Angular Momentum In Quantum Mechanics: A Survey

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Abstract

This article deals with the angular momentum viewpoint of quantum mechanics. It is a review of the chapters **Angular Momentum** in both **Landau and Lifshitz** as well as **V. Devanathan**. It deals with spin half particles in a bit of detail along with the explanation of the concepts of isospin and hypercharge in particle physics.

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Introduction

Any object moving under the influence of a central force exhibits certain characteristic properties. There are a few quantites that do not change when the object is undergoing motion here. Such quantities are called as **conserved quantities or conserved observables**. According to a standard theorem in classical mechanics known as **Noether's Theorem**, any conservation law is due to some sort of a symmetry in the spacetime structure of the system. In much deeper physics, breaking of such symmetries is believed to create new particles/energy quanta.

According to Noether's theorem, simply put, any symmetric coordinate or a variable leads to the conservation of its **canonically conjugate** coordinate or variable. In the case of linear momentum, it is due to the symmetries existant in the x, y and z directions that usually leads to the conservation of momentum in the respective direction. This property of a physical system is called as the **homogenity of space**. Similarly, in the case of angular momentum, if any angular coordinate shows some symmetry, perhaps the spherical symmetry of the central force field, then the angular momentum is said to be conserved due to the **isotropy of space.**

1 Angular Momentum Operator

Consider any infinitesimal change in a vector, the change being the vector is rotated by an angle of $\delta\phi$ in some plane. The new vector is then given by the simple cross product rule

$$\delta \vec{\mathrm{r_a}} = \delta\vec{\phi} \times \vec{\mathrm{r_a}}$$

. Now, when such a change in the coordinate is made, then there is a corresponding change in the value of any function ψ represented. This change, if it can be brought about by a unitary operator, then that is our required answer.

$$\psi\left(\vec{\mathrm{r_i}} + \delta\vec{\mathrm{r_i}}\right) = \psi\left(r_i\right) + \sum_j \delta\vec{\mathrm{r_j}} \bullet \nabla_j \psi$$

But as we know $\vec{\mathrm{r_i}}$ representation with a change of angle of $\delta\vec{\phi}$ as seen before was, $\delta\vec{\phi} \times \vec{\mathrm{r_a}}$. Hence, the above expression changes as:

$$\psi\left(\vec{\mathrm{r_i}} + \delta\vec{\mathrm{r_i}}\right) = \psi\left(r_i\right) + \sum_j \delta\vec{\phi} \times \vec{\mathrm{r}}_a \bullet \nabla_j \psi$$

$$= \left(1 + \delta\vec{\phi} \bullet \sum_j \vec{\mathrm{r}}_a \times \nabla_j\right) \psi\left(\vec{\mathrm{r}}_1, \vec{\mathrm{r}}_2, ...\right)$$

. The operator $= \left(1 + \delta\vec{\phi} \bullet \sum_j \vec{\mathrm{r}}_a \times \nabla_j\right)$ can be seen as an infinitesimal generator of rotation in space and it is also clearly seen that

$$\left(\sum_j \vec{\mathrm{r}}_j \times \nabla_j\right) \mathcal{H} - \mathcal{H} \left(\sum_j \vec{\mathrm{r}}_j \times \nabla_j\right) = 0$$

Here $\mathcal{H}$ is defined to be the **hamiltonian** of the system. This commutation rule conserves the canonically conjugate quantity to infinitesimal angular motion, that is, the angular momentum in the direction **perpendicular to the plane of rotation.** Now, it is clear that, then $\vec{\mathrm{r}}_j \times \nabla_j$ is the basic component of an angular momentum operator. This can also be seen as taking the quantum mechanical analog of the linear momentum operator and taking a cross product with $\vec{\mathrm{r}}$. Hence, we have

$$\vec{\mathrm{L}} = \vec{\mathrm{r}} \times \vec{\mathrm{p}}$$

$$\vec{\mathrm{L}} = \vec{\mathrm{r}} \times -i\hbar\nabla$$

. Therefore, we can now construct the values of L_x, L_y, L_z from the given vector equation above. This is quite simply

$$L_x = yp_z - zp_y; L_y = zp_x - xp_z; L_z = xp_y - yp_x$$

$$L_i = -i\hbar\left(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}\right)$$

is the most general version of the formulation of the angular momentum operator.

