Image Acquisition

Scene Representation

Technical University of Applied Sciences Würzburg-Schweinfurt

2D and 3D Geometry

- Euclidean Space
- Homogeneous Coordinates
- Line, Plane, Curve

Geometric Projections

- Central projection
- Intrinsics & Extrinsics
- Projections of Lines & Planes
- Camera Calibration I

Optics: The Lens

- Characteristic Values
- Thin Lense
- Imaging Errors
- Camera Calibration II

Scene Representation

Motivation

Task: Measurement, reconstruction and analysis of the 3D world using 2D images from a camera

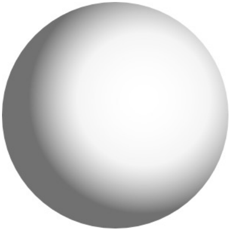
- Pose[1] and movement of objects
- Pose and movement of a camera
- 3D geometry of objects
- Relations between objects

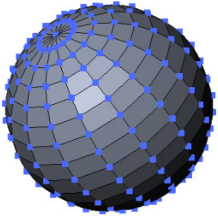
Basics:
Description of the geometrical relations of the 3D world and the camera by coordinate systems, points and vectors.

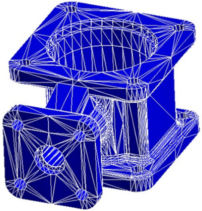
1 Pose = translational und rotational arrangement

Scene Representation

Motivation - Modeling tradeoffs







Requirements to:

- Accuracy of the model
- complexity of the model

Mostly trade-offs between:

- many, local, simple models $\leftrightarrow$ few global, complicated models

Euclidean 3D Space

Points

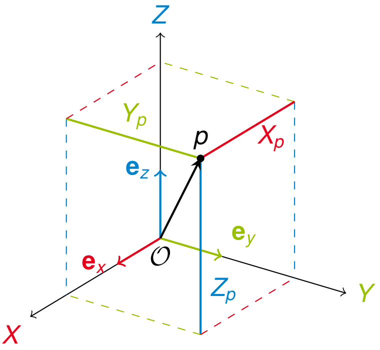
Cartesian (orthogonal) coordinate system with standard basis:

$$\mathbf{e}_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Coordinates of a point p in 3D space:

$$\mathbf{X}_p = \begin{bmatrix} X_p \\ Y_p \\ Z_p \end{bmatrix} \in \mathbb{R}^3$$

$$= X_p\mathbf{e}_x + Y_p\mathbf{e}_y + Z_p\mathbf{e}_z \,.$$



Euclidean 3D Space

Vectors

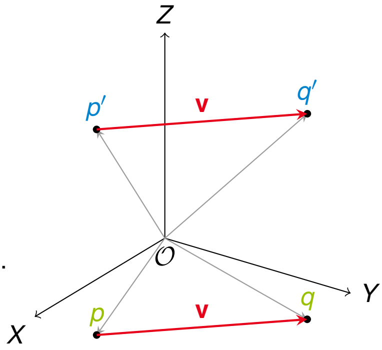
A vector is defined by a pair of points (p, q):

$$\mathbf{X}_p = \begin{bmatrix} X_p \\ Y_p \\ Z_p \end{bmatrix} \in \mathbb{R}^3 , \ \mathbf{X}_q = \begin{bmatrix} X_q \\ Y_q \\ Z_q \end{bmatrix} \in \mathbb{R}^3 .$$

coordinates of a vector **v** in 3D space:

$$\mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} X_q - X_p \\ Y_q - Y_p \\ Z_q - Z_p \end{bmatrix} = \mathbf{X}_q - \mathbf{X}_p \in \mathbb{R}^3 .$$

$$\overrightarrow{pq} = \mathbf{X}_q - \mathbf{X}_p = \mathbf{v} = \mathbf{X}_{q'} - \mathbf{X}_{p'} = \overrightarrow{p'q'} .$$



Euclidean 3D Space

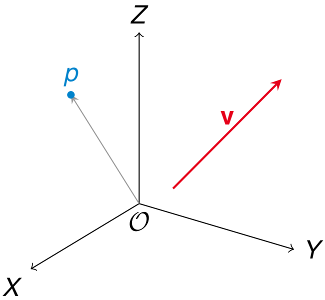
Points and Vectors

The definition of a coordinate system allows to describe points and vectors in space using coordinates.

