Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach

J.Daafouz, P. Riedinger, C. Iung

Anurag (15385)

March 28, 2018

Switched system

$$x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$$

$$y(t) = C_{\sigma(t)}x(t)$$

$$\sigma(t) \in \boldsymbol{\mathcal{J}} = \{1,2,\ldots,n\}, \forall t \in \boldsymbol{Z}$$

$$\boldsymbol{Assumption}: C_i\ is\ full\ row\ rank\ \forall i \in \boldsymbol{\mathcal{J}}$$

$$\boldsymbol{Static\ output\ feedback}: u(t) = K_{\sigma(t)}y(t)$$

$$x(t+1) = \left(A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)}C_{\sigma(t)}\right)x(t)$$

Introduction

- Existence of a **switched quadratic Lyapunov** function to check **asymptotic stability** of the **switched system** for **arbitrary switching** in **discrete time domain**
- Two equivalent conditions for existence
- Static output feedback design
- Investigation of static output feedback design (SOFD) problem
    - Necessary and sufficient condition (Non-convex)
    - Sufficient condition (Convex)

Following are equivalent

- There exists a Lyapunov function of the form

$$V(k, x_k) = x_k^T \left(\sum_{i=1}^{N} \xi_i(k) P_i \right) x_k$$

- $\begin{bmatrix} P_i & A_i^T P_j \\ P_j A_i & P_j \end{bmatrix} \succ 0;\ \forall\ (i,j) \in \boldsymbol{\mathcal{J}} \times \boldsymbol{\mathcal{J}} and\ P_i \in S_+^n\ \forall i$ (1)

- $\begin{bmatrix} G_i + G_i^T - S_i & G_i^T A_i^T \\ A_i G_i & S_j \end{bmatrix} \succ 0;\ \forall\ (i,j) \in \boldsymbol{\mathcal{J}} \times \boldsymbol{\mathcal{J}} and\ G_i \in \boldsymbol{R}^{nxn}, S_i \in S_+^n$ (2)
  - $P_i = S_i^{-1}\ in\ this\ case$

Example

- $A_1 = \begin{pmatrix} .5590 & .4778 \\ .2443 & .4976 \end{pmatrix}, A_2 = \begin{pmatrix} .4726 & .4525 \\ .5791 & .2000 \end{pmatrix}$
- $P_1 = \begin{pmatrix} .5064 & -.1446 \\ -.1446 & .5827 \end{pmatrix}, P_2 = \begin{pmatrix} .5146 & -.1465 \\ -.1465 & .5752 \end{pmatrix}$ Using (1)
- $S_1 = \begin{pmatrix} .6325 & -.1645 \\ -.1645 & .4753 \end{pmatrix}, S_2 = \begin{pmatrix} .4467 & -.1175 \\ -.1175 & .6786 \end{pmatrix}$
- $G_1 = \begin{pmatrix} .6325 & -.0361 \\ -.2930 & .4753 \end{pmatrix}, G_2 = \begin{pmatrix} .4467 & -.2484 \\ 0.0134 & .6786 \end{pmatrix}$

Using (2)

Static output feedback design (NASC)

- *Assumption:* $C \in R^{mxn},\ m \leq n,\ rank(C) = m$
- $u_k = K_{\alpha(k)} y_k$
  - $Replace\ A_i\ with\ (A_i + B_i K_i C_i)\ in\ previous\ NASC\ conditions$
- $$\begin{bmatrix} P_i & (A_i + B_i K_i C_i)^T P_j \\ P_j (A_i + B_i K_i C_i) & P_j \end{bmatrix} \succ 0$$

  $$\forall\ (i,j) \in \mathcal{I} \times \mathcal{I} and\ P_i \in S_+^n\ \forall i$$
- $$\begin{bmatrix} G_i + G_i^T - S_i & G_i^T (A_i + B_i K_i C_i)^T \\ (A_i + B_i K_i C_i) G_i & S_j \end{bmatrix} \succ 0$$

  $$\forall\ (i,j) \in \mathcal{I} \times \mathcal{I} and\ G_i \in \boldsymbol{R}^{nxn}, S_i \in S_+^n$$

  $P_i = S_i^{-1}\ in\ this\ case$

Non convex
problem

Static output feedback design

- Sufficient conditions (Convex problem)
  - $$\begin{pmatrix} S_i & (A_iS_i + B_iU_iC_i)^T \\ A_iS_i + B_iU_iC_i & S_j \end{pmatrix} \succ 0; S_i \in S^n \; and \; U_i, V_i \in \boldsymbol{R}^{nxn} \qquad (3)$$
    $$V_iC_i = C_iS_i; \; \forall \boldsymbol{\mathcal{J}}$$
    $$K_i = U_iV_i^{-1}; \; \forall \boldsymbol{\mathcal{J}}$$
  - $$\begin{pmatrix} G_i + G_i^T - S_i & (A_iG_i + B_iU_iC_i)^T \\ A_iG_i + B_iU_iC_i & S_j \end{pmatrix} \succ 0; S_i \in S^n \; and \; G_i, U_i, V_i \in \boldsymbol{R}^{nxn} \qquad (4)$$
    $$V_iC_i = C_iG_i; \; \forall \boldsymbol{\mathcal{J}}$$
    $$K_i = U_iV_i^{-1}; \; \forall \boldsymbol{\mathcal{J}}$$
  - Conservativeness comparison and constraint (Explanation and misnomer)