2 Commutators and Properties

Before discussing about commutators, we must first look at an important property of the **expectation** value of the angular momentum vector in quantum mechanics. Clearly by the expectation value, we can use the formula

$$\vec{\mathbf{L}} = -i\hbar \int \psi^* \left(\vec{\mathrm{r}}_j \times \nabla_j\right) \psi \, \mathrm{d}q$$

Note that from the physical definition of expectation value, it is a statistical **measure of average** and hence must be real. But on the right hand side, it is clearly visible that the integral is a Hilbert Schmidt inner product and is real, with an imaginary factor in front. Hence, the left side is purely imaginary. This can be possible iff both sides are zero. Hence, $\vec{\mathrm{L}}$ **has an expectation value of zero**.

Now, let us look at the commutation properties. It can be verified by taking test functions and proving that:

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$$

where ϵ_{ijk} is the Levi Civita symbol or the **the completely antisymmetric tensor of dimension 3**. This explains the **Heisenberg uncertainity** which is present in the angular momentum observables, i.e, **no two** L_i, L_j **can have simultaneous eigenvectors**, except in cases where in all three simultaneously vanish, but then there would be no point talking about angular momentum. Similar commutation rules can be verified:

$$[L_i, q_j] = i\hbar\epsilon_{ijk}q_k$$

$$[L_i, p_j] = i\hbar\epsilon_{ijk}p_k$$

There is however, the square of the angular momentum observable i.e, $\vec{\mathrm{L}}^2$. This quantity is special as it commutes with each of its components and hence, the two sets L_x, L_y, L_z and $\vec{\mathrm{L}}^2$ can have parallel eigenvectors. Mathematically stating this

$$[\vec{\mathrm{L}}^2, L_i] = 0$$

2.1 Ladder Operators

We shall now look at some interesting combinations of the angular momentum directional components. One such linear combination pair is the ladder operator pair. These are also sometimes called **raising and lowering** operators. The reasons for such terminology will be explained shortly. The operators are

$$L_+ = L_x + iL_y$$
$$L_- = L_x - iL_y$$

Also note the startling commutation relations given here. Note that, if you think hard enough, you might just understand why they are called ladder operators from the commutation rules itself(Hint: think of the quantum **harmonic oscillator**).

$$[L_+, L_-] = 2L_z$$
$$[L_z, L_+] = L_+$$
$$[L_z, L_-] = -L_-$$

Let us switch from the cartesian representation of L_x, L_y to the polar coordinates($x = r\sin\theta\cos\phi$ etc.), after switching it can be verified that, the new angular momentum observables are given by:

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$
$$L_+ = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$
$$L_- = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

Notice the exactness of $\vec{\mathrm{L}}^2$ and the **angular part of the Laplacian expressed in spherical coordinates**. This fact will also be impressed upon when we talk about Hydrogen atom and the Schrodinger equation there. If you can't wait, then here's the news, the eigenfunctions of the squared angular momentum operator are the **spherical harmonic** wavefunctions of the Hydrogen atom. The other variable separable solution is the radial equation which gives the radial wavefunction of the atom. The Laplacian is present in the Schrodinger's equation in three dimensions, to link all of the facts up.

3 Eigenvalues related to spherical coordinates

3.1 Finding out the ϕ based eigenvalues

Now, we shall look towards working out the eigenvalues for the L_z observable. The form of the observable in spherical polar coordinates is given in the previous section.Before, this, we shall mention a convention as to set $\hbar = 1$ in our calculations henceforth, as it is just a constant and dimensionalysis should set it right. Now, we have to solve

$$L_z\psi = m\psi$$

$$-i\hbar\frac{\partial\psi}{\partial\phi} = m\psi$$

The last differential equation has a very starighforward solution, and after normalising, the answer turns out to be

$$\psi\left(\phi\right) = \frac{e^{\pm im\phi}}{\sqrt{2\pi}}$$

The number m here must be determined. The simplest method is to look at the normalization condition again. It goes like

$$\int_0^{2\pi}\Phi_m^*\left(\phi\right)\Phi_{m'}\left(\phi\right)d\phi = \delta_{mm'}$$

. Now, it must be clear that m should be an integer to assign a valid normalization procedure for the given quantum system. It need not neccesarily be a positive integer as the exponent in the function has a $\pm$ symblol to it(as we shall see it can take both positive and negative integers, and if you are wondering, then yes, it is the same old **magnetic quantum number** that we are talking about here). Hence, both $\pm m$ are valid for the eigenfunction. Infact, the main point to be noted here is that, **if there exist stationary states with different values of m, then they need not neccesarily have different energies. Degenaracy is confirmed from the heuristic that the direction of z-axis is no way unique and hence, we can have states with the same energies whose z axes point in different directions**.