Now it is possible to calculate

- distances between points,
- angles between vectors,
- length of curves,
- volumes of regions.

This requires a metric*.



Points and vectors are different geometric objects!

* A mathematical function that assigns a non-negative real value = distance to each two elements of a space.

Euclidean 3D Space

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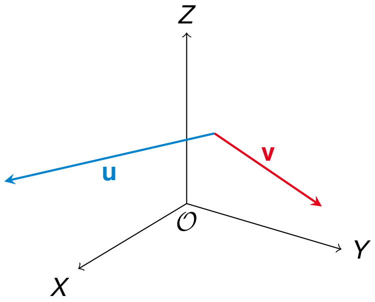
Scalar Product - distances

The metric of Euclidean space is defined via the scalar product between two vectors.

$$\mathbf{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}.$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\top \mathbf{v} = u_x v_x + u_y v_y + u_z v_z .$$



Thus, the Euclidean norm = length of a vector = distance between two points is given by

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{u_x^2 + u_y^2 + u_z^2} .$$

Example exam question: Why are distances between points on objects more robust features for describing the geometry of an object than the coordinates of the points?

Euclidean 3D Space

Scalar Product - angles

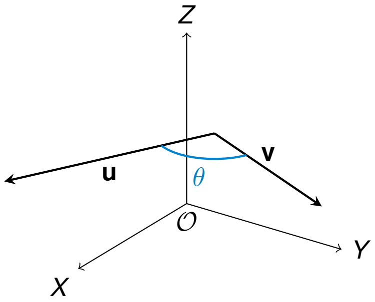
The angle between two vectors is given by the scalar product

$$\cos(\theta) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\sqrt{\langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle}} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} .$$

For vectors $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ it holds

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0 \, , \textit{if} \;\; \theta = 90^\circ \, ,$$

which means, the vectors are orthogonal to each other.



Euclidean 3D Space

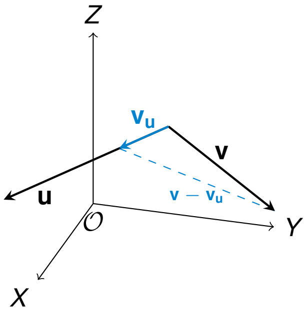
Scalar Product - projections

The orthogonal projection of vector $\mathbf{v}$ onto the direction of vector $\mathbf{u}$ is given by vector $\mathbf{v}_\mathbf{u}$. Vector $\mathbf{v} - \mathbf{v}_\mathbf{u}$ is orthogonal to $\mathbf{u}$.

$$\cos(\theta) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{\|\mathbf{v}_\mathbf{u}\|}{\|\mathbf{v}\|}\,.$$

This leads to:

$$\|\mathbf{v}_\mathbf{u}\| = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|}\,, \quad \mathbf{v}_\mathbf{u} = \|\mathbf{v}_\mathbf{u}\| \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}\,.$$



Euclidean 3D Space

Cross Product - surfaces

The cross product between two vectors leads to an orthogonal vector

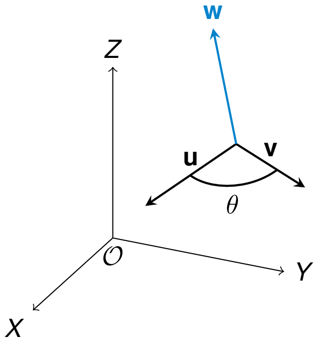
$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \widehat{\mathbf{u}}\mathbf{v} = \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix},$$

It holds: $\widehat{\mathbf{u}} = -\widehat{\mathbf{u}}^\top$, (skew symmetric)

$$\mathbf{u} \perp \mathbf{w} \;\wedge\; \mathbf{v} \perp \mathbf{w}.$$

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$$



Euclidean 3D Space

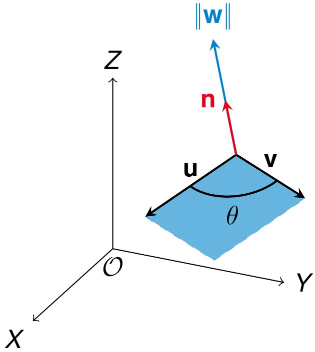
Cross-product - surfaces

The absolute value of the cross-product $\|\mathbf{w}\|$ equals the area of the parallelogram spanned by the vectors **u** and **v**

$$\|\mathbf{w}\| = \|\mathbf{u} \times \mathbf{v}\| = \|\widehat{\mathbf{u}}\mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin(\theta)\,.$$

The surface normal **n** is a unit vector:

$$\mathbf{n} = \mathbf{w}/\|\mathbf{w}\|\,, \quad \mathbf{u}\perp\mathbf{n} \wedge \mathbf{v}\perp\mathbf{n}\,.$$



Example exam question: You have modeled a surface in 3D space with a mesh of N triangles. How do you efficiently calculate the area using the cross product?

