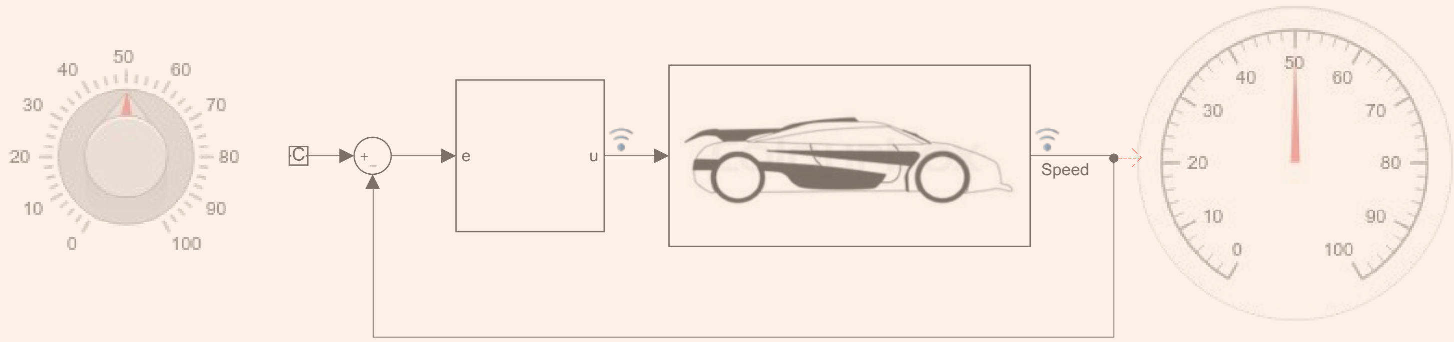
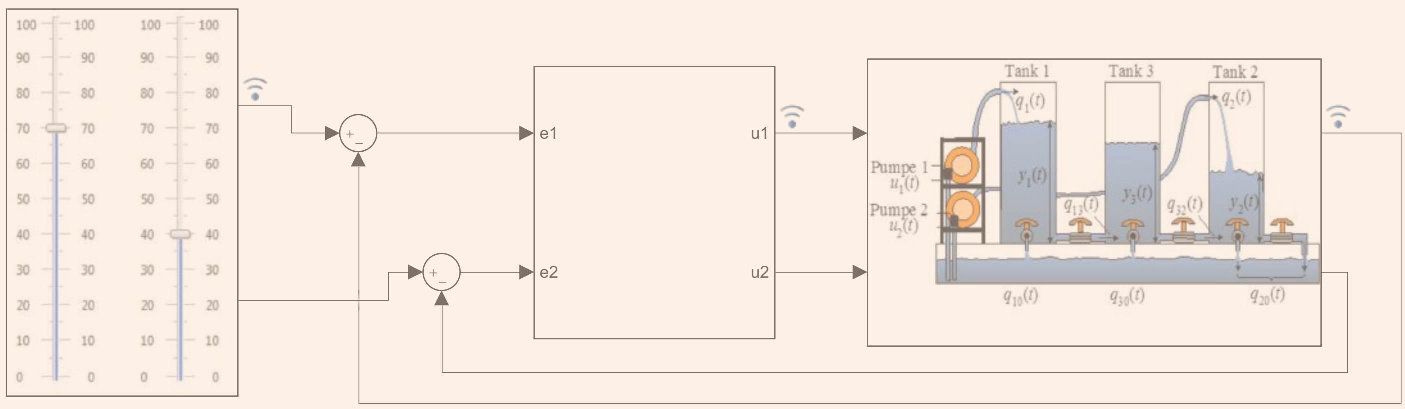


Control Systems I

Mechatronics

Prof Dr Abid Ali

SS 2021



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Attention: This document is not a complete documentation of the course. It is only an outline of the topics discussed during the lectures and should be used in combination with personal notes and literature mentioned in this document.

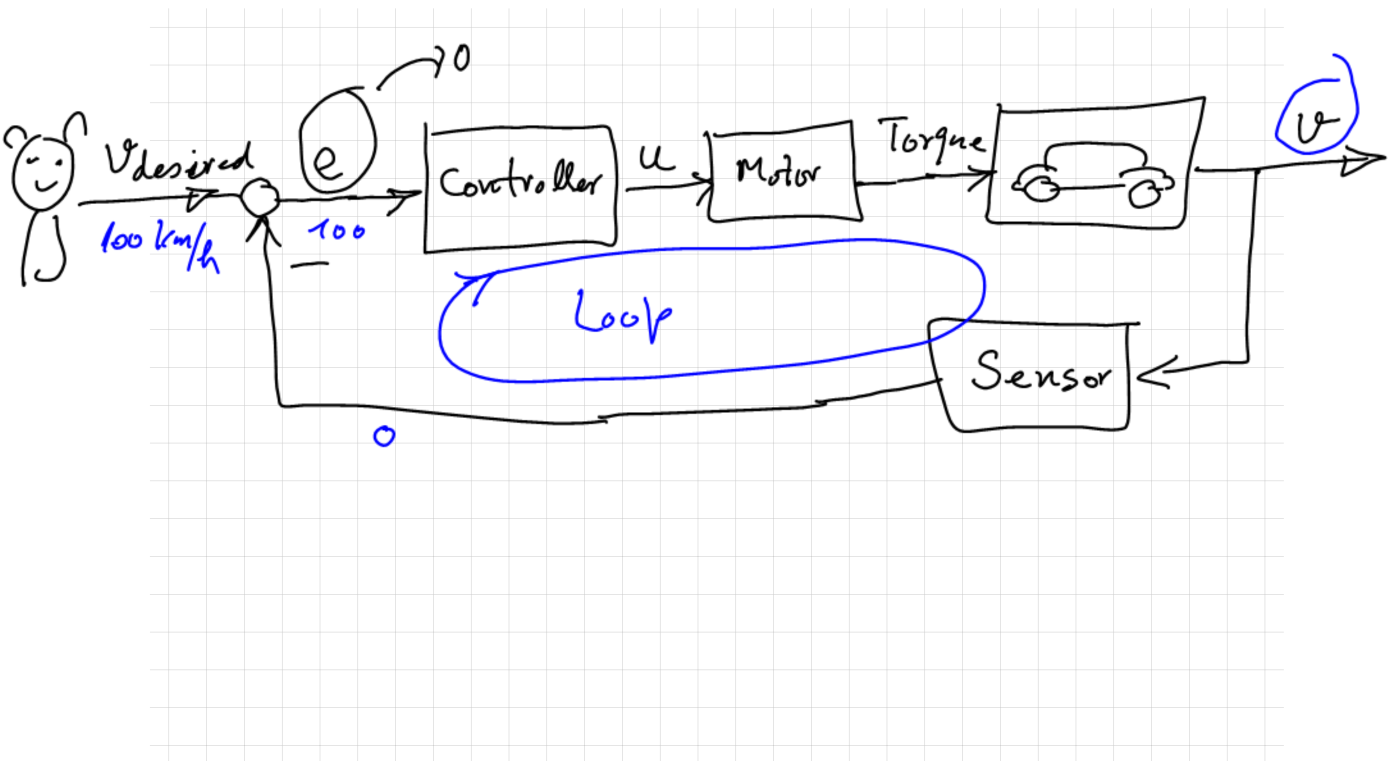
Summer semester 2020

Contents

1. Introduction

1.1 What is control?

1.1.1 Motivational Example



1.1.2 Formal Definition

Control is a mechanism, which manipulates the input of a system with the aim to bring or keep the output of the system to a desired value. The desired value, often known as reference signal or set-point, may be constant or a time-varying quantity. Control mechanisms can be divided into two categories: *feed-forward control* and *feedback control*.

1.1.3 Feed-forward control

As the name suggests, the feed-forward control or open-loop control does not use any feedback and tries to control the system using a static or dynamic inverse of the system model. This type of control performs poorly in the presence of disturbances and model uncertainties and fails absolutely for unstable plants.

1.1.4 Feedback control

A feedback control system continuously measures the output of the plant, compares it with the desired behaviour, computes the control action and manipulates the plant input in order to bring the plant output to the desired value.

The essential feature of such a control is the feedback loop. The difference between the desired value $r(t)$ and the actual plant output $y(t)$ is referred to as the control deviation $e(t)$. This deviation is used by the controller to calculate the proper value for the plant input.

1.2 Objectives of feedback control

1. Set-point tracking/servo control: The objective is to track the reference trajectories or varying set-points.

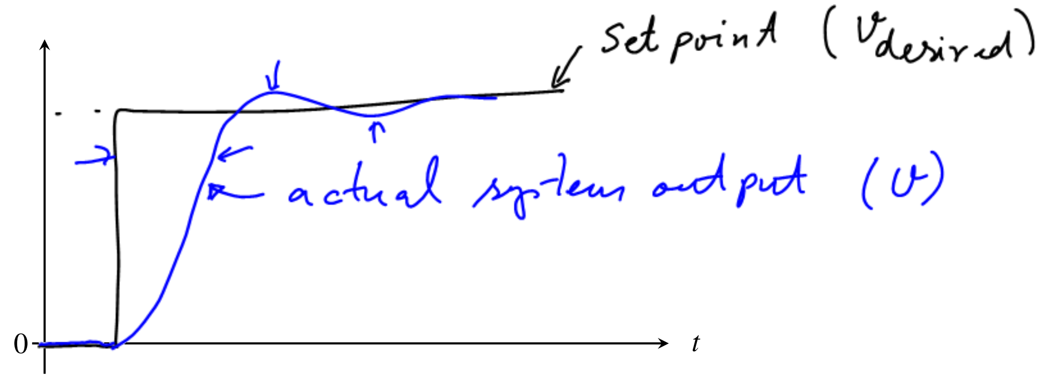


Figure 1.1: Set-point tracking

2. Regulation / disturbance rejection: Influence of the disturbances on the system output should be suppressed.

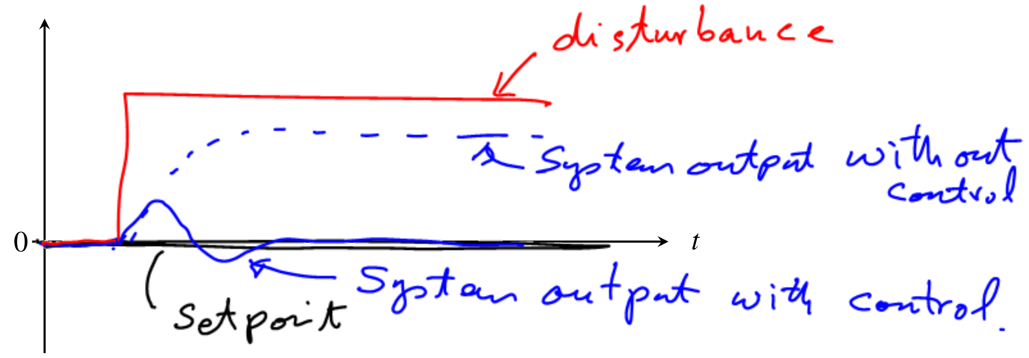


Figure 1.2: Disturbance rejection

A control system should react quickly to achieve these objectives as accurately as possible without oscillations.

1.3 Elements of a standard feedback control loop

The basic structure of a feedback control loop is given in Figure 1.3.

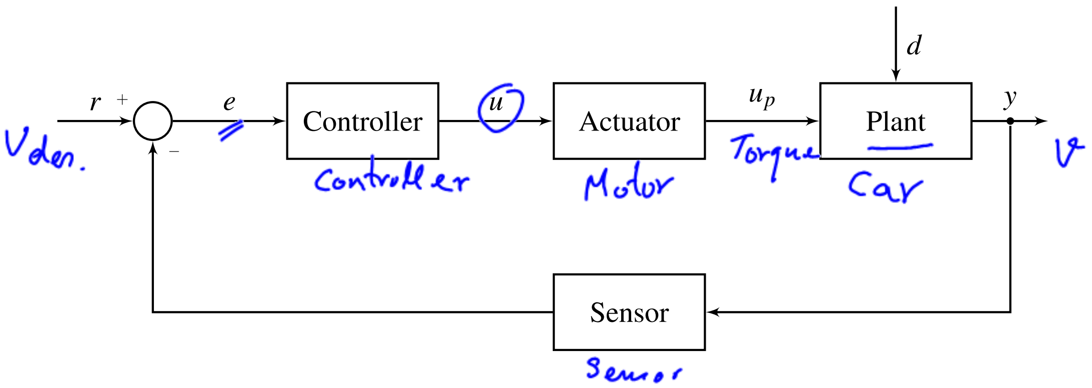


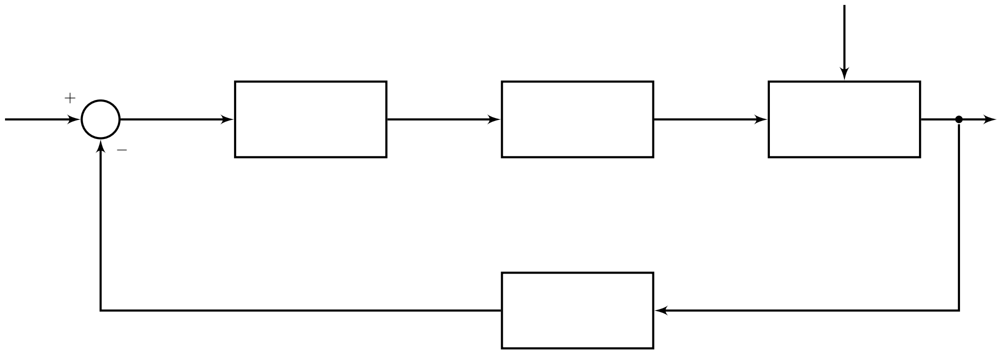
Figure 1.3: The basic structure of a standard control loop

r: Reference signal / desired value / set-point
e: Control deviation / control error
u: Controller output / control signal /control variable
u_p: Actuator output
d: Disturbance
y: Plant output / process output / controlled variable

R For the purpose of analysis and controller design, the blocks Actuator, Plant and Sensor are combined together to an extended plant model.

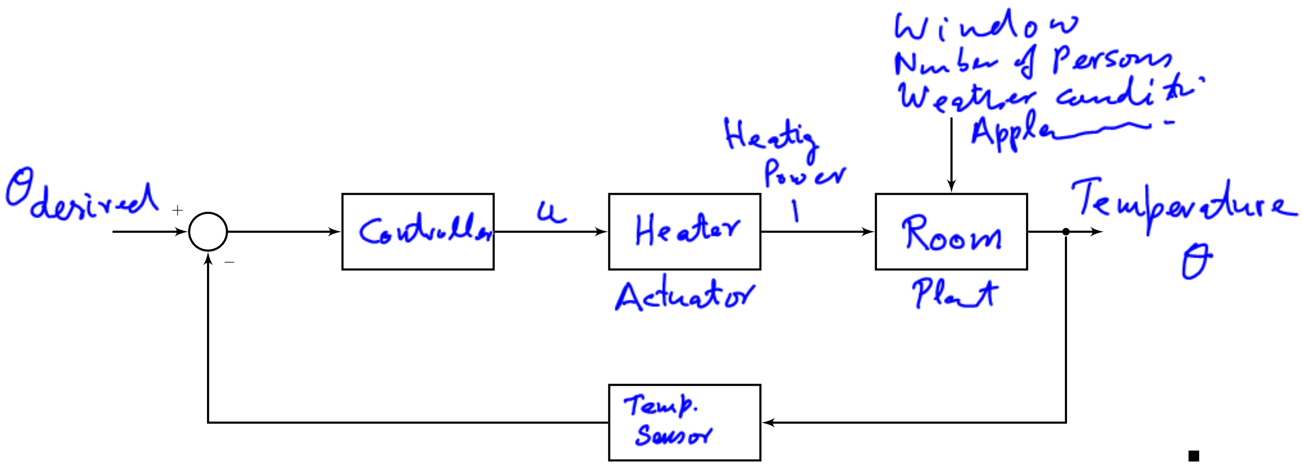
1.4 Examples of feedback control systems

■ **Example 1.1 — Vehicle speed control.**



■

■ **Example 1.2 — Room temperature control.**



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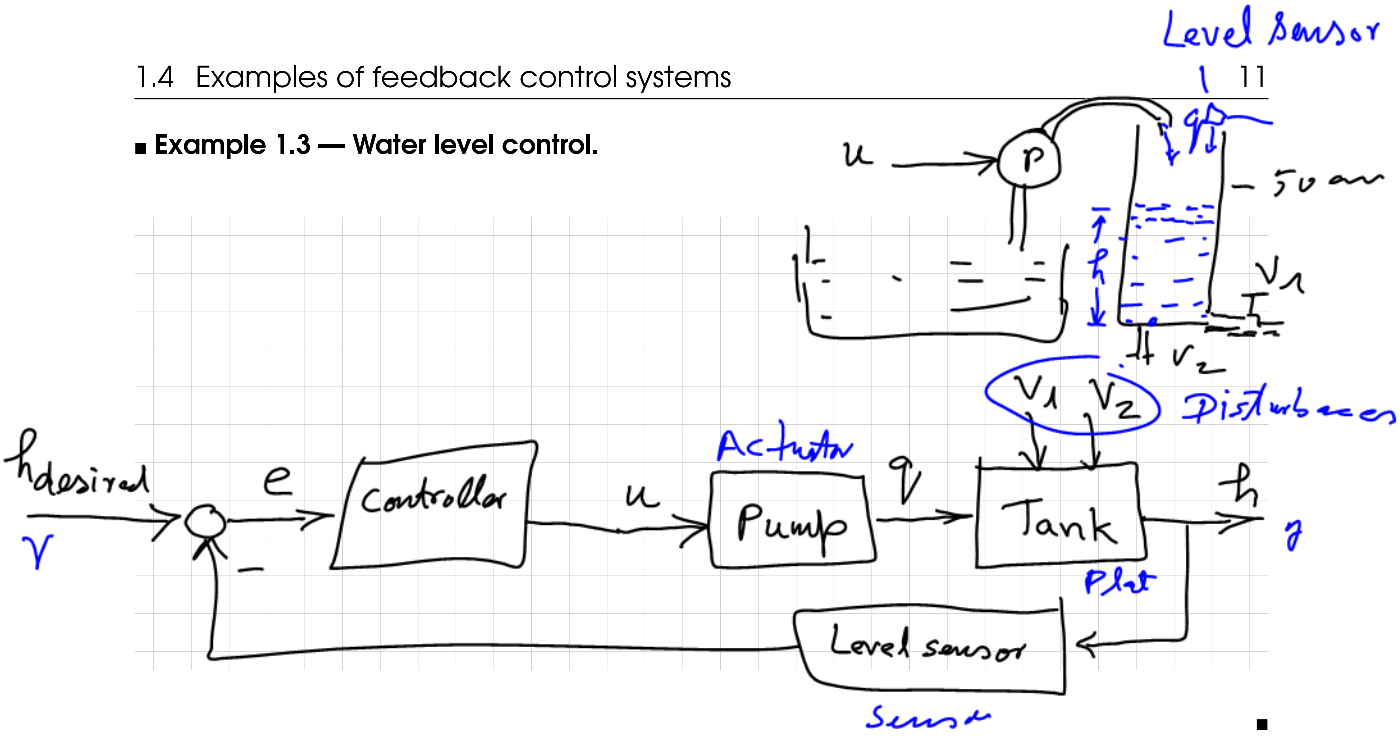
■ **Example 1.3 — Water level control.**



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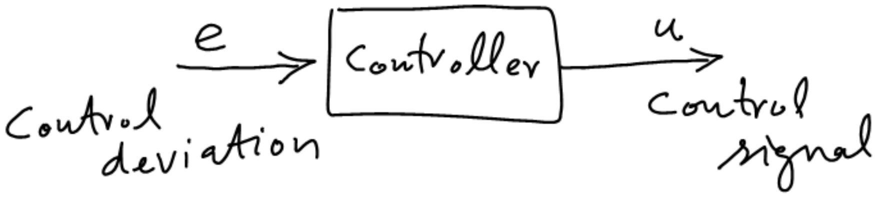
Exercise 1.1 — Think about at least two control systems and draw their block diagrams.

■

1.5 Classical controller components

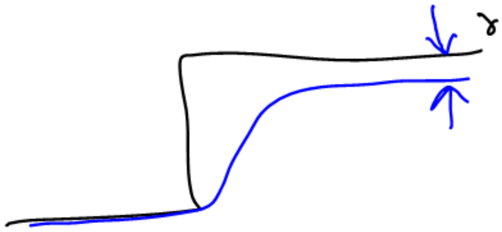


1.5.1 P Controller

The controller output u_P is directly proportional to control deviation e.

$$u_P(t) = K_P \cdot e(t)$$

$$u_p \sim e$$

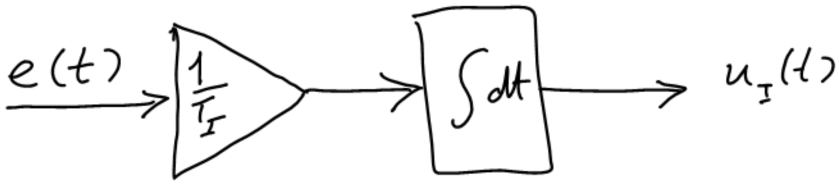


1.5.2 I Controller

The controller output u_I is directly proportional to the time integral of control deviation e.

$$u_I(t) = \frac{1}{T_I} \int_0^t e(\tau) d\tau$$

$$u_I(t) \sim \int e(t)\, dt$$

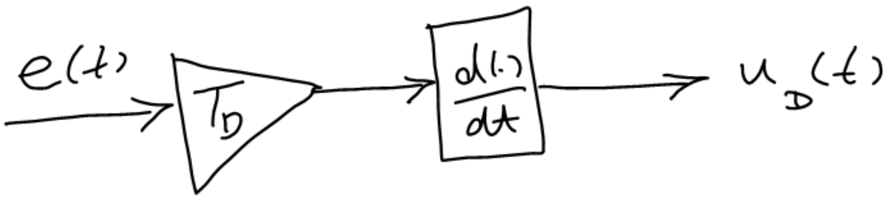


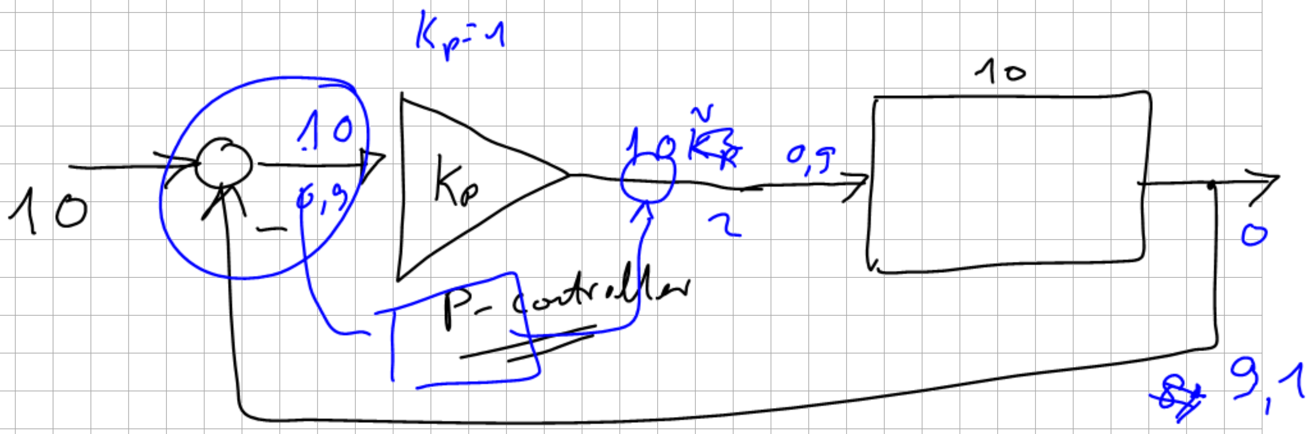
1.5.3 D Controller

The controller output u_D is directly proportional to the derivative of control deviation e.

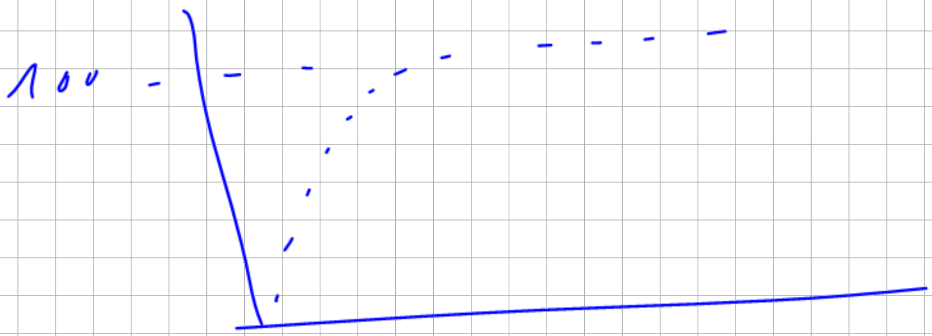
$$u_D(t) = T_D \cdot \frac{de(t)}{dt}$$

$$u_D \sim \frac{de}{dt}$$





Kp=1
10
10
0,9
Kp
10 ~~Kp~~
0,9
2
P-controller
0
9,1



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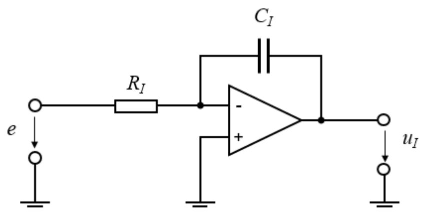
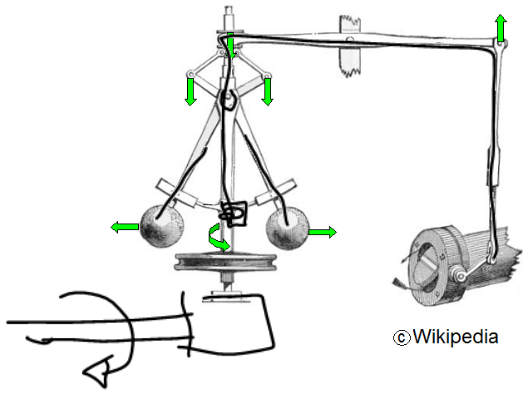
1.6 Controller implementation

1.6.1 Analog implementation

1. Electronic circuits involving operational amplifiers, resistor, capacitors etc.



2. Mechanical systems like centrifugal governor etc.



3. Fluid systems involving pneumatic and hydraulic components

1.6.2 Digital implementation

The control algorithm is implemented as software running on a microprocessor, micro-controller or digital signal processor.

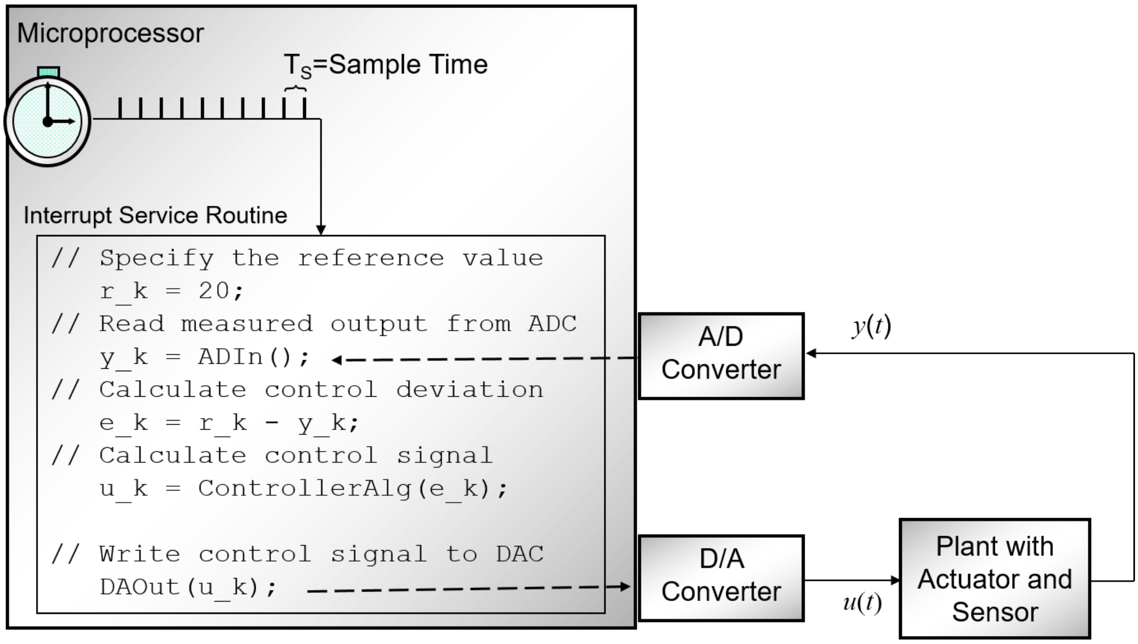


Figure 1.4: Basic mechanism of a digital controller

1.7 Controller types and design techniques

1. What is meant by controller types?
2. Which types of controllers are commonly used?
3. What is a control design technique?
4. Which information is needed as a pre-requisite of control design?

1.8 Job of a control engineer

- Modeling and identification of the plant/process
- Analysis and simulation of the plant model
- Controller design
- Implementation of control system
- Validation, test and tuning of control algorithm

1.9 Literature

1. K. J. Aström, R. M. Murray: *Feedback Systems*. Princeton University Press. Online: www.cds.caltech.edu/ murray/amwiki
2. R.C. Dorf, R.H. Bishop: *Modern Control Systems*. Pearson Education Inc.
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4. Unbehauen, H.: *Regelungstechnik I*. 14. Auflage. Vieweg-Verlag, Wiesbaden.
5. Föllinger, O.: Regelungstechnik, Hüthig Buch Verlag Heidelberg.
6. Lunze, J.: *Regelungstechnik 1*. Springer-Verlag, Berlin.

2.1 System dynamics

2.1.1 What is a system?

A system consists of elements or objects, which interact with each other as a part of a mechanism and are enclosed by a physical or an imaginary system boundary. The system theory is particularly interested in how a given system responds to external influences. The variables of a system, which can be changed directly from outside, are called **inputs** (or causes). The internal variables of a system, which represent the current state of the system are called *state variables*. Some of the internal variables of a system can be measured from outside. These variables which can be used to observe the effects of the input variables on system behaviour are known as **outputs**.

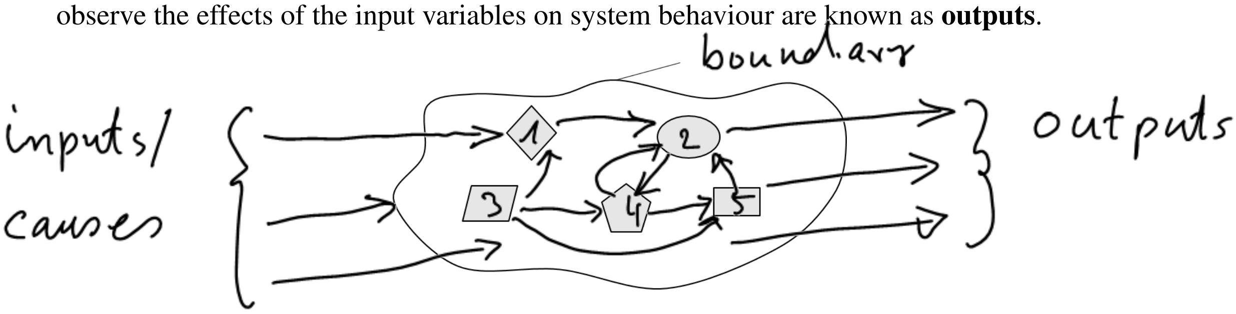


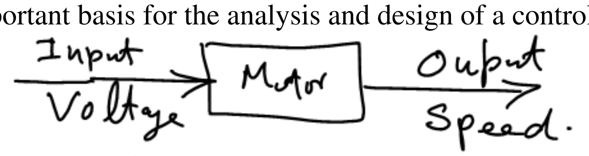
Figure 2.1: System definition

The *inputs* carry the *information* or influence from outside of the system *into the system*, while the *outputs* carry information *out of the system*. In the control and system theory we are primarily interested in the relationship between the inputs and outputs of a system.

2.1.2 What is a model?

A model is an abstracted representation of the reality. In control engineering and system theory, a description of the input-output relationship of a system is called a model. Often the model is formulated in the form of mathematical equations. Such model is called mathematical model. A model serves as a very important basis for the analysis and design of a control system.

example.



The model will describe a relationship between voltage & speed.

2.1.3 Static and dynamic systems

The input-output relationship of a *static system* can be described by an algebraic equation. Whereas the input-output behaviour of a *dynamic system* in time domain can not be described by an algebraic equation. We need differential equations in order to describe behaviour of such systems. The presence of energy-storing elements e.g. capacitors, inductors, springs, masses etc. in a system cause dynamic behaviour.

■ **Example 2.1** Consider a voltage divider without any load as shown in Figure 2.2. The output voltage u_a can be calculated as:

$$u_a(t) = \frac{R}{R+R} u_e(t) = 0.5 u_e(t) \qquad u_a(t) = f(u_e(t)) \tag{2.1}$$

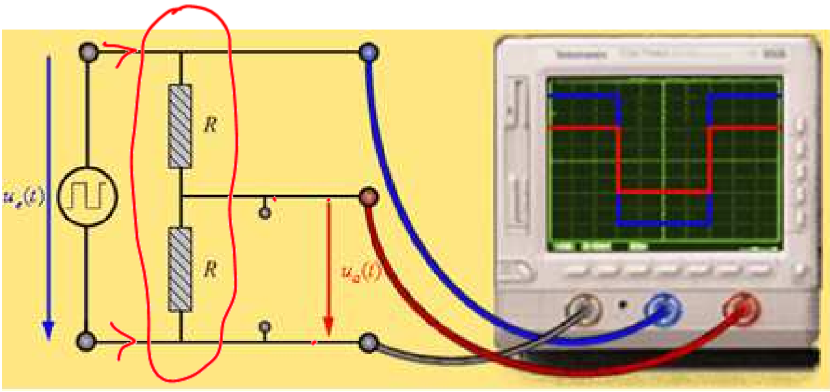


Figure 2.2: A voltage divider without load or with pure resistive load is a static system.