Examples

- $A_1 = \begin{pmatrix} .8100 & .2879 \\ .4544 & .3114 \end{pmatrix}, A_2 = \begin{pmatrix} .5495 & ..7079 \\ .2555 & .5984 \end{pmatrix}$

- $B_1 = \begin{pmatrix} .4944 & .0373 \\ .1487 & .8900 \end{pmatrix}, B_2 = \begin{pmatrix} .0552 & .6943 \\ .7375 & .9044 \end{pmatrix}$

- $C_1 = (.0872 \quad .1872), C_2 = (.9202 \quad .3166)$

- Stable under static output feedback but not without it

Numerical Evaluation

- Method 1: Common Lyapunov function
  - $P_i = P_j = S_i^{-1} = S_j^{-1}; \forall i, j \in \mathcal{J}$ in

$$\begin{pmatrix} S_i & (A_i S_i + B_i U_i C_i)^T \\ A_i S_i + B_i U_i C_i & S_j \end{pmatrix} \succ 0; S_i \in S^n \text{ and } U_i, V_i \in \boldsymbol{R}^{nxn} \qquad (3)$$

$$V_i C_i = C_i S_i;\ \forall \boldsymbol{\mathcal{J}}$$

$$K_i = U_i V_i^{-1};\ \forall \boldsymbol{\mathcal{J}}$$

- Method 2: Sufficient condition (3)
  - $$\begin{pmatrix} S_i & (A_i S_i + B_i U_i C_i)^T \\ A_i S_i + B_i U_i C_i & S_j \end{pmatrix} \succ 0; S_i \in S^n \text{ and } U_i, V_i \in \boldsymbol{R}^{nxn} \qquad (3)$$

$$V_i C_i = C_i S_i;\ \forall \boldsymbol{\mathcal{J}}$$

$$K_i = U_i V_i^{-1};\ \forall \boldsymbol{\mathcal{J}}$$

- Method 3: Less conservative sufficient condition (4)
  - $$\begin{pmatrix} G_i + G_i^T - S_i & (A_i G_i + B_i U_i C_i)^T \\ A_i G_i + B_i U_i C_i & S_j \end{pmatrix} \succ 0; S_i \in S^n \text{ and } G_i, U_i, V_i \in \boldsymbol{R}^{nxn} \qquad (4)$$

$$V_i C_i = C_i G_i;\ \forall \boldsymbol{\mathcal{J}}$$

$$K_i = U_i V_i^{-1};\ \forall \boldsymbol{\mathcal{J}}$$

System	Success	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$
$n = 3$	CLF	67	43	25	14	12
$m = 1$	$Th3$	93	85	79	68	52
$p = 1$	$Th4$	**96**	**96**	**89**	**82**	**79**
$n = 4$	CLF	23	6	1	0	0
$m = 1$	$Th3$	69	49	27	17	6
$p = 1$	$Th4$	**88**	**70**	**49**	**30**	**17**
$n = 5$	CLF	14	2	0	0	0
$m = 1$	$Th3$	59	20	10	1	1
$p = 1$	$Th4$	**82**	**39**	**15**	**4**	**1**
$n = 3$	CLF	93	74	60	62	47
$m = 2$	$Th3$	100	97	99	97	92
$p = 2$	$Th4$	**100**	**98**	**100**	**98**	**95**
$n = 4$	CLF	42	12	9	4	1
$m = 2$	$Th3$	86	77	68	54	40
$p = 2$	$Th4$	**97**	**87**	**85**	**77**	**67**
$n = 5$	CLF	8	1	0	0	0
$m = 2$	$Th3$	60	38	23	4	4
$p = 2$	$Th4$	**85**	**69**	**42**	**26**	**19**
$n = 3$	CLF	77	56	34	26	18
$m = 2$	$Th3$	96	95	91	82	83
$p = 1$	$Th4$	**96**	**96**	**93**	**90**	**90**
$n = 4$	CLF	47	8	4	2	0
$m = 2$	$Th3$	89	71	47	28	16
$p = 1$	$Th4$	**95**	**78**	**65**	**46**	**32**
$n = 5$	CLF	22	0	0	0	0
$m = 2$	$Th3$	57	29	13	3	1
$p = 1$	$Th4$	**76**	**51**	**26**	**6**	**4**
$n = 3$	CLF	78	46	42	16	17
$m = 1$	$Th3$	98	87	93	80	70
$p = 2$	$Th4$	**100**	**95**	**98**	**94**	**88**
$n = 4$	CLF	27	10	0	0	0
$m = 1$	$Th3$	79	50	28	26	8
$p = 2$	$Th4$	**97**	**83**	**64**	**48**	**34**
$n = 5$	CLF	6	1	0	0	0
$m = 1$	$Th3$	48	11	4	1	1
$p = 2$	$Th4$	**81**	**49**	**25**	**7**	**4**

Numerical Evaluation

- Method 1: Common
  - $P_i = P_j; \forall i, j \in \mathcal{J}$ in
- Method 3: Less cons

