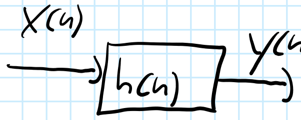
z-Transform: Partial Fraction Decomposition (PFD)

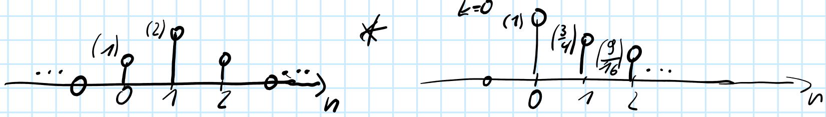
Motivation: filter / convolution



$y(n) = x(n) * h(n)$

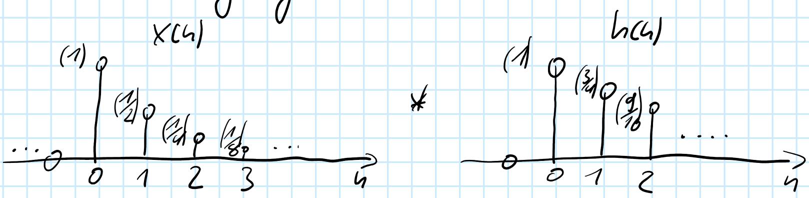
if x(n) or h(n) are of finite duration, then estimation / simulation of y(n) is simple.

$h(n) = \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k \delta(n-k) = \left(\frac{3}{4}\right)^n \varepsilon(n)$



$$y(n) = x(n) * h(n) = [\delta(n) + 2\delta(n-1) + \delta(n-2)] * h(n)$$
$$= h(n) + 2h(n-1) + h(n-2)$$

But: if both signals/filter are of infinite duration (IIR), then the estimation of y(n) is challenging.



$$x(n) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \delta(n-k) \qquad h(n) = \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k \delta(n-k)$$

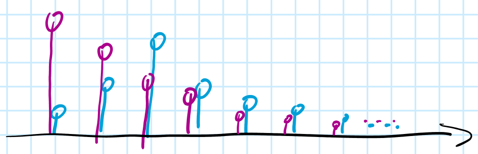
$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m) \cdot h(n-m)$$

$$= \sum_{m=-\infty}^{\infty} \left(\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^n \delta(m-k)\right) \cdot \left(\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k \delta(n-m-k)\right)$$

to complicated

$\Rightarrow$ solution via z-Transform

1) Typically just the very first samples are of importance.

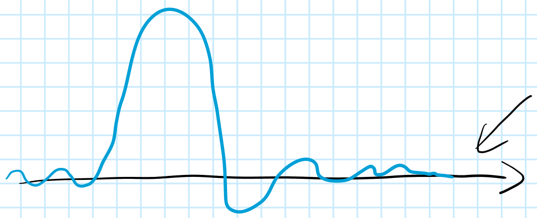


$y_1(n) = ?$

$y_2(n) = ?$

2) In reality a lot of processes can be described as impulse or decay process.

- Piano tone
- room impulse response
- impulse responce of RC-LP
- discharge



At some point the theoretically infinite signal disapears within the measurement noise.

What happens after convolution of two IIR processes?

Typically we represent such processes via the pole-null description:

$$\frac{(z-z_{N1})(z-z_{N2})}{(z-z_{p1})(z-z_{p2})} \cdot \frac{(z-z_{N3})}{(z-z_{p3})(z-z_{p4})} = \frac{(z-z_{N1})(z-z_{N2})(z-z_{N3})}{(z-z_{p1})(z-z_{p2})(z-z_{p3})(z-z_{p4})}$$

$$\underbrace{x(n) * h(n)}_{\text{up to a certain complexity}} = \underbrace{y(n)}_{\text{at a certain}}$$

up to a certain complexity we can find them in the formula sheet.

at a certain degree we cannot find them $\Rightarrow$ PFD!

