Task 17

We consider the LTI SISO system

$$\dot{\underline{x}} = \begin{bmatrix} -1 & 5 \\ 7 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x}$$

a) The system is <u>not</u> asymptotically stable, since the eigenvalues of the system matrix are $\lambda_1 = 4$ and $\lambda_2 = -8$ (see Task 11 on Exercise Sheet 3), i.e. one of the eigenvalues is positive.

b) The total solution of the initial value problem is given by the formula

$$\underline{x}(t) = \underbrace{\underline{e}^{At} \underline{x}_0}_{\underline{x}_h(t)} + \underbrace{\int_0^t \underline{e}^{A(t-\tau)} \underline{b} u(\tau) d\tau}_{\underline{x}_i(t)}$$

<u>1st step:</u> Response to the initial state $\underline{x}_h(t)$

Using the result of Task 12 on Exercise Sheet 3 we get

$$\underline{e}^{At} = \underline{e}^{\begin{bmatrix} -1 & 5 \\ 7 & -3 \end{bmatrix} t} = \frac{1}{12} \begin{bmatrix} 7e^{4t} + 5e^{-8t} & 5(e^{4t} - e^{-8t}) \\ 7(e^{4t} - e^{-8t}) & 5e^{4t} + 7e^{-8t} \end{bmatrix}$$

such that $\underline{x}_h(t) = \underline{e}^{At} \underline{x}_0$ can be written as

$$\underline{x}_h(t) = \underline{e}^{At} \underline{x}_0 = \frac{1}{12} \begin{bmatrix} 7e^{4t} + 5e^{-8t} & 5(e^{4t} - e^{-8t}) \\ 7(e^{4t} - e^{-8t}) & 5e^{4t} + 7e^{-8t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 19e^{4t} + 5e^{-8t} \\ 19e^{4t} - 7e^{-8t} \end{bmatrix}$$

<u>2nd step:</u> Response to the input $\underline{x}_i(t)$

$$\underline{x}_i(t) = \int_0^t \underline{e}^{A(t-\tau)} \underline{b} u(\tau) d\tau = \underline{e}^{At} \int_0^t \underline{e}^{-A\tau} \underline{b} u(\tau) d\tau =$$

$$= \underline{e}^{At} \int_0^t \underline{e}^{-A\tau} \underline{b} s(\tau - 2\mathrm{sec}) d\tau =$$

$$= \underline{e}^{At} \int_0^2 \underline{e}^{-A\tau} \underline{b} \cdot 0 \, d\tau + s(t - 2\mathrm{sec}) \underline{e}^{At} \int_2^t \underline{e}^{-A\tau} \underline{b} d\tau =$$

$$= s(t - 2\mathrm{sec}) \underline{e}^{At} \int_2^t \underline{e}^{-\begin{bmatrix} -1 & 5 \\ 7 & -3 \end{bmatrix} \tau} \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau$$

Using

$$\underline{e}^{\begin{bmatrix}-1 & 5\\ 7 & -3\end{bmatrix}(-\tau)} = \frac{1}{12}\begin{bmatrix}7e^{-4\tau}+5e^{8\tau} & 5(e^{-4\tau}-e^{8\tau})\\ 7(e^{-4\tau}-e^{8\tau}) & 5e^{-4\tau}+7e^{8\tau}\end{bmatrix}$$

we obtain

$$\underline{x}_i(t) = \frac{1}{12}\mathrm{s}(t-2\mathrm{sec})\underline{e}^{\underline{A}t}\int_2^t \begin{bmatrix}7e^{-4\tau}+5e^{8\tau}\\ 7\left(e^{-4\tau}-e^{8\tau}\right)\end{bmatrix}d\tau =$$

$$= \frac{1}{12}\mathrm{s}(t-2\mathrm{sec})\underline{e}^{\underline{A}t}\begin{bmatrix}-\frac{7}{4}e^{-4\tau}+\frac{5}{8}e^{8\tau}\\ -\frac{7}{4}e^{-4\tau}-\frac{7}{8}e^{8\tau}\end{bmatrix}_2^t =$$

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$$= \frac{1}{12}\mathrm{s}(t-2\mathrm{sec})\underline{e}^{\underline{A}t}\begin{bmatrix}-\frac{7}{4}e^{-4t}+\frac{5}{8}e^{8t}+\frac{7}{4}e^{-8}-\frac{5}{8}e^{16}\\ -\frac{7}{4}e^{-4t}-\frac{7}{8}e^{8t}+\frac{7}{4}e^{-8}+\frac{7}{8}e^{16}\end{bmatrix} =$$

$$= \frac{1}{12^2}\mathrm{s}(t-2\mathrm{sec})\begin{bmatrix}7e^{4t}+5e^{-8t} & 5\left(e^{4t}-e^{-8t}\right)\\ 7\left(e^{4t}-e^{-8t}\right) & 5e^{4t}+7e^{-8t}\end{bmatrix}\cdot$$

$$\begin{bmatrix}-\frac{7}{4}e^{-4t}+\frac{5}{8}e^{8t}+\frac{7}{4}e^{-8}-\frac{5}{8}e^{16}\\ -\frac{7}{4}e^{-4t}-\frac{7}{8}e^{8t}+\frac{7}{4}e^{-8}+\frac{7}{8}e^{16}\end{bmatrix} =$$

$$= \frac{1}{16}\mathrm{s}(t-2\mathrm{sec})\begin{bmatrix}\frac{7}{3}e^{4t-8}-\frac{5}{6}e^{-8t+16}-\frac{3}{2}\\ \frac{7}{3}e^{4t-8}+\frac{7}{6}e^{-8t+16}-\frac{7}{2}\end{bmatrix}$$

<u>3rd step:</u> Superimpose $\underline{x}_h(t)$ and $\underline{x}_i(t)$ to obtain total response

Using the results from the 1st and the 2nd step we finally get

$$\underline{x}(t) = \underline{x}_h(t)+\underline{x}_i(t) =$$

$$= \frac{1}{12}\begin{bmatrix}19e^{4t}+5e^{-8t}\\ 19e^{4t}-7e^{-8t}\end{bmatrix} + \frac{1}{16}\mathrm{s}(t-2\mathrm{sec})\begin{bmatrix}\frac{7}{3}e^{4t-8}-\frac{5}{6}e^{-8t+16}-\frac{3}{2}\\ \frac{7}{3}e^{4t-8}+\frac{7}{6}e^{-8t+16}-\frac{7}{2}\end{bmatrix}$$

$$y(t) = x_2(t) = \frac{1}{12}(19e^{4t}-7e^{-8t}) + \frac{1}{16}\mathrm{s}(t-2\mathrm{sec})\left(\frac{7}{3}e^{4t-8}+\frac{7}{6}e^{-8t+16}-\frac{7}{2}\right)$$