Euclidean 3D Space

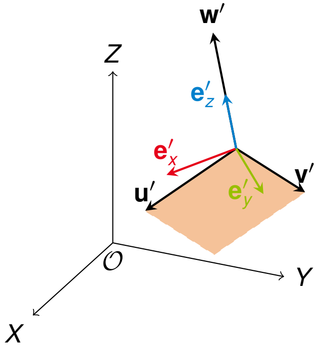
Planar 2D subspaces in 3D

We can define planar 2D subspaces in 3D spanned by two directions **u** and **v** including a local coordinate system given by the basis vectors $\mathbf{e}'_x$, $\mathbf{e}'_y$ and $\mathbf{e}'_z$. All vectors in that subspace reduce to 2D vectors in 3D space:

$$\mathbf{u}' = [u'_x, u'_y, 0]\,, \quad \mathbf{v}' = [v'_x, v'_y, 0]\,.$$

Now, the 2D determinant equals the absolute value of the cross-product:

$$\|\mathbf{u}' \times \mathbf{v}'\| = |\, \mathbf{u}', \mathbf{v}', \mathbf{e}'_z \,| = u'_x \cdot v'_y - u'_y \cdot v'_x\,.$$



Homogeneous Coordinates

2D position vectors

The coordinates of 2D position vectors can be mapped to homogeneous coordinates by multiplying each 2D component with a constant x_3 and adding a third dimension where this constant lives:

$$\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 x \\ x_3 y \\ x_3 \end{bmatrix} \in \mathbb{R}^3 .$$

The inverse (nonlinear) mapping from homogeneous coordinates to Euclidean coordinates reads as follows:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 / x_3 \\ x_2 / x_3 \end{bmatrix} \in \mathbb{R}^2 .$$

The origin $\bar{\mathbf{o}} = [0, 0, 0]^\top$ is not defined for homogeneous coordinates!

Questions: Why should c be chosen to be $x_3 = 1$? Why is the origin not defined?

Homogeneous Coordinates

2D direction vectors

The homogeneous coordinates of a 2D direction vector $\mathbf{v} \in \mathbb{R}^2$ are given as follows:

$$\overline{\mathbf{v}} = \overline{\mathbf{x}}_p - \overline{\mathbf{x}}_q = \begin{bmatrix} x_3(x_p - x_q) \\ x_3(y_p - y_q) \\ x_3 - x_3 \end{bmatrix} = \begin{bmatrix} x_3 v_x \\ x_3 v_y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} \in \mathbb{R}^3 .$$

The inverse mapping from homogeneous coordinates to Euclidean coordinates reads:

$$\mathbf{v} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \overline{v}_x \\ \overline{v}_y \end{bmatrix} \in \mathbb{R}^2 .$$

Now, position vectors and direction vectors can be differentiated by the last dimension. For position vectors $x_3 = 1$ and for direction vectors $v_3 = 0$!

Homogeneous Coordinates

Visualization

- Position vectors:

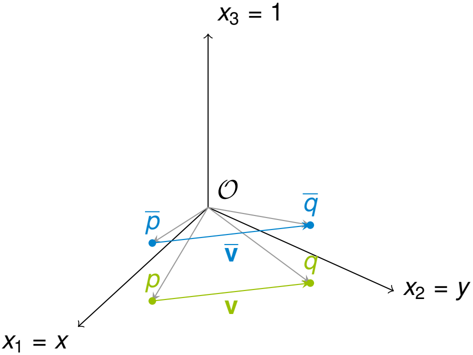
  $$\bar{\mathbf{p}} = \left[p_x, p_y, p_z\right]^\top \rightarrow \; p_z \neq 0 \,\forall\, \bar{\mathbf{p}}$$

- Positions at infinity:

  $$\bar{\mathbf{p}}_\infty = \left[p_x, p_y, 0\right]^\top \rightarrow \; p_z \overset{!}{=} 0 \,\forall\, \bar{\mathbf{p}}_\infty$$