Partial Fraction Decomposition

$$\frac{(z-z_{N1})(z-z_{N2})(z-z_{N3})}{(z-z_{p1})(z-z_{p2})(z-z_{p3})(z-z_{p4})} = \underbrace{\frac{A}{z-z_{p1}} + \frac{B}{z-z_{p2}} + \frac{C}{z-z_{p3}} + \frac{D}{z-z_{p4}}}$$

- can be individually transformed into time-domain
- y(n) representable
- Allows to simulate the system

For each pole: one term at the PFD

a) single poles:

$$\cdot \frac{1}{z-z_p} \xrightarrow{PFD} + \frac{A}{z-z_p}$$

b) complex conjugate poles:

$$\cdot \frac{1}{z-z_p} \cdot \frac{1}{z-z_p^*} = \frac{1}{z^2 - 2Re\{z_p\}z + |z_p|^2} \xrightarrow{PFD} \frac{Az+B}{z^2 - 2Re\{z_p\}z + |z_p|^2}$$

c) multiple poles:

$$\cdot \frac{1}{(z-z_p)^3} \xrightarrow{PFD} + \frac{A}{z-z_p} + \frac{B}{(z-z_p)^2} + \frac{C}{(z-z_p)^3}$$

Examples are in the script!

example: $Y(z) = \frac{z^3 - 2z^2 + 3z - 4}{\ldots} = \frac{z^3 - 2z^2 + 3z - 4}{\ldots}$

example: $Y(z) = \dfrac{z^3 - 2z^2 + 3z - 4}{(z-1)(z+1)} = \dfrac{z^3 - 2z^2 + 3z - 4}{z^2 - 1}$

looking for: $y(n) = ?$

Degree Numerator $\geq$ Degree Denominator

$\Rightarrow$ Polynome division

$$
\begin{array}{l}
(z^3 - 2z^2 + 3z - 4) : (z^2 - 1) = z - 2 + \dfrac{4z - 6}{z^2 - 1} \\
-(z^3 \qquad\qquad - z) \\
\hline
\qquad -2z^2 + 4z - 4 \\
\quad -(-2z^2 \qquad + 2) \\
\hline
\qquad\qquad 4z - 6
\end{array}
$$

PFD

$$\frac{4z-6}{z^2-1} = \frac{A}{z-1} + \frac{B}{z+1} \qquad | \cdot (z^2 - 1)$$

multiply with all poles.

$$4z - 6 = \frac{z^2-1}{z-1} A + \frac{z^2-1}{z+1} B$$

$$4z - 6 = (z+1)A + (z-1)B$$

Insert values for z:

a) poles

b) small integer: $z = 0,\ z = 1,\ z = -1 \ldots$

method a) $z = -1$: $4(-1) - 6 = (-1+1)A + (-1-1)B$

$\Rightarrow \quad -10 = -2B$

$\Rightarrow \quad \underline{\underline{B = 5}}$

$z = +1$: $4(1) - 6 = (1+1)A + (1-1)B$

$-2 = 2A$

$\underline{\underline{A = -1}}$

$$Y(z) = \frac{z^3-2z^2+3z-4}{(z-1)(z+1)} = z-2-\frac{1}{z-1}+\frac{5}{z+1}$$

$$= z-2-\frac{z^{-1}}{1-z^{-1}}+\frac{5z^{-1}}{1+z^{-1}}$$

$$= z - 2 - z^{-1}\cdot\frac{1}{1-z^{-1}} + 5\cdot z^{-1}\cdot\frac{1}{1+z^{-1}}$$

formula sheet : $z^{-n_0} \circ\!\!-\!\!\bullet\; \delta(n-n_0)$

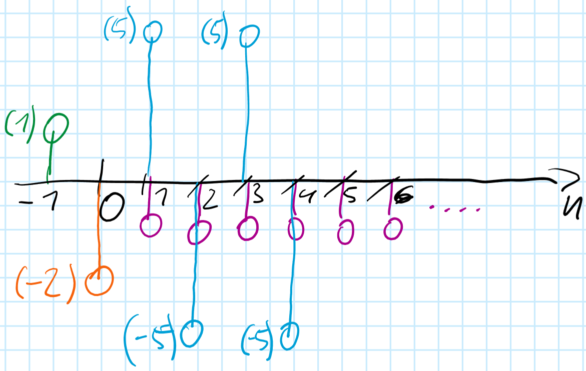
(S. 80) $\frac{1}{1-bz^{-1}} \bullet\!\!-\!\!\circ\; b^n\cdot\varepsilon(n)$

$$y(n) = \delta(n+1) - 2\delta(n) - \delta(n-1) * [1^n\cdot\varepsilon(n)] + 5\,\delta(n-1) * [(-1)^n\varepsilon(n)]$$

