b) Derive the real- and imaginary part of the Fourier Transform with $T = \frac{1}{48kHz}$

From the formular sheet:

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(nT) \cdot \delta(t-nT)$$

$$X_a(f) = \sum_{n=-\infty}^{\infty} x(nT) \cdot e^{-2\pi f nT}$$

$$h_{k=0}(n) = [1, 1, 1] = \delta(n) + \delta(n-1) + \delta(n-2)$$

$$H_{k=0}(f) = \sum_{n=0}^{2} h_{k=0}(n) \cdot e^{-j2\pi f nT} = \sum_{n=0}^{2} e^{-j2\pi f nT}$$

$$= e^{\underbrace{-j2\pi f \cdot 0 \cdot T}_{0}} + e^{-j2\pi f \cdot 1 \cdot T} + e^{-j2\pi f 2 \cdot T}$$

$$= 1 + e^{-j2\pi fT} + e^{-j4\pi fT}$$

$$= 1 + \cos(2\pi fT) - j\sin(2\pi fT) + \cos(4\pi fT) - j\sin(4\pi fT)$$

$$Re\{H_{k=0}(f)\} = 1 + \cos(2\pi fT) + \cos(4\pi fT)$$

$$Im\{H_{k=0}(f)\} = -\sin(2\pi fT) - \sin(4\pi fT)$$

$$h_1(n) = \left[\frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}\right] = \frac{\sqrt{3}}{2}\delta(n) - \frac{\sqrt{3}}{2}\delta(n-2)$$

$$H_1(f) = \sum_{n=0}^{2} h_1(n) \cdot e^{-j2\pi f nT}$$

$$= \frac{\sqrt{3}}{2} e^{-j2\pi f \cdot 0 \cdot T} + 0 - \frac{\sqrt{3}}{2} e^{-j2\pi f 2T}$$

$$= \frac{\sqrt{3}}{2}\left[1 - e^{-j4\pi fT}\right]$$

$$= \frac{\sqrt{3}}{2}\left[1 - e^{-j4\pi fT}\right]$$

$$= \frac{\sqrt{3}}{2}\left[1 - \cos(4\pi fT) + j\sin(4\pi fT)\right]$$

$$Re\{H_1(f)\} = \frac{\sqrt{3}}{2}\left[1 - \cos(4\pi fT)\right)$$

$$Im\{H_1(f)\} = \frac{\sqrt{3}}{2}\sin(4\pi fT)$$

$$h_2(n) = \left[\tfrac{1}{2}, -1, \tfrac{1}{2}\right] = \tfrac{1}{2}\delta(n) - \delta(n-1) + \tfrac{1}{2}\delta(n-2)$$

$$H_2(f) = \sum_{n=0}^{2} h_2(n)\cdot e^{-j2\pi f nT}$$

$$= \tfrac{1}{2}e^{-j2\pi f\cdot 0\cdot T} - e^{-j2\pi f\cdot 1\cdot T} + \tfrac{1}{2}e^{-j2\pi f\cdot 2\cdot T}$$

$\vdots$

$$= \underbrace{\tfrac{1}{2} - \cos(2\pi fT) + \tfrac{1}{2}\cos(4\pi fT)}_{\text{real part}} + j\,\underbrace{\sin(2\pi fT) - j\,\tfrac{1}{2}\sin(4\pi fT)}_{\text{imaginary part}}$$

real part   imaginary part

c) Draw $|H_n(f)|$ for $-\frac{1}{2T} \leq f \leq \frac{1}{2T}$ $\in$ Nyquist

(absolute value transfer function)

here: $T = \frac{1}{48kHz}$: $-\frac{1}{2\cdot\frac{1}{48kHz}} \leq f \leq \frac{1}{2\cdot\frac{1}{48kHz}}$

$$H_0(f) = 1 + \cos(2\pi fT) + \cos(4\pi fT) - j\sin(2\pi fT) - j\sin(4\pi fT)$$

$$|H_0(f)| = \sqrt{Re^2\{H_0(f)\} + Im^2\{H_0(f)\}}$$

$$= \sqrt{\left[1+\cos(2\pi fT)+\cos(4\pi fT)\right]^2 + \left[-\sin(2\pi fT) - \sin(4\pi fT)\right]^2}$$

$\vdots$

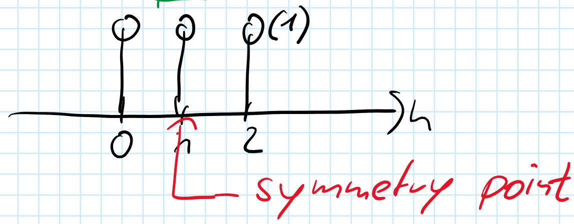
$\Rightarrow$ too complicate

Better way:

$$\ldots j2\pi f\cdot 0\cdot T \quad \boxed{-j2\pi f\cdot 1\cdot T} \quad -j2\pi f\cdot 2\cdot T$$