- Direction vectors:

  $$\bar{\mathbf{v}} = \left[v_x, v_y, 0\right]^\top \rightarrow \; \bar{v}_z \overset{!}{=} 0 \,\forall\, \bar{\mathbf{v}}$$



Homogeneous Coordinates

Representation of 2D lines

2D line equation using homogeneous coordinates:

$$\mathbf{l}^\top \bar{\mathbf{x}} = 0\,, \quad \lambda(ax + by + c) = 0\,, \quad \mathbf{l} = \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix} \forall \lambda \neq 0\,.$$

The line equation has three parameters $\mathbf{l}$ but only two DoF ($\frac{a}{b}, \frac{c}{b}$).
The parameters are given by two different points on the line $\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2$

$$\mathbf{l} = \bar{\mathbf{x}}_1 \times \bar{\mathbf{x}}_2\,.$$

The intersection point of two lines is given by:

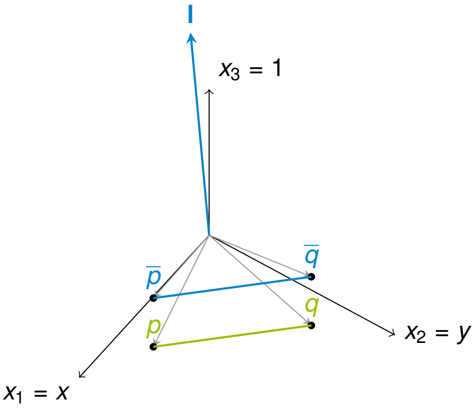
$$\bar{\mathbf{x}} = \mathbf{l}_1 \times \mathbf{l}_2\,.$$

Parallel lines intersect at infinity: $\bar{\mathbf{x}}_\infty = [b, -a, 0]^\top$.

Homogeneous Coordinates

Representation of 2D lines

Visualization of a line in a plane (in green) using homogeneous coordinates (in blue).



Representation of 2D lines - Duality

Duality principle for points and lines in the plane in homogeneous coordinates:

The terms

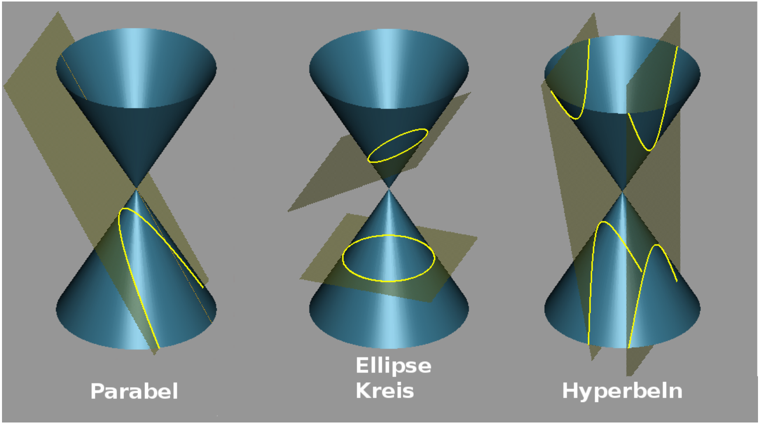
- point and line resp.
- connecting line of two points and intersection of two lines

can be interchanged in the following relations:

$$\mathbf{l}^{\top}\bar{\mathbf{x}} = 0 \quad \text{und} \quad \bar{\mathbf{x}}^{\top}\mathbf{l} = 0\,,$$

$$\mathbf{l} = \bar{\mathbf{x}}_1 \times \bar{\mathbf{x}}_2 \quad \text{und} \quad \bar{\mathbf{x}} = \mathbf{l}_1 \times \mathbf{l}_2\,.$$

Plane curves - Examples of conic sections



Question: What is the property of planes that create pairs of straight lines by intersecting a cone?

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Plane curves

Plane curves are conic sections and describe ellipses, parabolas, hyperbolas and pairs of straight lines in the plane. They are obtained by setting a quadratic function to zero:

$$\begin{aligned} ax^2 + bxy + cy^2 + dx + ey + f &= 0\,, \\ c_{11}x^2 + 2c_{12}xy + c_{22}y^2 + 2c_{13}x + 2c_{23}y + c_{33} &= 0\,, \end{aligned}$$

and are described by 5 points $\mathbf{x}_i$ with $i = 1 \dots 5$ in general position.
In homogeneous coordinates the following equivalent equation is obtained:

$$\bar{\mathbf{x}}^\top \mathbf{C} \bar{\mathbf{x}} = 0 \quad , \quad \text{mit} \qquad \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}\,,$$

where $\mathbf{C}$ is a symmetric matrix.

