

Fourier Series

Anurag Gopi

CONTENTS

1	Periodic Function	1
2	Fourier Series	1
3	Fourier Transform	3
4	Filter	4
5	Filter Design	5

Abstract—This manual provides a simple introduction to Fourier Series.

1 PERIODIC FUNCTION

Let

$$x(t) = A_0 |\sin(2\pi f_0 t)| \quad (1.1)$$

1.1 Plot $x(t)$.

Solution: The Python code `codes/1_1.py` plots $x(t)$ in Fig. 1.1.

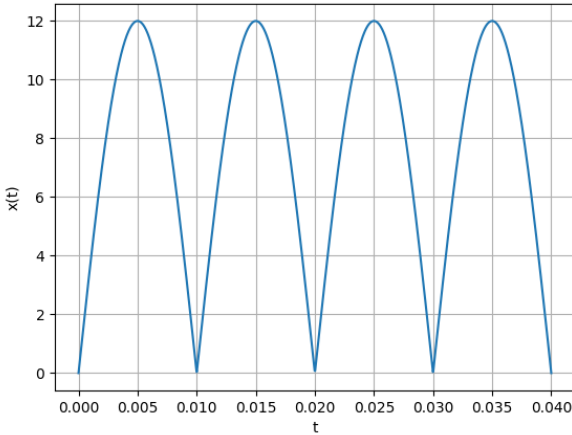


Fig. 1.1: $x(t)$

1.2 Show that $x(t)$ is periodic and find its period.

Solution: Note that,

$$x\left(t + \frac{1}{2f_0}\right) = A_0 \left| \sin\left(2\pi f_0 \left(t + \frac{1}{2f_0}\right)\right) \right| \quad (1.2)$$

$$= A_0 |\sin(2\pi f_0 t + \pi)| \quad (1.3)$$

$$= A_0 |\sin(2\pi f_0 t)| \quad (1.4)$$

Hence the period of $x(t)$ is $\frac{1}{2f_0}$.

2 FOURIER SERIES

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.1)$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t} dt \quad (2.2)$$

Solution: For some $n \in \mathbb{Z}$,

$$x(t) e^{-j2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-n)f_0 t} \quad (2.3)$$

But

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi k f_0 t} dt = \frac{1}{f_0} \delta_{0k} \quad (2.4)$$

where δ_{ij} denotes the Kronecker delta. Thus,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t} dt = \frac{c_n}{f_0} \quad (2.5)$$

$$\Rightarrow c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t} dt \quad (2.6)$$

2.2 Find c_k for (1.1)

Solution: Using (2.2),

$$c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 |\sin(2\pi f_0 t)| e^{-j2\pi n f_0 t} dt \quad (2.7)$$

$$= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 |\sin(2\pi f_0 t)| \cos(2\pi n f_0 t) dt$$

$$+ j f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 |\sin(2\pi f_0 t)| \sin(2\pi n f_0 t) dt \quad (2.8)$$

$$= 2f_0 \int_0^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 t) \cos(2\pi n f_0 t) dt \quad (2.9)$$

$$= f_0 A_0 \int_0^{\frac{1}{2f_0}} (\sin(2\pi(n+1)f_0 t)) dt - f_0 A_0 \int_0^{\frac{1}{2f_0}} (\sin(2\pi(n-1)f_0 t)) dt \quad (2.10)$$

$$= A_0 \frac{1 + (-1)^n}{2\pi} \left(\frac{1}{n+1} - \frac{1}{n-1} \right) \quad (2.11)$$

$$= \begin{cases} \frac{2A_0}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \quad (2.12)$$

2.3 Verify (2.1) using python.

Solution: The Python code codes/2_3.py verifies (2.13) by plotting Fig. 2.3.

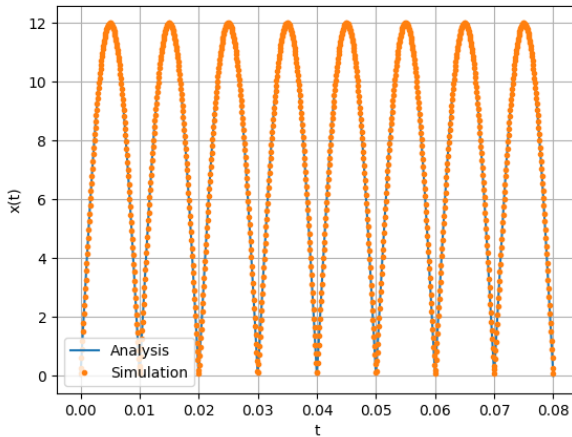


Fig. 2.3: Verification of (2.1).