As the relationship between the input voltage $u_e(t)$ and the output voltage $u_a(t)$ is an algebraic equation, the system is static.

■

■ **Example 2.2** Now consider the voltage divider circuit with capacitive shown in Figure 2.3. As it is clear from the shape of the input and output signals the relationship between $u_e(t)$ and $u_a(t)$ can not be described by means of an algebraic equation. This system is a dynamic system.

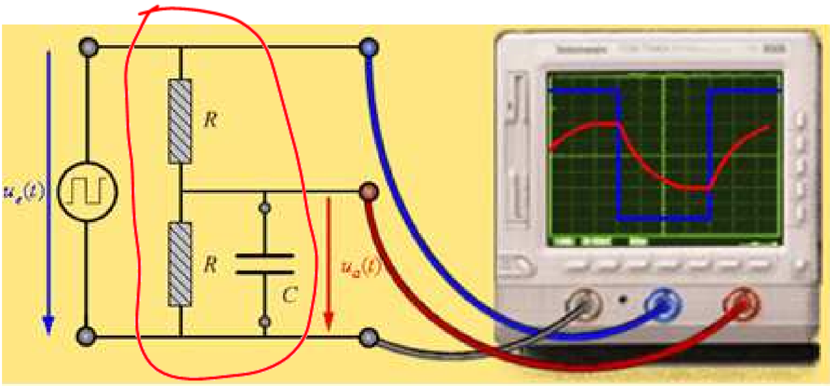
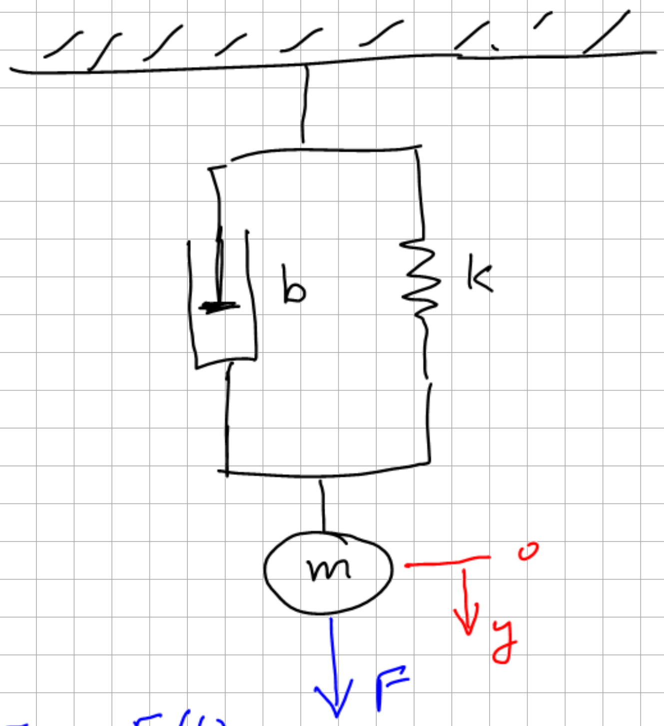


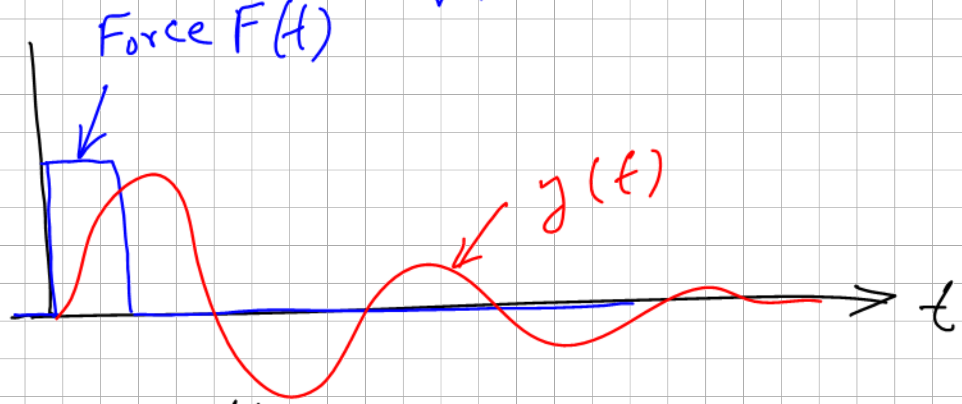
Figure 2.3: A voltage divider with a capacitive load is a dynamic system.

$u_a(t) \neq f(u_e(t))$

■

Example.





Input: $F(t)$
Output: $y(t)$

$$y(t) \neq f(F(t))$$

$\Rightarrow$ The System is not a static system

What kind of components cause dynamic behaviour?

Electrical.

C: can store energy $E_C = \frac{1}{2} C u_C^2$

L: $E_L = \frac{1}{2} L i^2$

Mechanical.

Spring: $E_s = \frac{1}{2} k x^2$

Mass:
- Potent. $\Rightarrow E_p = mgh$
- kinet. $\Rightarrow E_k = \frac{1}{2} m v^2$

2.2 Signals

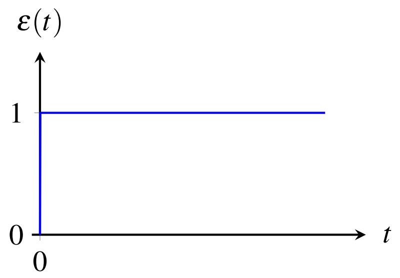
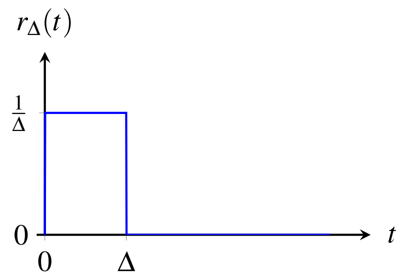
2.2.1 Introduction

A signal is the physical representation of information. It is usually a physical quantity which is represented as a function of time. In electrical engineering, the time-variant voltages and currents are usually the most common examples of signals. For control purposes, the physical quantities such as pressure, temperature, level, flow rate, speed, position, force, torque, angular velocity, frequency, voltage, current, power, concentration , pH value, light intensity etc. are measured by means of suitable measuring instruments known as sensors and are converted into electrical/electronic signals. Control signals generated by controllers are converted into physical quantities by using actuators.

2.2.2 Some useful signal models

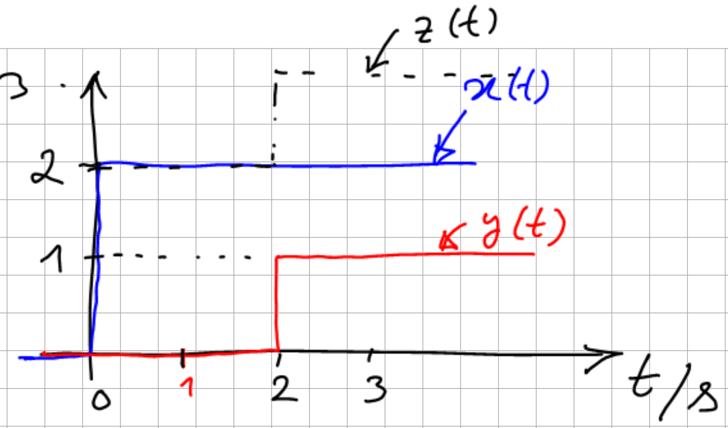
For the analysis of dynamic systems some basic signals play a very important role. These simple signals can not only be used as test signals but also provide a basis for the description of other signals. Some of these signal models are summarised in Table 2.1.

Table 2.1: Some useful signal models.

	Signal	**Graph**
1	Step function: $\varepsilon(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$	
2	Rectangular pulse function: $r_\Delta(t) = \begin{cases} \frac{1}{\Delta} & \text{for } 0 \leq t \leq \Delta \\ 0 & \text{otherwise} \end{cases}$	
3	Impulse function $\delta(t) = \lim\limits_{\Delta \to 0} r_\Delta(t)$	

Examples:

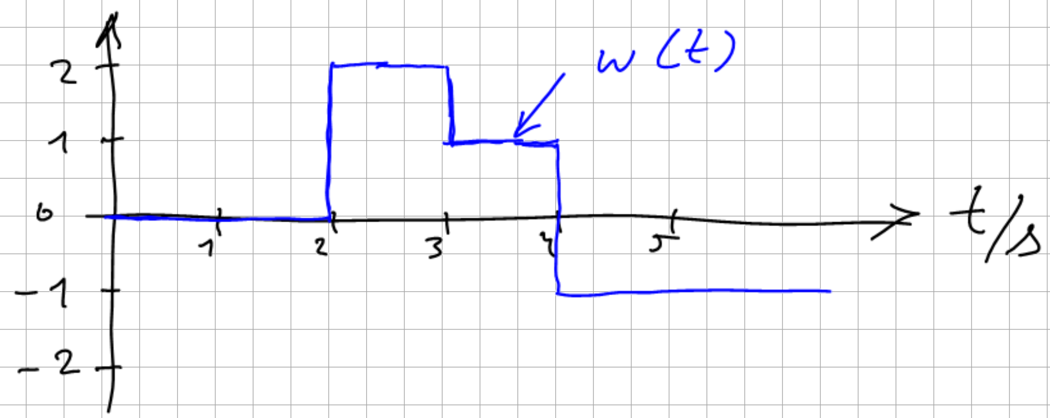
$$x(t) = 2\,\varepsilon(t)$$



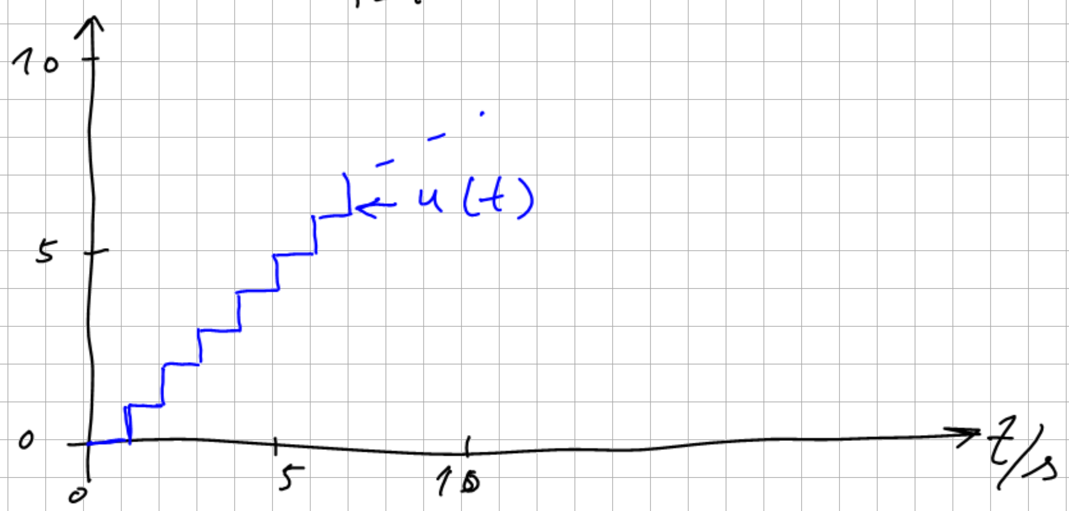
$$y(t) = \varepsilon(t-2)$$

$$z(t) = x(t) + y(t)$$
$$= 2\,\varepsilon(t) + \varepsilon(t-2)$$

Example 2: $w(t) = 2\,\varepsilon(t-2) - \varepsilon(t-3) - 2\,\varepsilon(t-4)$



Example 3: $u(t) = \sum_{i=1}^{\infty} \varepsilon(t-i)$



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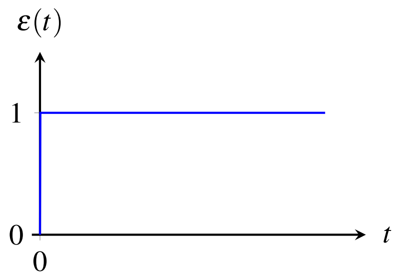
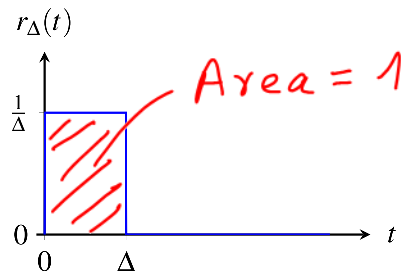
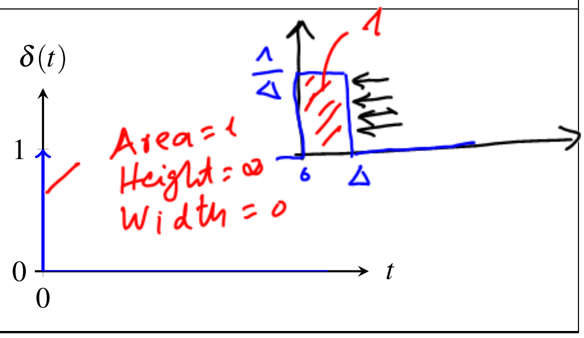
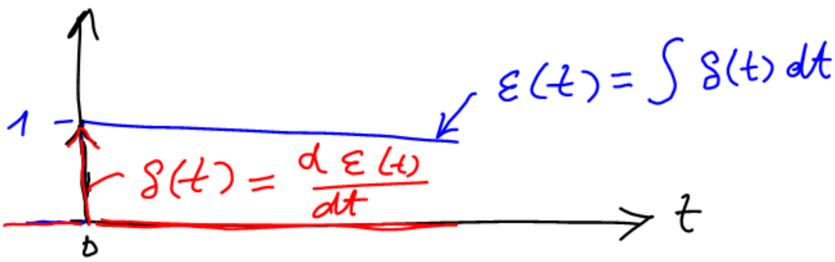
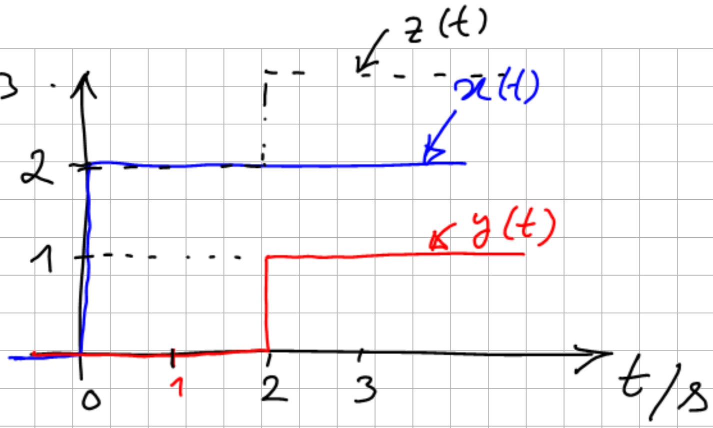
1	Step function: $\varepsilon(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$	
2	Rectangular pulse function: $r_\Delta(t) = \begin{cases} \frac{1}{\Delta} & \text{for } 0 \leq t \leq \Delta \\ 0 & \text{otherwise} \end{cases}$	
3	Impulse function $\delta(t) = \lim\limits_{\Delta \to 0} r_\Delta(t)$	



Examples:

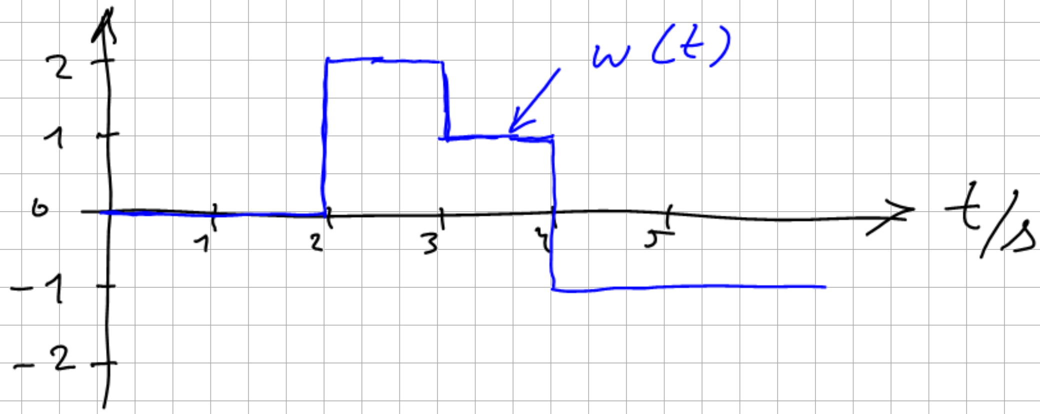
$$x(t) = 2\,\varepsilon(t)$$



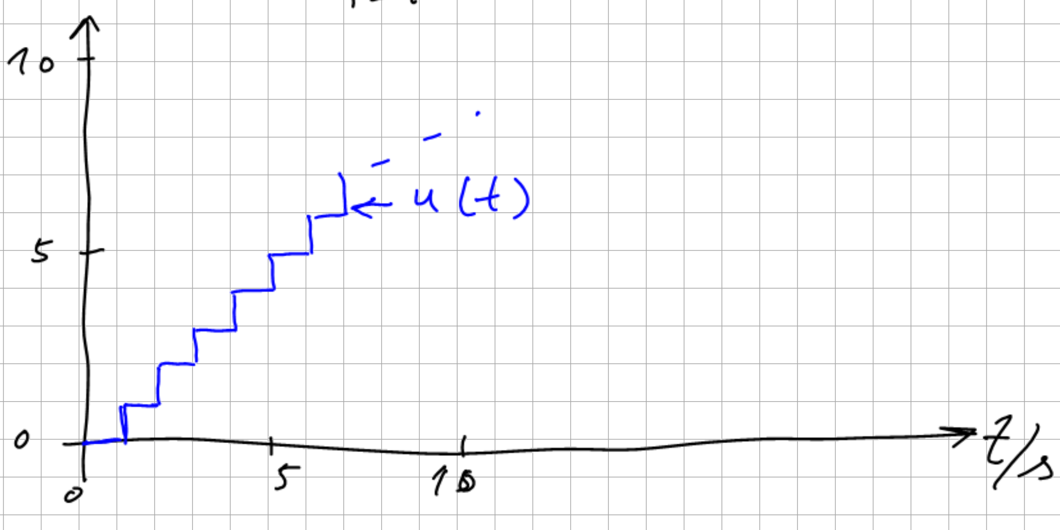
$$y(t) = \varepsilon(t-2)$$

$$z(t) = x(t) + y(t)$$
$$= 2\,\varepsilon(t) + \varepsilon(t-2)$$

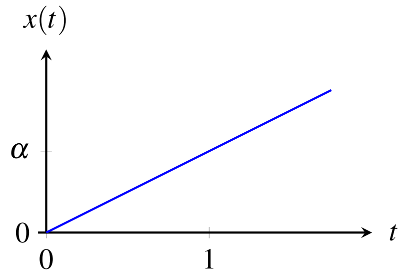
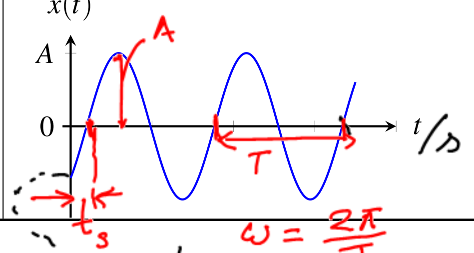
Example 2: $w(t) = 2\,\varepsilon(t-2) - \varepsilon(t-3) - 2\,\varepsilon(t-4)$



Example 3: $u(t) = \sum_{i=1}^{\infty} \varepsilon(t-i)$



Continuation of Table 2.1

	Signal	Graph
4	Ramp function $x(t) = \begin{cases} \alpha t & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$	
5	Sine wave $x(t) = \begin{cases} A\sin(\omega t + \phi) & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$ A: Amplitude ω: Angular frequency φ: Phase angle	

$\phi = -\omega \cdot t_s$ $\quad \omega = \frac{2\pi}{T}$

MATLAB® can be used to plot signals.

```
t=[-1:0.01:5];    % creates a time vector with Ts=0.01s.
u = t>0;          % generates a unit step function
r = (t>0)*2*t;    % generates a ramp function with slope 2 for t>0.
x = 10*sin(2*t);  % generates a sine function.
plot (t, x);      % plots the signal x against t.
```

Find out the significance of a semi-colon (;) at the end of a command.

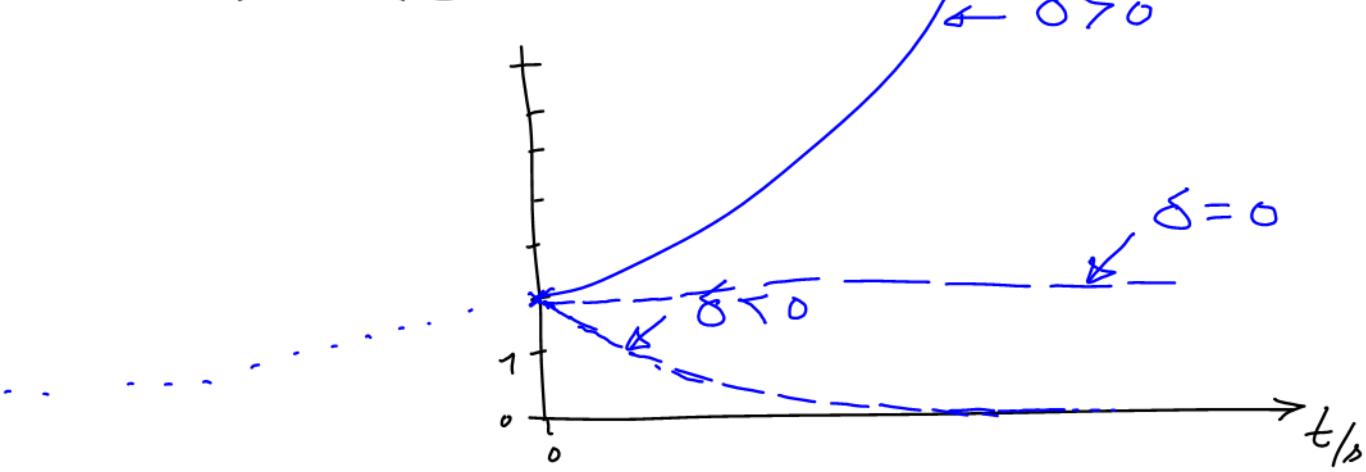
```
A = 1;
B = 2
```

Please find out the use of the following commands:

```
grid, xlabel, ylabel, legend, xlim, ylim, axis, hold
```

6. Exponential function

$x(t) = 2e^{\delta t}$



2.3 Laplace transform

2.3.1 Introduction

Definition 2.3.1 — Laplace Transform. The Laplace transform is a special integral transformation that transforms a signal $f(t)$ from time domain to complex frequency domain.

$$F(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t \tag{2.2}$$

$f(t)$: Original function in time domain,
$F(s)$: Laplace transform in s domain,
$s = \sigma + j\omega$: is a complex variable.

■ **Example 2.3** Function in time domain: $f(t) = e^{-3t}$
Laplace transform:

$$F(s) = \int_0^\infty e^{-st} e^{-3t}\mathrm{d}t = \int_0^\infty e^{-(s+3)t}\mathrm{d}t = \frac{1}{s+3}$$

■

Definition 2.3.2 — Inverse Laplace Transform. The inverse Laplace transform of the function $F(s)$ can be calculated by using the following integral:

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{st} F(s)\mathrm{d}s \tag{2.3}$$

where c describes the convergence range of the transformation.

■ **Example 2.4** Function in s-domain: $F(s) = \frac{1}{s^2}$
Inverse Laplace transform:

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{e^{st}}{s^2}\mathrm{d}s = t$$

■

The Laplace transform can be written in operator notation

$$F(s) = \mathscr{L}\{f(t)\} \qquad f(t) = \mathscr{L}^{-1}\{F(s)\} \tag{2.4}$$

$f(t) \circ\!\!-\!\!\bullet\, F(s)$

R For the calculation of Laplace transform, normally, it is not needed to compute these integrals explicitly. For a given function in time or s-domain the corresponding expression in the other domain can be determined by using a table of Laplace transforms (e.g. Table 2.2).

Table 2.2: Table of Laplace Transforms

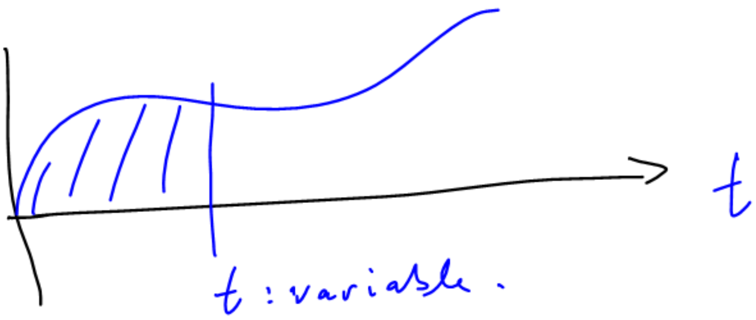
No.	$f(t)$ $(t \geq 0)$	$\mathscr{L}(f(t)) = F(s)$
1	$\delta(t)$	1
2	$\varepsilon(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	t^2	$\frac{2}{s^3}$
5	$t^n \ (n = 0, 1, 2, \dots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
9	$1 - e^{-at}$	$\frac{a}{s(s+a)}$
10	$\frac{1}{a^2}(e^{-at} - 1 + at)$	$\frac{1}{s^2(s+a)}$
11	$(1 - at)e^{-at}$	$\frac{s}{(s+a)^2}$
12	$\sin \omega_0 t$	$\frac{\omega_0}{s^2 + \omega_0^2}$
13	$\cos \omega_0 t$	$\frac{s}{s^2 + \omega_0^2}$
14	$e^{-at} \sin \omega_0 t$	$\frac{\omega_0}{(s+a)^2 + \omega_0^2}$
15	$e^{-at} \cos \omega_0 t$	$\frac{s+a}{(s+a)^2 + \omega_0^2}$
16	$\frac{e^{-at} - e^{-bt}}{a-b}$	$\frac{1}{(s+a)(s+b)}$
17	$\frac{ae^{-at} - be^{-bt}}{a-b}$	$\frac{s}{(s+a)(s+b)}$

2.3.2 Important properties of Laplace transform

Table 2.3: Properties of Laplace Transforms

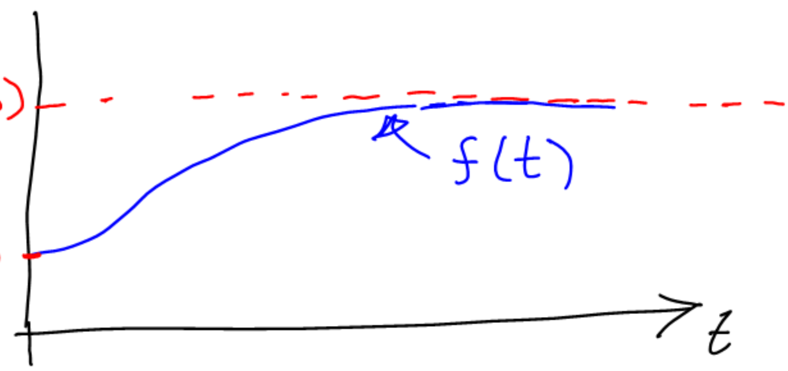
1	Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
2	Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
3	Time shifting	$f(t-a)$	$e^{-as} F(s)$
4	Frequency shifting	$e^{at} f(t)$	$F(s-a)$
5	Time differentiation	$f'(t)$	$sF(s) - f(0)$
6	Time integration	$\int_0^t f(\tau) d\tau$	$\frac{1}{s} F(s)$
7	Time convolution	$\int_0^t f_1(\tau) f_2(t-\tau) d\tau$	$F_1(s) F_2(s)$
8	Time multiplication	$f_1(t) f_2(t)$	$\int_0^s F_1(p) F_2(s-p) dp$
9	Initial value	$f(0^+) = \lim_{t \to 0^+} f(t)$	$\lim_{s \to \infty} sF(s)$
10	Final value	$f(\infty) = \lim_{t \to \infty} f(t)$	$\lim_{s \to 0} sF(s)$

$\int_0^t f(\tau)\, d\tau$



$\lim_{t \to \infty} f(t) = f(\infty)$

$\lim_{t \to 0_+} f(t) = f(0_+)$



Ex 1: $x(t) = 5 \cdot \varepsilon(t-2)$

$X(s) = ?$

$$X(s) = \mathcal{L}\{5 \cdot \varepsilon(t-2)\}$$

$$= 5 \cdot \mathcal{L}\{\varepsilon(t-2)\}$$

$$= 5 \cdot e^{-2s} \cdot \mathcal{L}\{\varepsilon(t)\}$$

$$= 5 \cdot e^{-2s} \cdot \frac{1}{s}$$

Ex 2: $F(s) = \dfrac{1}{s^2+s+1}$

$f(t) = ?$

$$F(s) = \frac{1}{s^2+s+1}$$

$$= \frac{1}{\underbrace{s^2+2\cdot\frac{1}{2}s+\frac{1}{4}}+\frac{3}{4}}$$

Line 14

$$= \frac{2}{\sqrt{3}} \frac{1 \quad \sqrt{3}/2}{\left(s+\underset{\sigma}{\frac{1}{2}}\right)^2+\left(\underset{\omega}{\frac{\sqrt{3}}{2}}\right)^2}$$

$$f(t) = \frac{2}{\sqrt{3}} \cdot e^{-\frac{1}{2}t} \cdot \sin\left(\frac{\sqrt{3}}{2}t\right)$$

2.4 Models of dynamic systems

2.4.1 Differential equation — the model in time domain

Dynamic systems can be modelled in continuous time domain using ordinary differential equations (ODEs). The behaviour of an nth order linear system with input u and output y can be described by using the following general differential equation:

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t) \tag{2.5}$$

a_i and b_i are the parameters and n is the order of the differential equation. For proper systems m must be less than or equal to n. $n \geq m$

Writing differential equations

The derivation of differential equations is usually done by analysing the system using laws of physics and chemistry. Models for electrical networks are normally derived using well-known rules like Ohm's law, Kirchhoff's voltage and current laws and Faraday's law etc.

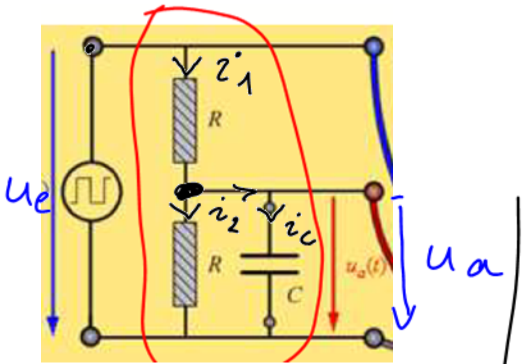
■ **Example 2.5** For the system of example 2.2:

$$\frac{u_e(t) - u_a(t)}{R} = \frac{u_a(t)}{R} + C\frac{du_a(t)}{dt} \tag{2.6}$$

After some simplification we get the following differential equation:

$$RC\frac{du_a(t)}{dt} + 2u_a(t) = u_e(t) \tag{2.7}$$

This differential equation describes the relationship between input and output quantities of the system in time domain. In order to calculate the system response $u_a(t)$ for a given value of the input variable $u_e(t)$, this differential equation has to be solved. ■



$$i_1 = \frac{u_e(t) - u_a(t)}{R}$$

$$i_2 = \frac{u_a(t) - 0}{R}$$

$$i_c = C \cdot \frac{du_a}{dt}$$

Node:

$$i_1 = i_2 + i_c$$

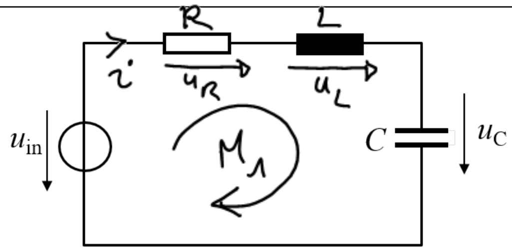
$$\frac{u_e(t) - u_a(t)}{R} = \frac{u_a(t) - 0}{R} + C\frac{du_a}{dt}$$

$$\frac{u_e(t)}{R} = 2\frac{u_a(t)}{R} + C\frac{du_a}{dt}$$

$$C \cdot \frac{du_a}{dt} + 2 \cdot \frac{u_a(t)}{R} = \frac{u_e(t)}{R}$$

$$\boxed{RC \cdot \frac{du_a}{dt} + 2u_a(t) = u_e(t)}$$

$$\frac{RC}{2}\frac{du_a}{dt} + u_a(t) = \frac{1}{2}u_e(t)$$



■ **Example 2.6 — An RLC circuit.**

M1:

$$-u_{in}(t) + u_R(t) + u_L(t) + u_c(t) = 0$$

$$-u_{in}(t) + R\,i(t) + L\cdot\frac{di}{dt} + u_c(t) = 0$$

$$-u_{in}(t) + R\cdot C\cdot\frac{du_c}{dt} + L\frac{d}{dt}\left(C\cdot\frac{du_c}{dt}\right) + u_c(t) = 0$$

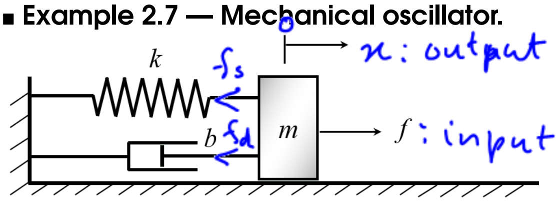
$$-u_{in}(t) + RC\frac{du_c}{dt} + LC\frac{d^2u_c}{dt^2} + u_c(t) = 0$$

$$\left|\begin{array}{l} u_R = R\cdot i \\ u_L = L\cdot\frac{di}{dt} \\ i\,? \\ i = C\cdot\frac{du_c}{dt} \end{array}\right.$$

$$\boxed{LC\frac{d^2u_c}{dt^2} + RC\frac{du_c}{dt} + u_c(t) = u_{in}(t)}$$

■

■ **Example 2.7 — Mechanical oscillator.**



$$f_s = kx$$

$$f_d = b\cdot\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} = a = \frac{f - f_s - f_d}{m}$$

$$\frac{d^2x}{dt^2} = \frac{f - kx - b\cdot\frac{dx}{dt}}{m}$$

$$\boxed{m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = f(t)}$$

$$\frac{d^2x}{dt^2} + \underbrace{\frac{b}{m}}_{a_1}\frac{dx}{dt} + \underbrace{\frac{k}{m}}_{a_0}x(t) = \underbrace{\frac{1}{m}}_{b_0}f(t)$$

■

Solution of linear differential equations

The classical method

This method is not dealt with in this course.

