Maximal Lyapunov Functions and Domain of Attraction for Autonomous Nonlinear Systems

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Notation

- Autonomous nonlinear system

$$\dot{x}(t) = f(x)$$

- Region of attraction

$$S = \{x_0 : x(t, x_0) \to 0 \ as \ t \to \infty\}$$

- ∂S : Boundary of a set S

Theorem

Suppose we can find a set $A \subset \boldsymbol{R}^n$ containing origin in its interior, a continuous function $V: A \to \boldsymbol{R}_+$ and a positive definite function ϕ s.t.

- $V(0) = 0; V(x) > 0 \; \forall x \in A \backslash \{0\}$
- $\nabla V(x)^T f(x) = -\phi(x) \; \forall x \in A$
- $V(x) \to \infty$ as $x \to \partial A$ and/or as $|x| \to \infty$

Then $A = S = Region\ of\ attraction$

Theorem

Suppose f is continuously differentiable in some neighbourhood of origin. Then ∃ a continuous function $V_m: S \to \boldsymbol{R}_+$ and a function γ of class K s.t.

- $V_m(0) = V(0), V_m(x) > 0\ \forall x \in S\backslash\{0\}$
- $\dot{V}_m$ is well defined at all $x \in S$, and
$$\dot{V}_m(x) = -\gamma(|x|)\ \forall x \in S$$
- $V_m(x) \to \infty$ as $x \to \partial S$

Moreover, if f is Lipschitz continuous on S, then V_m can be selected to be continuously differentiable on S, and

- $V_m(x) \to \infty$ as $|x| \to \infty$

Theorem

Suppose f is continuously differentiable in some neighbourhood of origin. Then $\exists$ a continuous function $V_m : S \rightarrow \boldsymbol{R}_+$ and a function γ of class K s.t.

- $V_m(0) = V(0)$
- $\dot{V}_m$ is well defi

- $V_m(x) \rightarrow \infty$ as

Moreover, if f is … can be selected to be continuously

- $V_m(x) \rightarrow \infty$ as $|x| \rightarrow \infty$

Existence of a Lyapunov function satisfying previous requirements

Computation of RoA

- For analytic function f, we can express (using Taylor series)

$$f(x) = \sum_{i=1}^{\infty} F_i(x)$$

where $F_i(x)$ is a homogeneous function of order i

- To obtain the desired effect of Lyapunov function blowing up at boundary, let

$$V(x) = \frac{N(x)}{D(x)} = \frac{\sum_{i=2}^{\infty} R_i(x)}{1 + \sum_{i=1}^{\infty} Q_i(x)}$$

where $R_i(x)$ and $Q_i(x)$ are homogeneous function of order i

Ques. Justify indices

Algorithm

$$\dot{V}(x) = -x^T Q x = -\phi(x)$$

$$\Rightarrow \left[\left(1 + \sum_{i=1}^{\infty} Q_i\right)\sum_{i}^{\infty} \nabla R_i^T - \left(\sum_{i=1}^{\infty} \nabla Q^T\right)\sum_{i=2}^{\infty} R_i\right]\sum_{i}^{\infty} F_i = -x^T Q x \left(1 + \sum_{i=1}^{\infty} Q_i\right)^2$$

$$Equating\ coefficients, for\ degree\ 2, we\ have,$$

$$\nabla R_2^T F_1 = -x^T Q x$$

$$For\ higher\ degrees\ (i.e.\ k \geq 3), we\ obtain\ the\ recursive\ relation,$$

$$\sum_{i=2}^{k} \nabla R_i^T F_{k+1-i} + \sum_{i=1}^{k-2}\sum_{j=2}^{k-1}\left(Q_i \nabla R_j^T - \nabla Q_i^T R_j\right) F_{k+1-i-j} = -x^T Q x \left(2Q_{k-2} + \sum_{i=1}^{k-3} Q_i Q_{k-2-i}\right)$$

$$A_n y = b_n\ (where\ y\ is\ the\ coefficients\ \ vector\ for\ polynomials\ R_i, Q_i)$$