N = number of modes
n = system order
m = number of inputs
p = number of outputs

System	Success	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$
$n=3$	CLF	67	43	25	14	12
$m=1$	$Th3$	93	85	79	68	52
$p=1$	$Th4$	**96**	**96**	**89**	**82**	**79**
$n=4$	CLF	23	6	1	0	0
$m=1$	$Th3$	69	49	27	17	6
$p=1$	$Th4$	**88**	**70**	**49**	**30**	**17**
$n=5$	CLF	14	2	0	0	0
$m=1$	$Th3$	59	20	10	1	1
$p=1$	$Th4$	**82**	**39**	**15**	**4**	**1**
$n=3$	CLF	93	74	60	62	47
$m=2$	$Th3$	100	97	99	97	92
$p=2$	$Th4$	**100**	**98**	**100**	**98**	**95**
$n=4$	CLF	42	12	9	4	1
$m=2$	$Th3$	86	77	68	54	40
$p=2$	$Th4$	**97**	**87**	**85**	**77**	**67**
$n=5$	CLF	8	1	0	0	0
$m=2$	$Th3$	60	38	23	4	4
$p=2$	$Th4$	**85**	**69**	**42**	**26**	**19**

Numerical Evaluation

- Method 1: Common
  - $P_i = P_j; \forall i, j \in \mathcal{I}$ in
- Method 3: Less cons

N = number of modes
n = system order
m = number of inputs
p = number of outputs

System	Success	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$
$n=3$	CLF	77	56	34	26	18
$m=2$	$Th3$	96	95	91	82	83
$p=1$	$Th4$	**96**	**96**	**93**	**90**	**90**
$n=4$	CLF	47	8	4	2	0
$m=2$	$Th3$	89	71	47	28	16
$p=1$	$Th4$	**95**	**78**	**65**	**46**	**32**
$n=5$	CLF	22	0	0	0	0
$m=2$	$Th3$	57	29	13	3	1
$p=1$	$Th4$	**76**	**51**	**26**	**6**	**4**
$n=3$	CLF	78	46	42	16	17
$m=1$	$Th3$	98	87	93	80	70
$p=2$	$Th4$	**100**	**95**	**98**	**94**	**88**
$n=4$	CLF	27	10	0	0	0
$m=1$	$Th3$	79	50	28	26	8
$p=2$	$Th4$	**97**	**83**	**64**	**48**	**34**
$n=5$	CLF	6	1	0	0	0
$m=1$	$Th3$	48	11	4	1	1
$p=2$	$Th4$	**81**	**49**	**25**	**7**	**4**

Q & A?

Appendix

Stability Analysis and Control Synthesis for Switched Systems: A switched Lyapunov function approach

Jamal Daafouz, Pierre Riedinger and Claude Iung

Abstract— **This paper addresses the problem of stability analysis and control synthesis of switched systems in the discrete time domain. The approach followed in this paper looks at the existence of a switched quadratic Lyapunov function to check asymptotic stability of the switched system under consideration. Two different LMI based conditions allow to check the existence of such a Lyapunov function. The first one is classical while the second one is new and uses a slack variable which makes it useful for design problems. These two conditions are proved to be equivalent for stability analysis. Investigating the static output feedback control problem we show that the second condition is in this case less conservative. The reduction of the conservatism is illustrated by a numerical evaluation.**

Keywords— **Switched Systems, Switched Lyapunov function, Hybrid Systems, Static Output Feedback.**

I. Introduction

In recent years, the study of switched systems has received a growing attention. Switched systems are a class of hybrid dynamical systems consisting of a family of continuous (or discrete) time subsystems and a rule that orchestrates the switching between them [1], [2], [3], [4], [5]. A survey of basic problems in stability and design of switched systems has been proposed recently in [6]. Among the large variety of problems encountered in practice, one can study the existence of a switching rule that ensures stability of the switched system. One can also assume that the switching sequence is not known a priori and look for stability results under arbitrary switching sequences. One can also consider some useful class of switching sequences (see for instance [6], [7] and the references therein).

positive definite matrix M, $\lambda_{min}(M)$ ($\lambda_{max}(M)$) stands for the minimal (maximum) eigenvalue of M.

II. Problem formulation

We consider the class of switched hybrid systems given by

$$\begin{aligned} x_{k+1} &= A_{\alpha(k)}x_k + B_{\alpha(k)}u_k \\ y_k &= C_{\alpha(k)}x_k \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input and $y_k \in \mathbb{R}^p$ is the output vector. $\alpha(k)$ is a switching rule defined by $\alpha(k) : \mathbb{N} \to \mathcal{I}$ with $\mathcal{I} = \{1, ..., N\}$. This means that the matrices $\left(A_{\alpha(k)}, B_{\alpha(k)}, C_{\alpha(k)}\right)$ are allowed to take values, at an arbitrary discrete time, in the finite set

$$\left\{(A_1, B_1, C_1), ..., (A_N, B_N, C_N)\right\}$$

Such systems are said to be switched and belong to the class of hybrid systems. The following assumptions are made:

- H_1: matrix $C_{\alpha(k)}$ is of full row-rank,
- H_2: $\alpha(k)$ in not known a priori but we assume that the its instantaneous value is available in real time.