Solution with the help of Laplace transform

This solution method consists of three steps:

1. A differential equation is transformed into an algebraic equation using the Laplace transform.
2. The algebraic equation is solved.
3. The solution of the differential equation is calculated from the solution of the algebraic equation using the inverse Laplace transform.

This approach is illustrated in Figure 2.4

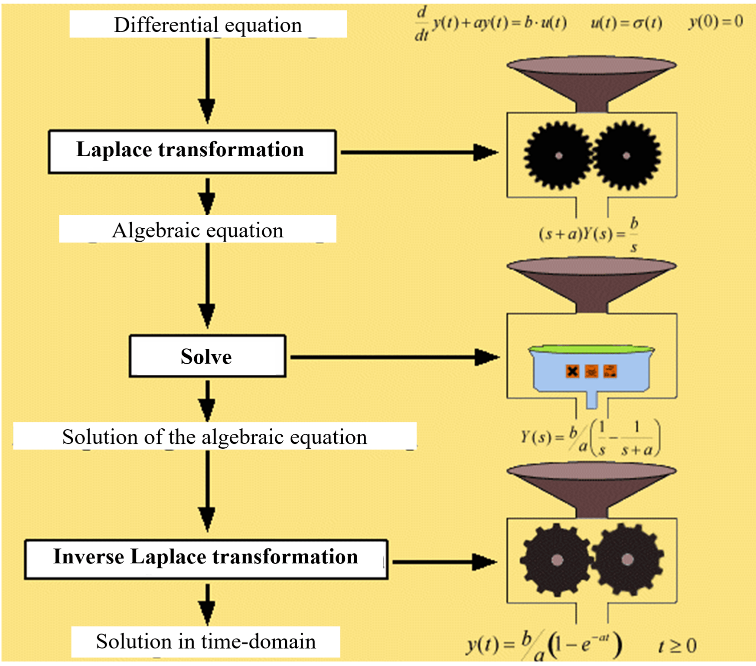


Figure 2.4: Solution of differential equations using Laplace transform

■ **Example 2.8 Step response of a first order system**

Given: $2\frac{dy}{dt} + y(t) = 5\,u(t)$; $u(t) = \varepsilon(t)$; $y(0) = 0$

Step 1: $\mathcal{L}\left\{2\frac{dy}{dt} + y(t)\right\} = \mathcal{L}\left\{5 \cdot \varepsilon(t)\right\}$

$$2\left(sY(s) - y(0)\right) + Y(s) = 5 \cdot \frac{1}{s}$$

$$2sY(s) + Y(s) = 5/s$$

Step 2: $Y(s) = \frac{5}{s} \cdot \frac{1}{(2s+1)}$

Step 3: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$\boxed{y(t) = 5 \cdot \left(1 - e^{-t/2}\right)}$$

■

■ **Example 2.9 Impulse response of a second order system**

Given: $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 6y(t) = 0{,}5\,u(t)$; $u(t) = \delta(t)$; $y(0) = 0$
$\dot{y}(0) = 0$

Required: $y(t) \quad \forall \; t \geq 0$

Step 1: $\mathcal{L}\left\{\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 6y(t)\right\} = \mathcal{L}\left\{0{,}5\,\delta(t)\right\}$

$$s^2 Y(s) + 4sY(s) + 6Y(s) = 0{,}5 \cdot 1$$

Step 2: $Y(s) = \frac{0{,}5}{s^2 + 4s + 6}$

Step 3:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{0{,}5}{s^2+4s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{(s+\underbrace{2}_{a})^2 + (\underbrace{\sqrt{2}}_{\omega_0})^2}\right\}\frac{0{,}5}{\sqrt{2}}$$

■

$$\boxed{y(t) = \frac{0{,}5}{\sqrt{2}} \cdot e^{-2t} \sin\left(\sqrt{2}\,t\right)}$$

2.4.2 Transfer function — the model in s-domain

Definition 2.4.1 — Transfer function. The transfer function $G(s)$ describes the ratio of the output variable y (effect) to the input variable u (cause) in s domain.

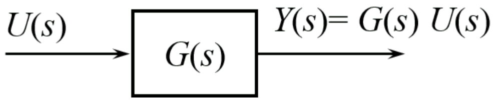
$$G(s) = \frac{Y(s)}{U(s)}, \tag{2.8}$$

where $U(s)$ and $Y(s)$ represent input and output signals in s domain.

For systems that have no transport delays, $G(s)$ is a rational function of the complex variable $s = \sigma + j\omega$.

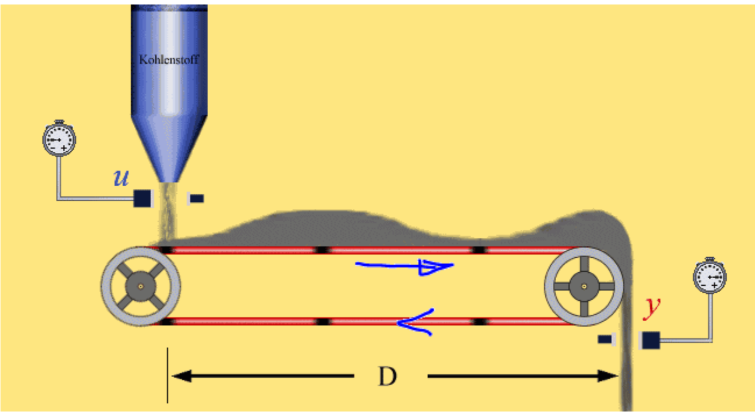
$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} = \frac{B(s)}{A(s)} \tag{2.9}$$

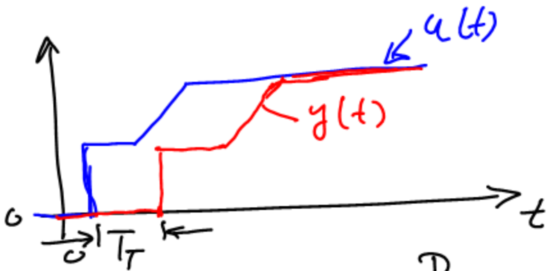
Polynomial $B(s)$ is called a numerator polynomial whereas the polynomial $A(s)$ is known as denominator polynomial or characteristic polynomial. Symbolically, a transfer function is represented as a block.



The transfer function $G(s)$ of a system that has a pure transport delay (T_T) is not a rational but a transcendental function of the complex variable s.

$$G(s) = e^{-sT_T} \tag{2.10}$$





$T_T = \frac{D}{v}$

$y(t) = u(t - T_T)$

$Y(s) = e^{-T_T s} \cdot U(s)$

$$G(s) = \frac{Y(s)}{U(s)} = e^{-T_T s}$$

T_T: Transport delay

Dead time

If the differential equation of a system is given, then its transfer function can be determined by applying the Laplace transform on both sides of the differential equation and setting all initial conditions equal to zero.
Transfer functions of some important systems are summarized in Section **??**.

■ **Example 2.10** Given is the following differential equation of a system with $u(t)$ as input $y(t)$ as output. Determine the associated transfer function in s domain:

$$\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y(t)}{\mathrm{d}t} + 4y(t) = 8u(t)$$

By applying Laplace transform on both sides of the equation we get (all initial conditions are set equal to zero):

$$s^2Y(s) + 4sY(s) + 4Y(s) = 8U(s)$$

$$(s^2+4s+4)\cdot Y(s) = 8\,U(s) \qquad :U(s)\,(s^2+4s+4)$$

The transfer function of the system is.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\overset{B(s)}{8}}{\underset{A(s)}{s^2+4s+4}}$$

■

Exampl 2.10a

Given: $y(t) = 4\cdot(1-e^{-2t})$

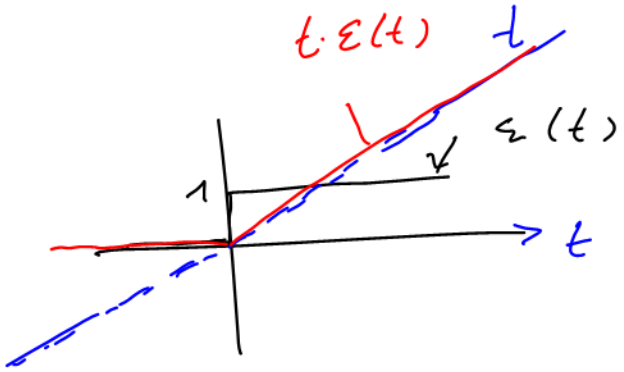
$u(t) = t\cdot\varepsilon(t)$

Required: Transfer fuction $G(s)$

$$Y(s) = \mathcal{L}\left\{4(1-e^{-2t})\right\} = 4\cdot\frac{2}{s(s+2)}$$

$$U(s) = \mathcal{L}\left\{t\cdot\varepsilon(t)\right\} = \frac{1}{s^2}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{4\cdot 2}{s(s+2)}\cdot\frac{s^2}{1} = \frac{8s}{s+2}$$



$\frac{B(s)}{A(s)}$

Poles, zeros and steady-state gain of a system

Definition 2.4.2 — Zeros. Roots of the numerator polynomial $B(s)$ are called zeros of the transfer function $G(s)$. These are the values of the variable s, where the transfer function becomes zero.

Definition 2.4.3 — Poles. Roots of the denominator polynomial $A(s)$ are called poles of the transfer function $G(s)$.These are the values of the variable s, where the magnitude of the transfer function becomes ∞.

Poles and zeros of a transfer function are plotted in an s plane. The resulting image is known as pole-zero map (pz map).

Definition 2.4.4 — Steady-state gain. The steady-state gain (or dc-gain) K_0 of a system is the ratio of the steady-state value of the output variable $y(\infty)$ to the steady-state value of the input variable $u(\infty)$.

$$K_0 = \frac{y(\infty)}{u(\infty)} = G(s)|_{s=0}$$

■ **Example 2.11** Consider the following transfer function:

$$G(s) = \frac{2s+2}{s^2+4s+5}$$

B(s)
A(s)

This system has only one zero and two poles:

$$2s_0 + 2 = 0 \Rightarrow s_0 = -1$$

$$s_\infty^2 + 4s_\infty + 5 = 0 \Rightarrow s_{\infty 1,2} = \frac{-4 \mp \sqrt{16-20}}{2} = -2 \pm j$$

The steady-state gain of the system is

$$K_0 = G(0) = \frac{2}{5} = 0.4$$

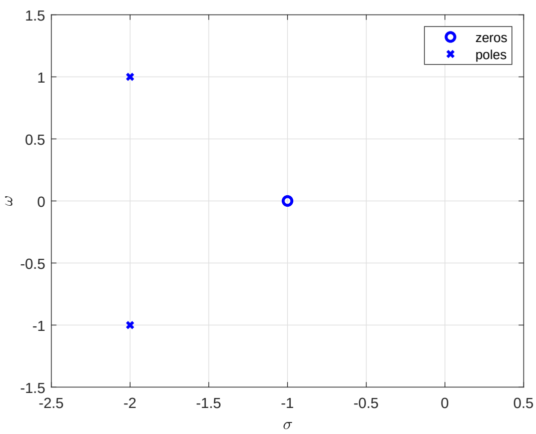


Figure 2.5: Pole zero map of the system

■

Using transfer function to calculate system response

If the input signal of a system is given then its output signal can be calculated using the following steps:

1. Calculate the Laplace transform $U(s)$ of the input signal $u(t)$
2. Calculate $Y(s)$ by multiplying $G(s)$ with $U(s)$
3. Calculate $y(t)$ by using inverse Laplace transform.

■ **Example 2.12** Calculate and draw the step response of a system with the following transfer function:

$$G(s) = \frac{3}{s+2}$$

Solution:

$$u(t) = \varepsilon(t)$$

$$U(s) = \mathcal{L}\left(\varepsilon(t)\right) = \frac{1}{s}$$

$$Y(s) = G(s)U(s) = \frac{3}{s+2}\cdot\frac{1}{s}$$

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{3}{s(s+2)}\right) = 1.5\left(1-e^{-2t}\right)$$

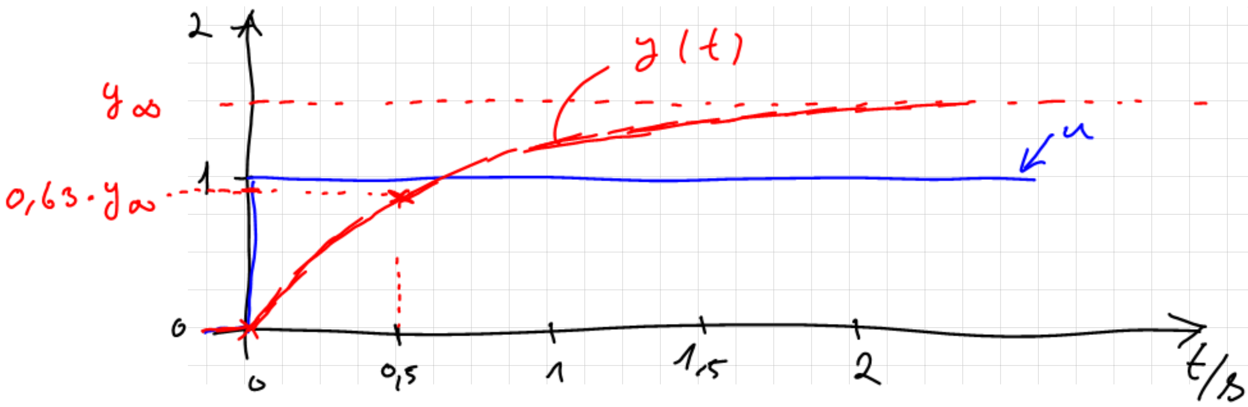
$t=0$; $y(0)=0$

$t=\infty$; $y(\infty)=1{,}5$

$t=0{,}5\,s$; $y(0{,}5)=1{,}5\cdot\left(1-e^{-2\cdot 0{,}5}\right)$

$=1{,}5\,(1-0{,}37)$

$=0{,}63\cdot 1{,}5$



■

[2 2]

$\frac{2s+2}{s^2+4s+5}$ [1 4 5]

In MATLAB® you can:

- create a transfer function object with the command `tf(num, den)`
  `G=tf([2 2],[1 4 5]);`
- calculate poles and zeros by using commands `pole, zero` respectively.
  `P=pole(G)`
  `Z=zero(G)`
- use `pzmap` to generate a pole-zero map.
  `pzmap(G)`
- calculate and plot step and impulse response using `step` and `impulse`.
  `step(G)`
  `impulse(G)`

$G(s)=\frac{s^7+1}{s^8+2s^2+5}$ $\frac{[1\ 0\ 0\ 0\ 0\ 0\ 0\ 1]}{[1\ 0\ 0\ 0\ 0\ 0\ 2\ 0\ 5]}$

[0 2 2]
[1 4 5]

2.6 Commonly used models

2.6.1 First-order systems (PT1)

→ proportional with first order time lag.

A fairly large number of technical systems can be modelled as a first order transfer function without any zero. The generalised model of such a system, known in German literature as PT1(proportional behaviour with a first-order time delay), is given by the following transfer function:

$$G(s) = \frac{K}{\tau s + 1},$$

$$\frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1} \Rightarrow (\tau s + 1)Y(s) = K \cdot U(s)$$

where K is the steady-state gain and τ is the time constant of the system. The associated differential equation can be written as:

$$\tau \frac{\mathrm{d}y}{\mathrm{d}t} + y(t) = Ku(t).$$

$$Y(s) = \frac{K}{\tau s + 1} \cdot \frac{u_\infty}{s}$$

$$y(t) = K \cdot \left(1 - e^{-t/\tau}\right) \cdot u_\infty$$

The response of the system to step input $u(t) = u_\infty \varepsilon(t)$ is

$$y(t) = Ku_\infty \left(1 - e^{-t/\tau}\right)$$

The steady-state value of the output is

$$y_\infty = Ku_\infty \Rightarrow K = \frac{y_\infty}{u_\infty}$$

The value of y after one time constant is

$$y(\tau) = 0.63y_\infty = 0.63Ku_\infty$$

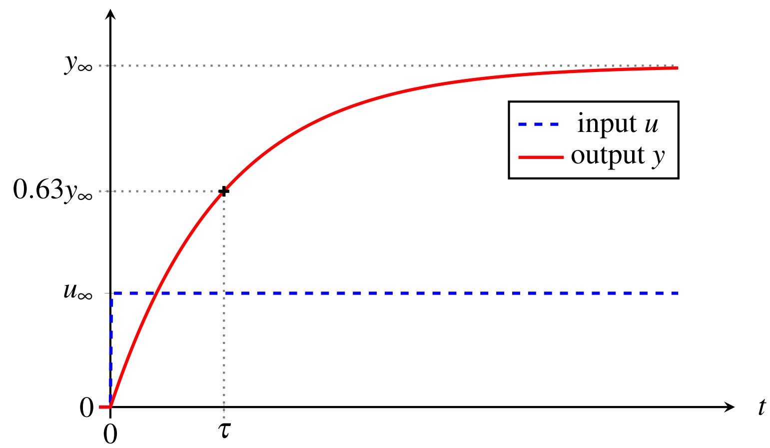
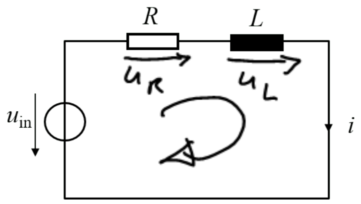


Figure 2.10: Typical step response of a first order system (PT1)

Examples

■ **Example 2.15 Current in an RL circuit**



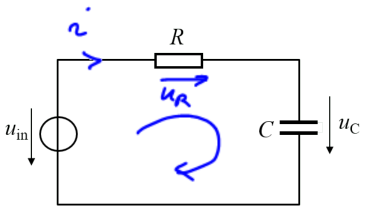
input: u_{in}
output: i

$$u_R = R \cdot i$$
$$u_L = L\frac{di}{dt}$$

$$-u_{in}(t) + u_R + u_L = 0$$
$$-u_{in}(t) + R\,i(t) + L\frac{di}{dt} = 0$$
$$L\frac{di}{dt} + R\,i(t) = u_{in}(t)$$
$$\boxed{\frac{L}{R}\frac{di}{dt} + i(t) = \frac{1}{R}u_{in}(t)}$$
$$\circ\!-\!\bullet$$
$$\frac{L}{R}s\,I(s) + I(s) = \frac{1}{R}\cdot U_{in}(s)$$
$$G(s) = \frac{I(s)}{U_{in}(s)} = \frac{\overset{K}{1/R}}{\underset{\tau}{\frac{L}{R}}s + 1}$$

■

■ **Example 2.16 Charging a capacitor in an RC circuit**



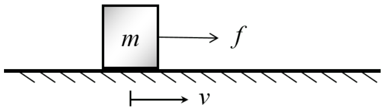
input: u_{in}
output: u_c

$$u_R = R \cdot i$$
$$i = C\cdot\frac{du_c}{dt}$$
$$u_R = R\cdot C\cdot\frac{du_c}{dt}$$

$$-u_{in}(t) + u_R + u_c = 0$$
$$-u_{in}(t) + RC\frac{du_c}{dt} + u_c(t) = 0$$
$$\boxed{RC\cdot\frac{du_c}{dt} + u_c(t) = u_{in}(t)}$$
$$\circ\!-\!\bullet$$
$$RC\cdot s\,U_c(s) + U_c(s) = U_{in}(s)$$
$$\frac{U_c(s)}{U_{in}(s)} = \frac{\overset{K}{1}}{\underset{\tau}{RC}\,s + 1}$$

■

■ **Example 2.17 Speed of an object with viscous friction**



input:force f
output:velocity v

$f_R \sim v$

$f_R = b \cdot v$

coefficient of viscous friction.

$$m \cdot \frac{dv}{dt} = f - f_R$$

$$m \cdot \frac{dv}{dt} = f - b v$$

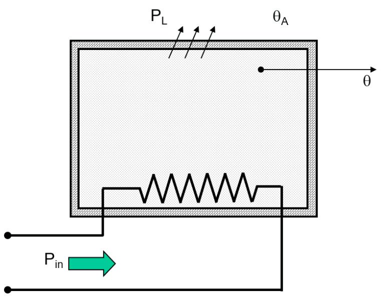
$$m \cdot \frac{dv}{dt} + b\, v(t) = f(t)$$

$$\boxed{\frac{m}{b} \frac{dv}{dt} + v(t) = \frac{1}{b} f(t)}$$

$$G(s) = \frac{V(s)}{F(s)} = \frac{\overset{K}{1/b}}{\underset{T}{m/b}\, s + 1}$$

■

■ **Example 2.18 Room heating**



input:Heating Power P_{in}
output:Room temperature θ

$$m \cdot c \cdot \frac{d\theta}{dt} = P_{in} - P_L$$

$P_L \sim \theta - \theta_A$

$P_L \simeq k \cdot (\theta - \theta_A)$

$$m \cdot c \cdot \frac{d\theta}{dt} = P_{in} - k \cdot (\theta - \theta_A)$$

(θ_A neglected)

Specific heat = c

J/kg/K

$$\frac{dW}{dt} = \frac{d(c \cdot m \cdot \theta)}{dt}$$

$$P = c\, m \cdot \frac{d\theta}{dt}$$

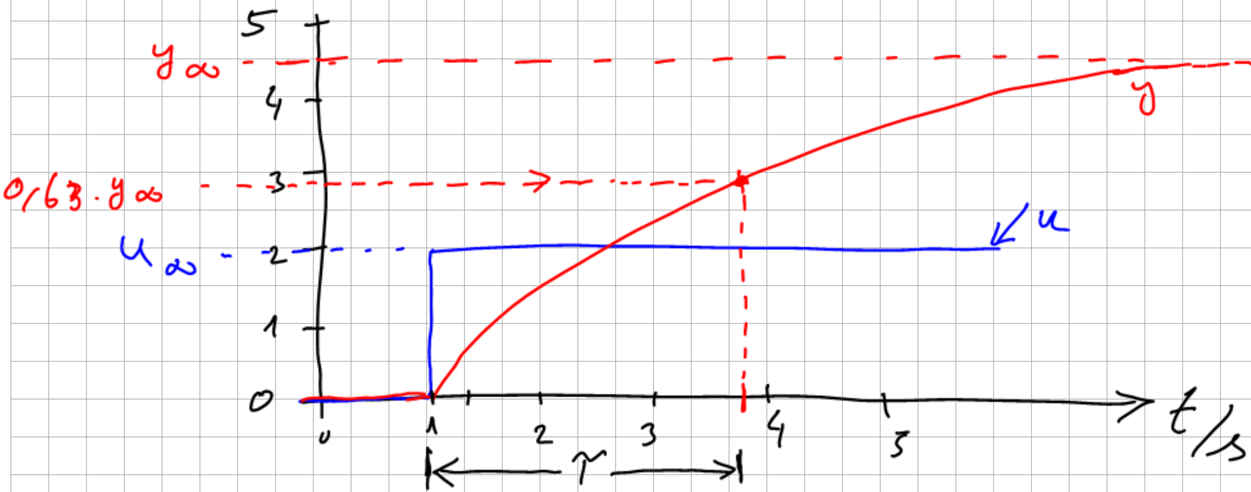
$$m c \frac{d\theta}{dt} + k \theta = P_{in}$$

$$\boxed{\frac{mc}{k} \frac{d\theta}{dt} + \theta(t) = \frac{1}{k} \cdot P_{in}}$$

■

$$\frac{\Theta(s)}{P_{in}(s)} = \frac{\overset{K}{1/k}}{\underset{T}{\frac{mc}{k}}\, s + 1}$$

Example: Given



Required: Transfer function of the system.

$$K = \frac{y_\infty}{u_\infty} = \frac{4{,}5}{2} = 2{,}25$$

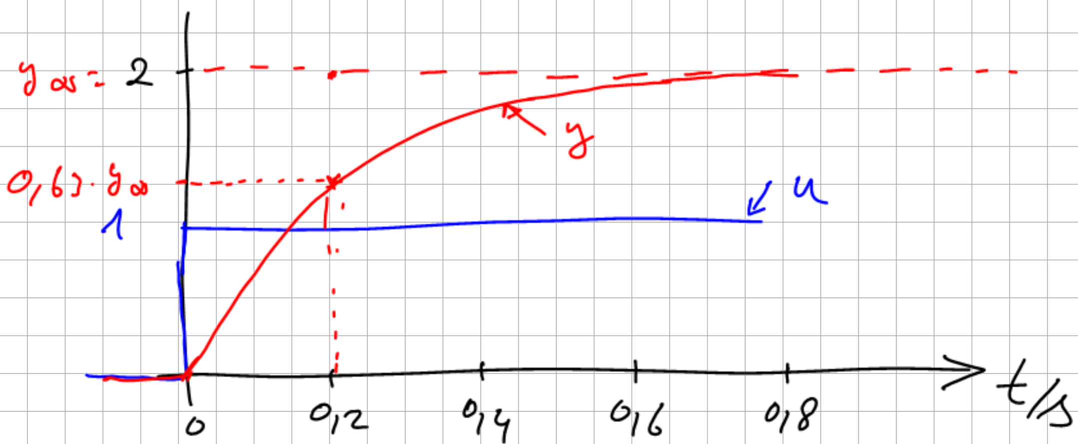
$$\tau = 2{,}75\,s$$

$$G(s) = \frac{K}{\tau s + 1} = \frac{2{,}25}{2{,}75\,s + 1}$$

Example: Given $G(s) = \dfrac{10}{s+5} = \dfrac{\cancel{10}\,2}{\cancel{5}(0{,}2s+1)}$

$$K = 2$$

$$\tau = 0{,}2 \quad s$$



2.6.2 Second-order systems (PT2)

Most technical systems involving two energy storing elements can be modelled as a transfer function which involves two poles and no zeros. The generalised model of such a system, known in German literature as PT2 (proportional behaviour with a second-order time delay), is given by the following transfer function:

$$G(s) = \frac{K}{\left(\frac{s}{\omega_0}\right)^2 + 2D\left(\frac{s}{\omega_0}\right) + 1},$$

where K is the steady-state gain,
ω_0 is the natural frequency and
D is the damping ratio of the system. In English language textbooks it is normally denoted by ζ. But for the sake of compatibility with the German system, the symbol D will be used.

The associated differential equation can be written as:

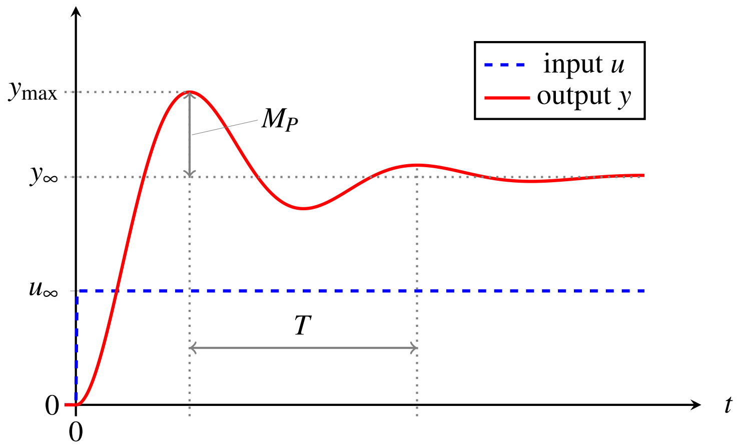
$$\frac{1}{\omega_0^2} \cdot \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{2D}{\omega_0} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + y(t) = Ku(t).$$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

The response of the system to step input $u(t) = u_\infty \cdot \varepsilon(t)$ is

$$y(t) = K \cdot u_\infty \left(1 - e^{-D\omega_0 t} \cos\left(\omega_0 \sqrt{1-D^2} \cdot t\right) - e^{-D\omega_0 t} \frac{D}{\sqrt{1-D^2}} \sin\left(\omega_0 \sqrt{1-D^2} \cdot t\right)\right)$$

Typical step response of such a system is drawn in Figure 2.11.



$$K = \frac{y_\infty}{u_\infty}$$

Figure 2.11: Typical step response of an under-damped second order system (PT2)

Significance of the parameters

- Parameter K is responsible for the steady-state response of the system.

$$y_\infty = Ku_\infty \Rightarrow K = \frac{y_\infty}{u_\infty}$$

- The damping ratio D describes how oscillations in a system decay after the system is disturbed.
  - The system is *undamped* if $D = 0$. In this case the system response will oscillate with a constant amplitude. This oscillation will never decay. The system is marginally stable.

- The system is *underdamped* if $0 < D < 1$. In this case the amplitude of the system response will decay with the passage of time.
- The system is *critically damped* if $D = 1$. In this case the system response will not oscillate. The system has two repeated real poles in the left half plane.
- The system is *overdamped* if $D > 1$. In this case the system response will not oscillate. The system has two different real poles in the left half plane.
- The system is *unstable* if $D < 0$.

In control systems the overshoot M_P defined as

$$M_P = \frac{y_{\max} - y_\infty}{y_\infty}$$

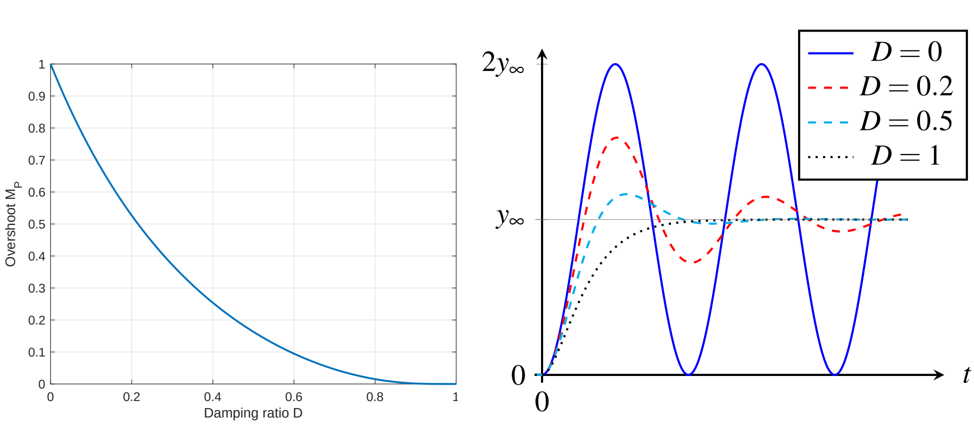
is of particular importance. The overshoot is directly related with the damping ratio.

$$M_P = e^{-\pi\frac{D}{\sqrt{1-D^2}}} \qquad \Rightarrow \qquad D = \frac{1}{\sqrt{1+\left(\frac{\pi}{\ln M_p}\right)^2}}$$

This relationship is shown in Figure 2.12.