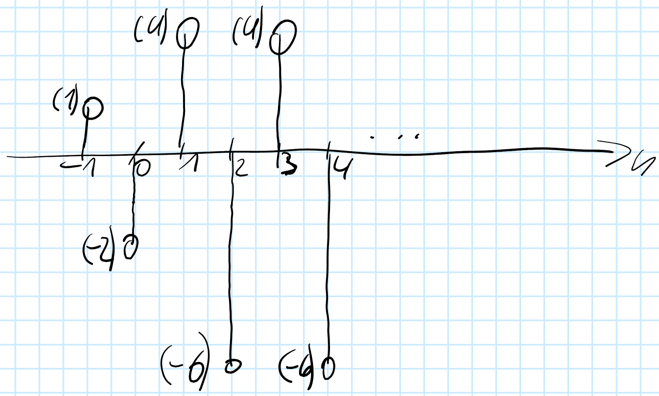
$$= \delta(n+1) - 2\delta(n) - \underbrace{\delta(n-1)}_{\text{delay}} * \varepsilon(n) + 5\,\delta(n-1) * [(-1)^n\varepsilon(n)]$$

alternating sign

$$= \underline{\delta(n+1)}\; \underline{-2\delta(n)}\; \underline{-\varepsilon(n-1)} + 5(-1)^{(n-1)}\varepsilon(n-1)$$



y(n)



2. example: multiple poles (2)

2. example: multiple poles (2)

$$Y(z) = \frac{1}{(z-1)(z+2)^2} = \frac{A}{z-1} + \frac{B}{z+2} + \frac{C}{(z+2)^2}$$

degree Numerator < degree Denominator

multiplication with poles

$$1 = (z+2)^2 A + (z-1)(z+2)B + (z-1)C$$

insert:

$z=1$: $\quad 1 = 9A + 0B + 0C$

$$\Rightarrow \underline{\underline{A = 1/9}}$$

$z=-2$: $\quad 1 = 0A + 0B + (-3)C$

$$\Rightarrow \underline{\underline{C = -1/3}}$$

pick integer $z=0$: $\quad 1 = 4A + (-2)B + (-1)C$

$$1 = 4/9 - 2B + 1/3$$

$$-2B = 1 - 4/9 - 1/3$$

$$B = -1/2 \cdot (1 - 4/9 - 1/3)$$

$$\underline{\underline{B = -1/9}}$$

$$Y(z) = \frac{1/9}{z-1} - \frac{1/9}{z+2} - \frac{1/3}{(z+2)^2} \cdot \frac{z^{-2}}{z^{-2}}$$

$$= \frac{1}{9}z^{-1} \cdot \frac{1}{1-z^{-1}} - \frac{1}{9}z^{-1} \cdot \frac{1}{1+2z^{-1}} - \frac{1}{3}z^{-2} \cdot \frac{1}{(1+2z^{-1})^2}$$

amplification

delay

[formula sheet: $\quad \frac{bz^{-1}}{(1-bz^{-1})^2} \bullet\!\!-\!\!\circ \; n b^n \varepsilon(n)$

$$\frac{1}{\;\;} \bullet\!\!-\!\!\circ \; b^n \varepsilon(n)$$

$$\frac{1}{1-bz^{-1}} \bullet\!\!-\!\!\circ\; b^n \varepsilon(n)$$

$$Y(z) = \frac{1}{9} z^{-1} \frac{1}{1-z^{-1}} - \frac{1}{9} z^{-1} \cdot \frac{1}{1+2z^{-1}} + \frac{1}{6} z^{-1} \frac{-2z^{-1}}{(1+2z^{-1})^2}$$