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} (a_k \cos j2\pi k f_0 t + b_k \sin j2\pi k f_0 t) \quad (2.13)$$

and obtain the formulae for a_k and b_k .

Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.14)$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{j2\pi k f_0 t} + c_{-k} e^{-j2\pi k f_0 t} \quad (2.15)$$

$$= c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k}) \cos(2\pi k f_0 t)$$

$$+ \sum_{k=0}^{\infty} j(c_k - c_{-k}) \sin(2\pi k f_0 t) \quad (2.16)$$

Hence, for $k \geq 0$,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \quad (2.17)$$

$$= \begin{cases} f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) dt & k = 0 \\ 2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \cos(2\pi k f_0 t) dt & k > 0 \end{cases} \quad (2.18)$$

$$b_k = \frac{c_k - c_{-k}}{j} = 2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \sin(2\pi k f_0 t) dt \quad (2.19)$$

2.5 Find a_k and b_k for (1.1)

Solution: From (2.1), we see that since $x(t)$ is even,

$$x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-j2\pi k f_0 t} \quad (2.20)$$

$$= \sum_{k=-\infty}^{\infty} c_{-k} e^{j2\pi k f_0 t} \quad (2.21)$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.22)$$

where we substitute $k := -k$ in (2.21). Hence, we see that $c_k = c_{-k}$. So, from (2.17) and (2.19), for $k \geq 0$,

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi(1-k^2)} & k > 0, k \text{ even} \\ 0 & \text{otherwise} \end{cases} \quad (2.23)$$

$$b_k = 0 \quad (2.24)$$

2.6 Verify (2.13) using python.

Solution: The Python code codes/2_6.py

verifies (2.13) by plotting Fig. 2.6.

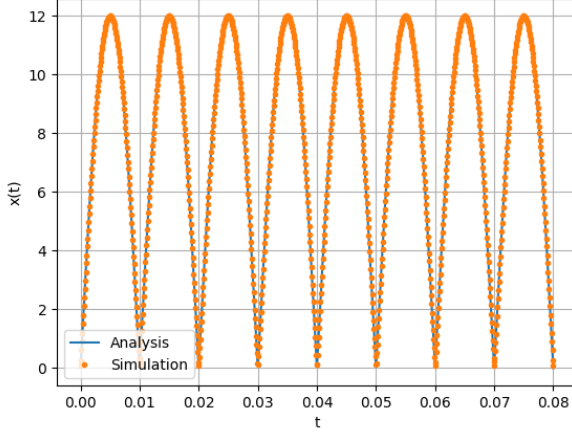


Fig. 2.6: Verification of (2.13).

3 FOURIER TRANSFORM

3.1

$$\delta(t) = 0, \quad t \neq 0 \quad (3.1)$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (3.2)$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} dt \quad (3.3)$$

3.3 Show that

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} G(f) e^{-j2\pi f t_0} \quad (3.4)$$

Solution: We write, substituting $u := t - t_0$,

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi ft} dt \quad (3.5)$$

$$= \int_{-\infty}^{\infty} g(u) e^{-j2\pi f(u+t_0)} du \quad (3.6)$$

$$= G(f) e^{-j2\pi f t_0} \quad (3.7)$$

where the last equality follows from (3.3).