- The undamped natural frequency ω_0 is a measure of the reaction speed of a system. Systems with higher values of ω_0 react faster to the changes in the inputs. This parameter is directly related with the damped natural frequency or the oscillation frequency ω.

$$\omega = \omega_0\sqrt{1-D^2} \quad \Rightarrow \quad \omega_0 = \frac{\omega}{\sqrt{1-D^2}} \quad \text{where} \quad \omega = \frac{2\pi}{T}$$



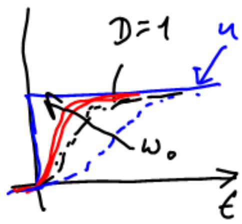


Figure 2.12: Relationship between damping ratio D and overshoot M_P of a system

$$s_{\infty 1,2} = -D\omega_0 \pm j\omega_0\sqrt{1-D^2}$$

$$|s_{\infty 1,2}| = \omega_0$$

$$D = \sin\phi_D$$

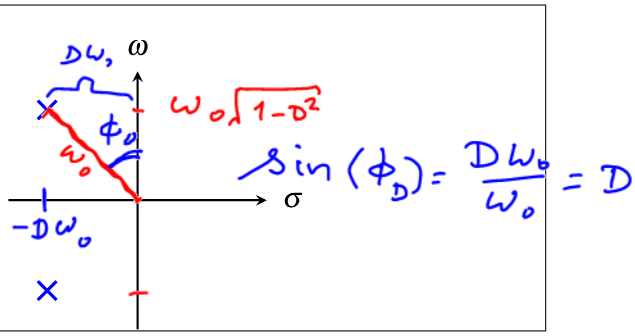
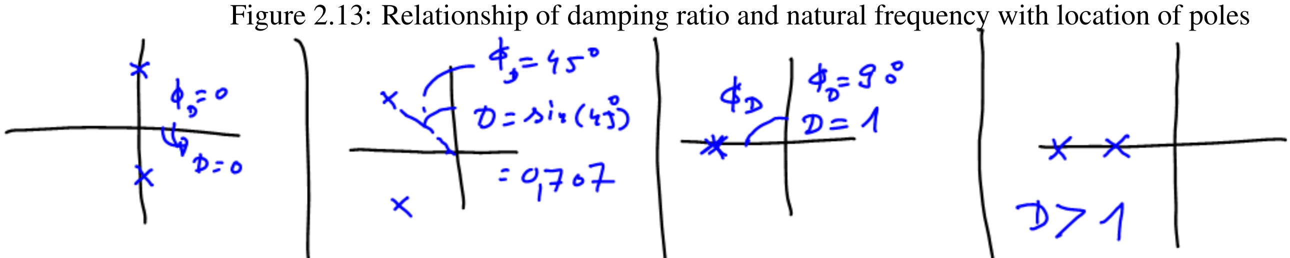


Figure 2.13: Relationship of damping ratio and natural frequency with location of poles



Examples

■ **Example 2.19** **RLC circuit** from Example 2.6.

$$LC\frac{d^2u_c}{dt^2}+RC\frac{du_c}{dt}+u_c(t)=u_{in}(t)$$

$$LC\,s^2U_c(s)+RC\,s\cdot U_c(s)+U_c(s)=U_{in}(s)$$

$$G(s)=\frac{U_c(s)}{U_{in}(s)}=\frac{\overset{K}{1}}{\underbrace{LC}_{\frac{1}{\omega_0^2}}s^2+\underbrace{RC}_{2D/\omega_0}s+1} \qquad \left|\quad \frac{K}{\frac{1}{\omega_0^2}s^2+\frac{2D}{\omega_0}s+1}\right.$$

$$\omega_0=\frac{1}{\sqrt{LC}} \qquad \left|\quad \frac{2D}{\omega_0}=RC \Rightarrow D=\frac{RC\omega_0}{2}\right.$$

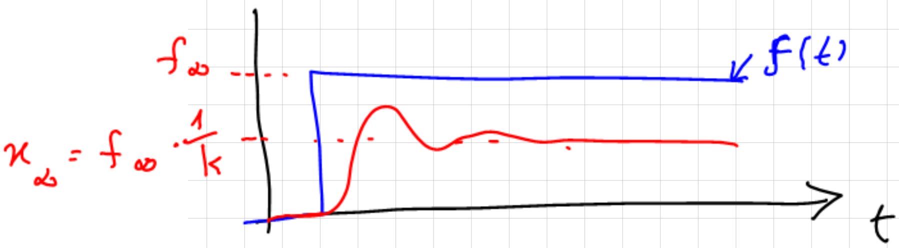
$$D=\frac{R}{2}\cdot\sqrt{\frac{C}{L}}$$

■

■ **Example 2.20** **Mechanical oscillator** from Example 2.7

$$\boxed{m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=f(t)}$$

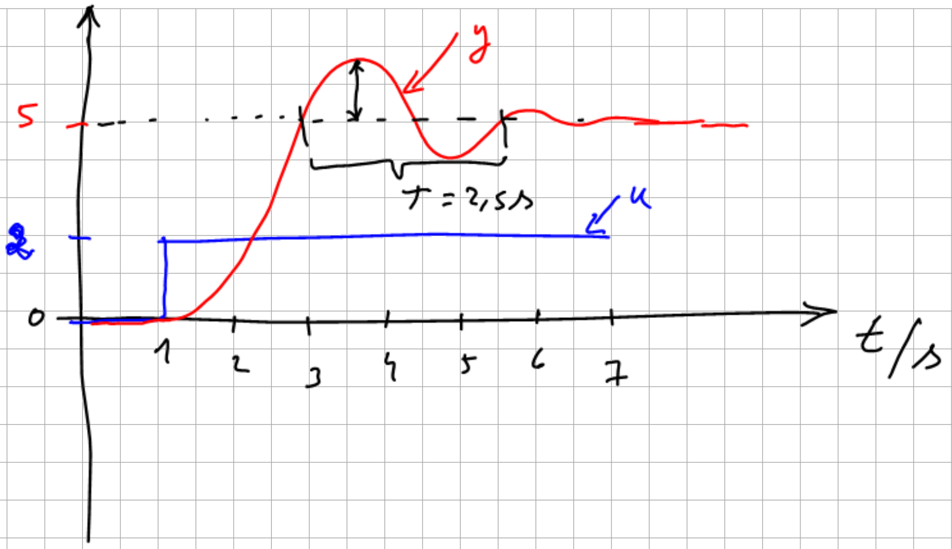
$$G(s)=\frac{X(s)}{F(s)}=\frac{1}{m\cdot s^2+bs+k}=\frac{\overset{K}{1/k}}{\underbrace{\frac{m}{k}}_{\frac{1}{\omega_0^2}}s^2+\underbrace{\frac{b}{k}}_{\frac{2D}{\omega_0}}s+1}$$



$$\omega_0=\sqrt{\frac{k}{m}} \qquad \left|\quad 2D=\frac{b}{k}\cdot\omega_0 \qquad D=\frac{b}{2}\frac{1}{\sqrt{km}}\right.$$

■

Exam:



Required: $G(s) = ?$

$$G(s) = \frac{K}{\frac{s^2}{\omega_0^2} + \frac{2D}{\omega_0}s + 1}$$

$$K = \frac{y_\infty}{u_\infty} = \frac{5}{2} = 2,5$$

$D = ?$

$$M_p = \frac{y_{max} - y_\infty}{y_\infty} = \frac{6,7-5}{5} = \frac{1,7}{5} = 34\%$$

From $D - M_p$ curve: $D = 0,32$

$$T = 2,5\,s \Rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{2,5} = \frac{4\pi}{5}$$

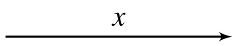
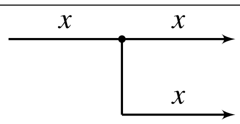
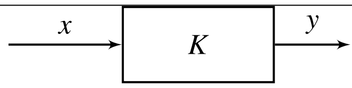
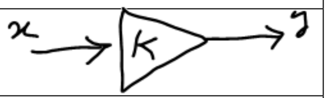
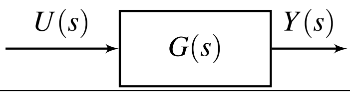
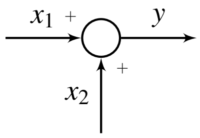
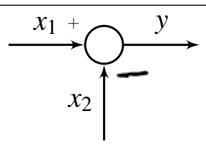
$$\omega_0 = \frac{\omega}{\sqrt{1-D^2}} = \frac{4\pi}{5} \cdot \frac{1}{\sqrt{1-(0,32)^2}}$$

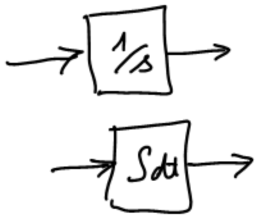
2.7 Block diagrams

2.7.1 Introduction

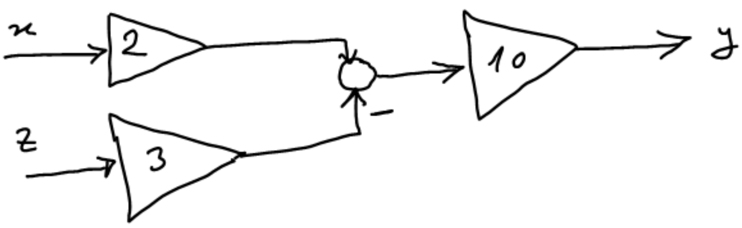
Structure of a dynamic system can be described by means of block diagrams. A block diagram consists of arrows and blocks (Table 2.5). The arrows represent time-variable quantities i.e. signals. The arrows have a clear direction of action, which is indicated by the direction of the arrow. The blocks represent processing units, ie dynamic systems or mathematical functions. A block can only influence its output signals. The input signals of a block are not changed within the block.

Table 2.5: Basic components of a block diagram

	Signal Information flow only in the direction of arrow "one-way traffic only"
	Branch
	Gain $y = Kx$ 
	Transfer function (complex gain) $Y(s) = G(s)U(s)$
	Sum $y = x_1 + x_2$
	Subtraction $y = x_1 - x_2$

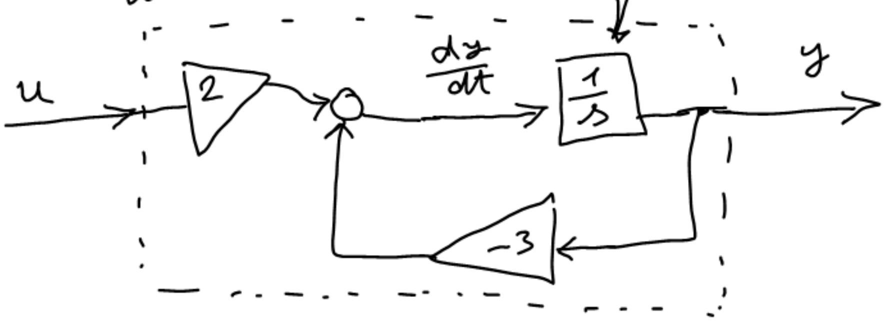


Ex1: $y = 10(2x - 3z)$



Ex2: $\frac{dy}{dt} = 2u - 3y$

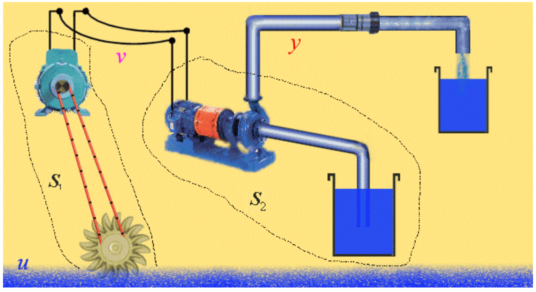
Integrator.



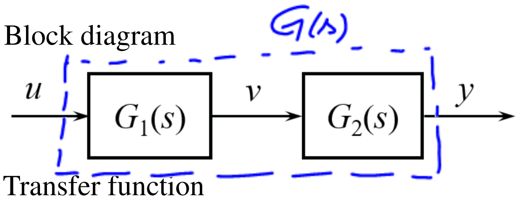
2.7.2 Interconnected systems

Series connection (Cascaded connection)

Example



Block diagram



Transfer function

$G(s) = \frac{Y(s)}{U(s)} = G_1(s) \cdot G_2(s)$

$V(s) = G_1(s) \cdot U(s)$ --(1)

$Y(s) = G_2(s) \cdot V(s)$ --(2)

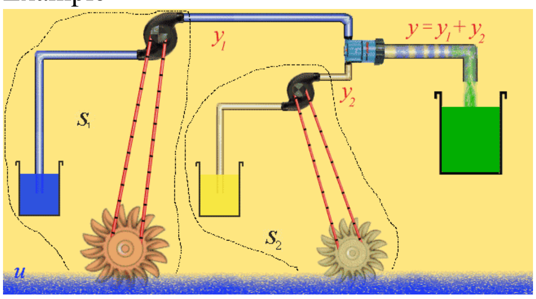
Put (1) into (2)

$Y(s) = G_2(s) \cdot G_1(s) \cdot U(s)$

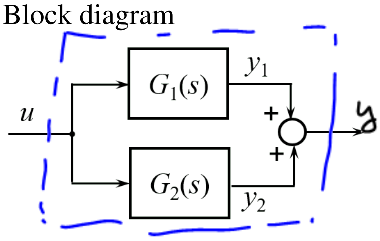
$G(s) = \frac{Y(s)}{U(s)} = G_2(s) \cdot G_1(s)$

Parallel connection

Example



Block diagram



Transfer function

$G(s) = \frac{Y(s)}{U(s)} = G_1(s) + G_2(s)$

$Y_1(s) = G_1(s) \cdot U(s)$

$Y_2(s) = G_2(s) \cdot U(s)$

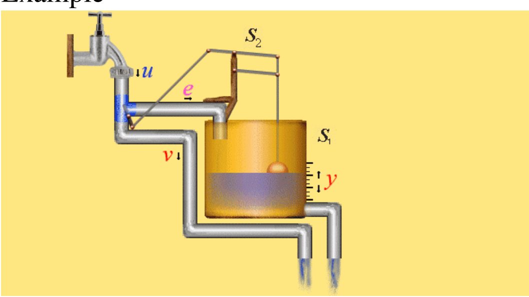
$Y(s) = Y_1(s) + Y_2(s)$

$Y(s) = G_1(s) \cdot U(s) + G_2(s) \cdot U(s)$

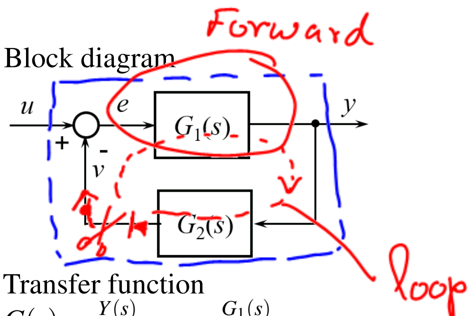
$G(s) = \frac{Y(s)}{U(s)} = G_1(s) + G_2(s)$

Feedback connection

Example



Block diagram



Transfer function

$G(s) = \frac{Y(s)}{U(s)} = \frac{G_1(s)}{1+G_1(s) \cdot G_2(s)}$

$Y(s) = G_1(s) \cdot E(s)$ -1

$V(s) = G_2(s) \cdot Y(s)$ -2

$E(s) = U(s) - V(s)$ -3

Put (3) into (1)

$Y(s) = G_1(s) \cdot (U(s) - V(s))$

Eqn (2)

$Y(s) = G_1(s) \cdot (U(s) - G_2(s) \cdot Y(s))$

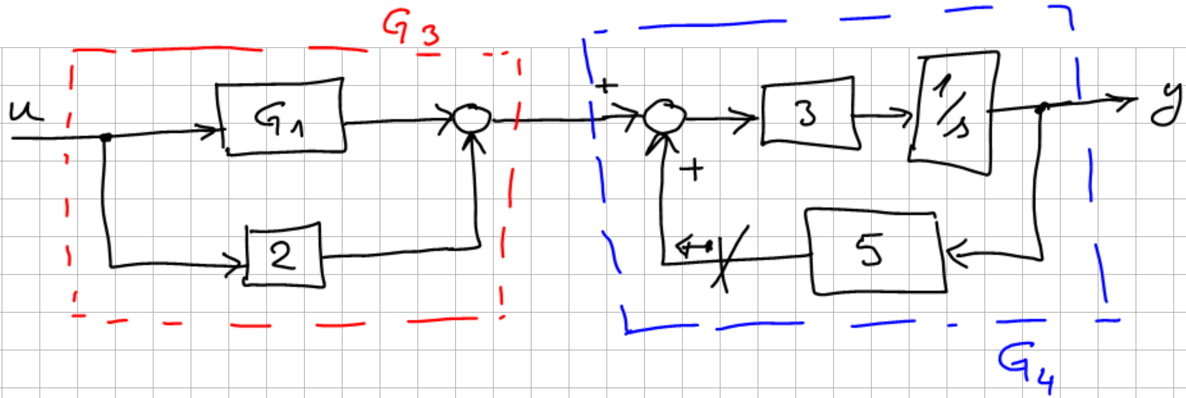
$Y(s) = G_1(s) \cdot U(s) - G_1(s) \cdot G_2(s) \cdot Y(s)$

$Y(s) + G_1(s) \cdot G_2(s)\, Y(s) = G_1(s) \cdot U(s)$

$G(s) = \frac{Y(s)}{U(s)} = \frac{G_1(s)}{1 + G_1(s) \cdot G_2(s)}$

$= \frac{\text{Forward branch}}{1 - (\text{loop TF})}$

Ex 1:



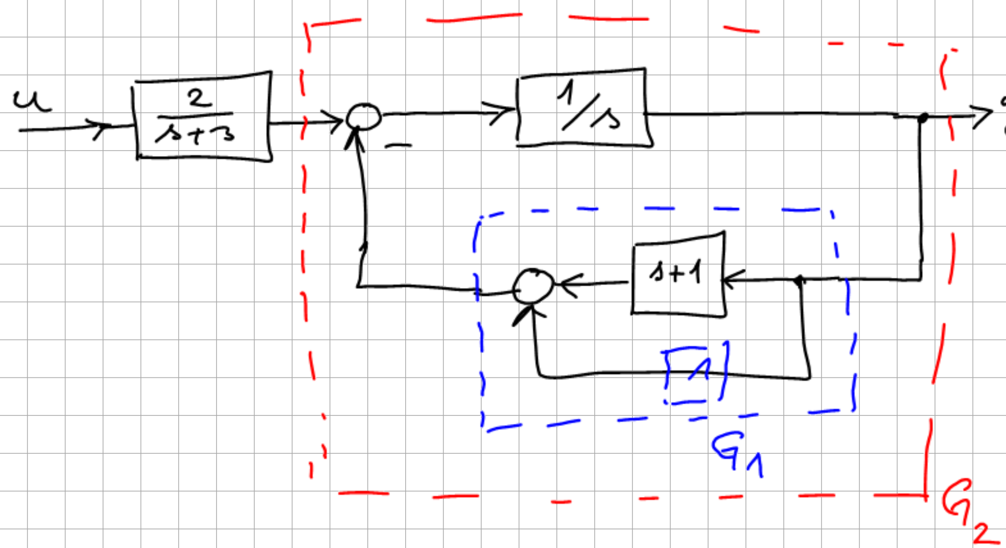
$$G_3 = G_1 + 2$$

$$G_4 = \frac{\text{Forward branch}}{1-(\text{looTF})} = \frac{3\cdot\frac{1}{s}}{1-\left(3\cdot\frac{1}{s}\cdot 5\right)} = \frac{3}{s-15}$$

$$G(s) = \frac{Y(s)}{U(s)} = G_3 \cdot G_4$$

$$= (G_1+2)\cdot\frac{3}{s-15}$$

<u>Ex. 2</u>

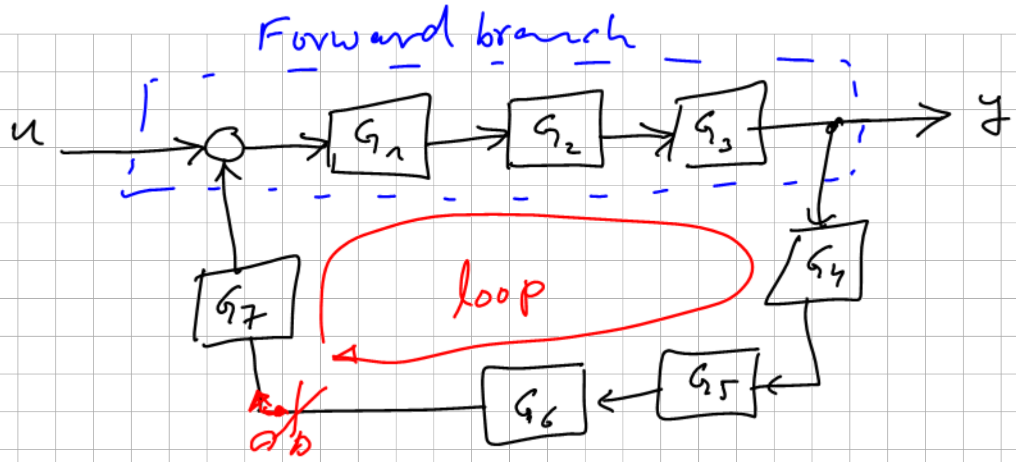


$$G_1(s) = (s+1) + (\ 1\ ) = s+2$$

$$G_2(s) = \frac{1/s}{1-\left(-\frac{1}{s}\cdot G_1(s)\right)} = \frac{1/s}{1+\frac{1}{s}\cdot(s+2)} = \frac{1}{s+s+2} = \frac{1}{2s+2}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2}{s+3}\cdot G_2(s) = \frac{2}{(s+3)(2s+2)} = \frac{1}{(s+3)(s+1)}$$

Ex 3

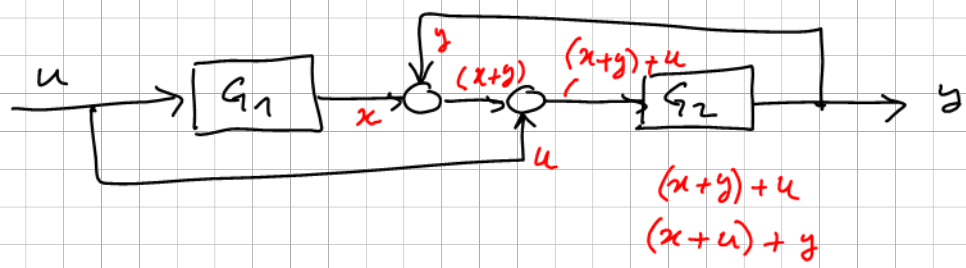


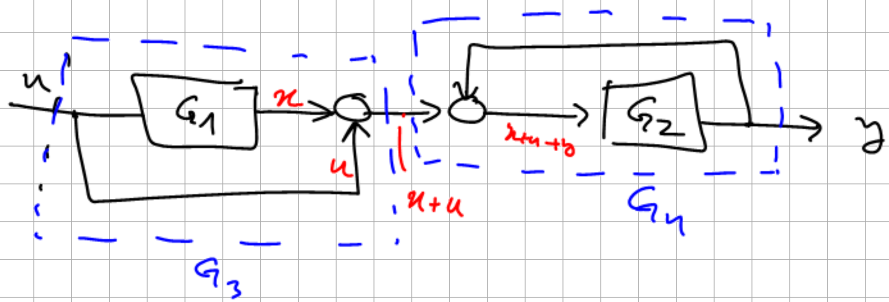
$$TF_{forward} = G_1 \cdot G_2 \cdot G_3$$

$$TF_{loop} = G_7 \cdot G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot G_5 \cdot G_6$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{TF_{forw.br.}}{1 - TF_{loop}} = \frac{G_1 \cdot G_2 \, G_3}{1 - G_7 \cdot G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot G_5 \cdot G_6}$$

Ex. 4



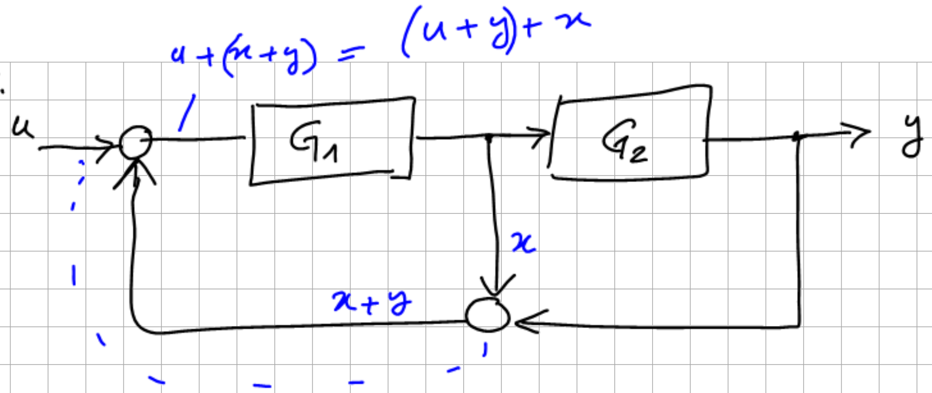


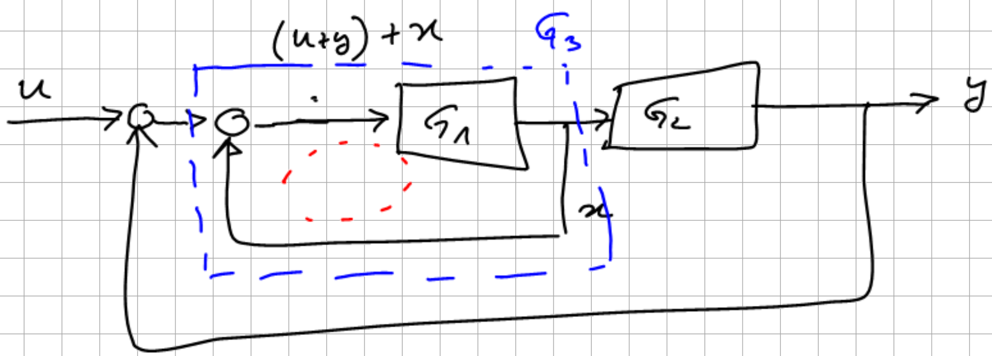
$$G_3 = G_1 + 1$$

$$G_4 = \frac{G_2}{1 - G_2}$$

$$G(s) = G_3 \cdot G_4 = (G_1 + 1) \cdot \frac{G_2}{1 - G_2}$$

Ex:



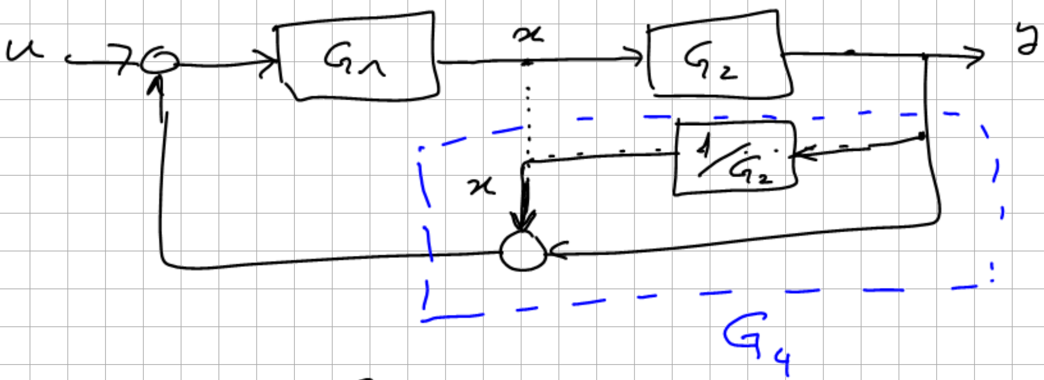


$$G_3 = \frac{G_1}{1 - G_1}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{G_3 \cdot G_2}{1 - G_3 G_2} = \frac{\frac{G_1}{1-G_1} \cdot G_2}{1 - \frac{G_1}{1-G_1} G_2}$$

$$= \frac{G_1 G_2}{1 - G_1 - G_1 G_2}$$

Alternative:



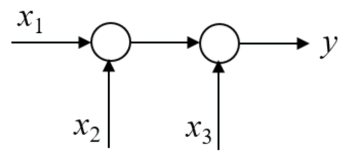
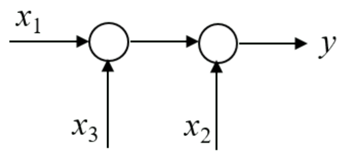
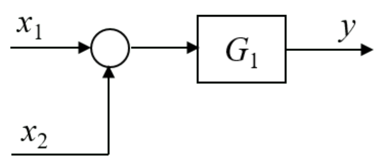
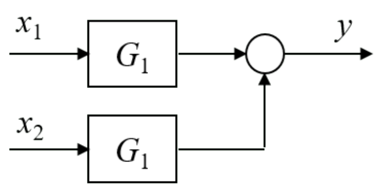
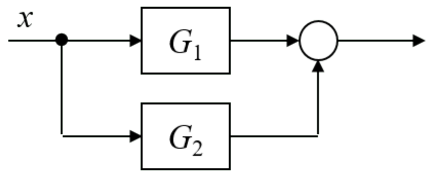
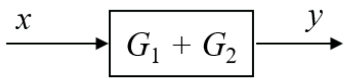
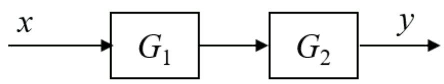
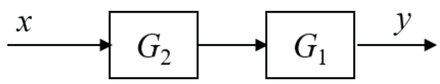
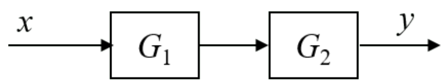
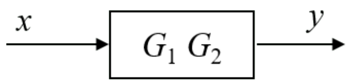
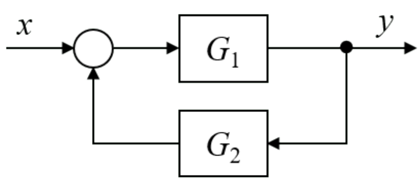
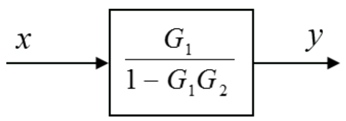
$$G_4 = \frac{1}{G_2} + 1 = \frac{1+G_2}{G_2}$$

$$G(s) = \frac{G_1 \cdot G_2}{1 - G_1 \cdot G_2 \cdot G_4} = \frac{G_1 G_2}{1 - G_1 \cancel{G_2} \left(\frac{1+G_2}{\cancel{G_2}}\right)} = \frac{G_1 G_2}{1 - G_1 - G_1 G_2}$$

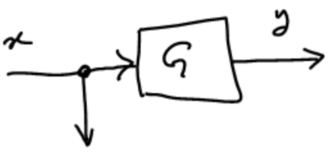
2.7.3 Simplification of complex block diagrams

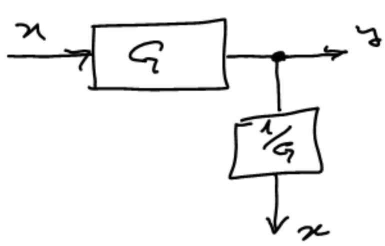
To determine the transfer function of complex diagrams, these block diagrams can be simplified. This results in various combinations of parallel, series or feedback connections. The original block diagrams should be modified carefully,so that the functionality of the diagram remain unchanged. Some of the allowable simplification rules are shown in Table 2.7.3.

Table 2.6: Some simplification rules for block diagram.

Block diagram	**Equivalent Representation**
	
	
	
	
	
	

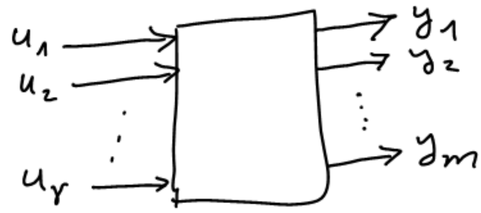
$(x_1 + x_2) \cdot G_1$
$= G_1 x_1 + G_1 x_2$





2.7.4 Multiple-variable systems

Multiple-input multiple-output (MIMO) systems with r inputs $u_1, u_2, \ldots, u_r$ and m outputs $y_1, y_2, \ldots, y_m$ can be modelled by the following equation:



$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s),$$

where

$$\mathbf{U}(s) = \begin{pmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_r(s) \end{pmatrix} \quad \text{and} \quad \mathbf{Y}(s) = \begin{pmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_m(s) \end{pmatrix}$$

are the vectors of input and output signals in s domain and

$$\mathbf{G}(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1r}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2r}(s) \\ \vdots & \vdots & \ddots & \vdots \\ G_{m1}(s) & G_{m2}(s) & \cdots & G_{mr}(s) \end{pmatrix}$$

is the matrix of transfer functions of the system.
Each element $G_{ij}(s)$ of this matrix represents the transfer function between input u_j and output y_i

$$G_{ij}(s) = \frac{Y_i(s)}{U_j(s)}.$$

R In order to compute $G_{ij}(s)$, all the inputs other than u_j are considered to be zero.

