Linear Algebra II

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Class 2, January 7 2026 (Substitution)

We first show what is called the *Fundamental Theorem of Linear Algebra*:

Theorem 1. *If $M \in M_n(\mathbb{R})$, then $\mathbb{R}^n = C(A) \oplus N(A^T)$.*

Proof. Simple, just use the fact that for any $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x}^T\mathbf{x} = 0 \implies \mathbf{x} = \mathbf{0}$ $\square$

Another result:

Theorem 2 (Fredholm Alternative). $\mathbf{a} \in C(A) \iff \mathbf{a}^T\mathbf{b} = 0\ \forall\ \mathbf{b} \in N(A^T)$.

Proof. Simple, just rewrite $\mathbf{a}$ as in the previous theorem : $\mathbf{a} = A\mathbf{y} + \mathbf{x}$ for $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x} \in N(A^T)$. Then this follows easily. The other direction is trivial. dghnhndhnhn $\square$

A good application of this is to check when a system of equations has a solution. The system $A\mathbf{x} = \mathbf{b}$ has a solution iff for any $\mathbf{x} \in N(A^T)$ we have $\mathbf{b}^T\mathbf{x} = 0$.