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \quad (3.8)$$

Solution: Using the definition of the Inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} df \quad (3.9)$$

Hence, setting $t := -f$ and $f := t$, which implies $df = dt$,

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-j2\pi ft} dt \quad (3.10)$$

$$\Rightarrow G(t) \xleftrightarrow{\mathcal{F}} g(-f) \quad (3.11)$$

3.5 $\delta(t) \xleftrightarrow{\mathcal{F}} ?$

Solution: We have, from the definition of $\delta(t)$,

$$\delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt \quad (3.12)$$

$$= \int_{-\infty}^{\infty} \delta(0) dt \quad (3.13)$$

$$= \int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (3.14)$$

3.6 $e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} ?$

Solution: Suppose $g(t) \xleftrightarrow{\mathcal{F}} G(f)$. Then,

$$g(t) e^{j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t) e^{-j2\pi(f-f_0)t} dt \quad (3.15)$$

$$= F(f - f_0) \quad (3.16)$$

Using (3.9) in (3.14), $1 \xleftrightarrow{\mathcal{F}} \delta(-f)$. Hence, applying (3.16),

$$e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \delta(-(f + f_0)) = \delta(f + f_0) \quad (3.17)$$

3.7 $\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} ?$

Solution: Using the linearity of the Fourier Transform and (3.17),

$$\cos(2\pi f_0 t) = \frac{1}{2} (e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) \quad (3.18)$$

$$\xleftrightarrow{\mathcal{F}} \frac{1}{2} (\delta(f + f_0) + \delta(f - f_0)) \quad (3.19)$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.

Solution: Substituting (2.12) in (2.1),

$$x(t) \xleftrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} c_k \delta(f + k f_0) \quad (3.20)$$

$$= \frac{2A_0}{\pi} \sum_{k=-\infty}^{\infty} \frac{\delta(f + 2k f_0)}{1 - 4k^2} \quad (3.21)$$

The python code `codes/3_8.py` verifies (3.21) while plotting Fig. 3.8

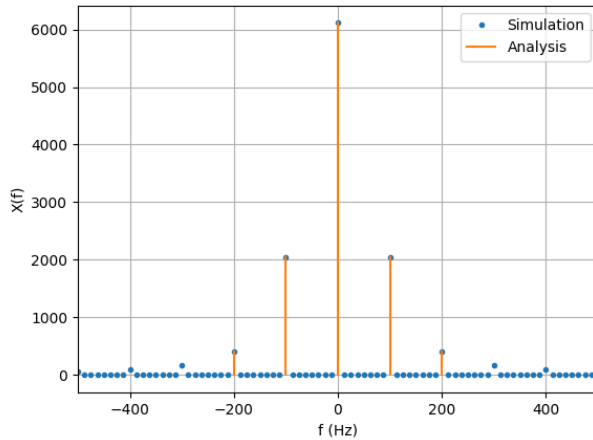


Fig. 3.8: Fourier Transform of $x(t)$.

3.9 Show that

$$\text{rect } t \xleftrightarrow{\mathcal{F}} \text{sinc } f \quad (3.22)$$

Verify using python.

Solution: We write

$$\text{rect } t \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \text{rect } t e^{-j2\pi f t} dt \quad (3.23)$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi f t} dt \quad (3.24)$$

$$= \frac{e^{j\pi f} - e^{-j\pi f}}{j2\pi f} = \frac{\sin \pi f}{\pi f} = \text{sinc } f \quad (3.25)$$

The python code `codes/3_9.py` verifies (3.25) by plotting Fig. 3.9.

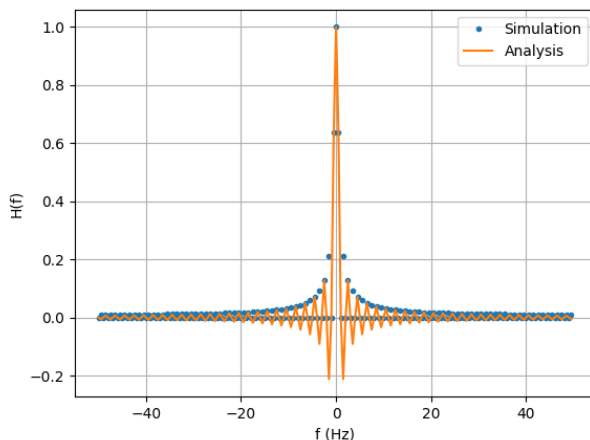


Fig. 3.9: Fourier Transform of $\text{rect}(t)$.

3.10 $\text{sinc } t \xleftrightarrow{\mathcal{F}} ?$ Verify using python.

Solution: From (3.9), we have

$$\text{sinc } t \xleftrightarrow{\mathcal{F}} \text{rect}(-f) = \text{rect } f \quad (3.26)$$

Since $\text{rect } f$ is an even function. The python code `codes/3_10.py` verifies (3.26) by plotting Fig. 3.10.