Getting back to the ladder operators, let us see what happens when we operate any one of $L_{\pm}$ on the state ψ_m. By definition

$$L_z \psi_m = m \psi_m$$

But we know that

$$[L_z, L_{\pm}] = \pm L_{\pm}$$

Hence,

$$L_z L_{\pm} - L_{\pm} L_z = \pm L_{\pm}$$
$$(L_z L_{\pm} \mp L_{\pm}) \psi_m = L_{\pm} L_z \psi_m$$
$$(L_z L_{\pm} \mp L_{\pm}) \psi_m = m L_{\pm} \psi_m$$
$$(L_z L_{\pm}) \psi_m = m L_{\pm} \psi_m \pm L_{\pm} \psi_m$$
$$(L_z L_{\pm}) \psi_m = (m \pm 1) L_{\pm} \psi_m$$

This should show us that, the action of $L_{\pm}$ is to raise and lower the eigenvalue of ψ_m by one $\hbar$ unit and to make the shift of the state

$$\psi_m \to \psi_{m \pm 1}$$

Hence, put more succintly,

$$L_{\pm} \psi_m = (m \pm 1) \psi_{m+1}$$

A standard theorem

This theorem is regarding the m degeneracy of the states ψ_{lm}. As we have emphasized before, the states with differing values of m need not have different energies. Infact, there always exist degenerate states, and this is assured by this theorem. **Given a Hilbert Space $\mathbb{H}$, if there exist two or more conserved quantities whose operators do not commute, then there always exist levels that are degenerate.**

We have so far discussed the eigenvalues of L_z only and have seen that there exist degenerate states ψ_m. But are there any bounds for m? To answer this question, we must first discover the eigenvalues of the remaining angular momentum observables. This has been discussed very well in the book by Prof. **Devanathan**. Consider the following:

$$\left(L_x^2 + L_y^2\right) \psi_{lm} = \left(L^2 - m^2\right) \hbar^2 \psi_{lm}$$

Let us assume, for the sake of discussion, the eigenvalues of $\vec{L}^2$ to be some η_l. Then,

$$\left(L^2 - m^2\right)\hbar^2\psi_{lm} = \left(\eta_l - m^2\right)\hbar^2\psi_{lm}$$

Now the argument goes like this. Since, both L_x, L_y are hermitian, the eigenvalues of the observable represented by the sum of their squares have to be positive. This implies that $\eta_j - m^2 \geq 0$. This should explain the existance of upper and lower bounds. It has already been discussed before that m must take integer values.

$$m = m_1, m_1 + 1, m_1 + 2, ..., m_2 - 1, m_2$$

. This then should imply that, there cannot exist a state above m_2 and below m_1. Hence

$$L_-\psi_{jm_1} = 0; L_+\psi jm_2 = 0$$

$$L_+L_-\psi_{lm_1} = 0; L_-L_+\psi_{lm_2} = 0$$

Now from the definitions, we can see,

$$L_+L_- = L^2 - L_z\left(L_z - \hbar\right)$$

$$L_-L_+ = L^2 - L_z\left(L_z + \hbar\right)$$

These should help realise the following expressions below,

$$\eta_l = m_1\left(m_1 - 1\right)$$

$$\eta_l = m_2\left(m_2 + 1\right)$$

The equality that follows ensues the fact

$$m_1 = -m_2 = l$$

Now its simple, the eigenvalues of $\vec{L}^2$ are nothing but $\eta_l = l\left(l+1\right)\hbar^2$ and the values of m range from $-l$ to l through 0, i.e., a total of $2l+1$ values.