■ **Example 2.21** Consider the following block diagram.

$$G_x = \frac{\frac{1}{1{,}5s}}{1+\frac{1}{1{,}5s}} = \frac{1}{1{,}5s+1}$$

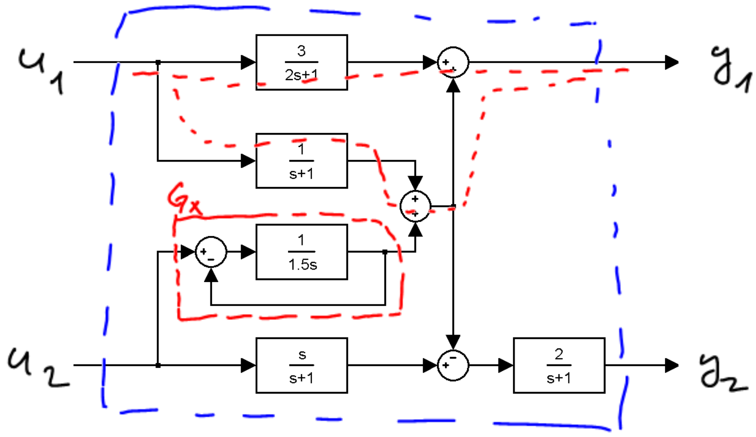


Figure 2.14: Example of a multiple-input multiple-output system

The transfer function matrix of this system is:

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

$$\mathbf{G}(s) = \begin{pmatrix} \frac{3}{2s+1} + \frac{1}{s+1} & \frac{1}{1.5s+1} \\ \frac{-2}{s+1} \cdot \frac{1}{s+1} & \frac{2}{s+1} \cdot \left(\frac{s}{s+1} - \frac{1}{1.5s+1}\right) \end{pmatrix}$$

■

$$G_{11}(s) = \frac{Y_1(s)}{U_1(s)} = \frac{3}{2s+1} + \frac{1}{s+1}$$

$$G_{21}(s) = \frac{Y_2(s)}{U_1(s)} = -\frac{1}{s+1} \cdot \frac{2}{s+1}$$

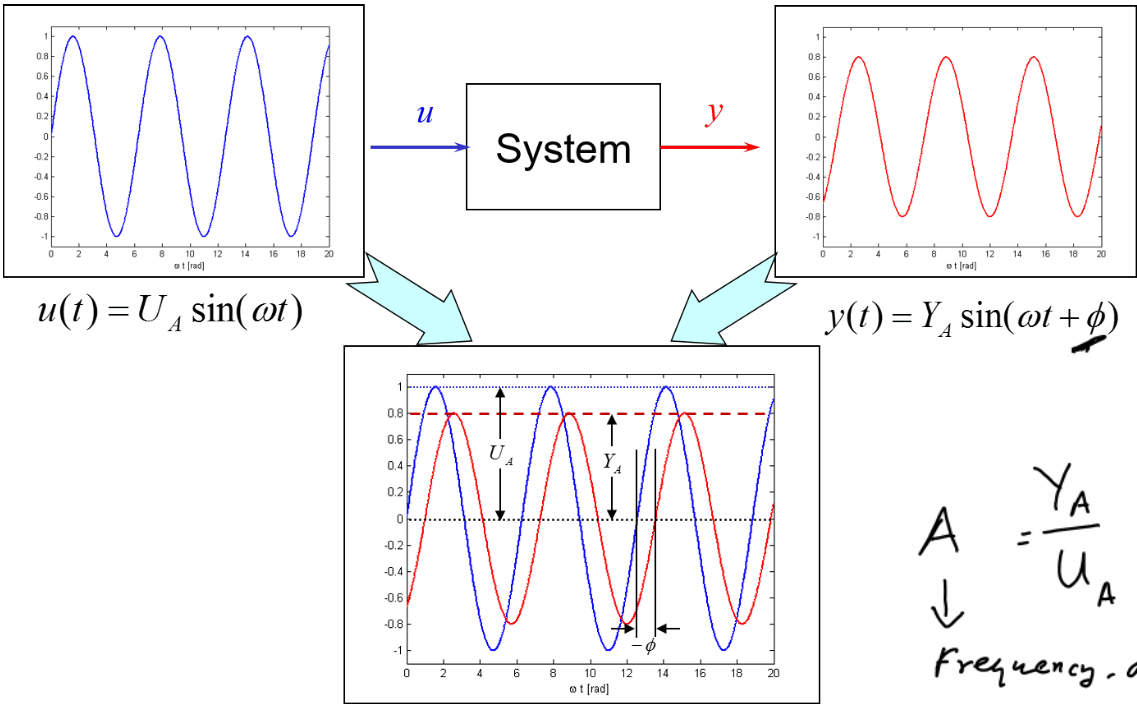
$$G_{12}(s) = \frac{Y_1(s)}{U_2(s)} = G_x = \frac{1}{1{,}5s+1}$$

$$G_{22}(s) = \frac{Y_2(s)}{U_2(s)} = \left(\frac{s}{s+1} - G_x\right)\left(\frac{2}{s+1}\right)$$

$$= \left(\frac{s}{s+1} - \frac{1}{1{,}5s+1}\right)\frac{2}{s+1}$$

2.8 Frequency response

2.8.1 Definition



$A = \frac{Y_A}{U_A}$

$\downarrow$

Frequency-dependent $A(\omega)$

$\phi = \phi_y - \phi_u \longrightarrow \phi(\omega)$

Figure 2.15: Basic idea of the frequency response

Frequency response of a system is a complex function of frequency ω.

$$G(j\omega) = A(\omega) \cdot e^{j\phi(\omega)} \tag{2.11}$$

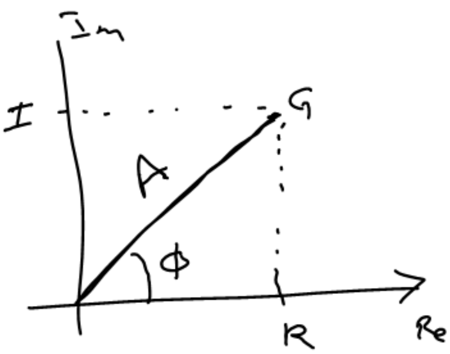
$A(\omega)$: Magnitude response
$\phi(\omega)$: Phase response
The frequency response can also be expressed in rectangular coordinates.

$$G(j\omega) = R(\omega) + j \cdot I(\omega) \tag{2.12}$$

$R(\omega)$: Real part of the frequency response
$I(\omega)$: Imaginary part of the frequency response

Basic mathematics of complex numbers:

$$A(\omega) = \sqrt{R^2(\omega) + I^2(\omega)} \tag{2.13}$$



$$\phi(\omega) = \arctan\left(\frac{I(\omega)}{R(\omega)}\right) \tag{2.14}$$

$$R(\omega) = A(\omega) \cdot \cos(\phi(\omega)) \tag{2.15}$$

$$I(\omega) = A(\omega) \cdot \sin(\phi(\omega)) \tag{2.16}$$

$$z = \frac{z_1 \cdot z_2}{z_3 \cdot z_4}$$

$$|z| = \frac{|z_1| \cdot |z_2|}{|z_3| \cdot |z_4|}$$

2.8.2 Relationship between frequency response and transfer function

$$\phi_z = \phi_{z_1} + \phi_{z_2} - \phi_{z_3} - \phi_{z_4}$$

The frequency response of system can be calculated from its transfer function by setting $s = j\omega$

$$G(s)|_{s=j\omega} = G(j\omega)$$

■ Example 2.22

Transfer function: $G(s) = \frac{s+1}{s(4s+5)}$

Frequency response: $G(j\omega) = \frac{j\omega+1}{j\omega(j4\omega+5)}$

Magnitude of the gain: $A(\omega) = |G(j\omega)| = \frac{\sqrt{\omega^2+1}}{\omega\sqrt{16\omega^2+25}}$

Phase: $\phi(\omega) = \arg\{G(j\omega)\} = \arctan(\omega) - 90^\circ - \arctan\left(\frac{4\omega}{5}\right)$

■

2.8.3 Graphical representation of frequency response

Nyquist plot

The Nyquist plot is the locus of the frequency response $G(j\omega)$ in the complex plane as ω is increased from 0 to ∞

Axis	Variable	Scale
x-Axis	Real part of $G(j\omega)$	Linear
y-Axis	Imaginary part of $G(j\omega)$	Linear

Bode plot

The Bode plot consists of two curves:

Curve 1: Magnitude curve

Axis	Variable	Units	Scale
x-Axis	Angular frequency ω	rad/s	logarithmic
y-Axis	Magnitude A_{dB}	dB	linear

Curve 2: Phase curve

Axis	Variable	Units	Scale
x-Axis	Angular frequency ω	rad/s	logarithmic
y-Axis	Phase ϕ	deg	linear

$$z = |z| \cdot e^{j\theta}$$

Ex. $$G(s) = \frac{2s+1}{s} \cdot e^{-2s}$$

$$A(\omega) = ? \qquad \phi(\omega) = ?$$

$$G(j\omega) = \frac{(j2\omega+1) \cdot e^{-j2\omega}}{j\omega}$$

$$A(\omega) = |G(j\omega)| = \frac{\sqrt{4\omega^2+1} \cdot 1}{\omega}$$

$$\phi(\omega) = \arg(G(j\omega)) = \arctan\left(\frac{2\omega}{1}\right) - 90^\circ - 2\omega \cdot \frac{180^\circ}{\pi}$$

conversion from rad to degree.

■ **Example 2.23**

Transfer function: $G(s) = \frac{10}{2s+1}$

Frequency response: $G(j\omega) = \frac{10}{j2\omega+1}$

Magnitude: $A(\omega) = |G(j\omega)| = \frac{10}{\sqrt{4\omega^2+1}}$

Phase: $\phi(\omega) = \arg\{G(j\omega)\} = -\arctan(2\omega)$

Real part: $R(\omega) = \frac{10}{4\omega^2+1}$

Imaginary part: $I(\omega) = \frac{-20\omega}{4\omega^2+1}$

$A_{dB} = 20 \cdot \log_{10}(A)$

Table 2.7: Frequency response in tabular form

ω in rad/s	$A(\omega)$	$\phi(\omega)$ in [°]	$A_{dB}(\omega)$	$R(\omega)$	$I(\omega)$
0	10	0	20	10	0
0.05	9.95	-5.7	19.95	9.9	-0.99
0.1	9.8	-11.3	19.83	9.62	-1.92
0.3	8.57	-30.96	18.66	7.35	-4.41
0.5	7.07	-45	16.99	5	-5
0.7	5.81	-54.46	15.29	3.38	-4.73
1	4.47	-63.43	13.01	2	-4
10	0.5	-87.14	-6.03	0.025	-0.5
100	0.05	-89.71	-26.02	0.0002	-0.05
∞	0	-90	-∞	0	0

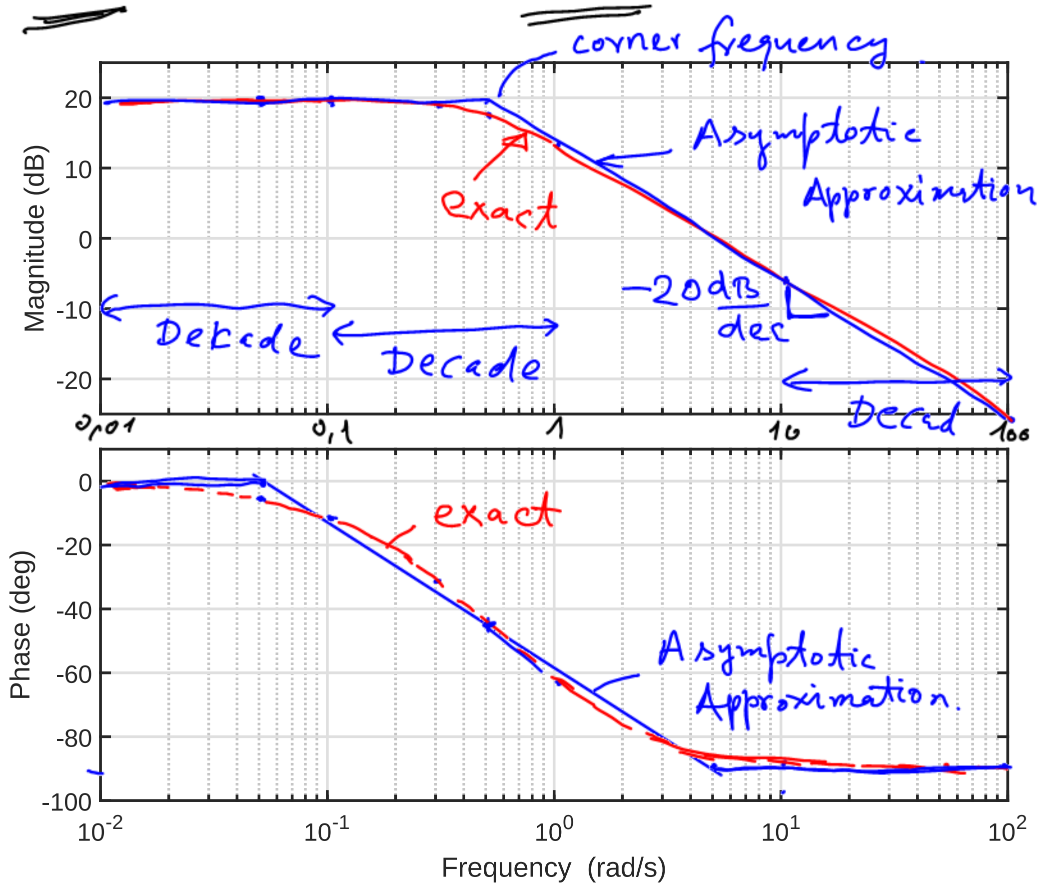
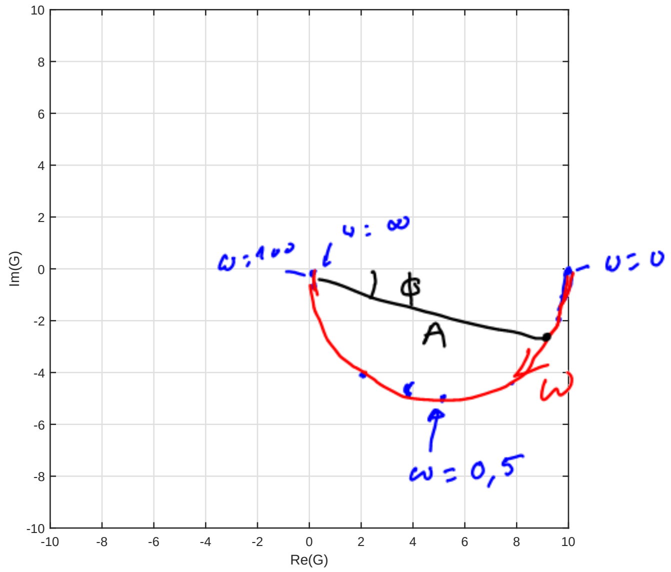


Figure 2.16: Bode plot of the system

$R(\omega)$	$I(\omega)$
10	0
9.9	-0.99
9.62	-1.92
7.35	-4.41
5	-5
3.38	-4.73
2	-4
0.025	-0.5
0.0002	-0.05
0	0



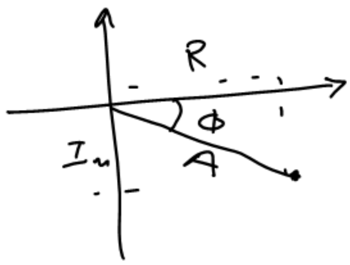


Figure 2.17: Nyquist plot of the system

■

2.8.4 Construction of Bode plots

Manually sketching the Bode diagram of a complicated transfer function can be done in three steps

1. Factorise $G(s)$ or $G(j\omega)$ in basic factors
2. Draw amplitude and phase curves of the basic factors
3. Add the curves to get Bode diagram of the whole system

Basic factors of $G(s)$

Any transfer function $G(s)$ can be represented as a product of the following basic factors. For reasons of simplicity, only stable systems with zeros in the left s-half plane are considered in this course. Of course, this discussion can be extended to other systems.

1. Gain K
2. Integrator or differentiator:
$$\left(\frac{1}{s}\right) \text{ oder } (s)$$
3. First order polynomial:
$$\left(\frac{1}{\frac{s}{\omega_e}+1}\right) \text{ or } \left(\frac{s}{\omega_e}+1\right) \quad \text{with} \quad \omega_e > 0$$
4. Second order polynomial:
$$\left(\frac{1}{\left(\frac{s}{\omega_0}\right)^2+2d\left(\frac{s}{\omega_0}\right)+1}\right) \text{ or } \left(\left(\frac{s}{\omega_0}\right)^2+2d\left(\frac{s}{\omega_0}\right)+1\right)$$
with $\omega_0 > 0$ and $0 \leq d < 1$
5. Exponential function (transport delay):
$$\mathrm{e}^{-sT_T} \text{ with } \quad T_T > 0$$

■ **Example 2.24** The transfer function

$$G_1(s) = \frac{(5s+1)e^{-3s}}{(s^2+s+4)(s+1)s}$$

$$s^2+s+4 = 4\left(\frac{s^2}{4}+\frac{s}{4}+1\right) = 4\cdot\left(\left(\frac{s}{\underbrace{2}_{\omega_0}}\right)^2+2\cdot\underbrace{\left(1/4\right)}_{d}\cdot\frac{s}{2}+1\right)$$

can be factorised as:

$$G_1(s) = \left(\frac{s}{0.2}+1\right)\cdot\left(\frac{1}{s}\right)\cdot\left(\frac{1}{\frac{s}{1}+1}\right)\cdot e^{-3s}\cdot\frac{1}{4}\cdot\left(\frac{1}{\left(\frac{s}{2}\right)^2+2\cdot\frac{1}{4}\cdot\left(\frac{s}{2}\right)+1}\right)$$

■

■ **Example 2.25** The transfer function

$$G_2(s) = \frac{(s+10)s}{(s+100)(s+20)}$$

can be written as:

$$G_2(s) = \frac{1}{200}\cdot\left(\frac{s}{10}+1\right)\cdot\left(\frac{1}{\frac{s}{20}+1}\right)\left(\frac{1}{\frac{s}{100}+1}\right)\cdot s$$

■

Exercise 2.2 Factorise the following transfer function to get the basic factors in standard form:

$$G_a(s) = 10\left(1+\frac{1}{0.2s}+\frac{0.1s}{0.01s+1}\right)$$

$$= 10\,\frac{0{,}2s(0{,}01s+1)+0{,}01s+1+0{,}1s\cdot 0{,}2s}{0{,}2s(0{,}01s+1)}$$

$$= 10\,\frac{0{,}002s^2+0{,}2s+0{,}01s+1+0{,}02s^2}{0{,}2s(0{,}01s+1)}$$

$$= 10\,\frac{0{,}022s^2+0{,}21s+1}{0{,}2s(0{,}01s+1)}$$

$$= 50\cdot\left(\frac{1}{s}\right)\cdot\frac{1}{\left(\frac{s}{100}+1\right)}\cdot\left(\left(\frac{s}{\sqrt{0{,}022}}\right)^2+2\cdot\left(\frac{\overset{0{,}21}{\sqrt{0{,}022}}}{2}\right)\cdot\frac{s}{\sqrt{0{,}022}}+1\right)$$

■

Bode plots of basic factors

Gain K

$K \in \mathbb{R}$

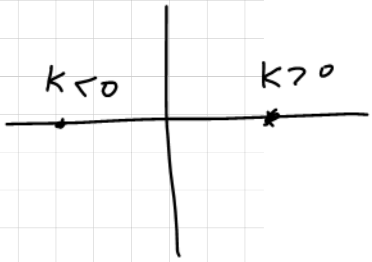
$$G(s) = K$$

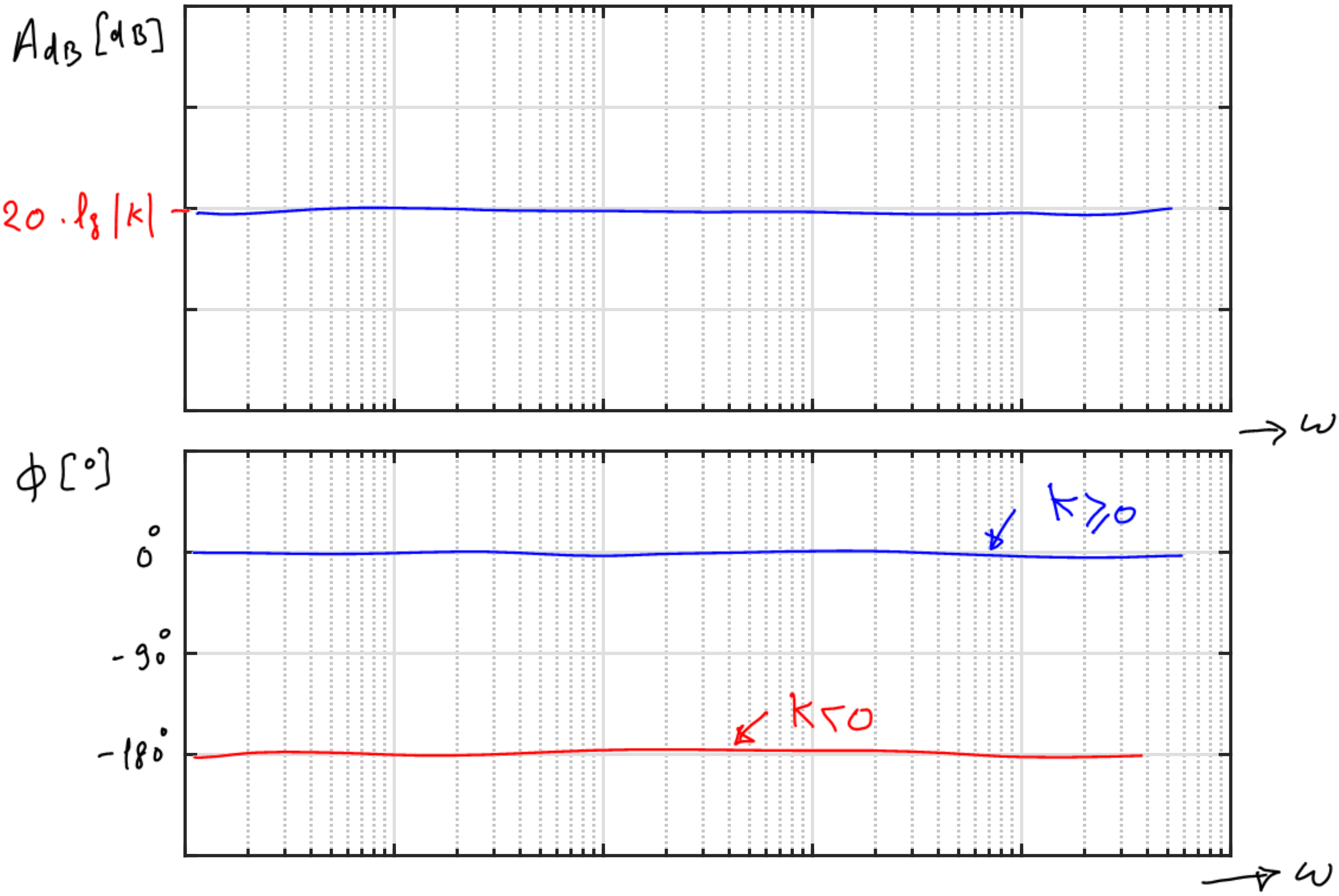
$$G(j\omega) = K$$

$$A(\omega) = |K|$$

$$\phi(\omega) = \begin{cases} 0^\circ & \text{if } K \geq 0 \\ \pm 180^\circ & \text{if } K < 0 \end{cases}$$

$$A_{dB} = 20 \cdot \lg |K|$$





Integrator

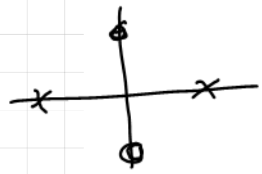
$$G(s) = \frac{1}{s} \qquad \omega \geq 0$$

$$G(j\omega) = \frac{1}{j\omega}$$

$$\phi = \arctan\left(\frac{I}{R}\right)$$

$$G(j\omega) = j\omega = 0 + j\omega$$

$$\phi = \arctan\left(\frac{\omega}{0}\right) = \arctan(\infty) = 90^\circ$$

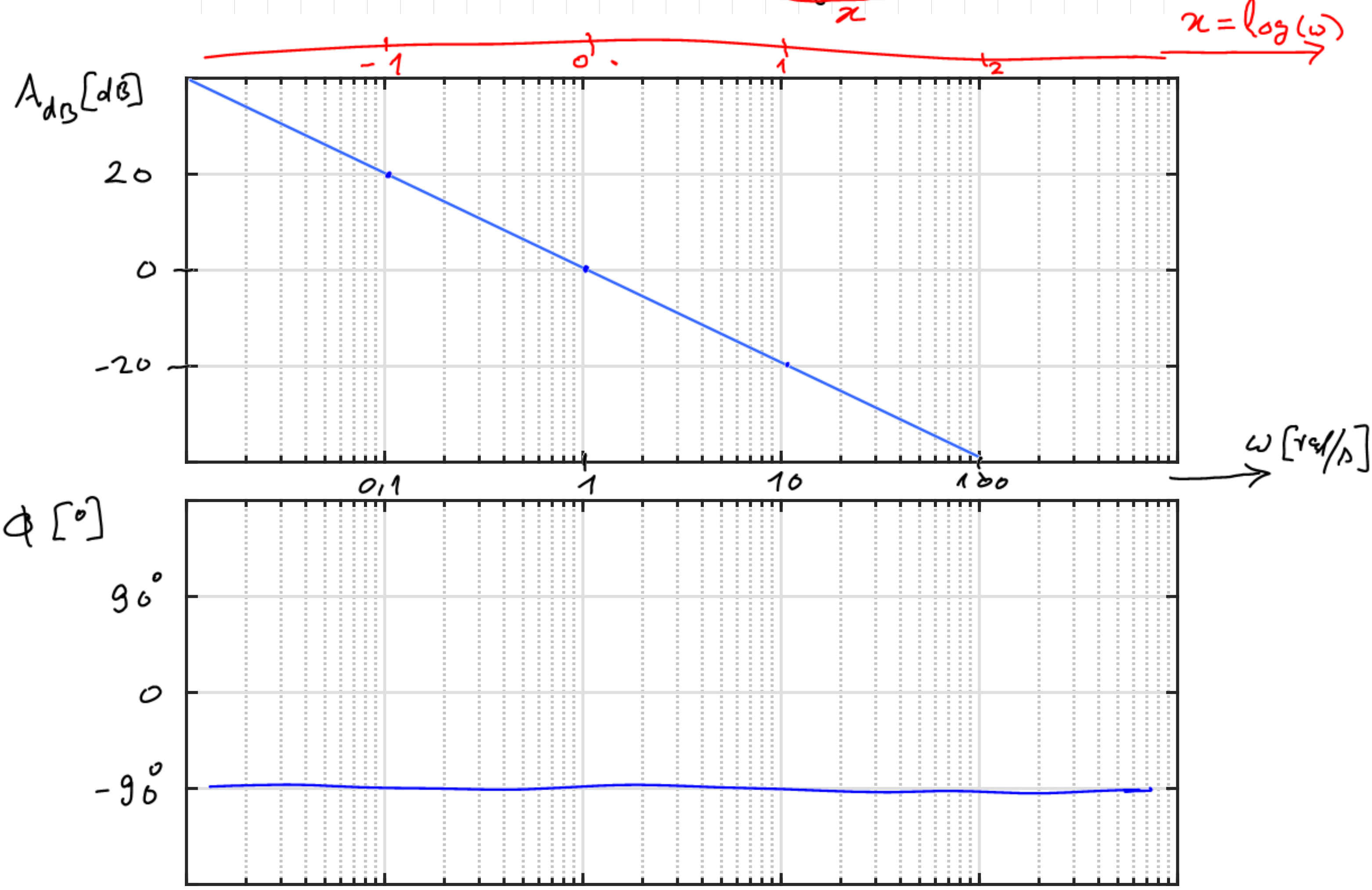


$$A(\omega) = |G(j\omega)| = \frac{1}{\omega}$$

$$\phi(\omega) = 0 - 90^\circ = -90^\circ$$

$$A_{dB} = 20 \cdot \log\left(\frac{1}{\omega}\right) = 20 \cdot \log 1 - 20 \log(\omega)$$

$$= -20 \underbrace{\log(\omega)}_{x}$$



Bode plot of a differentiator can be drawn by vertically mirroring both curves across ω-axis.

First order polynomial in denominator (a real pole)

$\omega_e > 0$

$$G(s) = \frac{1}{\frac{s}{\omega_e}+1}$$

$$G(j\omega) = \frac{1}{\frac{j\omega}{\omega_e}+1}$$

$$A(\omega) = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_e}\right)^2+1}}$$

$$\phi(\omega) = -\arctan\left(\frac{\omega}{\omega_e}\right)$$

$\omega = \omega_e$: $\phi(\omega) = -\mathrm{atan}\left(\frac{\omega_e}{\omega_e}\right) = -45^\circ$

$\omega = 0{,}1\omega_e$: $\phi(\omega) = -\mathrm{atan}(0{,}1) = -5{,}7^\circ$

$\omega = 10\omega_e$: $\phi(\omega) = -\mathrm{atan}(10) = -84{,}3^\circ$

$$\Rightarrow A_{dB} = 20\log\left(\frac{1}{\sqrt{\left(\frac{\omega}{\omega_e}\right)^2+1}}\right) = -20\log\sqrt{\left(\frac{\omega}{\omega_e}\right)^2+1}$$

case $\omega \ll \omega_e$: $\left(\frac{\omega}{\omega_e}\right)^2+1 \approx 1$

$$A_{dB} \approx -20\log\sqrt{1} = 0\,dB$$

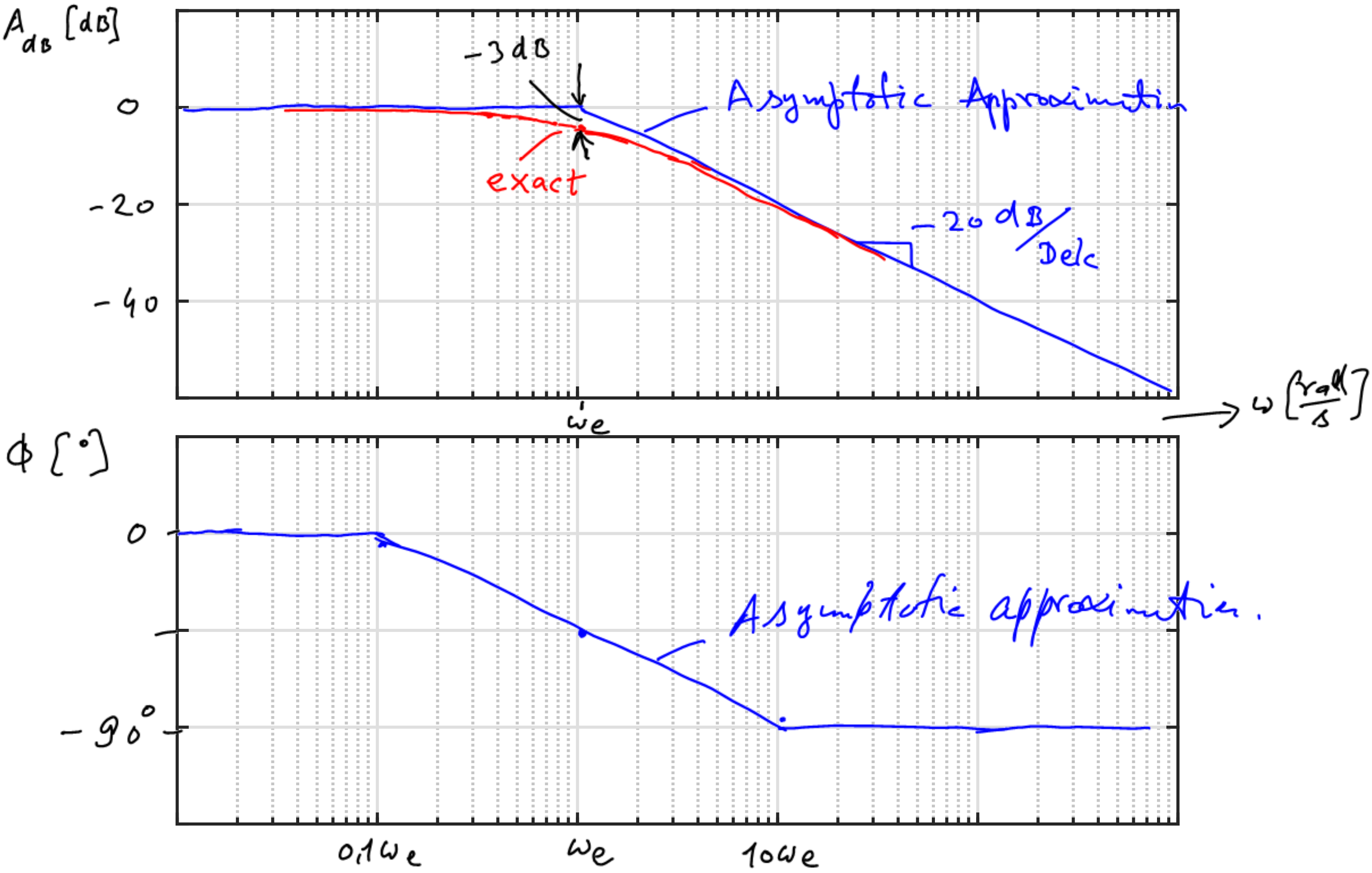
case $\omega \gg \omega_e$: $\left(\frac{\omega}{\omega_e}\right)^2+1 \approx \left(\frac{\omega}{\omega_e}\right)^2$

$$A_{dB} = -20\cdot\log\sqrt{\left(\frac{\omega}{\omega_e}\right)^2} = -20\log\frac{\omega}{\omega_e}$$

$$= -20\underbrace{\log\omega}_{x} + 20\log\omega_e$$

Case $\omega = \omega_e$

$$A_{dB} = -20\log\sqrt{\left(\frac{\omega_e}{\omega_e}\right)^2+1} = -20\log\sqrt{2} = -3\,dB$$



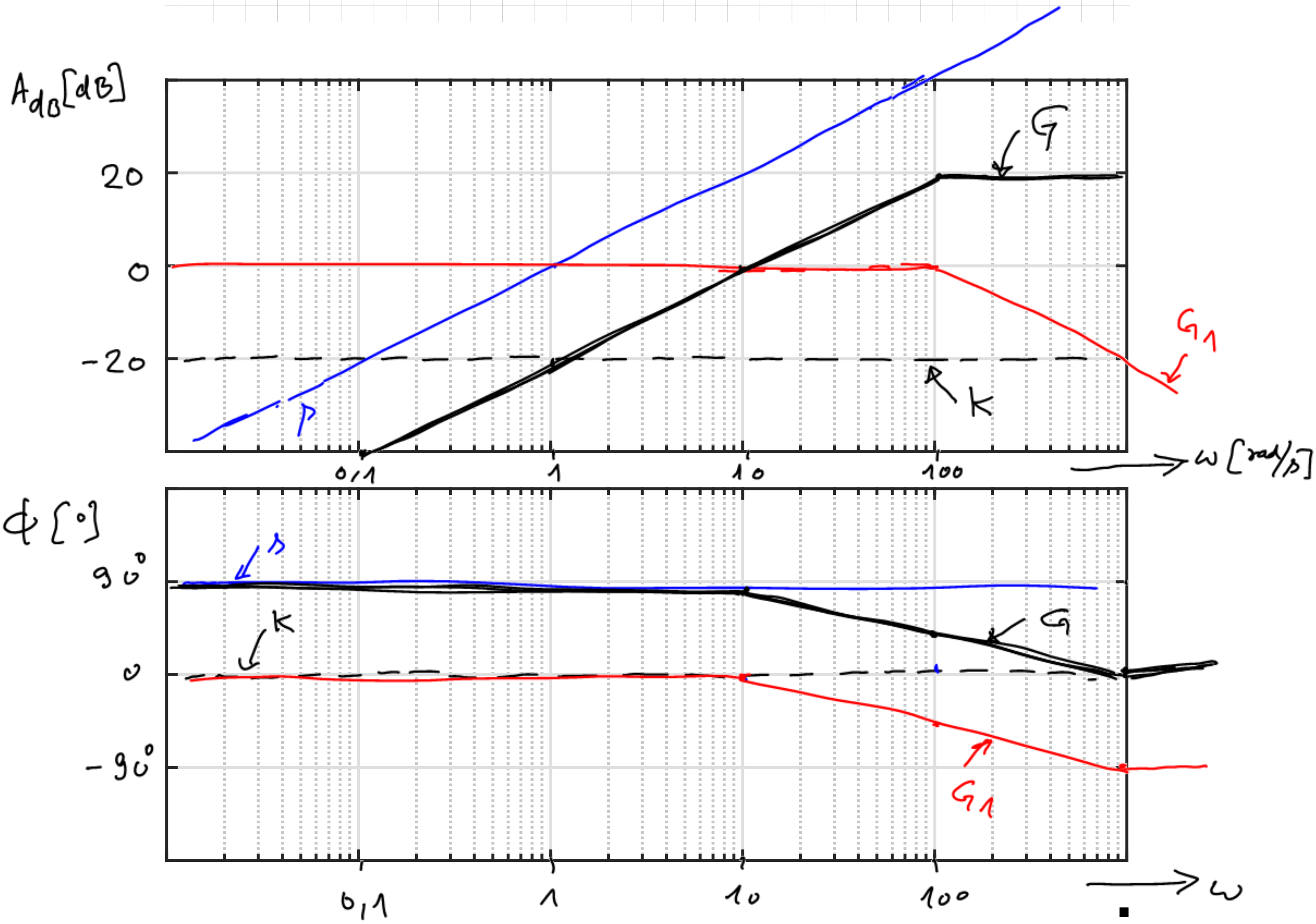
R Bode plot of the first order polynomial in numerator (real zero) can be drawn by vertically mirroring both curves across ω-axis.