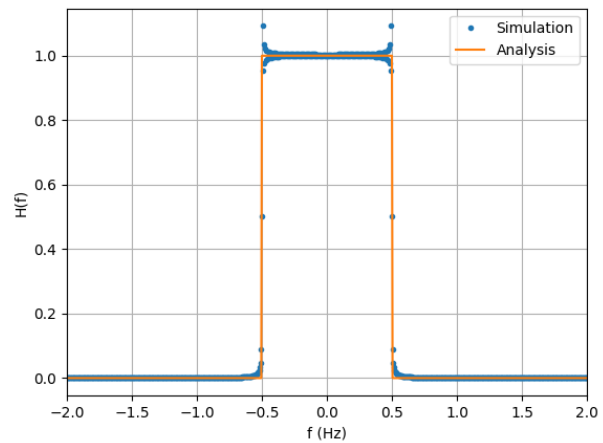


Fig. 3.10: Fourier Transform of $\text{sinc}(t)$.

4 FILTER

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.

Solution: The function $H(f)$ is a low pass filter which filters out even harmonics and leaves the zero frequency component behind. The rectangular function represents an ideal low pass filter. Suppose the cutoff frequency is $f_c = 50$ Hz, then

$$H(f) = \text{rect} \frac{f}{2f_c} = \begin{cases} 1 & |f| < f_c \\ 0 & \text{otherwise} \end{cases} \quad (4.1)$$

Multiplying by a scaling factor to get DC 5V,

$$H(f) = \frac{\pi V_0}{2A_0} \text{rect} \left(\frac{f}{2f_c} \right) \quad (4.2)$$

where $V_0 = 5$ V.

4.2 Find $h(t)$.

Solution: Suppose $g(t) \xleftrightarrow{\mathcal{F}} G(f)$. Then, for

some nonzero $a \in \mathbb{R}$

$$g(at) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(at) e^{-j2\pi ft} dt \quad (4.3)$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} g(u) e^{-j2\pi \frac{f}{a} t} dt \quad (4.4)$$

$$= \frac{1}{a} G\left(\frac{f}{a}\right) \quad (4.5)$$

where we have substituted $u := at$. Using (4.5) of the Fourier Transform in (4.1),

$$h(t) = \frac{2\pi V_0}{A_0} f_c \text{sinc}(2f_c t) \quad (4.6)$$

4.3 Verify your result using convolution.

Solution: The Python code `codes/4_3.py` verifies the result by plotting the graph below.

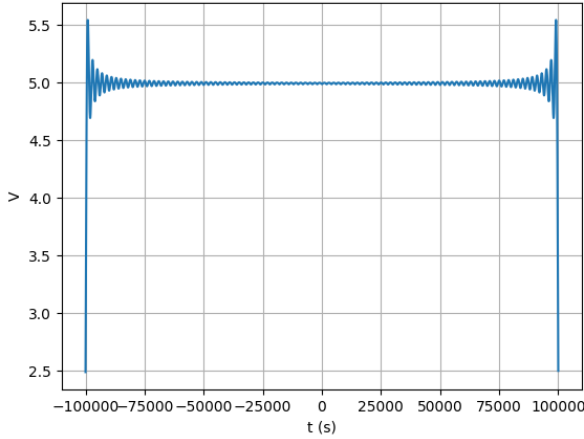


Fig. 4.3: Convolution of the two signals.

5 FILTER DESIGN

5.1 Design a Butterworth filter for $H(f)$.

Solution: The Butterworth filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1 + \left(\frac{f}{f_c}\right)^{2n}\right)} \quad (5.1)$$

where n is the order of the filter and f_c is the cutoff frequency. The attenuation at frequency f is given by

$$A = -10 \log_{10} |H(f)|^2 \quad (5.2)$$

$$= -20 \log_{10} |H(f)| \quad (5.3)$$

We consider the following design parameters for our lowpass analog Butterworth filter:

- a) Passband edge, $f_p = 50$ Hz
- b) Stopband edge, $f_s = 100$ Hz
- c) Passband attenuation, $A_p = -1$ dB
- d) Stopband attenuation, $A_s = -20$ dB

