Task 22

We want to design a state feedback controller for the LTI SISO system

$$\dot{\underline{x}} = \begin{bmatrix} 1 & -4 \\ 2 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x}$$

<u>Solution:</u>

a) In Taks 20 on Ex. Sheet 8 we showed that the system is completely controllable. As a consequence, we can move all eigenvalues of the closed-loop system to arbitrary places by a state feedback controller.

b) Design of state feedback controller such that the closed-loop eigenvalues $\lambda_{C,1} = \lambda_{C,2} = -5$ result:

Design equation:

$$\det\left(\lambda \underline{I} - \underline{A} + \underline{b}\underline{k}^T\right) = (\lambda + 5)^2$$

$$\Leftrightarrow \quad \det\begin{pmatrix} \lambda - 1 + k_1 & 4 + k_2 \\ k_1 - 2 & \lambda + 3 + k_2 \end{pmatrix} = \lambda^2 + 10\lambda + 25$$

$$\Leftrightarrow \quad \lambda^2 + (2 + k_1 + k_2)\lambda + (k_2 - k_1 + 5) = \lambda^2 + 10\lambda + 25$$

Equating the coefficients:

$$\Rightarrow \quad \begin{aligned} 2 + k_1 + k_2 &= 10 \\ k_2 - k_1 + 5 &= 25 \end{aligned}$$

Solution of this linear system of two equations:

$$k_1 = -6 \text{ and } k_2 = 14$$

$$\Rightarrow \quad \underline{k}^T = \begin{bmatrix} -6 & 14 \end{bmatrix}$$

Thus, the state feedback control law we are looking for is

$$u = -\underline{k}^T \underline{x} = -\begin{bmatrix} -6 & 14 \end{bmatrix} \underline{x} = 6x_1 - 14x_2$$

c) In principle, we can move the closed-loop eigenvalues to $\lambda_{C,1} = \lambda_{C,2} = +5$ (see solution to subtask a)). However, the resulting closed-loop would be unstable such that this choice of eigenvalues does not make any sense.

Task 23:

The resulting state equations from Task 1 are (see solution to Ex. Sheet 1):

$$\dot{x}_1 = -\frac{1}{L}x_2 + \frac{1}{L}u_1 - \frac{1}{L}u_2$$

$$\dot{x}_2 = \frac{1}{C}x_1 - \frac{1}{RC}x_2 - \frac{1}{RC}u_2$$

$$y = \frac{1}{R}x_2 + \frac{1}{R}u_2$$

We want to find the state feedback law acting on u_1 on the condition that $u_2 = 0$ such that desired closed-loop eigenvalues $\lambda_{C,1}$ and $\lambda_{C,2}$ result. The control law must be stated in general form depending on the plant parameters and on the particular choice of $\lambda_{C,1}$ and $\lambda_{C,2}$.

Solution:

a)

$$u_2 = 0 \;\Rightarrow\quad \underline{\dot{x}} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}\underline{x} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}u_1$$

$$y = \begin{bmatrix} 0 & \frac{1}{R} \end{bmatrix}\underline{x}$$

Controllability:

$$\underline{Q}_c = [\underline{b} \quad \underline{A}\underline{b}] = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{LC} \end{bmatrix} \Rightarrow \det\left(\underline{Q}_c\right) = \frac{1}{L^2C} > 0,\ \text{ since } L \text{ and } C \text{ are positive parameters}$$

→ Even when using only the input u_1, the system is completely controllable (for all positive values of the electric component parameters).

d) Design of state feedback controller such that the closed-loop eigenvalues $\lambda_{C,1}$ and $\lambda_{C,2}$ result:

Design equation:

$$\det(\lambda\underline{I} - \underline{A} + \underline{b}\underline{k}^T) = (\lambda - \lambda_{C,1})(\lambda - \lambda_{C,2})$$

$$\Leftrightarrow \quad \det\begin{pmatrix} \lambda + \frac{k_1}{L} & \frac{1}{L} + \frac{k_2}{L} \\ -\frac{1}{C} & \lambda + \frac{1}{RC} \end{pmatrix} = \lambda^2 + (-\lambda_{C,1} - \lambda_{C,2})\lambda + \lambda_{C,1}\lambda_{C,2}$$

$$\Leftrightarrow \quad \lambda^2 + (\frac{1}{RC} + \frac{k_1}{L})\lambda + (\frac{k_2}{LC} + \frac{k_1}{RLC} + \frac{1}{LC}) = \lambda^2 + (-\lambda_{C,1} - \lambda_{C,2})\lambda + \lambda_{C,1}\lambda_{C,2}$$

Equating the coefficients:

$$\Rightarrow \qquad \frac{1}{RC} + \frac{k_1}{L} = -\lambda_{C,1} - \lambda_{C,2}$$
$$\frac{k_2}{LC} + \frac{k_1}{RLC} + \frac{1}{LC} = \lambda_{C,1}\lambda_{C,2}$$

Solution of this linear system of two equations:

$$k_1 = L(-\lambda_{C,1} - \lambda_{C,2} - \frac{1}{RC}) \text{ and } k_2 = LC\lambda_{C,1}\lambda_{C,2} + \frac{L}{R}\left(\lambda_{C,1} + \lambda_{C,2}\right) + \frac{L}{R^2C} - 1$$

$$\Rightarrow \qquad \underline{k}^T = \left[L(-\lambda_{C,1} - \lambda_{C,2} - \frac{1}{RC}) \quad LC\lambda_{C,1}\lambda_{C,2} + \frac{L}{R}\left(\lambda_{C,1} + \lambda_{C,2}\right) + \frac{L}{R^2C} - 1\right]$$

Thus, the state feedback control law we are looking for is

$$u = -\underline{k}^T\underline{x} = -\left[L(-\lambda_{C,1} - \lambda_{C,2} - \frac{1}{RC}) \quad LC\lambda_{C,1}\lambda_{C,2} + \frac{L}{R}\left(\lambda_{C,1} + \lambda_{C,2}\right) + \frac{L}{R^2C} - 1\right]\underline{x}$$