3.2 Finding out the θ based eigenvalues

So far, we have seen how to derive the azimuthal wavefunction based on angular momentum considerations. Now, we shall solve for the wavefunction related to θ coordinate. The procedure is again, very innovative, as described by **Landau and Lifshitz**. Let us define a combined wavefunction for both θ and ϕ

$$Y_{lm} = \Phi(\phi)\,\Theta(\theta)$$

Now, as we have discussed before, there are bounds on the number m, the maximum being l. This implies

$$L_+ Y_{ll} = 0$$

Using the spherical polar coordinate representation of the L_+ operator and also we know $\Phi(\phi) = \frac{e^{\pm im\phi}}{\sqrt{2\pi}}$ This can all be condensed to

$$L_+ Y_{ll} = L_+\left(\frac{e^{\pm im\phi}}{\sqrt{2\pi}}\right) = 0$$

$$\frac{\mathrm{d}\Theta_{ll}}{\mathrm{d}\theta} - l\cot\theta\Theta_{ll} = 0$$

This is a standard equation, whose solution can be seen as $\Theta_{ll} = \lambda \sin^l \theta$ where λ is a constant, function of l which is to be determined by normalization. The normalization condition is

$$\int_0^\pi |\Theta_{lm}|^2 \sin\theta d\theta = 1$$

Now we have to derive the value of Θ_{lm} from Θ_{ll}. Using the equation

$$L_- Y_{l,m+1} = (L_-)_{m,m+1} Y_{lm} = \sqrt{(l-m)(l+m+1)}\, Y_{lm}$$

,we can achieve the required conversion by its repeated application. The final answer shall be stated

$$\Theta_{lm}(\theta) = (-i)^l \sqrt{\frac{(2l+1)(l+m)!}{2(l-m)!}}\, \frac{1}{2^l l! \sin^m\theta}\, \frac{d^{l-m}}{d(cos\theta)^{l-m}} sin^2 l\theta$$

4 Matrix Mechanical Formulation of Angular Momentum

We shall now discuss about another formulation of angular momentum in quantum mechanics. Till now, we were using **wave mechanics** or

Schrodinger formalism or picture of quantum mechanics which depended on Schrodinger's equation as the basic equation of motion and used ideas such as normalization and Born Interpretation to establish factors related to the wavefunction. Now, we move on to the **Heisenberg-Jordan-Kramers** picture or simply Matrix Mechanics. In this picture, the representation is as follows

$$L^2 \left|l, m\right\rangle = l\left(l+1\right)\hbar^2 \left|l, m\right\rangle ; L_z \left|l, m\right\rangle = m\hbar \left|l, m\right\rangle$$

Now from basic braket calculus, we know that, for any observable A in some state ψ_j, the matrix elements for the linear transformation is given by

$$\left\langle\psi_i\right| A \left|\psi_j\right\rangle = A_{ij}$$

Hence,

$$\left\langle l', m'\right| L^2 \left|l, m\right\rangle = l\left(l+1\right)\delta_{ll'}\delta_{mm'}$$

$$\left\langle l', m'\right| L_z \left|l, m\right\rangle = m\delta_{ll'}\delta_{mm'}$$

Let us now work out the matrix elements for L_-, L_+. Now,

$$L_\pm \left|l, m\right\rangle = \Lambda_\pm \left|l, m \pm 1\right\rangle$$

Taking the length of the vector on both sides and also using the fact$L_+^\dagger = L_-$, we have

$$\left\langle l, m\right| L_\mp L_\pm \left|l, m\right\rangle = l\left(l+1\right) - m\left(m \pm 1\right)$$

This should condense to the following matrix representations,

$$\left\langle l', m'\right| J_\pm \left|l, m\right\rangle = \sqrt{l\left(l+1\right) - m\left(m \pm 1\right)}\delta_{ll'}\delta m'm \pm 1$$

From now onwards, the discussion would continue based on what values of l the particle can take. In case of $l = 1/2$, we say we are talking about the family of particles having **spin one half**. Electrons, Protons, Neutrons all enter this family of half integral spin particles, called as **Fermions**. There are particles with integer spin also. Particles like the Graviton, Photons all form another group of particles called as **Bosons**. The naming is so because each ensemble of particles follows either the **Fermi-Dirac Statistics** or the **Bose Einstein Statistics** respectively. Let us discuss first about the spin half particles. In this case, the particles can take two spin orientations (described by m) as either spin up or spin down. In the general case, when we try constructing the angular momentum matrices, they are of the order of $2l+1$ square matrices. In this case, using the matrix formulae described above, we can construc the matrices as given below

$$L^2 = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$$

$$L_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix}$$

$$L_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$L_- = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$$

$$L_x = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

$$L_y = \begin{pmatrix} 0 & \frac{-i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}$$