We are required to find a desirable order n and cutoff frequency f_c for the filter. From (5.3),

$$A_p = -10 \log_{10} \left[1 + \left(\frac{f_p}{f_c}\right)^{2n} \right] \quad (5.4)$$

$$A_s = -10 \log_{10} \left[1 + \left(\frac{f_s}{f_c}\right)^{2n} \right] \quad (5.5)$$

Thus,

$$\left(\frac{f_p}{f_c}\right)^{2n} = 10^{-\frac{A_p}{10}} - 1 \quad (5.6)$$

$$\left(\frac{f_s}{f_c}\right)^{2n} = 10^{-\frac{A_s}{10}} - 1 \quad (5.7)$$

Therefore, on dividing the above equations and solving for n ,

$$n = \frac{\log\left(10^{-\frac{A_s}{10}} - 1\right) - \log\left(10^{-\frac{A_p}{10}} - 1\right)}{2(\log f_s - \log f_p)} \quad (5.8)$$

In this case, making appropriate substitutions gives $n = 4.29$. Hence, we take $n = 5$. Solving for f_c in (5.6) and (5.7),

$$f_{c1} = f_p \left[10^{-\frac{A_p}{10}} - 1 \right]^{-\frac{1}{2n}} = 57.23 \text{ Hz} \quad (5.9)$$

$$f_{c2} = f_s \left[10^{-\frac{A_s}{10}} - 1 \right]^{-\frac{1}{2n}} = 63.16 \text{ Hz} \quad (5.10)$$

Hence, we take $f_c = \sqrt{f_{c1} f_{c2}} = 60$ Hz approximately.

5.2 Design a Chebyshev filter for $H(f)$.

Solution: The Chebyshev filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1 + \epsilon^2 C_n^2\left(\frac{f}{f_c}\right)\right)} \quad (5.11)$$

where

- a) n is the order of the filter
- b) ϵ is the ripple
- c) f_c is the cutoff frequency
- d) $C_n = \cosh^{-1}(n \cosh x)$ denotes the n^{th} order

Chebyshev polynomial, given by

$$c_n(x) = \begin{cases} \cos(n \cos^{-1} x) & |x| \leq 1 \\ \cosh(n \cosh^{-1} x) & \text{otherwise} \end{cases} \quad (5.12)$$

We are given the following specifications:

- Passband edge (which is equal to cutoff frequency), $f_p = f_c$
- Stopband edge, f_s
- Attenuation at stopband edge, A_s
- Peak-to-peak ripple δ in the passband. It is given in dB and is related to ϵ as

$$\delta = 10 \log_{10}(1 + \epsilon^2) \quad (5.13)$$

and we must find a suitable n and ϵ . From (5.13),

$$\epsilon = \sqrt{10^{\frac{\delta}{10}} - 1} \quad (5.14)$$

At $f_s > f_p = f_c$, using (5.12), A_s is given by

$$A_s = -10 \log_{10} \left[1 + \epsilon^2 c_n^2 \left(\frac{f_s}{f_p} \right) \right] \quad (5.15)$$

$$\Rightarrow c_n \left(\frac{f_s}{f_p} \right) = \frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon} \quad (5.16)$$

$$\Rightarrow n = \frac{\cosh^{-1} \left(\frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon} \right)}{\cosh^{-1} \left(\frac{f_s}{f_p} \right)} \quad (5.17)$$

We consider the following specifications:

- Passband edge/cutoff frequency, $f_p = f_c = 60$ Hz.
- Stopband edge, $f_s = 100$ Hz.
- Passband ripple, $\delta = 0.5$ dB
- Stopband attenuation, $A_s = -20$ dB

$\epsilon = 0.35$ and $n = 3.68$. Hence, we take $n = 4$ as the order of the Chebyshev filter.

5.3 Design a circuit for your Butterworth filter.

Solution: Looking at the table of normalized element values L_k, C_k , of the Butterworth filter for order 5, and noting that de-normalized values L'_k and C'_k are given by

$$C'_k = \frac{C_k}{\omega_c} \quad L'_k = \frac{L_k}{\omega_c} \quad (5.18)$$

De-normalizing these values, taking $f_c = 60$

Hz,

$$C'_1 = C'_5 = 1.64 \text{ mF} \quad (5.19)$$

$$L'_2 = L'_4 = 4.29 \text{ mH} \quad (5.20)$$

$$C'_3 = 5.31 \text{ mF} \quad (5.21)$$

$$(5.22)$$

The L-C network is shown in Fig. 5.3.