Assumption H_1 does not impose any loss of generality since it can be achieved by discarding redundant measurement components of the output y_k. Assumption H_2 corresponds to practical implementations where the switched system is supervised by a discrete event system and the discrete state value is available in real time. Here, we are interested in stability analysis and control synthesis problems for this class of switched systems. By stability analysis we mean stability analysis of the origin of the autonomous switched system. The control synthesis is related to the design of a switched output feedback control

$$u_k = K_{\alpha(k)}y_k, \tag{2}$$

ensuring stability of the closed loop switched system:

Appendix

In this paper, we are interested in stability analysis and control synthesis of switched systems under arbitrary switching sequences. The approach followed in this paper looks at the existence of a switched quadratic Lyapunov function to check asymptotic stability of the switched system under consideration. To evaluate the interest of this approach for control design problems, we concentrate on the output feedback design problem. By output feedback we mean the design of output feedback gains for each subsystem such that the closed loop switched system is asymptotically stable. The results proposed in this paper can be considered as a tradeoff between highly conservative results (those using a single quadratic Lyapunov function) and less conservative ones but numerically hard to check.

This paper is organized as follows. In the next section we give the problem formulation. Section III is dedicated to stability analysis of switched systems by mean of a switched quadratic Lyapunov function. In Section IV, switched static output feedback control design is investigated. A numerical evaluation is given at the end of this paper.

Notations: We use standard notations throughout the paper. M^T is the transpose of the matrix M. $M > \mathbf{0}$ ($M < \mathbf{0}$) means that M is positive definite (negative definite). For a

Corresponding author: Jamal.Daafouz@ensem.inpl-nancy.fr, Tel: (+33) 3.83.59.57.13, Fax: (+33) 3.83.59.56.44
CRAN CNRS UMR 7039 - INPL - ENSEM, 2 av. de la forêt de Haye, 54516 Vandœuvre Cedex - France

$$x_{k+1} = \left(A_{\alpha(k)} + B_{\alpha(k)} K_{\alpha(k)} C_{\alpha(k)}\right) x_k \tag{3}$$

The switched state feedback control problem can be considered as a particular case for which a solution can be provided from the one given in this paper letting $C_{\alpha(k)} = \mathbf{I}$.

III. Stability analysis

In this section, we investigate the stability of the origin of an autonomous switched system given by

$$x_{k+1} = A_{\alpha(k)} x_k \tag{4}$$

Define the indicator function

$$\xi(k) = \begin{bmatrix} \xi_i(k), & \dots, & \xi_N(k) \end{bmatrix}^T$$

with $\forall i = 1, \dots, N$:

$$\xi_i(k) = \begin{cases} 1 & \text{when the switched system is described by} \\ & \text{the } i\text{-th mode } A_i \\ 0 & \text{otherwise} \end{cases}$$

Then the switched system (4) can also be written as

$$x_{k+1} = \sum_{i=1}^{N} \xi_i(k) A_i x_k \tag{5}$$

Here we are interested in checking stability by mean of particular quadratic Lyapunov functions taking into account the

Appendix

switching nature of our system. Recall that polytopic time varying systems are systems where the dynamical matrix evolves in a polytope defined by its vertices. Switched systems can be viewed as polytopic systems with the particularity that the allowable values for the dynamical matrix are those corresponding to the vertices of the polytope. Stability analysis results proposed in [9], when adapted to switched systems, allow to provide interesting results from the conservatism point of view. Results in [9] use parameter dependent Lyapunov functions to check stability of polytopic time varying systems. In the case of switched systems as (5), this corresponds to the switched Lyapunov function defined as

$$V(k, x_k) = x_k^T P(\xi(k)) x_k = x_k^T \Big(\sum_{i=1}^{N} \xi_i(k) P_i\Big) x_k \qquad (6)$$

with $P_1, \ldots P_N$ symmetric positive definite matrices. If such a positive definite Lyapunov function exists and is such that $\Delta V(k, x_k) = V(k+1, x_{k+1}) - V(k, x_k)$ is negative definite[1] along the solutions of (5), then the origin of the switched system given by (4) is globally asymptotically stable as shown by the following general Theorem:

Theorem 1: [10] (Chap.5). The equilibrium $\mathbf{0}$ of

$$x_{k+1} = f_k(x_k) \qquad (7)$$

is globally uniformly asymptotically stable if there is a function $V : \mathbb{Z}^+ \times \mathbb{R}^n \to \mathbb{R}$ such that

- V is a positive definite function, decrescent and radially unbounded
- $\Delta V(k, x_k) = V(k+1, x_{k+1}) - V(k, x_k)$ is negative definite along the solutions of (7).

The Lyapunov function (6) is a positive definite function, decrescent and radially unbounded since $V(k, 0) = 0$, $\forall k \geq 0$ and

$$\alpha \|x_k\|^2 \leq V(k, x_k) = x_k^T \Big(\sum_{i=1}^{N} \xi_i(k) P_i\Big) x_k \leq \beta \|x_k\|^2$$

for all $x_k \in \mathbb{R}^n$ and $k \geq 0$ with $\alpha = \min_{i \in \mathcal{I}} \lambda_{min}(P_i)$ and $\beta = \max_{i \in \mathcal{I}} \lambda_{max}(P_i)$ positive scalars.