■ **Example 2.26** Sketch the Bode plot of the following transfer function:

$$G(s) = \frac{10s}{s+100}$$

$$= \frac{10 \cdot s}{100\left(\frac{s}{100}+1\right)} = \underbrace{0{,}1}_{K} \cdot \underbrace{s}_{} \cdot \underbrace{\frac{1}{\frac{s}{100}+1}}_{G_1}$$

$$K_{dB} = 20 \log(0{,}1) = -20 \text{ dB}$$



■

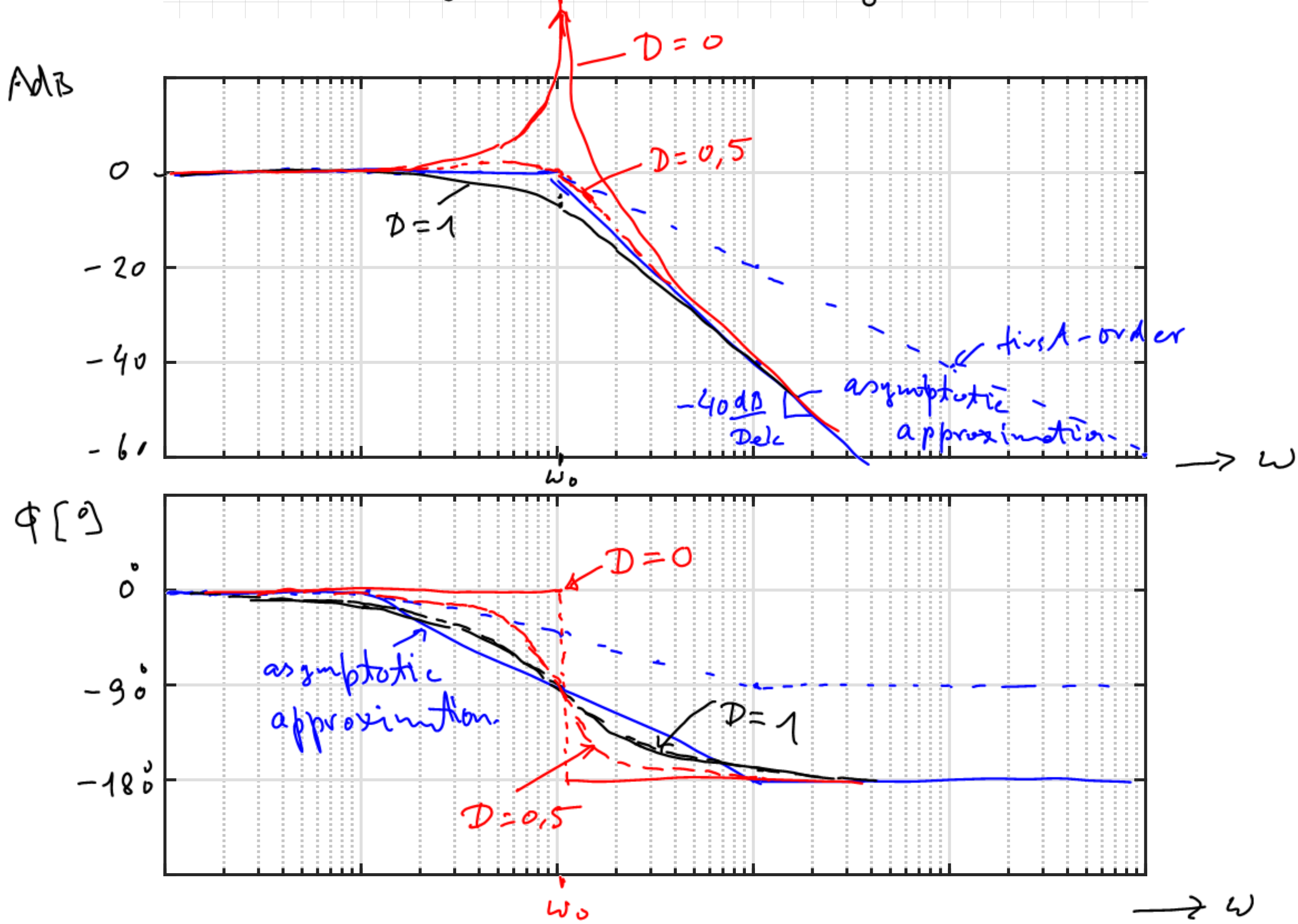
Second order polynomial in denominator (a pair of poles)

$$G(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2 \cdot D \cdot \frac{s}{\omega_0} + 1} \qquad \omega_0 > 0$$

if $D > 1$ then

$$\frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2D\frac{s}{\omega_0} + 1} = \frac{1}{\left(\frac{s}{\omega_{e1}} + 1\right)\left(\frac{s}{\omega_{e2}} + 1\right)}$$

$\Rightarrow$ No need to consider here, as we know how to skech curves of first order polynomials.



Bode plot of the second order polynomial in numerator (a pair of zeros) can be drawn by vertically mirroring both curves across ω-axis.

$1 \geq D \geq 0$

if $D = 1$

$$G(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 2 \cdot \left(\frac{s}{\omega_0}\right) + 1} = \frac{1}{\left(\frac{s}{\omega_0} + 1\right)^2}$$

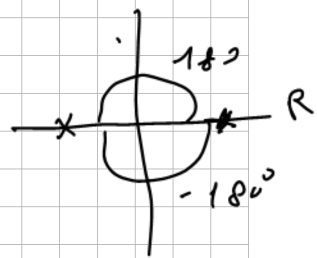
$\Rightarrow$ two first-order polynomials with same corner frequency.

if $D = 0$

$$G(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + 1}$$

$$G(j\omega) = \frac{1}{\left(\frac{j\omega}{\omega_0}\right)^2 + 1} = \frac{1}{-\frac{\omega^2}{\omega_0^2} + 1} = \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2}$$

$$A(\omega) = |G(j\omega)| = \left| \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2} \right|$$



$$\phi(\omega) = \begin{cases} 0^\circ & \omega < \omega_0 \\ -180^\circ & \omega > \omega_0 \end{cases}$$

$$A_{dB} = 20 \cdot \log A(\omega) = 20 \cdot \log \left| \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2} \right|$$

case 1: $\omega \ll \omega_0 \qquad 1 - \left(\frac{\omega}{\omega_0}\right)^2 \approx 1$

$$A_{dB} = 20 \cdot \log 1 = 0 \text{ dB}$$

case 2: $\omega \gg \omega_0 \Rightarrow 1 - \left(\frac{\omega}{\omega_0}\right)^2 \approx -\left(\frac{\omega}{\omega_0}\right)^2$

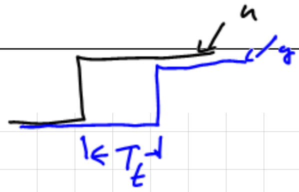
$$A_{dB} = 20 \cdot \log \left| \frac{1}{-\left(\frac{\omega}{\omega_0}\right)^2} \right| = 20 \cdot \lg \left(\frac{\omega_0}{\omega}\right)^2 =$$

$$= 40 \log \omega_0 - 40 \underbrace{\lg \omega}_{x}$$

Case 3: $\omega = \omega_0$

$$A_{dB} = 20 \cdot \log \left| \frac{1}{1-\left(\frac{\omega_0}{\omega_0}\right)^2} \right| = \infty$$

Transport delay



$$G(s) = e^{-sT_t}$$

$$G(j\omega) = 1 \cdot e^{-j\omega T_t}$$

$$A(\omega) = 1 \Longrightarrow A_{dB} = 20 \cdot \lg(1) = 0\,dB$$

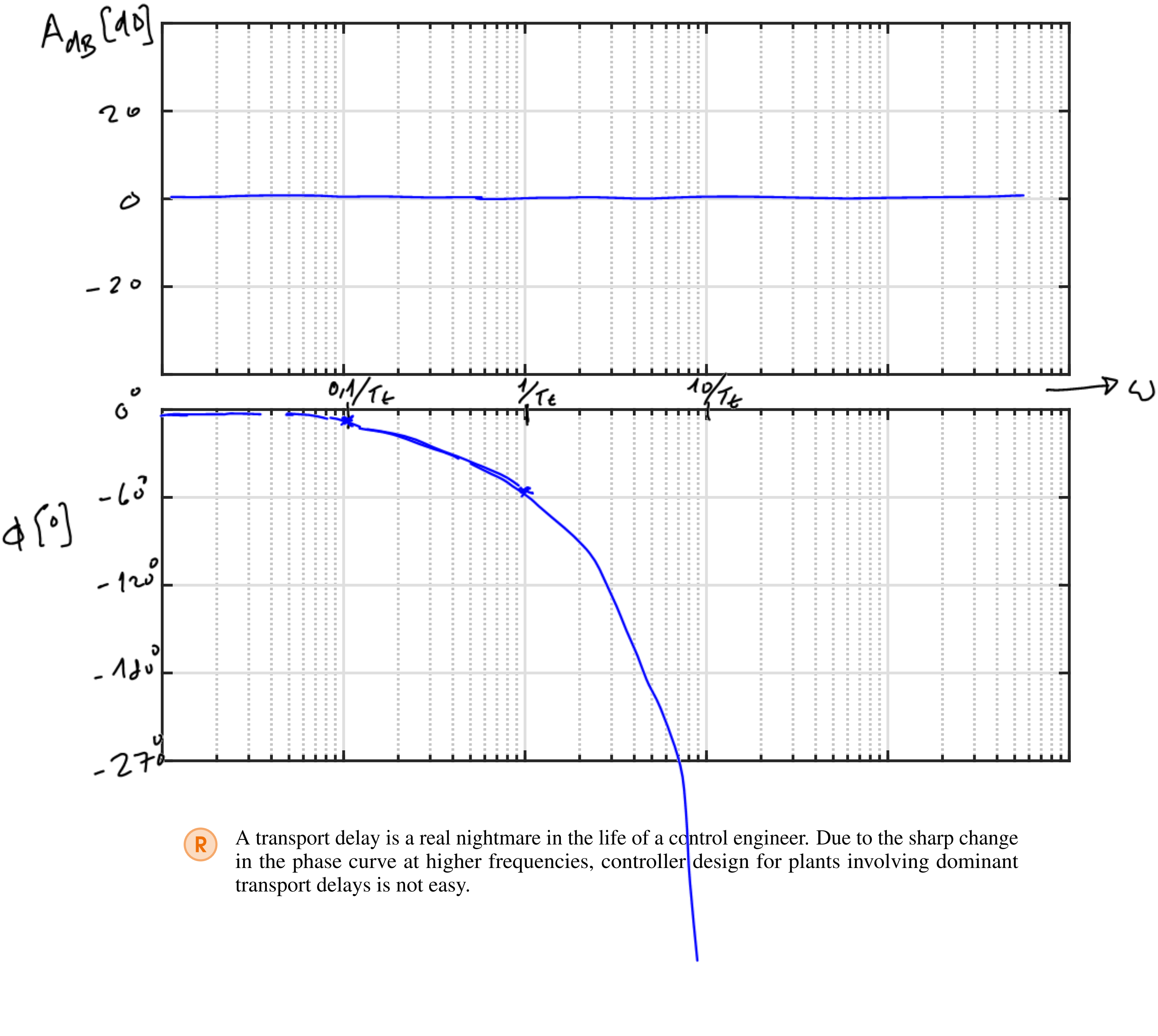
$$\phi(\omega) = -\omega \cdot T_t \cdot \frac{180^\circ}{\pi} = -\frac{180^\circ}{\pi} \cdot T_t \cdot \omega$$

$$\omega = 0 \Rightarrow \phi(\omega) = 0$$

$$\omega = \frac{1}{T_t} = \phi(\omega) = -\frac{180^\circ}{\pi} = 57^\circ$$

$$\omega = \frac{0.1}{T_t} \Rightarrow \phi(\omega) = -\frac{180}{\pi} \cdot 0.1 = 5.7^\circ$$

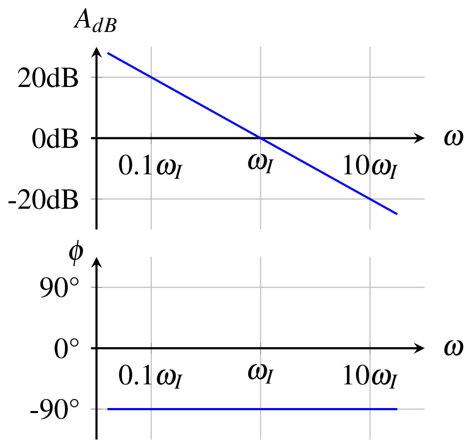
$$\omega = \frac{10}{T_t} \Rightarrow \phi(\omega) = -\frac{180^\circ}{\pi} \cdot 10 = 570^\circ$$



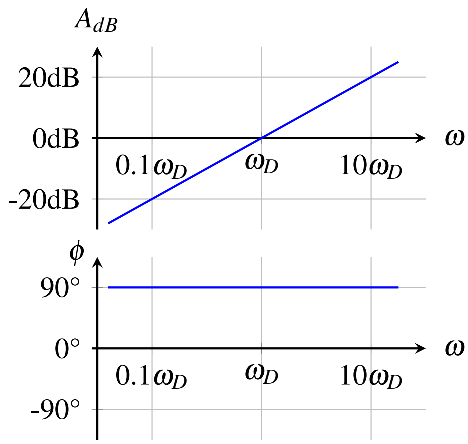
R A transport delay is a real nightmare in the life of a control engineer. Due to the sharp change in the phase curve at higher frequencies, controller design for plants involving dominant transport delays is not easy.

Table 2.8: Bode plots of some basic factors.

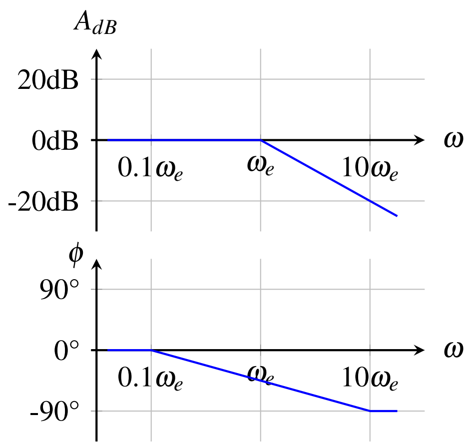
$$G(s) = \frac{\omega_I}{s}$$



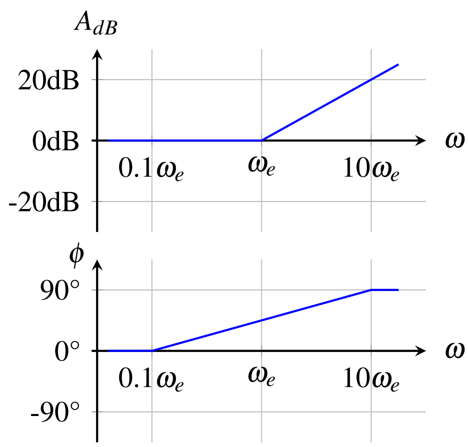
$$G(s) = \frac{s}{\omega_D}$$



$$G(s) = \frac{1}{\frac{s}{\omega_e} + 1}$$



$$G(s) = \frac{s}{\omega_e} + 1$$

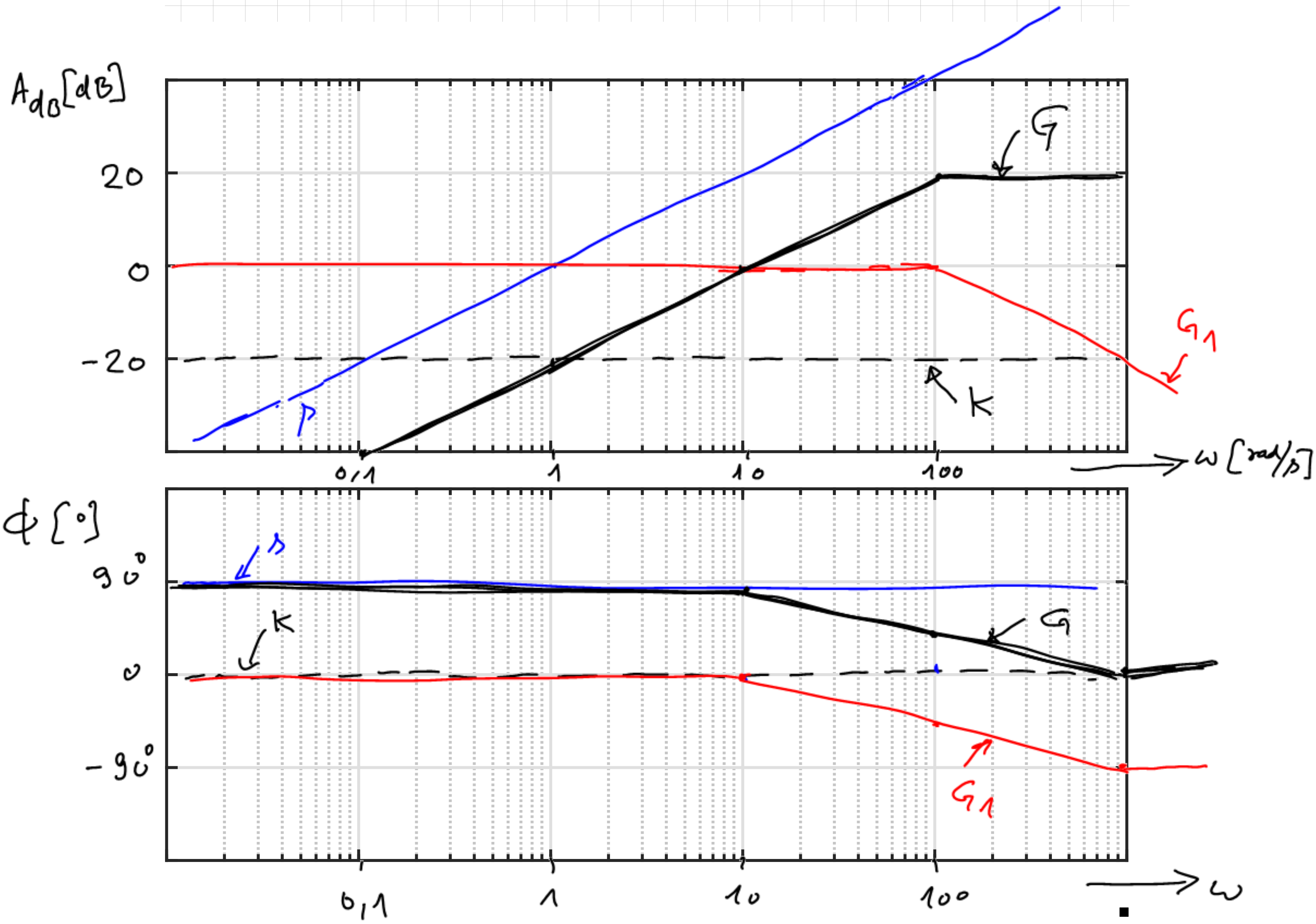


■ **Example 2.26** Sketch the Bode plot of the following transfer function:

$$G(s) = \frac{10s}{s+100}$$

$$= \frac{10\cdot s}{100\left(\frac{s}{100}+1\right)} = \underbrace{0,1}_{K} \cdot \underbrace{s} \cdot \underbrace{\frac{1}{\frac{s}{100}+1}}_{G_1}$$

$$K_{dB} = 20\log(0,1) = -20\ dB$$

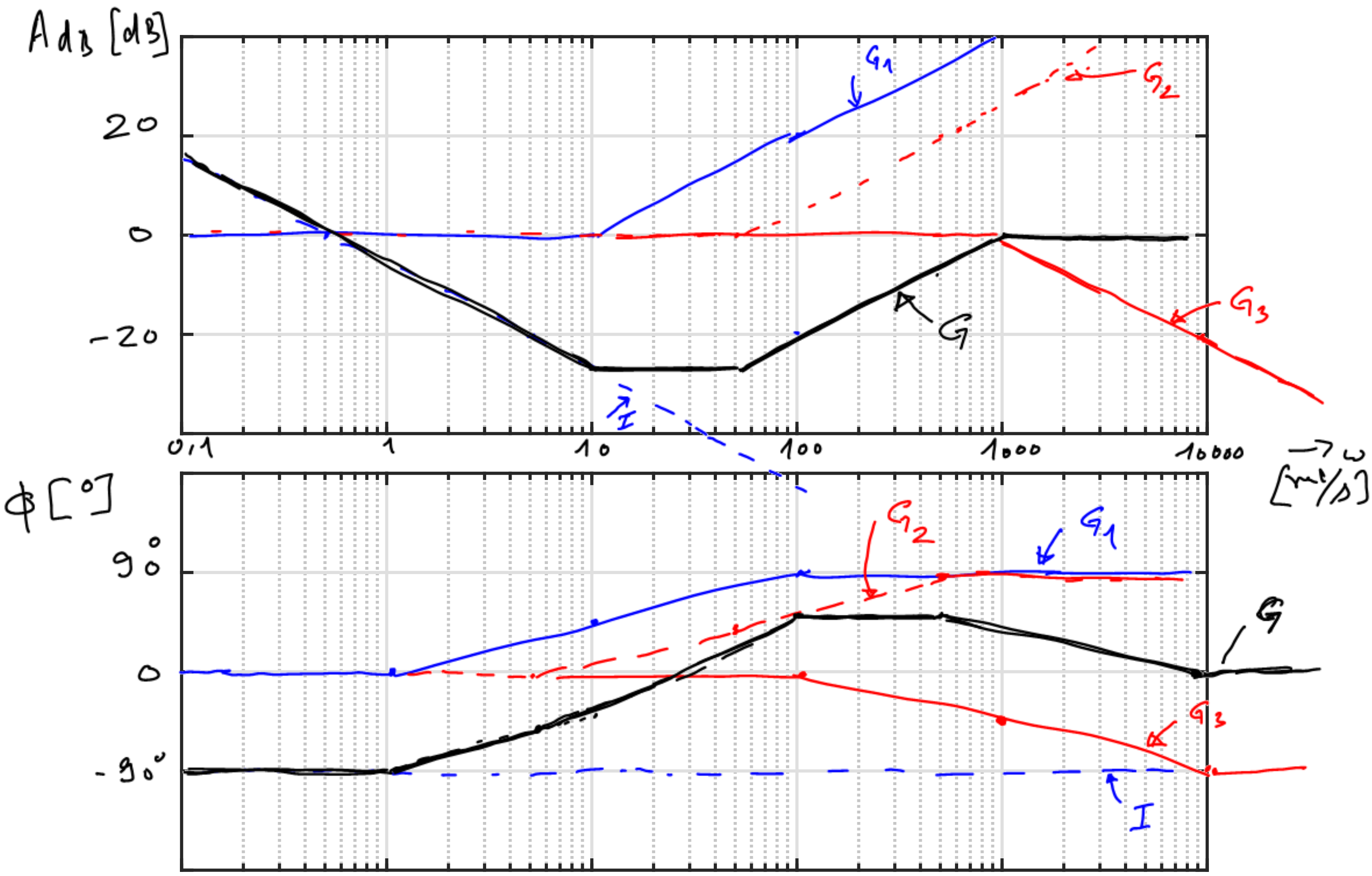


■

■ **Example 2.27** Sketch the Bode plot of the following transfer function:

$$G(s) = \frac{(s+10)(s+50)}{s^2+1000s} = \frac{10\left(\frac{s}{10}+1\right)\cdot 50\left(\frac{s}{50}+1\right)}{1000\,s\left(\frac{s}{1000}+1\right)}$$

$$= \underbrace{\frac{0{,}5}{s}}_{I}\cdot\underbrace{\left(\frac{s}{10}+1\right)}_{G_1}\cdot\underbrace{\left(\frac{s}{50}+1\right)}_{G_2}\cdot\underbrace{\frac{1}{\frac{s}{1000}+1}}_{G_3}$$



■

3.1 Basic structure of a control loop

Control is a mechanism, which manipulates the input of a system with the aim to bring or keep the output of the system to a desired value. The desired value, often known as reference signal or set-point, may be constant or a time-varying quantity.
The essential feature of such a control is the feedback loop. The output $y(t)$ of the plant is continuously measured. It is compared with the desired behaviour $r(t)$ to calculate control deviation $e(t)$. This deviation is used by the controller to calculate the proper value for the plant input.

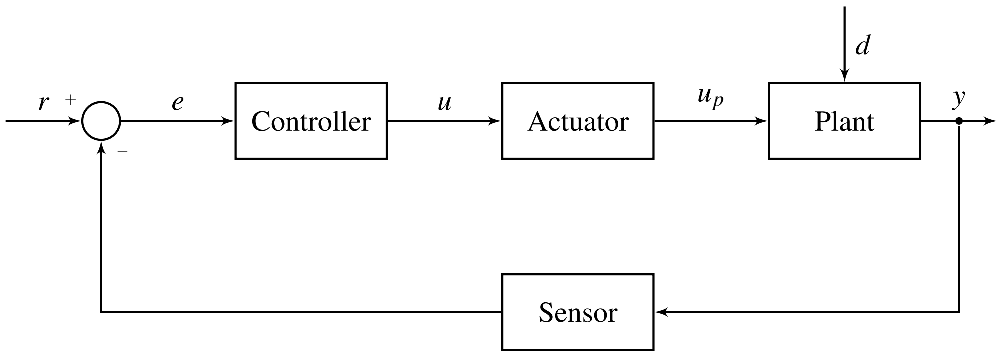


Figure 3.1: The basic structure of a standard control loop

r: Reference signal / desired value / set-point
e: Control deviation / control error
u: Controller output / control signal /control variable
u_p: Actuator output
d: Disturbance
y: Plant output / process output / controlled variable

A simplified model of the control loop is drawn as block diagram in Figure 3.2. The controller is represented by the transfer function $G_C(s)$ whereas $G_P(s)$ describes the behaviour of the extended plant including actuator, plant and sensor. Die disturbance d is transformed to the output of the plant.

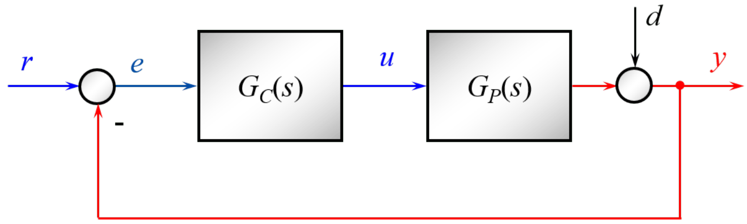


Figure 3.2: Simplified model of the closed loop

If we remove the feedback connection for instant then we the block diagram of an open-loop system shown in Figure 3.3. The open-loop transfer function $G_0(s)$ can be calculated as:

$$G_0(s) = G_C(s)G_P(s). \tag{3.1}$$

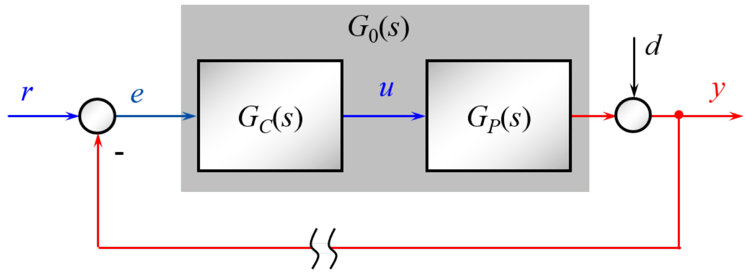


Figure 3.3: The open loop

The system output of the closed-loop system can be calculated using the following expression:

$$Y(s) = \frac{G_P(s)G_C(s)}{1+G_P(s)G_C(s)}R(s) + \frac{1}{1+G_P(s)G_C(s)}D(s) \tag{3.2}$$

The transfer function

$$G_{YR}(s) = \frac{G_P(s)G_C(s)}{1+G_P(s)G_C(s)} \tag{3.3}$$

describes the influence of the reference value (or set-point) $R(s)$ on the system output $Y(s)$ in a closed loop. The tracking performance of the closed loop can be described by using this model. Whereas, the transfer function

$$G_{YD}(s) = \frac{1}{1+G_P(s)G_C(s)} \tag{3.4}$$

describes the influence of the disturbance $D(s)$ on the system output $Y(s)$ in a closed loop. The regulation capability of the closed loop can be described by using this model.

3.2 Performance evaluation

Control systems are designed to perform specific tasks. In order to evaluate how good a control system is performing these tasks some performance specifications are formulated. These specifications are not only used for analysis and comparison purposes, but can also flow directly into the controller design procedures. For the sake of performance evaluation, the control objectives can be divided into two categories:

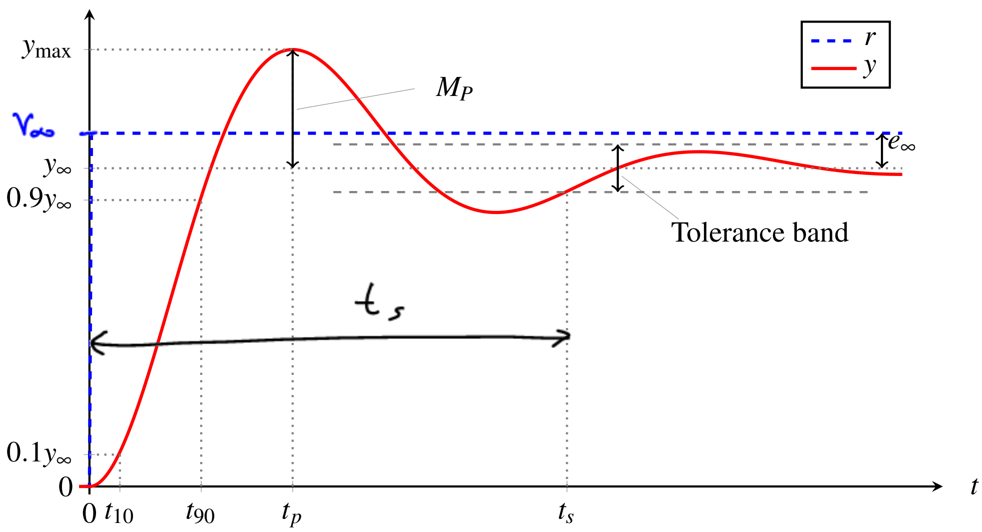
- Set-point tracking/servo control: The objective is to track the reference trajectories or varying set-points.
- Regulation / disturbance rejection: Influence of the disturbances on the system output should be suppressed.

A control system should react quickly to achieve these objectives as accurately as possible without oscillations.

The requirements for a control system are normally expressed as performance specifications. One can differentiate between two groups of specifications: transient response requirements (such as the maximum overshoot and settling time in step response) and steady-state requirements (such as steady-state error). For routine design problems, the performance specifications (which relate to accuracy, relative stability, and speed of response) may be given in terms of precise numerical values. In other cases they may be given partially in terms of precise numerical values and partially in terms of qualitative statements. Some requirement specification related to the step response of a control system are explained in the following subsections.

3.2.1 Set-point tracking / Servo control

Typical response of a control system to a step like change in the set-point is drawn in Figure 3.4.



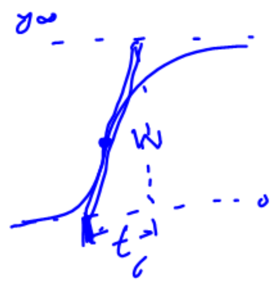


Figure 3.4: Control performance specifications for set-point tracking

- Rise time t_r is the time required for the response to rise from 10% to 90% of its final value.

$$t_r = t_{90} - t_{10}$$

- Peak time t_p is the time required for the response to reach the first peak of the overshoot.
- Maximum overshoot M_p is the maximum peak value of the response curve measured in percent of the final value

$$M_p = \frac{y_{\max} - y_\infty}{y_\infty} \cdot 100\%$$

- Settling time t_s is the time required for the system output to reach and stay within a pre-defined tolerance band (e.g. 5%) about the final value.
- Steady-state control error e_∞ is the deviation of the system output from the set-point in steady-state measured in percent of the set-point.

$$e_\infty = \frac{r_\infty - y_\infty}{r_\infty} \cdot 100\%$$

3.2.2 Regulation / disturbance rejection

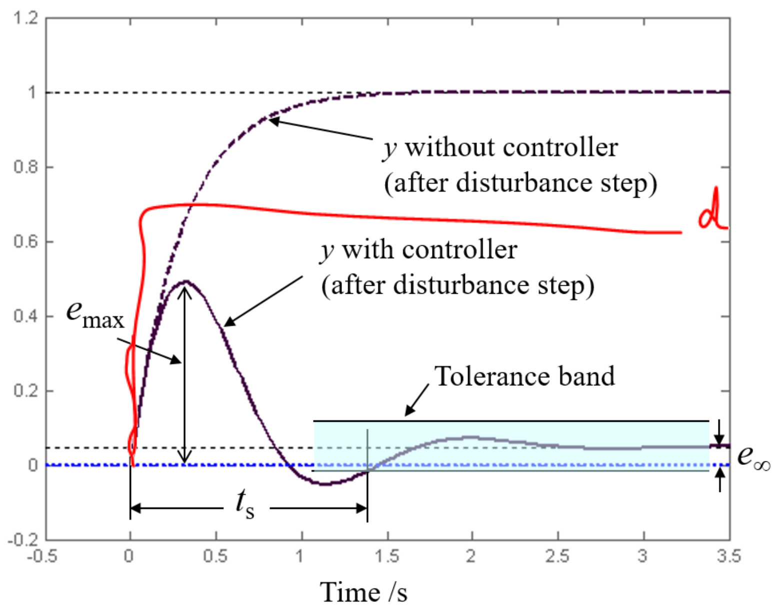


Figure 3.5: Control performance specifications for disturbance rejection

- Maximum error e_{max} is the maximum value of the control deviation.
- Settling time t_s is the time required for the system output to reach and stay within a pre-defined tolerance band (e.g. 5%) about the final value.
- Steady-state control error e_∞ is the deviation of the system output from the set-point in steady-state.

