Task 24

We want to design an output feedback controller

$$u = -k \cdot y$$

for the first-order LTI SISO system

$$\begin{aligned} \dot{x} &= -10x + u \\ y &= 2x \end{aligned}$$

such that the quadratic objective function

$$J = \frac{1}{2}\int_0^\infty 11 \cdot y^2(t) + u^2(t)\, dt$$

is minimized.

<u>Solution:</u>

- Formulation as state feedback design problem

For this first-order example, any output feedback $u = -k \cdot y$ can also be expressed as state feedback $u = -k_x \cdot x = -2k \cdot x$ and vice versa, which directly follows from the 1-to-1 identity $y = 2x$. Thus, the task is equivalent to designing a state feedback controller using the standard LQR method. To this end, the objective function can be rewritten in the usual form

$$J = \frac{1}{2}\int_0^\infty 11 \cdot (2x(t))^2 + u^2(t)\, dt = \frac{1}{2}\int_0^\infty 44 \cdot x^2(t) + u^2(t)\, dt$$

by substituting $y = 2x$.

- Calculation of Riccati “matrix” $\underline{P} = p$

Since $n = 1$, the symmetric and positive definite Riccati matrix $\underline{P}$ simplifies to a real positive value $p > 0$ and the Riccati equation

$$\frac{1}{s}\underline{Pbb^T}\underline{P} - \underline{PA} - \underline{A}^T\underline{P} - \underline{Q} = \underline{0}$$

reads

$$\begin{aligned} & \tfrac{1}{1}p \cdot 1 \cdot 1 \cdot p + 10p + 10p - 44 = 0 \\ \Leftrightarrow \quad & p^2 + 20p - 44 = 0 \end{aligned}$$

This quadratic equation has the two solutions

$$p_1 = \frac{-20+\sqrt{400+176}}{2} = -10 + \sqrt{144} = 2$$
$$p_2 = \frac{-20-\sqrt{400+176}}{2} = -10 - \sqrt{144} = -22$$

The general condition that the Riccati matrix must be positive definite means that the scalar parameter p must be positive such that the desired solution coincides with the positive of these two values

$$p = p_1 = 2$$

- Calculation of state feedback gain k_x

Having calculated the Riccati matrix, the corresponding state feedback gain k_x is obtained from the general formula

$$\underline{k}_x^T = \frac{1}{s}\underline{b}^T\underline{P}$$

according to

$$k_x = \frac{1}{s} \cdot b \cdot p$$
$$\Rightarrow \qquad k_x = \frac{1}{1} \cdot 1 \cdot 2 = 2$$

- Calculation of output feedback gain

Using $y = 2x \Leftrightarrow x = 0.5y$ the optimal state feedback $u = -k_x x = -2x$ can be converted into the required output feedback

$$u = -2 \cdot (0.5y) = -y$$

with the corresponding gain factor

$$k = 1$$