Problem Set 1

2.1 Coin Flips

(a)

The inputs to the random variable X mentioned in the question are in the following form:

H, TH, TTH, TTTH, ...

These inputs represent sequences of coin tosses performed until an H is observed. X outputs the length of a given such sequence. For instance, the outputs corresponding to the above sequences are:

1, 2, 3, 4, ...

Because a sequence of coin tosses are independent, the probability for this random variable can be defined as:

$$P(X = x) = \frac{1}{2^x} \tag{1}$$

where x is the number of tosses performed to get an H. Remember that for a fair coin, the probability of getting a head is the same as the probability of getting a tail i.e. $\frac{1}{2}$. And as the tosses are independent, you just multiply the probabilites of each individual outcome to get the probability of a whole sequence.

As the entropy is to be calculated in bits, logarithm of base 2 will be used. Now, we know:

$$\begin{aligned}
H(X) &= -\sum_{x=1}^{\infty} P(X = x) log_2(P(X = x)) \\
&= -\sum_{x=1}^{\infty} \frac{1}{2^x} log_2(\frac{1}{2^x}) && [(1)] \\
&= \sum_{x=1}^{\infty} \frac{1}{2^x} log_2(2^x) && [log(\frac{1}{x}) = -log(x)] \\
&= \sum_{x=1}^{\infty} \frac{x}{2^x} log_2(2) && [log(a^x) = xlog(a)] \\
&= \sum_{x=1}^{\infty} \frac{x}{2^x} && [2^1 = 2 \implies log_2(2) = 1] \\
&= 2
\end{aligned}$$

You can show that the series $\sum_{x=1}^{\infty} \frac{x}{2^x}$ converges to 2. Look at the link here.