2.5 Important properties of dynamic systems

2.5.1 Stability

Three types of equilibrium

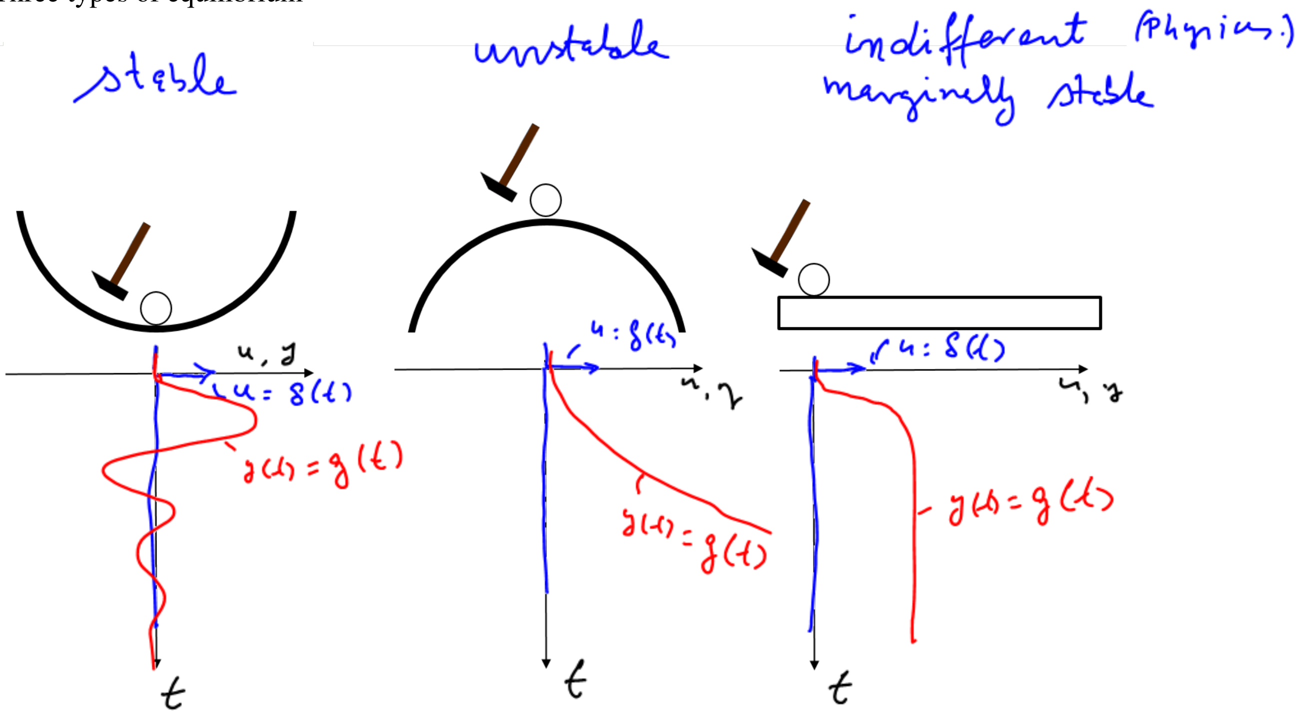


Figure 2.6: Three types of equilibrium

g(t): Impulse response

Stability analysis using the impulse response

For linear systems:

Asymptotically stable	$\lim\limits_{t\to\infty} g(t) = 0$
Marginally stable	$\lim\limits_{t\to\infty} g(t) \neq 0$ aber $\lvert g(t)\rvert < \infty$ for alle t
Unstable	$\lim\limits_{t\to\infty} \lvert g(t)\rvert \to \infty$

Stability analysis with help of pole locations

■ **Example 2.13** First order systems:

$$G(s) = \frac{1}{s-\sigma} \text{ with } \sigma \in \Re$$

Impulse response:

$$\frac{1}{s-\sigma} \qquad \mathcal{L}\{\delta(t)\} = 1$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-\sigma}\cdot 1\right\}$$

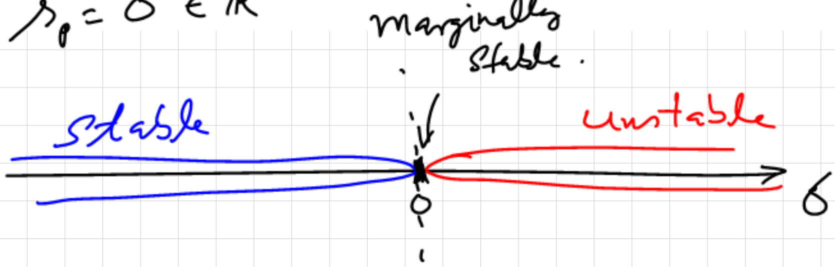
$$g(t) = e^{\sigma t}$$

$\sigma > 0$	$g(t) = e^{\sigma t} \Rightarrow g_\infty = e^{+\infty} = \infty \Rightarrow$ unstable.
$\sigma < 0$	$g(t) = e^{-\lvert\sigma\rvert t} \Rightarrow g_\infty = e^{-\infty} = 0 \Rightarrow$ stable.
$\sigma = 0$	$g(t) = e^{0t} = 1 \Rightarrow$ marginally stable.

Pole: $s_p = \sigma$

Statement about stability:

$s_p = \sigma \in \mathbb{R}$



■

■ **Example 2.14** Second order systems with complex poles:

$$G(s) = \frac{1}{(s-\sigma+j\omega)(s-\sigma-j\omega)} = \frac{1}{(s-\sigma)^2+\omega^2} \text{ with } \sigma,\omega \in \Re$$

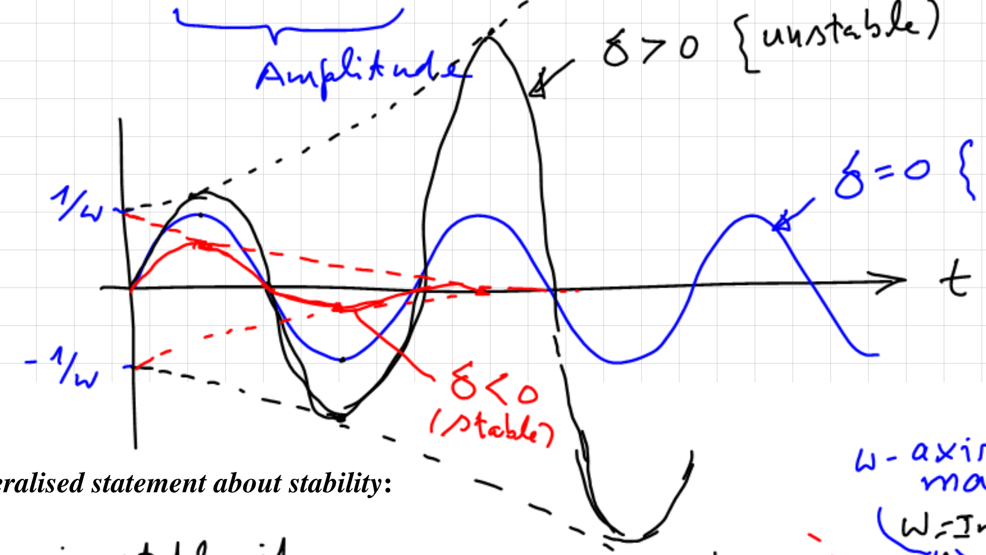
Impulse response:

Poles: $s_{1,2} = \underbrace{\sigma}_{\text{Real part}} \pm j \underbrace{\omega}_{\text{Imaginary}}$

$$g(t) = \mathcal{L}^{-1}\left\{ G(s) \cdot 1 \right\}$$

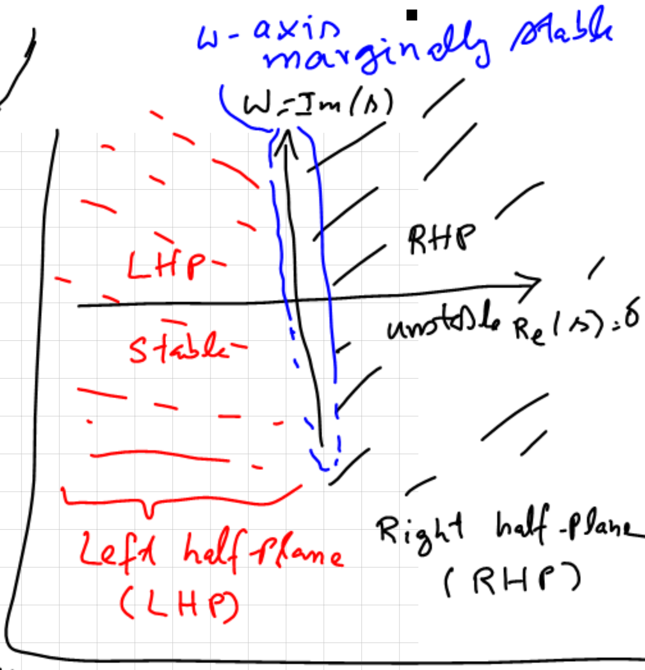
$$= \mathcal{L}^{-1}\left\{ \frac{1 \cdot \omega}{(s-\sigma)^2+\omega^2} \cdot \frac{1}{\omega} \right\}$$

$$g(t) = \underbrace{\underbrace{\frac{1}{\omega}}_{\text{const.}} \cdot \underbrace{e^{\sigma t}}_{f(t)}}_{\text{Amplitude}} \cdot \underbrace{\sin(\omega \cdot t)}_{\text{sin function with angular frequency } \omega}$$



***Generalised statement about stability*:**

1. System is stable if all poles are in LHP
2. System is unstable if
   - at least one pole is in RHP OR
   - at least one repeated pole is on ω-axis.
3. System is marginally stable if only non-repeated poles on ω-axis and no poles in RHP. (It may have some poles in LHP).



Exercise 2.1 Check if the following systems are stable:

$$G_1(s) = \frac{5}{s^2+5s+4} \qquad G_2(s) = \frac{1}{s^2}$$

$$G_3(s) = \frac{5}{s^2+2s+2} \qquad G_4(s) = \frac{5}{s^2+3s-4}$$

G_1: Poles: $s_{1,2} = -4$ & -1 both poles are in LHP
$\Rightarrow$ G_1 is stable.

G_2: Poles: $s_p^2 = 0 \Rightarrow s_{p_{1,2}} = 0$ $\Rightarrow$ a repeated pole on ω-axis.
$\Rightarrow$ unstable.

Impulse response:
$g_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$; $g_2(\infty) = \infty \Rightarrow$ unstable

G_4: Poles $s_{p_{1,2}} = -3/2 \pm \frac{\sqrt{9+16}}{2} = -3/2 \pm 5/2 = \underbrace{-4}_{LHP}$ & $\underbrace{1}_{RHP}$

One pole is in RHP $\Rightarrow$ unstable. ■

2.5.2 Oscillation behaviour

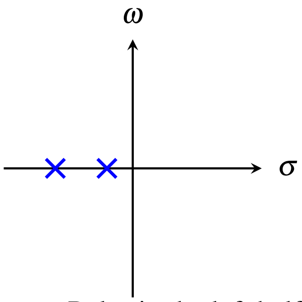
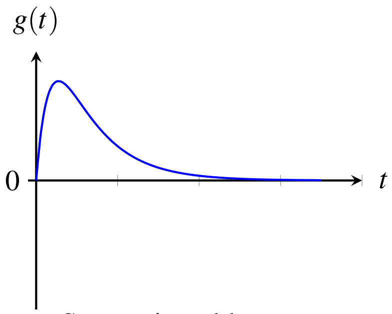
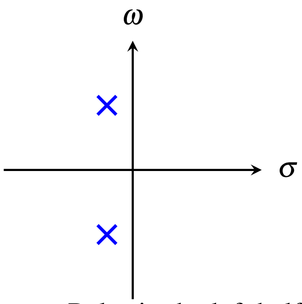
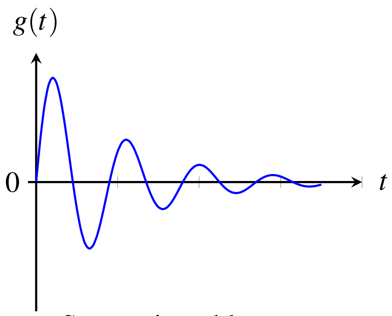
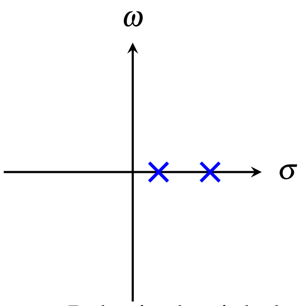
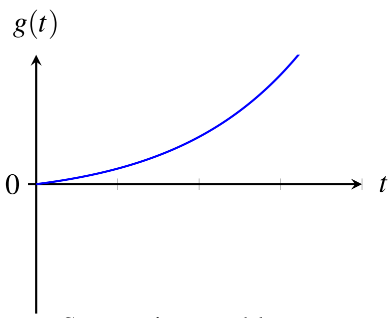
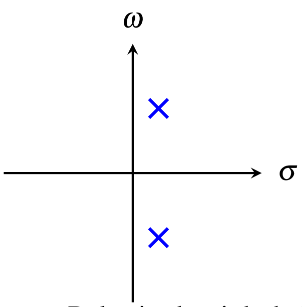
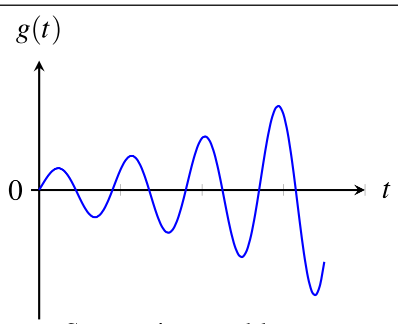
Physical causes for oscillation:

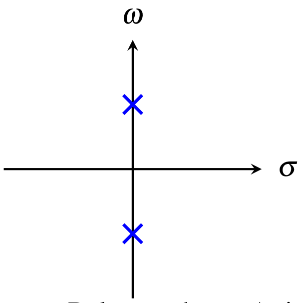
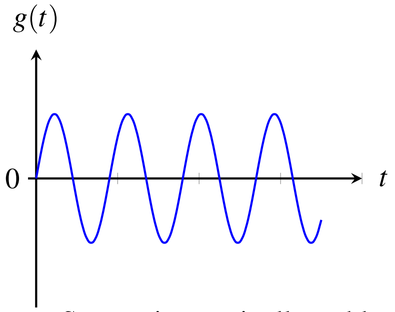
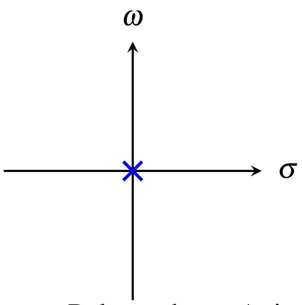
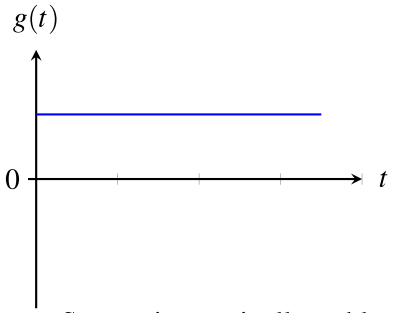
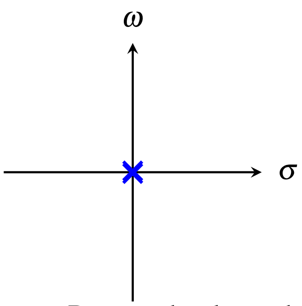
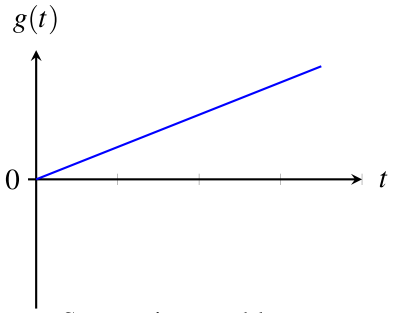
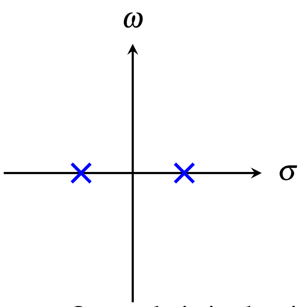
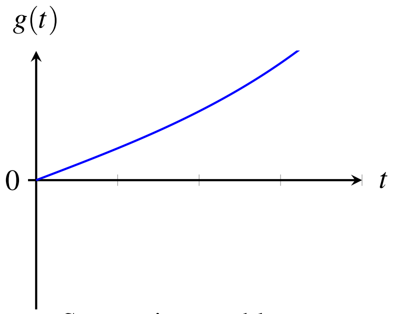
- two different types of energy storage
- energy is shifted from one storage to the other and back. This process repeats for ever.

Dependency on the location of poles:

- complex or imaginary poles cause oscillation.
- the angular frequency of oscillation is exactly equal to the imaginary part of poles.

Table 2.4: Stability and oscillation properties of linear systems.

Location of poles	**Impulse response**
 • Poles in the left half plane. • Poles are real.	 • System is stable. • System response does not oscillate.
 • Poles in the left half plane. • Poles are complex.	 • System is stable. • System response oscillates.
 • Poles in the right half plane. • Poles are real.	 • System is unstable. • System response does not oscillate.
 • Poles in the right half plane. • Poles are complex.	 • System is unstable. • System response oscillates.

Continuation of Table 2.4	
Location of poles	**Impulse response**
 • Poles on the ω-Axis. • Poles are complex.	 • System is marginally stable. • System response oscillates.
 • Pole on the ω-Axis. • Pole is not complex.	 • System is marginally stable. • System response does not oscillate.
 • Repeated pole on the ω-Axis. • Poles are not complex.	 • System is unstable. • System response does not oscillate.
 • One pole is in the right half plane. • Poles are not complex.	 • System is unstable. • System response does not oscillate.

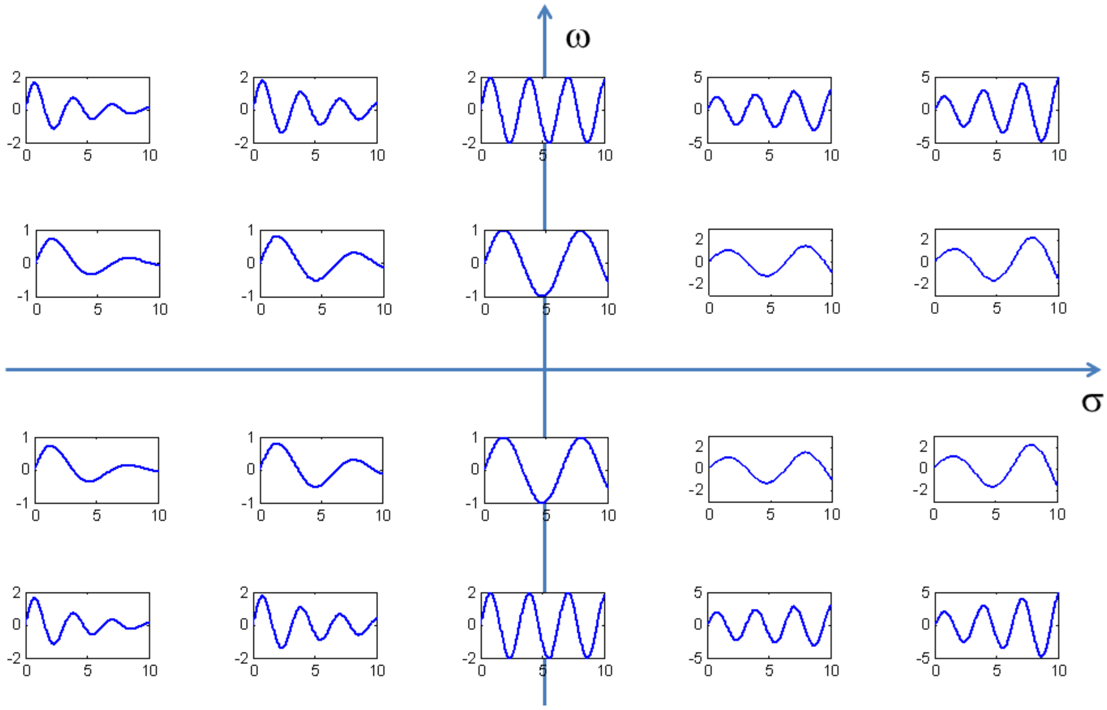


Figure 2.7: Location of poles and system behaviour

2.5.3 Linearity

The principle of superposition is valid for linear systems.

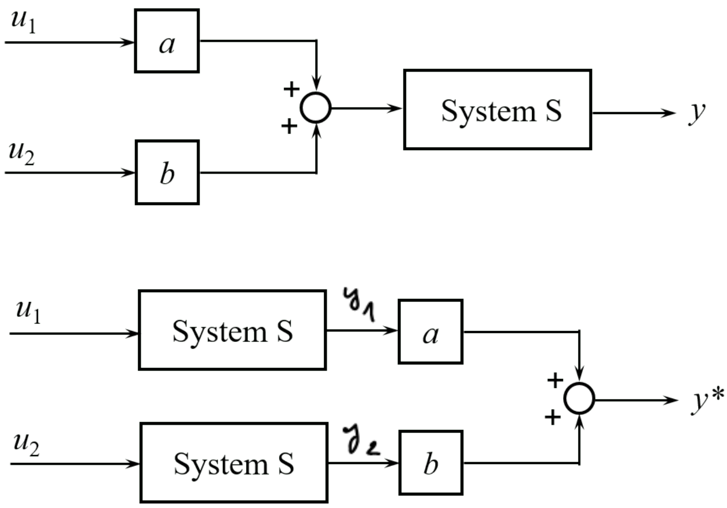
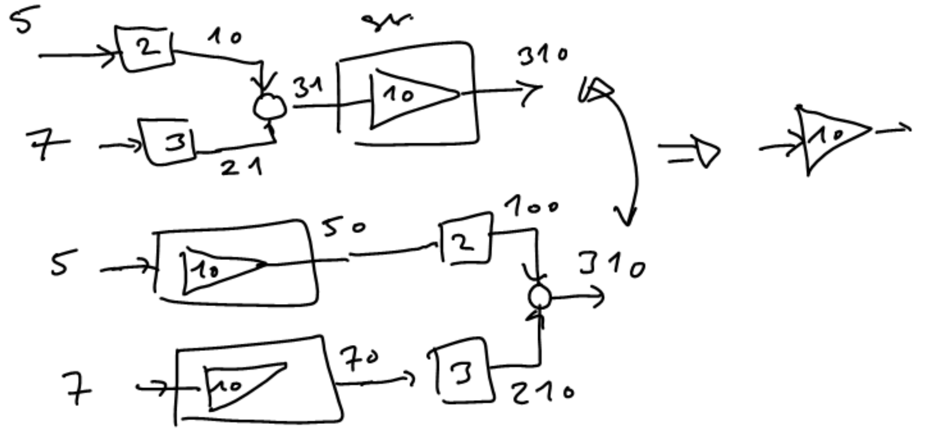
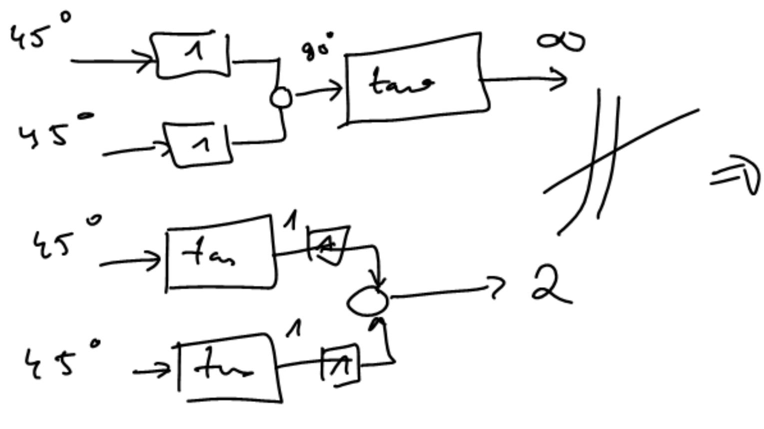


Figure 2.8: Principle of superposition

If for any values of the input variables u_1 and u_2 the output $y = y*$, then the system S is linear. Parameters a and b are any non-zero constants.



is linear.



⇒ tan is nonlinear.

2.5.4 Causality

Definition 2.5.1 — Causal systems. A causal system is one for which the output $y(t)$ at any instant t depends only on the value of the input $u(\tau)$ for $\tau \leq t$.

In other words, the value of the output at the present instant depends only on the past and present values of the input u, not on its future values. All physical systems are causal.

2.5.5 Minimum-phase behaviour

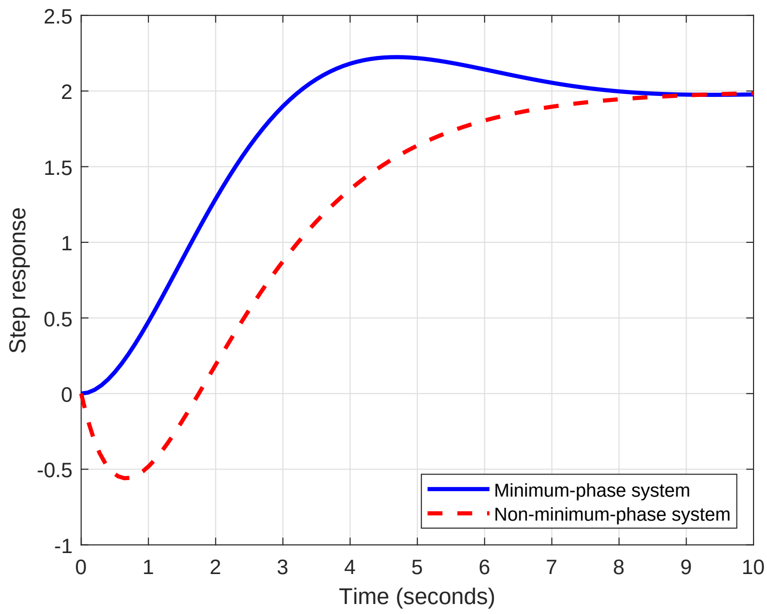


Figure 2.9: Step response of minimum-phase and non-minimum-phase systems

Definition 2.5.2 — Minimum-phase systems. Stable systems without transport delay which have no zeros in the right half plane are known as minimum-phase systems.

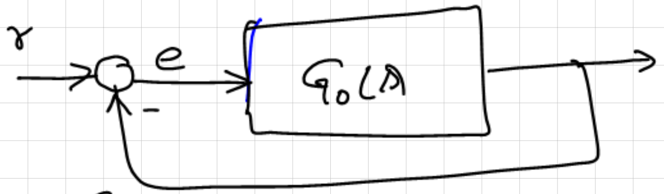
3.3 Control loop stability

The stability of a closed loop system is of extra ordinary importance. The stability depends on the location of closed loop poles. Explicit calculation of poles of the closed-loop system is sometimes not straightforward. This is the case, when the poles are dependent on many unknown controller parameters. Moreover, the closed-loop stability analysis of systems involving transport delays is not possible by these methods. Nyquist criterion offers a possibility to determine the stability of the closed loop by analysing the open-loop frequency response.

3.3.1 Nyquist criterion

1. determines the stability of a closed-loop system $G_{YR}(s) = \frac{G_0(s)}{1+G_0(s)}$ by analysing its open-loop frequency response $G_0(j\omega)$,
2. is a graphical method and
3. is also valid for systems involving transport delays (dead times).
4. can be used to calculate stability margins.

Basic idea



$$G_{YR}(s) = \frac{G_0(s)}{1+G_0(s)}$$

$G_0(s) = G_c(s)\cdot G_p(s)$

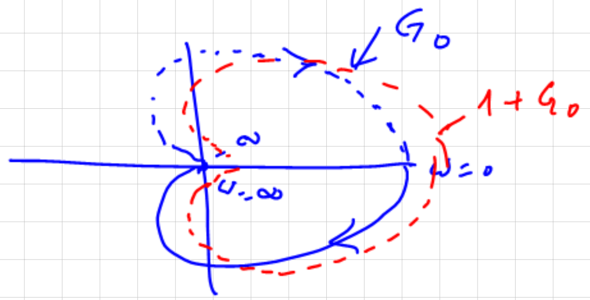
zeros of this are the poles of the closed loop.

Poles of $1+G_0(s)$

$G_0 = \frac{B}{A}$

$$1+G_0 = 1+\frac{B}{A} = \frac{A+B}{A}$$

Poles of G_0 and Poles of $1+G_0$ are same.



Theorem 3.3.1 — General formulation of the Nyquist stability criterion. If the open-loop transfer function $G_0(s)$ has k poles in the right-half s plane, then for the stability of the closed-loop system the modified Nyquist plot $G_0(j\omega)$ must encircle the $-1+j0$ point k times in the counter-clockwise direction.

The procedure shown in Figure 3.6 can be adopted to examine the stability of linear control systems using the Nyquist stability criterion. There are only the follwoing three cases [Ogata]:

1. There is no encirclement of the $-1+j0$ point. This implies that the system is stable if there are no poles of $G_0(s)$ in the right-half s plane; otherwise, the system is unstable.
2. There are one or more counter-clockwise encirclements of the $-1+j0$ point. In this case the system is stable if the number of counter-clockwise encirclements is the same as the number of poles of $G_0(s)$ in the right-half s plane; otherwise, the system is unstable.
3. There are one or more clockwise encirclements of the $-1+j0$ point. In this case the system is unstable.

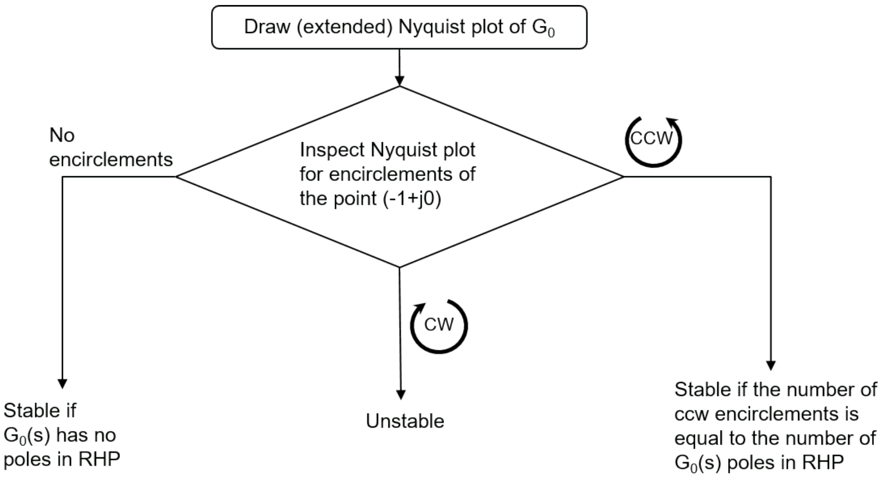
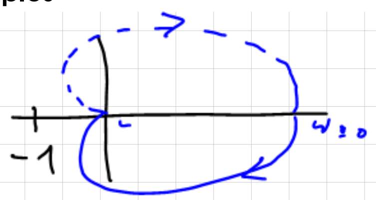


Figure 3.6: Analysing stability with Nyquist criterion

Examples

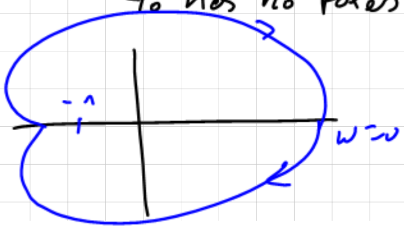
1.



⇒ no encirclements of (−1 ± j0) ⇒ The closed loop is stable

G_0 has no poles in RHP

2.

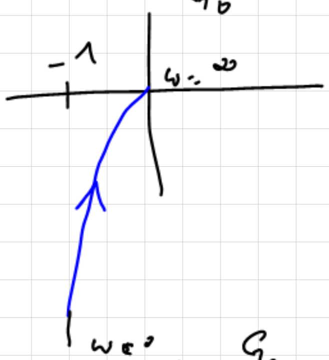


⇒ 1 CW encirclements of −1 + j0 } ⇒ Closed loop is unstable.

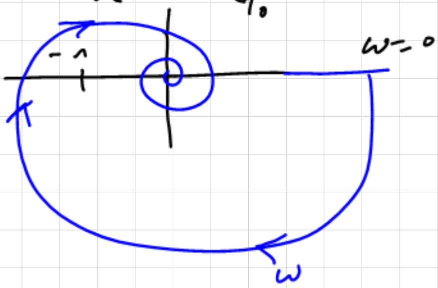
> **Theorem 3.3.2 — Simplified version of the Nyquist stability criterion.** If the open-loop transfer function $G_0(s)$ has only poles in LHP except one or two poles at $s = 0$ then the closed loop system will be asymptotically stable if and only if the critical point $(-1, j0)$ remains on left of the simple Nyquist plot of $G(j\omega)$ from $\omega = 0$ to $\omega = \infty$ [Unbehauen;RT1].

$= \frac{G_0}{1+G_0}$

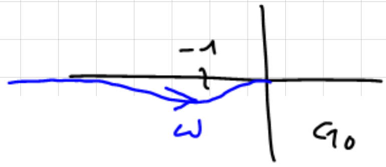
Examples

1. 

Closed loop will be stable

2. 

Closed ~~loop~~ loop is unstable.

3. 

Closed-loop will be stable.

C

Example. Given $G_0(s) = \dfrac{5e^{-2s}}{s+1}$

Check if the closed loop is stable.

1st attempt:

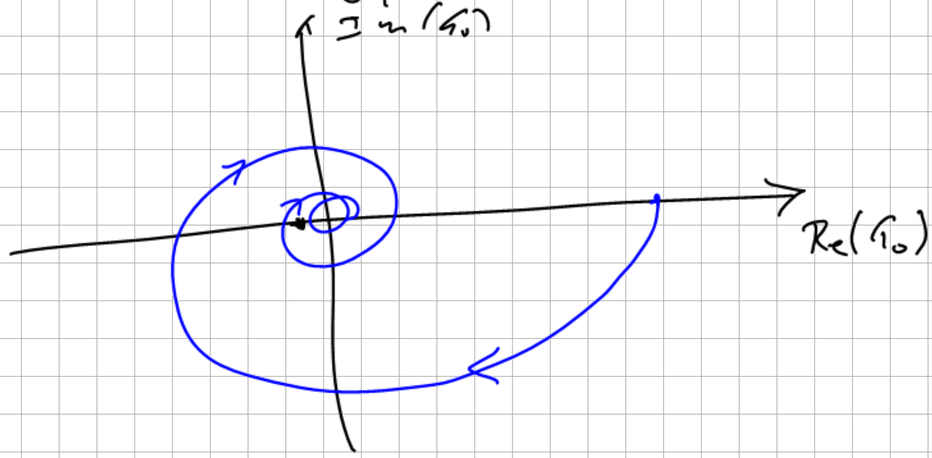
$$G_{YR} = \frac{G_0}{1+G_0} = \frac{\frac{5e^{-2s}}{s+1}}{1+\frac{5e^{-2s}}{s+1}}$$

$$= \frac{5 \cdot e^{-2s}}{s+1+5e^{-2s}}$$

Numerator is not a polynomial in s

$\Rightarrow$ We cannot calculate its poles

$\Rightarrow$ stability analysis is only possible with Nyquist criterion.