The Lyapunov function is then given by:

$$V(k, x_k) = x_k^T \Big(\sum_{i=1}^{N} \xi_i(k) S_i^{-1}\Big) x_k$$

Proof:

- To prove $i) \Rightarrow ii)$, assume that there exists a Lyapunov function of the form (6) whose difference is negative definite. Hence:

$$\begin{aligned} \Delta V &= V(k+1, x_{k+1}) - V(k, x_k) \\ &= x_{k+1}^T P(\xi(k+1)) x_{k+1} - x_k^T P(\xi(k)) x_k \\ &= x_k^T \Big(A^T(\xi(k)) P(\xi(k+1)) A(\xi(k)) - P(\xi(k))\Big) x_k \\ &< 0 \end{aligned}$$

As this has to be satisfied under arbitrary switching laws it follows that this has to hold for the special configuration $\xi_i(k) = 1, \xi_{l \neq i}(k) = 0, \xi_j(k+1) = 1, \xi_{l \neq j}(k+1) = 0$ and for all $x_k \in \mathbb{R}^n$. Then:

$$P_i - A_i^T P_j A_i > \mathbf{0}, \qquad \forall (i,j) \in \mathcal{I} \times \mathcal{I}$$

and condition ii) follows by the Schur complement formula.

- To prove $ii) \Rightarrow i)$, assume that condition (8) is satisfied for all $i = 1, \ldots, N$ and $j = 1, \ldots, N$. For each i, multiply the $j = 1, \ldots, N$ inequalities by $\xi_j(k+1)$ and sum. Multiply the resulting $i = 1, \ldots, N$ inequalities by $\xi_i(k)$ and sum. As $\sum_{i=1}^{N} \xi_i(k) = \sum_{j=1}^{N} \xi_j(k+1) = 1$, we get

$$\begin{bmatrix} P(\xi(k)) & A^T(\xi(k)) P(\xi(k+1)) \\ P(\xi(k+1)) A(\xi(k)) & P(\xi(k+1)) \end{bmatrix} > \mathbf{0}$$

which is equivalent by Schur complement to: $\forall x_k \in \mathbb{R}^n$

$$x_k^T \Big(P(\xi(k)) - A^T(\xi(k)) P(\xi(k+1)) A(\xi(k))\Big) x_k > \mathbf{0}$$

As:

$$\Delta V = -x_k^T \Big(P(\xi(k)) - A^T(\xi(k)) P(\xi(k+1)) A(\xi(k))\Big) x_k$$

Appendix

In the following theorem, we give two equivalent necessary and sufficient conditions for the existence of a Lyapunov function of the form (6) whose difference is negative definite proving asymptotic stability of the system (4).

Theorem 2: The following statements are equivalent

i) There exists a Lyapunov function of the form (6) whose difference is negative definite proving asymptotic stability of the system (4).

ii) There exist N symmetric matrices $P_1, ..., P_N$ satisfying

$$\begin{bmatrix} P_i & A_i^T P_j \\ P_j A_i & P_j \end{bmatrix} > \mathbf{0} \qquad \forall (i,j) \in \mathcal{I} \times \mathcal{I} \qquad (8)$$

The Lyapunov function is then given by:

$$V(k, x_k) = x_k^T \left(\sum_{i=1}^{N} \xi_i(k) P_i \right) x_k$$

iii) There exist N symmetric matrices $S_1, ..., S_N$ and N matrices $G_1, ..., G_N$ satisfying

$$\begin{bmatrix} G_i + G_i^T - S_i & G_i^T A_i^T \\ A_i G_i & S_j \end{bmatrix} > \mathbf{0} \qquad \forall (i,j) \in \mathcal{I} \times \mathcal{I} \quad (9)$$

[1] i.e. $\Delta V(k, 0) = 0 \ \ \forall k \geq 0$ and $\Delta V(k, x_k) \leq -\gamma(\|x_k\|)$, $\forall k \geq 0$, $\forall x_k \in \mathbb{R}^n$ where γ is of class K.
A function $\alpha : [0, \infty) \to [0, \infty)$ is of class K if it is continuous, strictly increasing, zero at zero and unbounded ($\alpha(s) \to \infty$ as $s \to \infty$).

we have $\Delta V \leq -\gamma(\|x_k\|)$ with

$$\gamma(\|x_k\|) = min_{(i,j) \in \mathcal{I} \times \mathcal{I}} \lambda_{min}(P_i - A_i^T P_j A_i) \|x_k\|^2$$

and i) follows by Theorem 1.

- To prove *iii)* $\Rightarrow$ *ii)*, assume that (9) is feasible. Then

$$G_i + G_i^T - S_i > \mathbf{0} \qquad \forall i \in \mathcal{I}$$

This means that G_i is of full rank. Moreover, as S_i is strictly positive definite, we have also:

$$(S_i - G_i)^T S_i^{-1} (S_i - G_i) \geq 0 \qquad \forall i \in \mathcal{I}$$

which is equivalent to

$$G_i^T S_i^{-1} G_i \geq G_i^T + G_i - S_i \qquad \forall i \in \mathcal{I}$$

Then, if (9) holds, it follows that

$$\begin{bmatrix} G_i^T S_i^{-1} G_i & G_i^T A_i{}^T \\ A_i G_i & S_j \end{bmatrix} > \mathbf{0} \qquad \forall (i,j) \in \mathcal{I} \times \mathcal{I}$$

which is equivalent to: $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$

$$\begin{bmatrix} G_i^T & \mathbf{0} \\ \mathbf{0} & S_j \end{bmatrix} \begin{bmatrix} S_i^{-1} & A_i{}^T S_j^{-1} \\ S_j^{-1} A_i & S_j^{-1} \end{bmatrix} \begin{bmatrix} G_i & \mathbf{0} \\ \mathbf{0} & S_j \end{bmatrix} > \mathbf{0} \qquad (10)$$