The formulae for L_x, L_y can be got from the relations with the ladder operators. Now, we shall introduce the **Pauli Spin Matrices**. Notice that we can write

$$L_x = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$L_y = \begin{pmatrix} 0 & \frac{-i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$L_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{-1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The matrices in the parantheses of the last expression form a group of representations called as the Pauli Spin Matrices. They can be easily remembered by the following notation simplified by the Kronecker delta

$$\sigma_j = \begin{pmatrix} \delta_{j3} & \delta_{j1} + i\delta_{j2} \\ \delta_{j1} - i\delta_{j2} & -\delta_{j3} \end{pmatrix}$$

Hence, for every

$$L_j = \frac{1}{2}\sigma_j$$

The Pauli matrices are restricted to the analysis of spin one half particles only. For other spins, we can find or construct other standard matrices based on the similarities of each angular momentum matrix. Using spin half particles, we can now construct a new spin space with two basis kets $|\alpha\rangle$ and $|\beta\rangle$ defined as the matrices (10) and (01). In this new notation, we can simply express the previous properties as

$$L^2 |\alpha\rangle = \frac{3}{4} |\alpha\rangle ; L^2 |\beta\rangle = \frac{3}{4} |\beta\rangle$$

$$L_z |\alpha\rangle = \frac{1}{2} |\alpha\rangle ; L_z |\beta\rangle = -\frac{1}{2} |\beta\rangle$$

The above relation is the base case for discrimination of spin up and spin down particles. The next sections will discuss the quantum systems with two spin half particles and we shall end the discussion with another section on **Isospin** and a few advanced topics.

5 Two Spin $\frac{1}{2}$ particles in a quantum state

The central theme of discussion will be the various types of combinations that can be possible with two spin half particles. This topic is usually discussed under a wide variety of headings such as **Identical Particles** or **Fermi Statistics** as we are ultimately studying the properties of Fermions. The basic uncoupled state vectors in the system will be represented as $|s, m\rangle$ where s is the spin of the particle. There are four different situations that can rise here

$$\chi (3, 1) = \alpha_1 \alpha_2$$

$$\chi (3, -1) = \beta_1 \beta_2$$

$$\chi (3, 0) = \sqrt{\frac{1}{2}} (\alpha_1 \beta_2 + \beta_1 \alpha_2)$$

$$\chi (1, 0) = \sqrt{\frac{1}{2}} (\alpha_1 \beta_2 - \beta_1 \alpha_2)$$

If you are confused regarding the terminology of α and β, they are called just to identify different identical particles(!). Each of the above coupled states can be represented in general with the help of linear combinations of uncoupled $|s, m\rangle$ of independent particles. The coefficients in each expansion are the famous **Clebsch-Gordan Coefficients**.

$$|S, m\rangle = \sum_{m_1, m_2} \mathcal{C}^{l,m}_{l_1, m_1, l_2, m_2} |l_1, m_1\rangle |l_2, m_2\rangle$$

There are standard tables that can be found anywhere online or in a standard quantum chemistry textbook. They are useful in determining the amplitudes of a system to be in one of the ket vectors described before. There are a few simple properties of these coefficients which will be described below, which include a few interesting symmetry properties. From the orthonormality of the kets

$$\langle j_1, j_2, m_1', m_2' \,|\, j_1, j_2, m_1, m_2\rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

$$\langle j_1, j_2, j', m' \,|\, j_1, j_2, j, m\rangle = \delta_{jj'} \delta_{mm'}$$

Using these, and the definitions of Clebsch Gordan coefficients in the uncoupled expansion of the left hand side kets we can get the following relation

$$\sum_{m_1, m_2} \mathcal{C}^{l,m}_{l_1, m_1, l_2, m_2} \mathcal{C}^{l',m'}_{l_1, m_1, l_2, m_2} = \delta_{jj'} \delta_{mm'}$$

$$\sum_{jm} \mathcal{C}^{l,m}_{l_1, m_1, l_2, m_2} \mathcal{C}^{l',m'}_{l_1, m_1, l_2, m_2} = \delta_{jj'} \delta_{mm'}$$

There are many symmetry properties of these coefficients, the most important ones given below