive APUtil()
    def AP(self):

        # Mark all the vertices as not visited
        # and Initialize parent and visited,
        # and ap(articulation point) arrays
        visited = [False] * (self.V)
        disc = [float("Inf")] * (self.V)
        low = [float("Inf")] * (self.V)
        parent = [-1] * (self.V)
        ap = [False] * (self.V) #To store articulation points

        # Call the recursive helper function
        # to find articulation points
        # in DFS tree rooted with vertex 'i'
        for i in range(self.V):
            if visited[i] == False:
                self.APUtil(i, visited, ap, parent, low, disc)

        for index, value in enumerate (ap):
            if value == True: print index,

 # Create a graph given in the above diagram
g1 = Graph(5)
g1.addEdge(1, 0)
g1.addEdge(0, 2)
g1.addEdge(2, 1)
g1.addEdge(0, 3)
g1.addEdge(3, 4)

print "\nArticulation points in first graph "
g1.AP()

g2 = Graph(4)
g2.addEdge(0, 1)
g2.addEdge(1, 2)
g2.addEdge(2, 3)
print "\nArticulation points in second graph "
g2.AP()


g3 = Graph (7)
g3.addEdge(0, 1)
g3.addEdge(1, 2)
```

```
g3.addEdge(2, 0)
g3.addEdge(1, 3)
g3.addEdge(1, 4)
g3.addEdge(1, 6)
g3.addEdge(3, 5)
g3.addEdge(4, 5)
print "\nArticulation points in third graph "
g3.AP()

#This code is contributed by Neelam Yadav
```

Run on IDE

Output:

```
Articulation points in first graph
0 3
Articulation points in second graph
1 2
Articulation points in third graph
1
```

Time Complexity: The above function is simple DFS with additional arrays. So time complexity is same as DFS which is O(V+E) for adjacency list representation of graph.

References:

https://www.cs.washington.edu/education/courses/421/04su/slides/artic.pdf
http://www.slideshare.net/TraianRebedea/algorithm-design-and-complexity-course-8
http://faculty.simpson.edu/lydia.sinapova/www/cmsc250/LN250_Weiss/L25-Connectivity.htm

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