Appendix

Letting $P_i = S_i^{-1}$ and $P_j = S_j^{-1}$, (10) is equivalent to:

$$\begin{bmatrix} P_i & A_i^T P_j \\ P_j A_i & P_j \end{bmatrix} > \mathbf{0} \qquad \forall (i,j) \in \mathcal{I} \times \mathcal{I} \tag{11}$$

- To prove $ii) \Rightarrow iii)$, assume that (8) is satisfied, then by Schur complement we have

$$P_i - A_i^T P_j A_i > \mathbf{0} \qquad \forall (i,j) \in \mathcal{I} \times \mathcal{I}$$

Letting $S_i = P_i^{-1}$ and $S_j = P_j^{-1}$ and using the Schur complement formula one gets

$$S_j - A_i S_i A_i^T = T_{ij} > \mathbf{0} \qquad \forall (i,j) \in \mathcal{I} \times \mathcal{I}$$

Let $G_i = S_i + g_i \mathbf{I}$ with g_i a positive scalar. There exists a sufficiently small g_i such that

$$g_i^{-2}(S_i + 2g_i \mathbf{I}) > A_i^T T_{ij}^{-1} A_i \qquad \forall (i,j) \in \mathcal{I} \times \mathcal{I}$$

which is equivalent, by Schur complement, to

$$\begin{bmatrix} S_i + 2g_i \mathbf{I} & -g_i A_i^T \\ -A_i g_i & T_{ij} \end{bmatrix} > \mathbf{0}, \qquad \forall (i,j) \in \mathcal{I} \times \mathcal{I}$$

which is nothing than

$$\begin{bmatrix} G_i + G_i^T - S_i & S_i A_i^T - G_i A_i^T \\ A_i S_i - A_i G_i & S_j - A_i S_i A_i^T \end{bmatrix} > \mathbf{0} \qquad \forall (i,j) \in \mathcal{I} \times \mathcal{I}$$

To end the proof of Theorem 2, one can notice that the latest LMI is equivalent to: $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -A_i & \mathbf{I} \end{bmatrix} \begin{bmatrix} G_i + G_i^T - S_i & G_i^T A_i^T \\ A_i G_i & S_j \end{bmatrix} \begin{bmatrix} \mathbf{I} & -A_i^T \\ \mathbf{0} & \mathbf{I} \end{bmatrix} > \mathbf{0}$$

■

Theorem 3: If there exist symmetric matrices S_i, matrices U_i and V_i ($\forall i \in \mathcal{I}$) such that: $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$

$$\begin{pmatrix} S_i & (A_i S_i + B_i U_i C_i)^T \\ A_i S_i + B_i U_i C_i & S_j \end{pmatrix} > \mathbf{0} \tag{15}$$

and

$$V_i C_i = C_i S_i \qquad \forall i \in \mathcal{I} \tag{16}$$

then the output feedback given by (2) with

$$K_i = U_i V_i^{-1} \qquad \forall i \in \mathcal{I} \tag{17}$$

stabilizes the system (1).

Proof: Assume that there exist S_i, U_i and V_i such that (15) and (16) are satisfied. As C_i is of full row rank and S_i is positive definite, it follows from (16) that V_i is of full rank for all $i = 1, ..., N$ and then invertible. From (16) and (17) we get

$$U_i C_i = K_i C_i S_i \qquad \forall i \in \mathcal{I}$$

Replacing $U_i C_i$ in (15) by $K_i C_i S_i$ and applying the Schur complement formula one gets: $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$,

$$S_i^{-1} - (A_i + B_i K_i C_i)^T S_j^{-1} (A_i + B_i K_i C_i) > \mathbf{0}$$

Letting $P_i = S_i^{-1}$ and $P_j = S_j^{-1}$ and using the Schur complement formula, the latest inequality is nothing than the stability condition (8) applied to the closed loop system (3). Hence, by Theorem 2 the closed loop system is asymptotically stable. ■

Theorem 4: If there exist symmetric matrices S_i, matrices G_i, U_i and V_i ($\forall i \in \mathcal{I}$) such that: $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$

$$\begin{pmatrix} G_i + G_i^T - S_i & (A_i G_i + B_i U_i C_i)^T \\ A_i G_i + B_i U_i C_i & S_j \end{pmatrix} > \mathbf{0} \tag{18}$$

and

$$V_i C_i = C_i G_i \qquad \forall i \in \mathcal{I} \tag{19}$$

Appendix

Condition (8) has also been proposed in [8] to check stability of piecewise affine systems. Notice that we can recover condition ii) by imposing $G_i = S_i$ and letting $S_i^{-1} = P_i$ in condition iii). However, Theorem 2 shows that both conditions are equivalent and then present the same level of conservatism. The next section shows that the contribution of Theorem 2, in addition to prove the equivalence i)$\Longleftrightarrow$ ii)$\Longleftrightarrow$ iii), is to propose a condition which is less conservative for constrained problems.