Plane curves

The type of curve is completely given by the following three quantities:

$$\Delta = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{vmatrix}, \quad \delta = \begin{vmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{vmatrix}, \quad s = c_{11} + c_{22}.$$

The following real curves result:

- $\delta > 0$ and $s\Delta < 0$: ellipse, if $c_{11} = c_{22}$ and $c_{12} = 0$: circle.
- $\delta < 0$ and $\Delta \neq 0$: hyperbole.
- $\delta = 0$ and $\Delta \neq 0$: parabola.
- $\delta < 0$ and $\Delta = 0$: pair of staight lines.
- $\delta = 0$ and $\Delta = 0$: parallel pair of straight lines.

Plane curves

If $\Delta \neq 0$ results in the center $\mathbf{m}$ to:

$$\mathbf{m} = \begin{bmatrix} \delta_2/\delta \\ \delta_3/\delta \end{bmatrix}, \quad \text{wobei} \quad \delta_2 = \begin{vmatrix} c_{12} & c_{13} \\ c_{22} & c_{23} \end{vmatrix}, \quad \delta_3 = \begin{vmatrix} c_{13} & c_{11} \\ c_{23} & c_{12} \end{vmatrix}.$$

If $c_{12} \neq 0$ the principal axes are not parallel to the coordinate axes but are rotated by an angle θ:

$$\tan(2\theta) = \frac{2c_{12}}{c_{11} - c_{22}}.$$

Each plane curve can be transformed to normal form $a^* x''^2 + b^* y''^2 + c^* = 0$ via a shift $\mathbf{x}' = \mathbf{x} - \mathbf{m}$ and a rotation $\mathbf{x}'' = \mathbf{R} = [\cos\theta, -\sin\theta; \sin\theta, \cos\theta]\,\mathbf{x}'$.

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Plane curves

A tangent $\mathbf{l}$ to the curve satisfies the equation

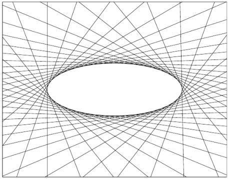
$$\mathbf{l} = \mathbf{C}\overline{\mathbf{x}}\,, \quad \text{da gelten muss} \quad \mathbf{l}^\top \overline{\mathbf{x}} = \overline{\mathbf{x}}^\top \mathbf{C}\overline{\mathbf{x}} = 0\,.$$

Thus a plane curve is given by 5 points as well as by 5 straight lines:

$$\overline{\mathbf{x}}^\top \mathbf{C}\overline{\mathbf{x}} = \underbrace{(\mathbf{C}^{-1}\mathbf{l})^\top}_{\overline{\mathbf{x}}^\top} \mathbf{C}\,\underbrace{(\mathbf{C}^{-1}\mathbf{l})}_{\overline{\mathbf{x}}} = \mathbf{l}^\top \mathbf{C}^{-1}\mathbf{l}\,, \quad \text{da} \quad \mathbf{C}^{-\top} = \mathbf{C}^{-1}\,.$$

Point-line duality of the conic section:

$$\begin{aligned} \overline{\mathbf{x}}^\top \mathbf{C}\overline{\mathbf{x}} &= 0\,, \\ \mathbf{l}^\top \mathbf{C}^{-1}\mathbf{l} &= 0\,. \end{aligned}$$



Scene Representation

3D positions & directions

3D positions in homogeneous coordinates:

$$\bar{\mathbf{X}} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} X_4 X \\ X_4 Y \\ X_4 Z \\ X_4 \end{bmatrix} \in \mathbb{R}^4, \quad \mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_1/X_4 \\ X_2/X_4 \\ X_3/X_4 \end{bmatrix} \in \mathbb{R}^3 .$$

3D direction vectors in homogeneous coordinates $X_4 = 1$:

$$\bar{\mathbf{v}} = \bar{\mathbf{x}}_p - \bar{\mathbf{x}}_q = \begin{bmatrix} X_p - X_q \\ Y_p - Y_q \\ Z_p - Z_q \\ 1 - 1 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ \\ \\ 0 \end{bmatrix} \in \mathbb{R}^4 .$$