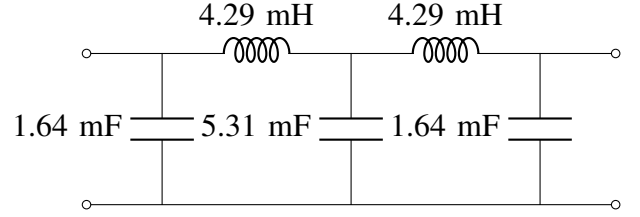


Fig. 5.3: L-C Butterworth Filter

This circuit is simulated in the ngspice code `codes/5_3.cir`. The Python code `codes/5_3.py` compares the amplitude response of the simulated circuit with the theoretical expression.

5.4 Design a circuit for your Chebyshev filter.

Solution: Looking at the table of normalized element values of the Chebyshev filter for order 3 and 0.5 dB ripple, and de-normalizing those values, taking $f_c = 50$ Hz,

$$C'_1 = 4.43 \text{ mF} \quad (5.23)$$

$$L'_2 = 3.16 \text{ mH} \quad (5.24)$$

$$C'_3 = 6.28 \text{ mF} \quad (5.25)$$

$$L'_4 = 2.23 \text{ mH} \quad (5.26)$$

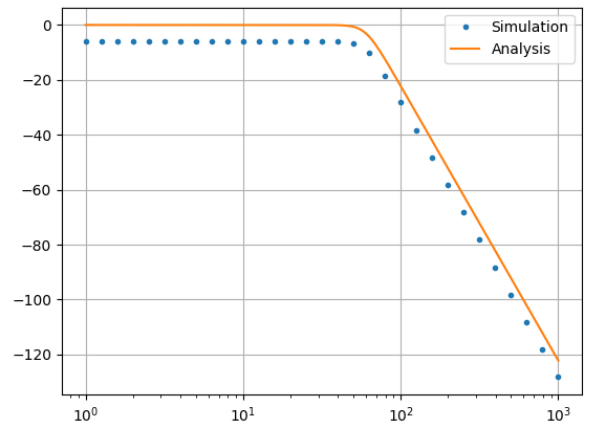


Fig. 5.4: Simulation of Butterworth filter.

The L-C network is shown in Fig. 5.4.

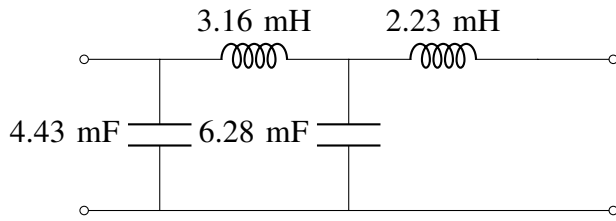


Fig. 5.4: L-C Chebyshev Filter

This circuit is simulated in the ngspice code `codes/5_4.cir`. The Python code `codes/5_4.py` compares the amplitude response of the simulated circuit with the theoretical expression.

5.5 Design a low pass digital Butterworth filter for your $H(f)$.

Solution:

5.6 Design a low pass digital Chebyshev filter for your $H(f)$.

Solution:

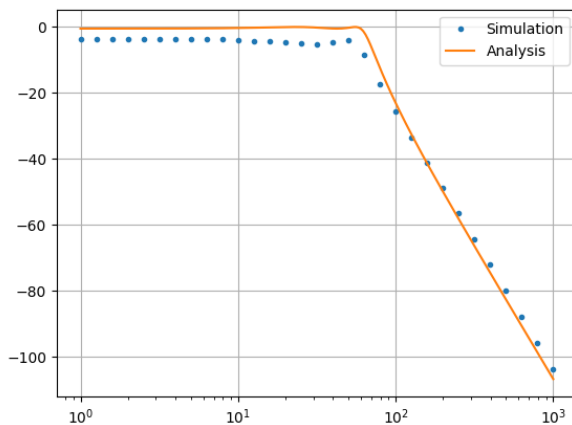


Fig. 5.4: Simulation of Chebyshev filter.