IV. Static output feedback design

Consider the synthesis of a switched static output feedback

$$u_k = K_{\alpha(k)} y_k, \tag{12}$$

ensuring stability of the closed loop switched system (3). This problem reduces to find P_i and K_i ($\forall i \in \mathcal{I}$) such that:

$$\begin{bmatrix} P_i & (A_i + B_i K_i C_i)^T P_j \\ P_j (A_i + B_i K_i C_i) & P_j \end{bmatrix} > \mathbf{0} \tag{13}$$

$\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, or equivalently find S_i, G_i and K_i such that:

$$\begin{bmatrix} G_i + G_i^T - S_i & G_i^T (A_i + B_i K_i C_i)^T \\ (A_i + B_i K_i C_i) G_i & S_j \end{bmatrix} > \mathbf{0} \tag{14}$$

$\forall (i,j) \in \mathcal{I} \times \mathcal{I}$. The problem of solving numerically (13) or (14) for (P_i, K_i) or (S_i, G_i, K_i) respectively is non-convex in general. This makes the problem of output feedback a very difficult one. A sufficient condition is given in the following Theorems. These conditions have the advantage to be convex and are numerically well tractable.

then the output feedback control given by (2) with

$$K_i = U_i V_i^{-1} \quad \forall i \in \mathcal{I} \tag{20}$$

stabilizes the system (1).

Proof: First notice that if (18) holds, then $G_i + G_i^T - S_i > \mathbf{0}$ and the matrices G_i are full-rank. Hence, matrices V_i satisfying (19) are non-singular. Following similar arguments as in the proof of Theorem 3 we find that satisfying conditions of Theorem 4 leads to $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$:

$$\begin{pmatrix} G_i + G_i^T - S_i & G_i^T (A_i + B_i K_i C_i)^T \\ (A_i + B_i K_i C_i) G_i & S_j \end{pmatrix} > \mathbf{0} \tag{21}$$

which is nothing than condition (9) applied to the closed system (3). Hence, by Theorem 2 the closed loop system is asymptotically stable. ■

The above Theorems deserve some comments. Theorem 3 and Theorem 4 state how to determine a stabilizing switched output feedback control by solving a simple convex problem. In Theorem 3, the Lyapunov matrices P_i ($\forall i \in \mathcal{I}$) are constrained to satisfy the equality constraint (16). In Theorem 4, the Lyapunov matrices $P_i = S_i^{-1}$ ($\forall i \in \mathcal{I}$) have to satisfy the stability condition only. The equality constraint is reported on the slack variables G_i (19). Hence, the equivalence between conditions given in Theorem 3 and Theorem 4 fails. As it is shown in the numerical evaluation, reporting the equality constraint on the slack variables G_i makes the conditions of Theorem (4) less conservative than those given in Theorem 3. Moreover, the proposed conditions can be adapted to the state feedback control design. This can be achieved by replacing in the corresponding conditions (15) or (18) the matrices C_i by $C_i = \mathbf{I}$ $\forall i \in \mathcal{I}$.

Appendix

V. Numerical evaluation

In this section, a numerical evaluation is proposed. Recall that our switched system is characterized by: the number of modes (N), the system order (n), the number of inputs (m) and number of outputs (p). For fixed values of (N, n, m, p), we generate randomly 100 switched systems of the form (1). For each switched system we try to compute a stabilizing output feedback control using three methods

- Method 1 uses constant Lyapunov functions (CLF) $V(x_k) = x_k^T P x_k$. This corresponds to conditions in Theorem 3 with $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, $S_i = S_j = S$ a constant matrix.
- Method 2 uses the conditions given in Theorem 3.
- Method 3 uses the conditions given in Theorem 4.

To each method we associate a counter (SuccessCLF, SuccessTh3 and SuccessTh4) which is increased if the corresponding method succeeds in providing an output feedback stabilizing control. The difficulty in performing this evaluation is to generate switched systems for which a static output feedback control is known to exist. Generating dynamical matrices A_1, ..., A_N stable, in the discrete time LTI sense, allows to have more chance to succeed. To check the feasibility of the LMI conditions, the LMI control toolbox for Matlab 5.3 has been used. The results of this evaluation are given in Table I. One can see that static feedback synthesis conditions given in Theorem 4 reduce significantly the conservatism. This shows the advantage in using the slack variable G_i to satisfy the constraint (19). Notice that the number of randomly generated systems that can not be stabilized by static output feedback grows as the number of modes and the system order increase.