Better way:

$$H_0(f) = e^{-j2\pi f\cdot 0\cdot T} + \boxed{e^{-j2\pi f\cdot 1\cdot T}} + e^{-j2\pi f\cdot 2\cdot T}$$

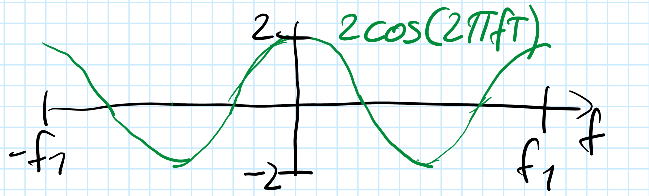


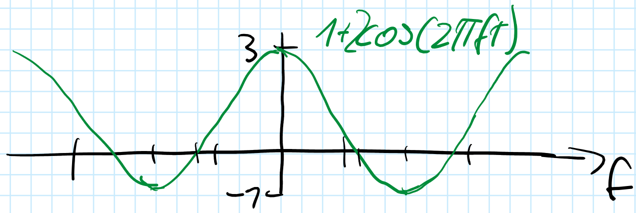
$$H_0(f) = e^{-j2\pi fT}\left(e^{+j2\pi fT} + 1 + e^{-j2\pi fT}\right)$$

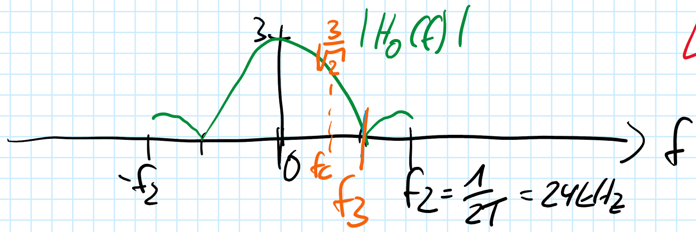
$$= e^{-j2\pi fT}\left(1 + \cos(2\pi fT) + j\sin(2\pi fT) + \cos(2\pi fT) - j\sin(2\pi fT)\right)$$

$$= e^{-j2\pi fT}\left(1 + 2\cos(2\pi fT)\right)$$

$$|H_0(f)| = |1 + 2\cos(2\pi fT)|$$







Low pass filter LP

f_3 $\quad 2\cos(2\pi f_3 T) + 1 = 0$

$$\Rightarrow f_3 = \arccos(-\tfrac{1}{2}) \cdot \frac{1}{2\pi T}$$

$$= \frac{24\,kHz}{\pi} \arccos(-\tfrac{1}{2}) = 16\,kHz$$

$f_{cut\,off}$: Border frequency / cut off frequency of the Filter:

Signal is attenuated by $\frac{1}{\sqrt{2}}$

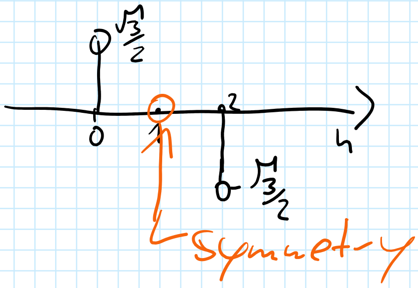
$$2\cos(2\pi f_c T) + 1 = \frac{3}{\sqrt{2}}$$

$$\cos(2\pi f_c T) = \frac{\frac{3}{\sqrt{2}} - 1}{2} \approx 0{,}56$$

$$2\pi f_c T = 0{,}98$$

$$\Rightarrow f_c = 0{,}98 \cdot \frac{1}{2\pi T} \approx \underline{\underline{7{,}49\,kHz}}$$

$$h_1(n) = \left[\frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}\right] = \frac{\sqrt{3}}{2}\delta(n) - \frac{\sqrt{3}}{2}\delta(n-2)$$

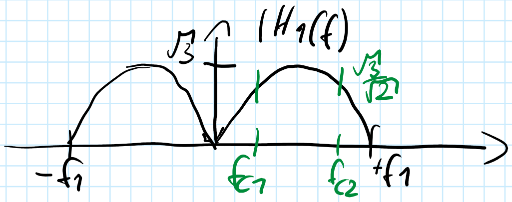


$$H_1(f) = \frac{\sqrt{3}}{2} e^{-j2\pi f 0 T} + 0 - \frac{\sqrt{3}}{2} e^{-j2\pi f 2T}$$

$$= e^{-j2\pi f \cdot 1 \cdot T}\left(\frac{\sqrt{3}}{2} e^{+j2\pi f T} - \frac{\sqrt{3}}{2} e^{-j2\pi f T}\right)$$

$$|H_1(f)| = \left|\frac{\sqrt{3}}{2}\left[\cos(2\pi f T) + j\sin(2\pi f T) - \cos(2\pi f T) + j\sin(2\pi f T)\right]\right|$$

$$= \left|\frac{\sqrt{3}}{2} \cdot 2j\sin(2\pi f T)\right| = \sqrt{3}\,|\sin(2\pi f T)|$$



Bandpass Filter BP