Scene Representation

3D line & plane

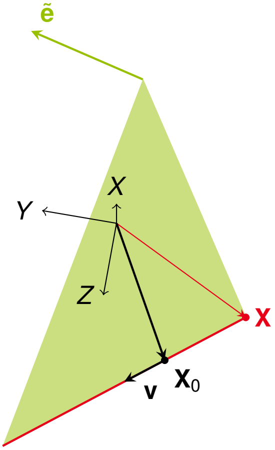
A line in homogeneous coordinates:

$$\overline{\mathbf{X}} = \overline{\mathbf{X}}_0 + \mu\overline{\mathbf{v}} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} v_x \\ v_y \\ v_z \\ 0 \end{bmatrix}, \ \mu \in \mathbb{R}.$$

A plane in homogeneous coordinates:

$$\begin{aligned} \tilde{\mathbf{e}}^\top(\mathbf{X} - \mathbf{X}_0) = \quad 0 \quad &= \tilde{e}_1 X + \tilde{e}_2 Y + \tilde{e}_3 Z \underbrace{-\tilde{\mathbf{e}}^\top \mathbf{X}_0} \\ \mathbf{e}^\top \overline{\mathbf{X}} = \quad 0 \quad &= e_1 X + e_2 Y + e_3 Z + e_4 . \end{aligned}$$

A 3D plane has 4 parameters but 3 degrees of freedom.



Scene Representation

3D points & plane

3D points and 3D planes are dual:

$$\mathbf{e}^\top \overline{\mathbf{X}} = 0 \quad \text{or} \quad \overline{\mathbf{X}}^\top \mathbf{e} = 0 \,.$$

Three points $\overline{\mathbf{X}}_i$, $i = 1 \ldots 3$ define a plane:

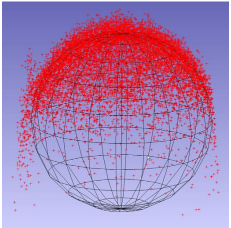
$$\begin{bmatrix} \overline{\mathbf{X}}_1^\top \\ \overline{\mathbf{X}}_2^\top \\ \overline{\mathbf{X}}_3^\top \end{bmatrix} \mathbf{e} = \mathbf{0} \,.$$

The intersection of three planes $\mathbf{e}_i$, $i = 1 \ldots 3$ defines a point:

$$\begin{bmatrix} \mathbf{e}_1^\top \\ \mathbf{e}_2^\top \\ \mathbf{e}_3^\top \end{bmatrix} \overline{\mathbf{X}} = \mathbf{0} \,.$$

Scene Representation

Application - 3D Reconstruction



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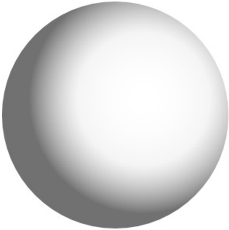


Two possible solutions:

- Fitting the model into a measured point cloud via adjustment theory. (e.g. method of least squares or RANSAC)
- Simultaneous estimation of the point cloud and the model parameters.

Scene Representation

Application - 3D Reconstruction

	Top Face	Front Face
α_x	$+48.48°$	$-36.92°$
α_y	$+8.44°$	$+15.11°$
z_a	496.52mm	498.76mm

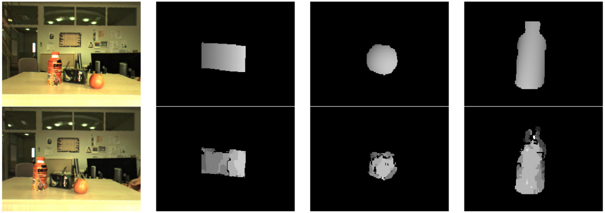
(a) Box

	Ground Truth	Estimated
r	49.30mm	53.98mm
z_a	$\sim$ 525mm	542.19mm

(b) Ball

	Ground Truth	Estimated
r	35.00mm	35.40mm
z_a	$\sim$ 540mm	542.84mm
α_x	—	$-23.64°$
α_z	—	$+0.30°$

(c) Can



(a) Office scene (b) Box disparity (c) Apple disparity (d) Bottle disparity

Quelle: Direct Surface Fitting, VISAPP 2010.