System	Success	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$
$n=3$	CLF	67	43	25	14	12
$m=1$	$Th3$	93	85	79	68	52
$p=1$	$Th4$	**96**	**96**	**89**	**82**	**79**
$n=4$	CLF	23	6	1	0	0
$m=1$	$Th3$	69	49	27	17	6
$p=1$	$Th4$	**88**	**70**	**49**	**30**	**17**
$n=5$	CLF	14	2	0	0	0
$m=1$	$Th3$	59	20	10	1	1
$p=1$	$Th4$	**82**	**39**	**15**	**4**	**1**
$n=3$	CLF	93	74	60	62	47
$m=2$	$Th3$	100	97	99	97	92
$p=2$	$Th4$	**100**	**98**	**100**	**98**	**95**
$n=4$	CLF	42	12	9	4	1
$m=2$	$Th3$	86	77	68	54	40
$p=2$	$Th4$	**97**	**87**	**85**	**77**	**67**
$n=5$	CLF	8	1	0	0	0
$m=2$	$Th3$	60	38	23	4	4
$p=2$	$Th4$	**85**	**69**	**42**	**26**	**19**
$n=3$	CLF	77	56	34	26	18
$m=2$	$Th3$	96	95	91	82	83
$p=1$	$Th4$	**96**	**96**	**93**	**90**	**90**
$n=4$	CLF	47	8	4	2	0
$m=2$	$Th3$	89	71	47	28	16
$p=1$	$Th4$	**95**	**78**	**65**	**46**	**32**
$n=5$	CLF	22	0	0	0	0
$m=2$	$Th3$	57	29	13	3	1
$p=1$	$Th4$	**76**	**51**	**26**	**6**	**4**
$n=3$	CLF	78	46	42	16	17
$m=1$	$Th3$	98	87	93	80	70
$p=2$	$Th4$	**100**	**95**	**98**	**94**	**88**
$n=4$	CLF	27	10	0	0	0
$m=1$	$Th3$	79	50	28	26	8
$p=2$	$Th4$	**97**	**83**	**64**	**48**	**34**
$n=5$	CLF	6	1	0	0	0
$m=1$	$Th3$	48	11	4	1	1
$p=2$	$Th4$	**81**	**49**	**25**	**7**	**4**

TABLE I
NUMERICAL EVALUATION

VI. Conclusion

Appendix

In this paper, a switched Lyapunov function approach is proposed for stability analysis and control synthesis for switched systems. Motivated by recent results developed for polytopic time varying uncertain systems, a condition has been proposed for stability analysis of switched systems. Even if this condition has been proved to be equivalent to a classical one, this condition is shown to be less conservative when constrained control design problems are investigated. The difficult problem related to switched static output feedback design has been addressed to illustrate such a conservatism reduction. As the proposed conditions are LMI based conditions, they can be easily extended to other control problems (Decentralized control, H_∞ control...). Moreover, the results presented in this paper can be easily applied to the class of piecewise affine and hybrid systems considered in [8] leading to less conservative results for constrained control problems.

Acknowledgement: The authors would like to thank the anonymous reviewers for their valuable comments which improved the final version of this paper.

References

[1] M. S. Branicky, "Multiple lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. Autom. Control*, vol. 43, no. 4, pp. 475–482, April 1998.

[2] P. Peleties and R. DeCarlo, "Asymptotic stability of m-switched systems using lyapunov-like functions," *Proceedings of the American Control Conference*, 1991.

[3] H. Ye, A. N. Michel, and L. Hou, "Stability analysis of switched systems," *Proceedings of the Conference on Decision and Control*, 1996.

[4] S. Pettersson and B. Lennartson, "Lmi for stability and robustness of hybrid systems," *Proceedings of the American Control Conference*, pp. 1714–1718, June 1997.

[5] R. DeCarlo, M. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and results on the stability and stabilizability of hybrid systems," *Proceedings of the IEEE*, pp. 1069–1082, July 2000.

[6] D. Liberzon and A. Stephen Morse, "Basic problems in stability and design of switched system," *IEEE Control Systems*, vol. 19, no. 5, pp. 59–70, October 1999.

[7] S. H. Lee, T. H. Kim, and J. T. Lim, "A new stability analysis of switched systems," *Automatica*, vol. 36, pp. 917–922, 2000.

[8] D. Mignone, G. Ferrari-Trecate, and M. Morari, "Stability and stabilization of piecewise affine and hybrid systems: An lmi approach," *Proccedings of Conference on Decision and Control*, vol. Sydney-Australia., December 12-15 2000.

[9] J. Daafouz and J. Bernussou, "Parameter dependent lyapunov functions for discrete time systems with time varying parametric uncertainties," *Systems and Control Letters*, vol. 43, pp. 355–359, 2001.

[10] M. Vidyasagar, *Nonlinear Systems Analysis*, Prentice-Hall International Editions, 1993.

[11] A. Bemporad, G. Ferrari-Trecate, and M. Morari, "Observability and controllability of piecewise affine and hybrid systems," *Proceedings of the Conference on Decision and Control*, 1999.

[12] M. Johansson and A. Rantzer, "Computation of piecewise quadratic lyapunov functions for hybrid systems," *IEEE Transations on Automatic Control*, vol. 43, no. 4, pp. 555–559, April 1998.

[13] M. C. De Oliveira, J. Bernussou, and J. C. Geromel, "A new discrete time robust stability condition," *System and Control Letters*, vol. 36, pp. 135–141, 1999.

[14] A. S. Mehr and T. Chen, "Properties of linear switching time-varying discrete-time systems with applications," *Systems and Control Letters*, vol. 39, pp. 229–235, 2000.

[15] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics, 1994.

[16] M. Johansson, *Piecewise Linear Control Systems*, Ph.d. thesis, Lund Institute of Technology, Sweden, 1999.