$$Equivalent\ to\ solving\ an\ undetermined\ set\ of\ linear\ equations,$$

Algorithm …

$$Finding\ the\ maximal\ Lyapunov\ function$$

$$V_n(x) = \frac{N(x)}{D(x)} = \frac{\sum_{i=2}^{n} R_i(x)}{1 + \sum_{i=1}^{n-2} Q_i(x)}$$

$$\dot{V}_n = \frac{1}{\left(1 + \sum_{i}^{n-2} Q_i\right)^2} \left(\sum_{i=2}^{n-2} \nabla R_i^T + \sum_{i=1}^{n-2} \sum_{j=2}^{n} \left(Q_i \nabla R_j^T - \nabla Q^T R_j \right) \right) \sum_{i=1}^{\infty} F_k$$

$$= -x^T Q x + \frac{\{terms\ of\ degree \geq n+1\}}{\left(1 + \sum_{i=1}^{n-2} Q_i\right)^2}$$

$$\boldsymbol{Minimization\ problem}$$

$$\min_{s.t.\ A_n y = b_n} \boldsymbol{e_n(y)}$$

Algorithm…

$$To\ obtain\ an\ estimate\ of\ Region\ of\ Attraction$$

$$\max C^*$$

$$s.t.\ \ V_n(x) = C^*$$

$$\dot{V}_n(x) < 0$$

Ques. Why level set is better than $\|x\| = C^*$ in constraint?

Results

$$\dot{x} = -y$$
$$\dot{y} = x - y + x^2 y$$

$$V_4(x) = \frac{R_2 + R_3 + R_4}{1 + Q_1 + Q_2} = \boldsymbol{C}^*$$

$$R_2 = \frac{3}{2}x^2 - xy + y^2$$

$$R_3 = 0$$

$$R_4 = -0.3186x^4 + 0.7124x^3y - 0.1459x^2y^2 + 0.1409xy^3 - 0.03769y^4$$

$$Q_1 = 0$$

$$Q_2 = -0.2362x^2 + 0.31747xy - 0.1091y^2$$

$$C^* = 5.4413$$

Results

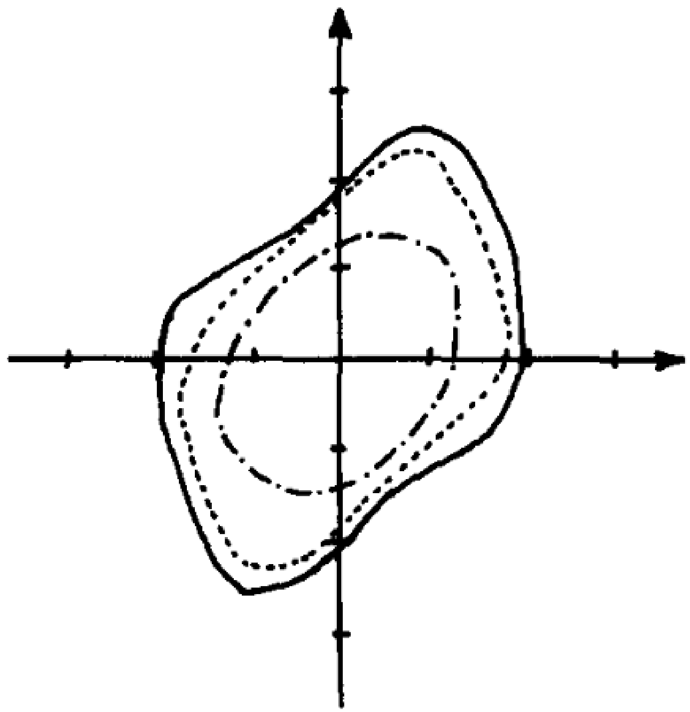

FIG. 1. Actual domain of attraction S ——; estimate of S obtained by Kurak and Davison (1971) —·—; estimate of S obtained by proposed method $\cdots$.