The closed-loop will be unstable.

3.3.2 Stability margins

Definition 3.3.1 — Gain crossover frequency. The frequency, where the magnitude of the loop gain becomes unity is known as gain crossover frequency ω_{gc}.

$$A(\omega_{gc}) = 1 \quad \Rightarrow \quad A_{dB}(\omega_{gc}) = 0\text{dB}$$

Definition 3.3.2 — Phase crossover frequency. The frequency, where the phase angle becomes $-180°$ is known as phase crossover frequency ω_{pc}.

$$\phi(\omega_{pc}) = -180°$$

Definition 3.3.3 — Phase margin. Additional negative phase at gain crossover frequency which would bring the control system to the verge of instability is called the phase margin ϕ_M.

$$\phi_M = \phi(\omega_{gc}) + 180°$$

Definition 3.3.4 — Gain margin. The gain margin A_M is defined as the minimum additional positive gain at phase crossover frequency which would bring the closed loop system to the verge of instability.

$$A_M = \frac{1}{A(\omega_{pc})} \quad \Rightarrow \quad A_{M\_dB} = 20 \cdot \log(A_M) = -A_{dB}(\omega_{pc})$$

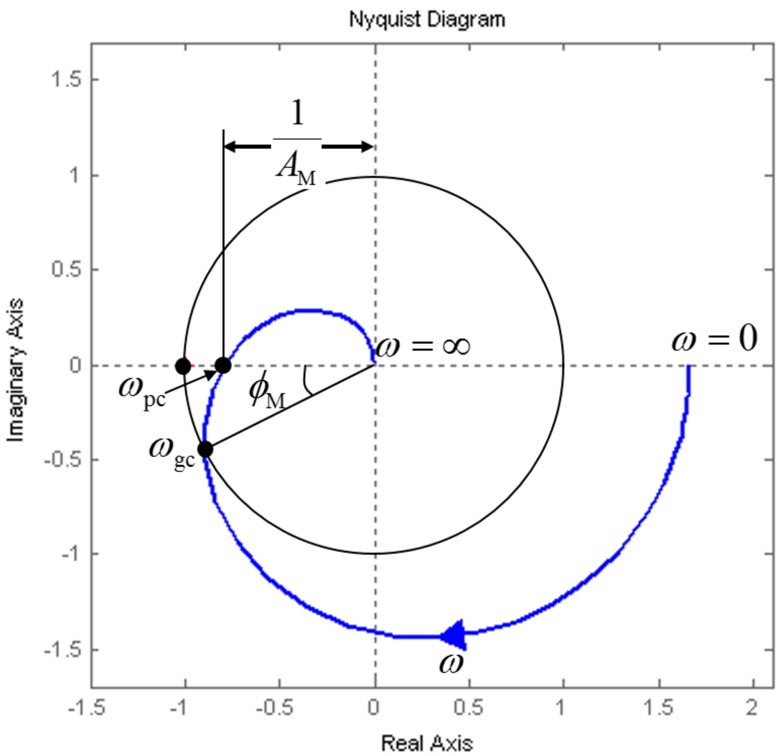


Figure 3.7: Definition of stability margins

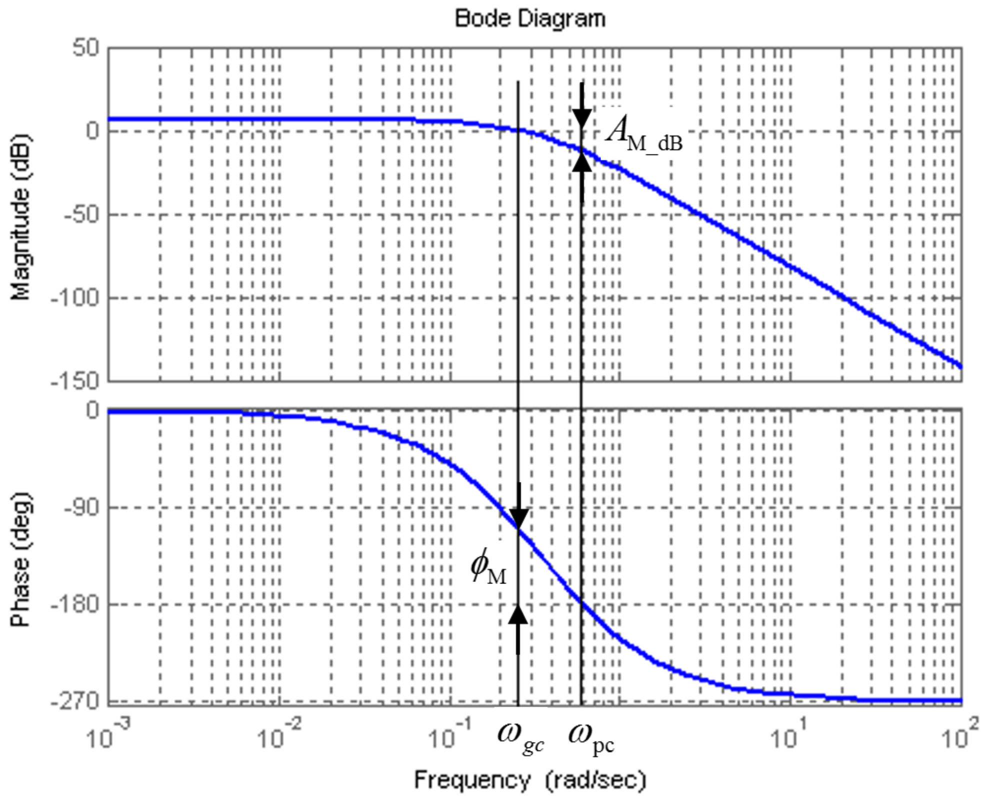


Figure 3.8: Definition of stability margins in Bode plot

Example: $G_0(s) = \dfrac{5}{s\,(s+2)}$

$\omega_{gc}\,,\ \omega_{pc}\,,\ \phi_M\,,\ A_M = ?$

$$G_0(j\omega) = \frac{5}{j\omega\,(j\omega+2)}$$

$$A(\omega) = |G_0(j\omega)| = \frac{5}{\omega\cdot\sqrt{\omega^2+4}}$$

$$\phi(\omega) = -90^\circ - \arctan\left(\frac{\omega}{2}\right)$$

$\omega_{gc} = ?$ $\quad A(\omega_{gc}) = 1$

$$\frac{5}{\omega_{gc}\cdot\sqrt{\omega_{gc}^2+4}} = 1$$

$$\omega_{gc}\cdot\sqrt{\omega_{gc}^2+4} = 5$$

$$\omega_{gc}^2\cdot\left(\omega_{gc}^2+4\right) = 25$$

Warning!

subs. $x = \omega_{gc}^2$

$$x(x+4) - 25 = 0$$

$$x^2 + 4x - 25 = 0$$

$$x = -\frac{4}{2} \pm \frac{\sqrt{16+100}}{2} = -2 \pm \frac{\sqrt{116}}{2} = -2 \pm \frac{10{,}78}{2}$$

$$= -2 \pm 5{,}39 = -7{,}39 \ \&\ 3{,}39$$

$$\omega_{gc} = \pm\sqrt{x} = \underbrace{\pm\sqrt{-7{,}39}}_{\text{wrong soln.}}\ ;\ \pm\sqrt{3{,}39}\ \checkmark$$

$$\omega_{gc} = \sqrt{3{,}39} = 1{,}84 \ \text{rad/s}$$

$\omega_{pc} = ?$ $\qquad \phi(\omega_{pc}) = -180^\circ$

$$-90^\circ - \text{atan}\left(\frac{\omega_{pc}}{2}\right) = -180^\circ$$

$$\frac{\omega_{pc}}{2} = \tan(90) \Rightarrow \omega_{pc} = \infty$$

$$\phi_M = \phi(\omega_{gc}) + 180^\circ$$

$$= -90^\circ - \text{atan}\left(\frac{1,84}{2}\right) + 180^\circ =$$

$$\phi_M = -90^\circ - 42,6^\circ + 180^\circ = \underline{\underline{47,4^\circ}}$$

$$A_M = \frac{1}{A(\omega_{pc})} = \frac{1}{\frac{5}{\omega_{pc}\sqrt{\omega_{pc}^2+4}}} = \frac{\omega_{pc} \cdot \sqrt{\omega_{pc}^2+4}}{5}$$

$$A_M = \frac{\infty \cdot \sqrt{\infty^2+4}}{5} = \infty$$

4.1 PID controller

4.1.1 Basic structure

A PID controller consists of a parallel combination of proportional, integral and derivative controllers. The control signal of a PID controller in time domain can be given as:

$$u(t) = \underbrace{K_P e(t)}_{u_P} + \underbrace{K_I \int_0^t e(\tau) d\tau}_{u_I} + \underbrace{K_D \frac{de(t)}{dt}}_{u_D} \tag{4.1}$$

K_c $1/T_I$

$$K_p\left(e(t) + \frac{K_I}{K_P}\int e\,dt + \frac{K_D}{K_P}\frac{de}{dt}\right)$$

T_D

K_P, K_I and K_D are the gains for each control action, whereas, e represents the control deviation and u is the control signal. The symbols u_P, u_I and u_D represent different P, I and D components of the control signal respectively. A block diagram of this controller is drawn in Figure 4.1

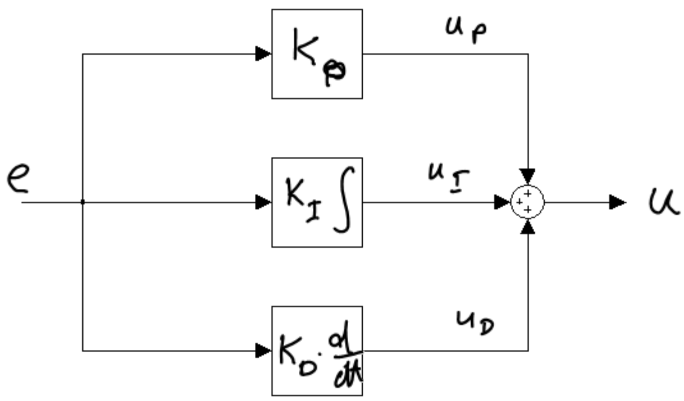


Figure 4.1: Block diagram of a PID controller

For a better theoretical analysis the PID controller may be implemented with a common gain as given in the following equation.

$$u(t) = K_C \left(e(t) + \frac{1}{T_I} \int_0^t e(\tau) d\tau + T_D \frac{de(t)}{dt} \right), \tag{4.2}$$

where K_C, T_I and T_D are the controller parameters.
The Laplace transform of the above equation delivers the control action in s domain:

$$U(s) = K_C \left(1 + \frac{1}{T_I s} + T_D s \right) E(s) \tag{4.3}$$

This form of the PID controller is shown in Figure 4.2.

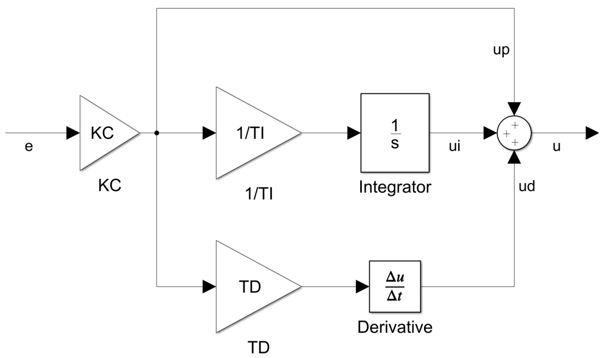


Figure 4.2: Simulink block diagram of an ideal PID controller

The transfer function of the controller:

$$G_C(s) = \frac{U(s)}{E(s)} = K_C \left(1 + \frac{1}{T_I s} + T_D s \right) \tag{4.4}$$

Disadvantages of an ideal PID controller:

1. Transfer function is not proper. That means it is not implementable.
2. The derivative part of the controller plays a role of a high-pass filter. The high frequency measurement noise is amplified by the D action, which can deteriorate the control performance and may cause excessive wear and tear of the actuator. This phenomenon is shown in Figure 4.3.

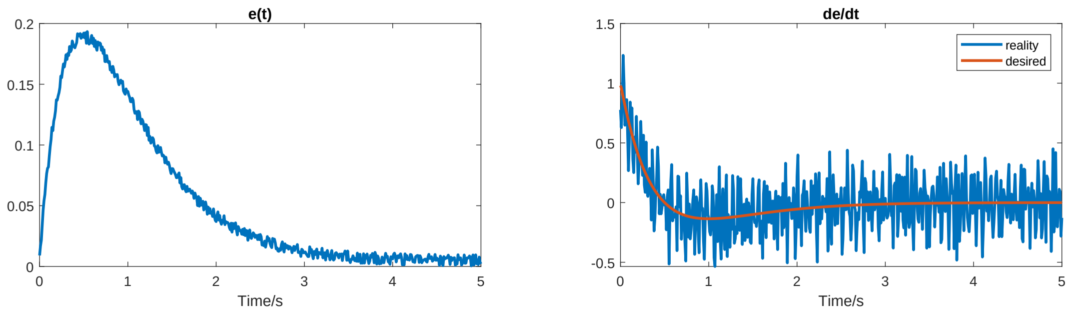
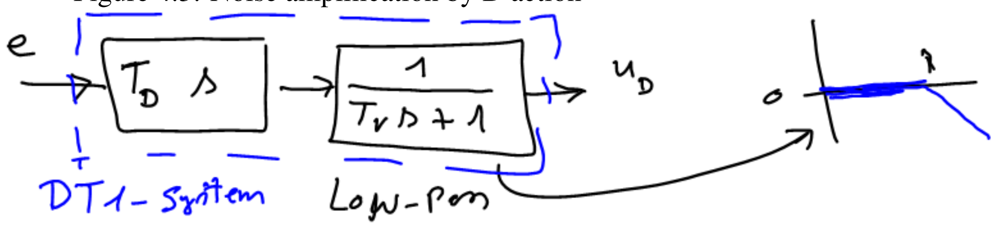


Figure 4.3: Noise amplification by D action



$$\rightarrow \boxed{\frac{T_D s}{T_V s + 1}} \rightarrow$$

DT1

Extending the derivative term by a first-order low-pass solves both problems. It filters the noise and the resulting transfer function also becomes proper. This modified PID controller is also known as PIDT1 controller.

$$G_C(s) = K_C \left(1 + \frac{1}{T_I s} + \frac{T_D s}{1 + T_V s}\right) \tag{4.5}$$

The following formulation shows that the PIDT1 controller can be implemented without using any derivative term. It can be implemented by using two integrators (see Figure 4.4).

$$G_C(s) = K_C \left(1 + \frac{1}{T_I}\frac{1}{s} + \frac{T_D}{T_V + \frac{1}{s}}\right) \tag{4.6}$$

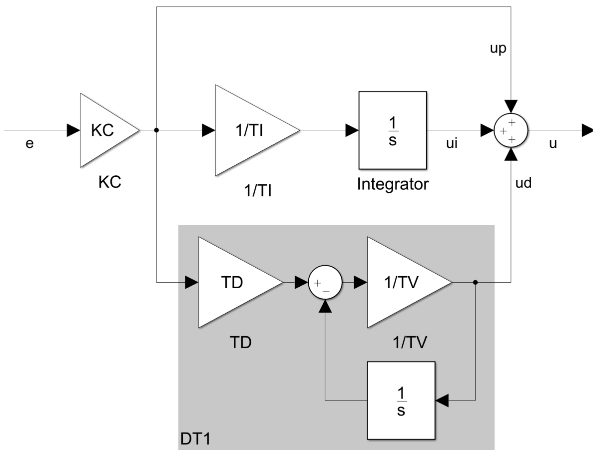
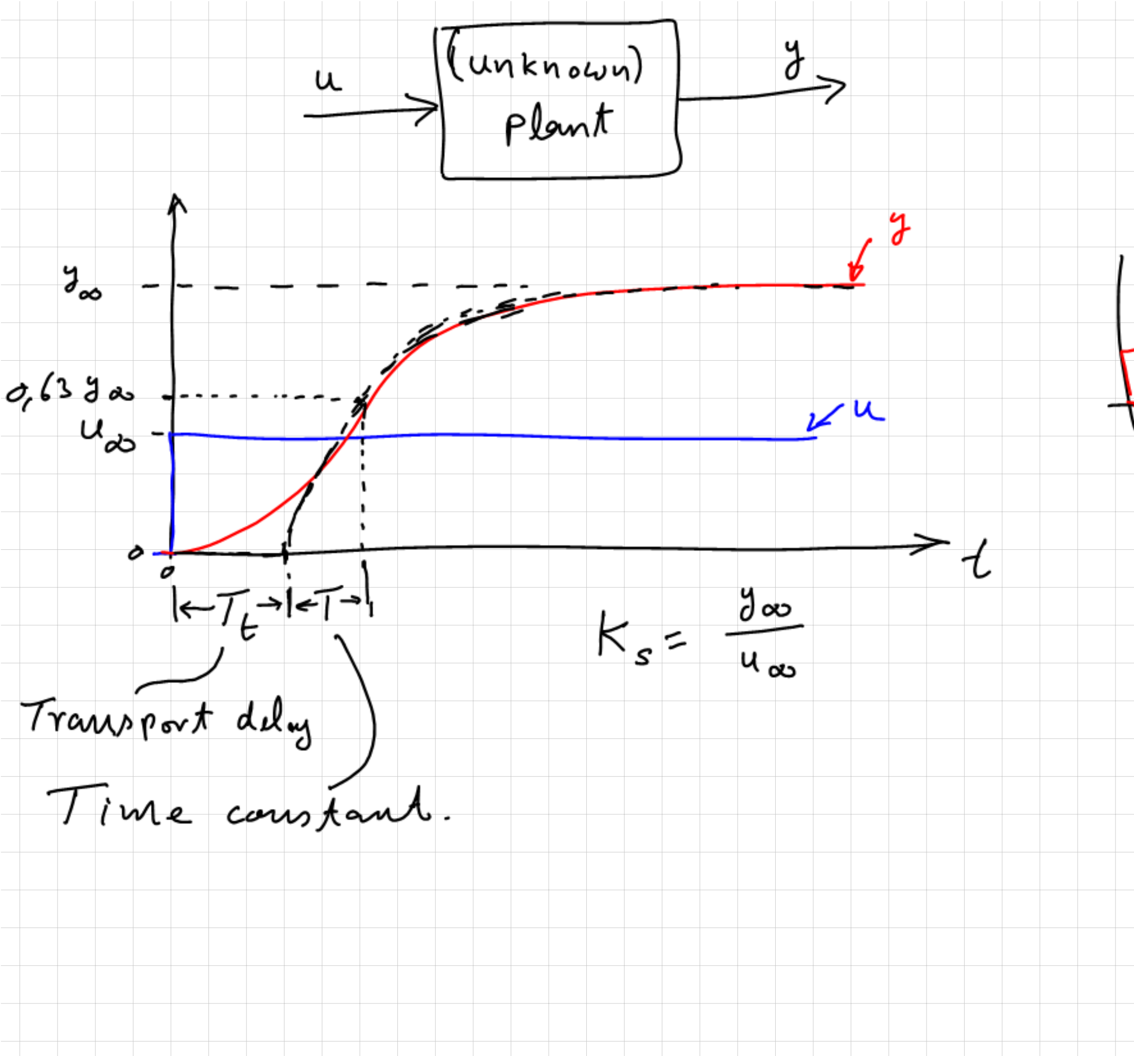


Figure 4.4: Simulink block diagram of the PIDT1 controller

4.2 Empirical methods for PID controllers

4.2.1 Ziegler-Nichols method 1: Step response method

Approximation of the step response of the plant by a first order transfer function with transport delay. $G_s(s) = \frac{K_s}{Ts+1}e^{-sT_t}$



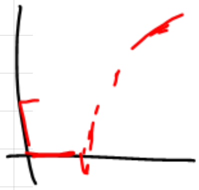
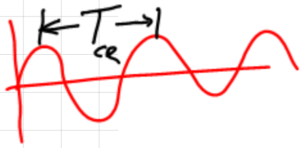


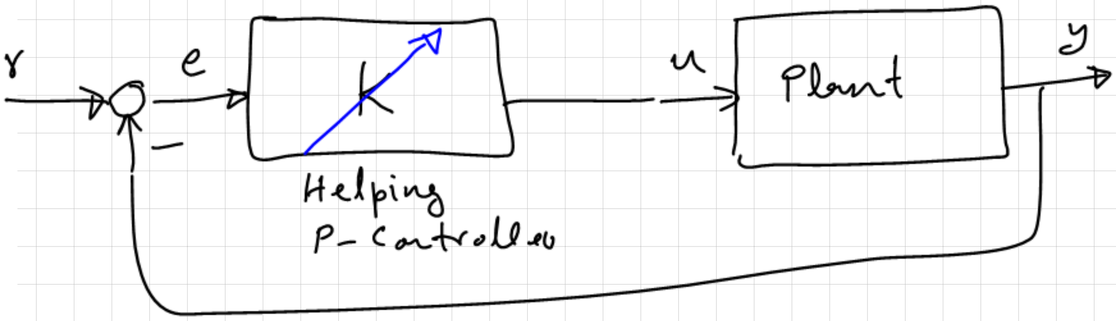
Table 4.1: Ziegler Nichols tuning rules based on the step response

Controller	Parameters
P	$K_C = \frac{1}{K_s}\frac{T}{T_t}$
PI	$K_C = \frac{0.9}{K_s}\frac{T}{T_t}$ $\quad T_I = 3.33T_t$
PID	$K_C = \frac{1.2}{K_s}\frac{T}{T_t}$ $\quad T_I = 2T_t$ $\quad T_D = 0.5T_t$

4.2.2 Ziegler-Nichols method 2: Marginal stability method

1. Control the plant using a helping controller (P-controller)





2. Increase controller gain K to a critical value K_{CR} so that the closed loop is marginally stable (oscillations with const amplitude)
3. Note values of K_{CR} and T_{CR}, the time period of the oscillation.
4. Calculate controller parameters using Table 4.2

Table 4.2: Ziegler Nichols tuning rules based on the marginal stability

Controller	Parameters
P	$K_C = 0.5K_{\text{cr}}$
PI	$K_C = 0.45K_{\text{cr}}$ $\quad T_I = 0.83T_{\text{cr}}$
PID	$K_C = 0.6K_{\text{cr}}$ $\quad T_I = 0.5T_{\text{cr}}$ $\quad T_D = 0.12T_{\text{cr}}$

4.3 Controller design in frequency domain (loop shaping)

The basic idea behind the frequency domain design techniques is to modify or reshape the frequency response of the open loop $G_0(s) = G_C(s)G_P(s)$ so that the closed-loop control satisfies the performance specifications. In order to be able to use these techniques, the relationship between the control loop performance specifications and the features of the open-loop frequency response must be understood. For this purpose, consider the standard control loop according to 4.9 with reference variable $R(s)$, controlled variable $Y(s)$, disturbance $D(s)$ and measurement noise $N(s)$.

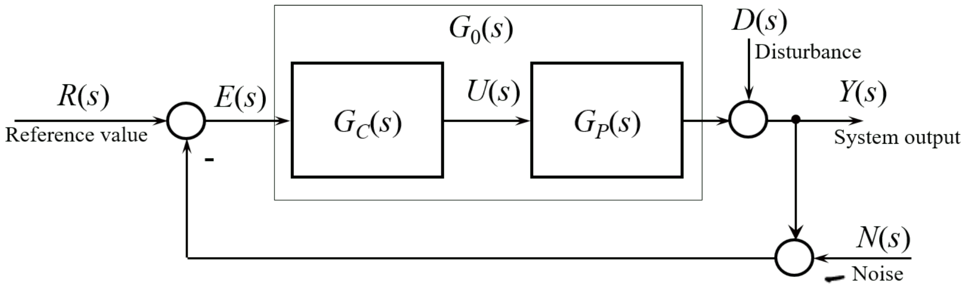


Figure 4.9: Standard feedback control loop with disturbance and noise

The system output $Y(s)$ can be calculated as:

$$Y(s) = \underbrace{\frac{1}{1+G_0(s)}}_{G_{YD}(s)} D(s) + \underbrace{\frac{G_0(s)}{1+G_0(s)}}_{G_{YR}(s)} R(s) + \underbrace{\frac{G_0(s)}{1+G_0(s)}}_{G_{YN}(s)} N(s) \tag{4.7}$$

In an ideal control loop, the system output $Y(s)$ should perfectly follow the set-point $R(s)$ and the effects of disturbance $D(s)$ and noise $N(s)$ must be suppressed. The following should be true for the individual transfer functions of Equation 4.7:

$$\begin{array}{lll} G_{YD}(s) = \frac{1}{1+G_0(s)} = 0 & \Rightarrow & G_0(s) \to \infty \\ G_{YR}(s) = \frac{G_0(s)}{1+G_0(s)} = 1 & \Rightarrow & G_0(s) \to \infty \\ G_{YN}(s) = \frac{G_0(s)}{1+G_0(s)} = 0 & \Rightarrow & G_0(s) \to 0 \end{array} \left.\vphantom{\begin{array}{l} a \\ a \\ a \end{array}}\right\} \text{Conflict}$$

Good tracking and proper disturbance rejection demand that the magnitude of the open-loop transfer function $G_0(s)$ should be very large (approaching to ∞), whereas for proper noise damping the open-loop magnitude needs to be very small (near to 0). A possible solution of the conflict can be achieved by considering frequency of the signals R, D and N. Set-point R and load disturbance D are generally low frequency signals, whereas, the sensor noise N is normally composed of high frequency components. A good compromise can be achieved, if the magnitude of G_0 is made large at low frequencies and kept low at higher frequencies. The magnitude curves of different transfer functions drawn in Figure 4.10 show that this trade-off would achieve good control performance at lower frequencies without amplifying the high frequency noise.

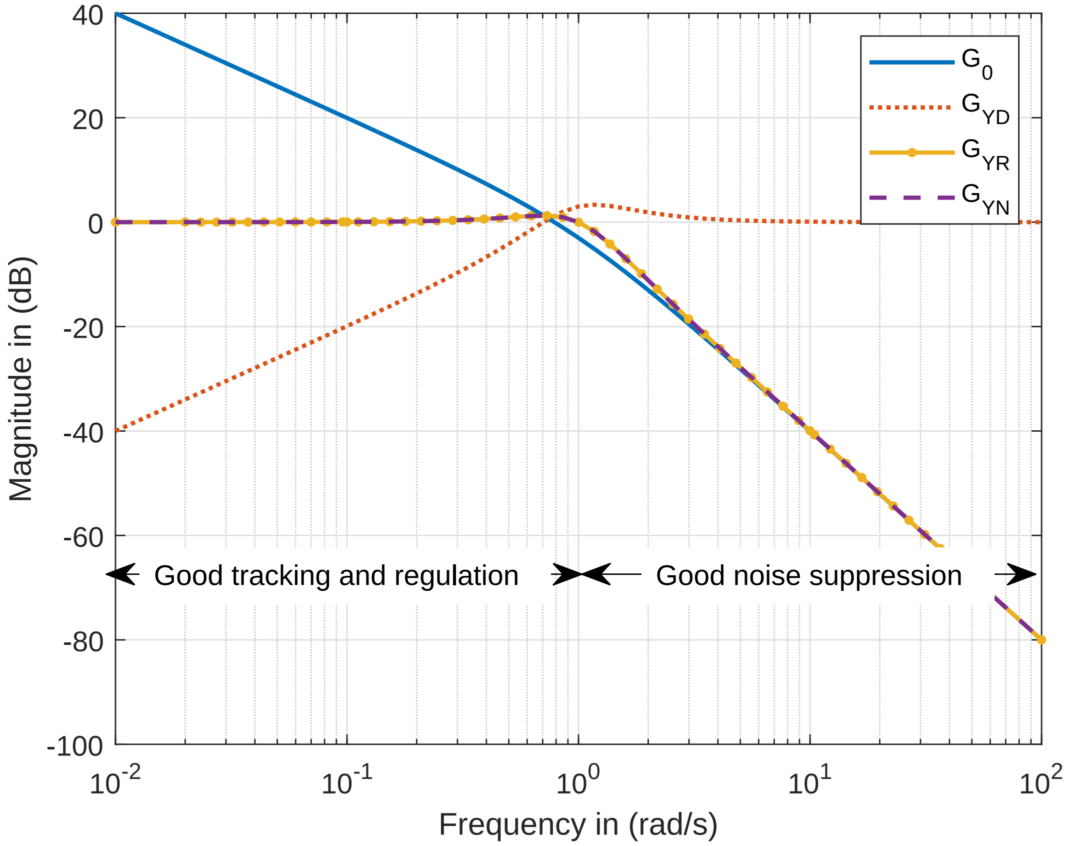


Figure 4.10: Magnitude curves of G_0, G_{YR}, G_{YD} and G_{YN}

4.3.1 Performance specifications in frequency domain

Performance indicators of a feedback control loop like maximum overshoot M_P, rise time t_r, steady-state control error e_∞ are directly related with certain characteristics of the open-loop frequency response. We can try to understand these relationships with the help the following example: Consider an open loop with the following transfer function:

$$G_0(s) = \frac{K}{s(T_1 s+1)}$$

(IT₁ system)

The corresponding closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{G_0(s)}{1+G_0(s)} = \frac{K}{s(T_1 s+1)+K} = \frac{K}{T_1 s^2 + s + K} = \frac{1}{\frac{T_1}{K} s^2 + \frac{1}{K} s + 1}$$

is a second order system (PT2). The natural frequency and the damping ratio of the closed-loop system are dependent on the parameters K and T_1 of the open-loop system. By varying the parameters K and T_1, the frequency response characteristics like phase margin, amplitude margin and crossover frequencies are changed. On the other hand, the closed-loop parameters like natural frequency and damping ratio will also change resulting a change in rise time and oscillation behaviour of the system.

$$\frac{E(s)}{R(s)} = \frac{1}{1+G_0(s)}$$

$$e_\infty = \frac{1}{1+G_0(0)} \cdot \underset{1}{w_\infty}$$

$$\frac{1}{1+A(0)}$$

$$A(0) = \frac{1}{e_\infty} - 1$$

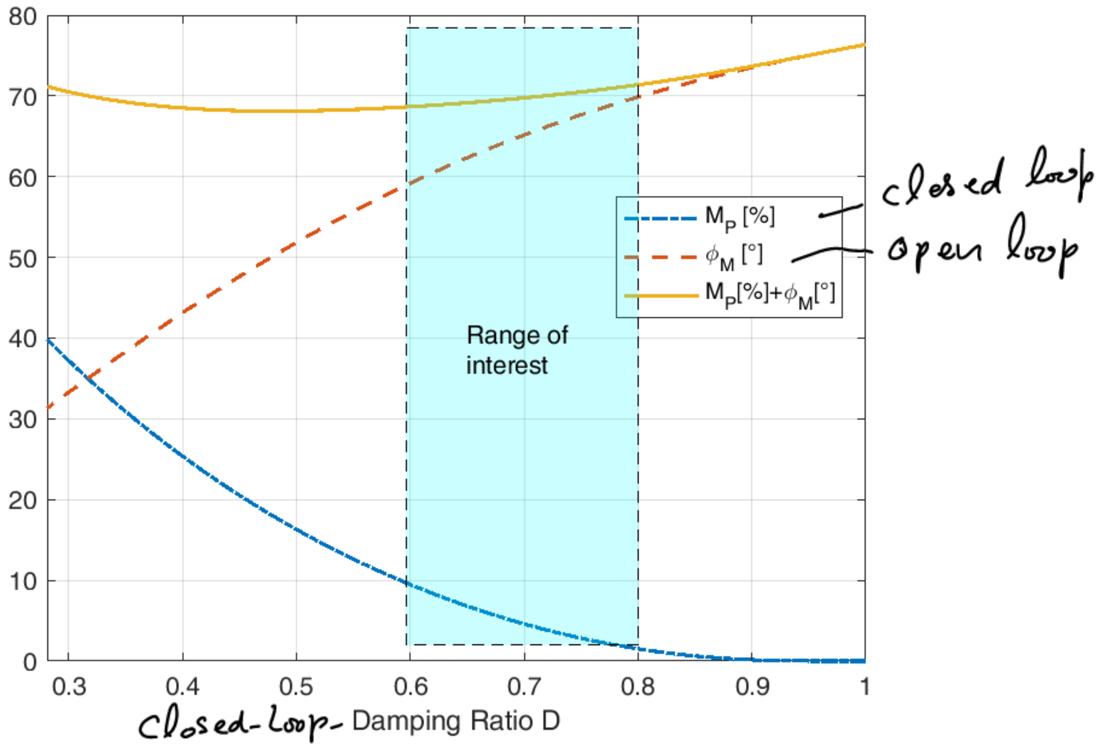


Figure 4.11: Relationship among phase margin, damping ratio and maximum overshoot

As the Figure 4.11 shows, the relationship among the phase margin ϕ_M, damping ratio D and overshoot can be expressed by the following thumb rule:

Oscillation behaviour: An increase in the phase margin causes an increase in the damping ratio of the closed-loop and a decrease in the overshoot.

$$\phi_M \uparrow \quad \Rightarrow \quad D \uparrow \quad \Rightarrow \quad M_P \downarrow$$

Thumb rule: $[M_P[\%] + \phi_M[°] \approx 70$

Similarly the relationship between the gain crossover frequency and the rise time (see Figure 4.12) can be described as follows:

Response speed: An increase in the gain crossover frequency causes an increase in the natural frequency of the closed-loop and a decrease in the rise time.

$$\omega_{gc} \uparrow \quad \Rightarrow \quad \omega_0 \uparrow \quad \Rightarrow \quad t_r \downarrow$$

Thumb rule: $t_r \cdot \omega_{gc} \approx 1.5$

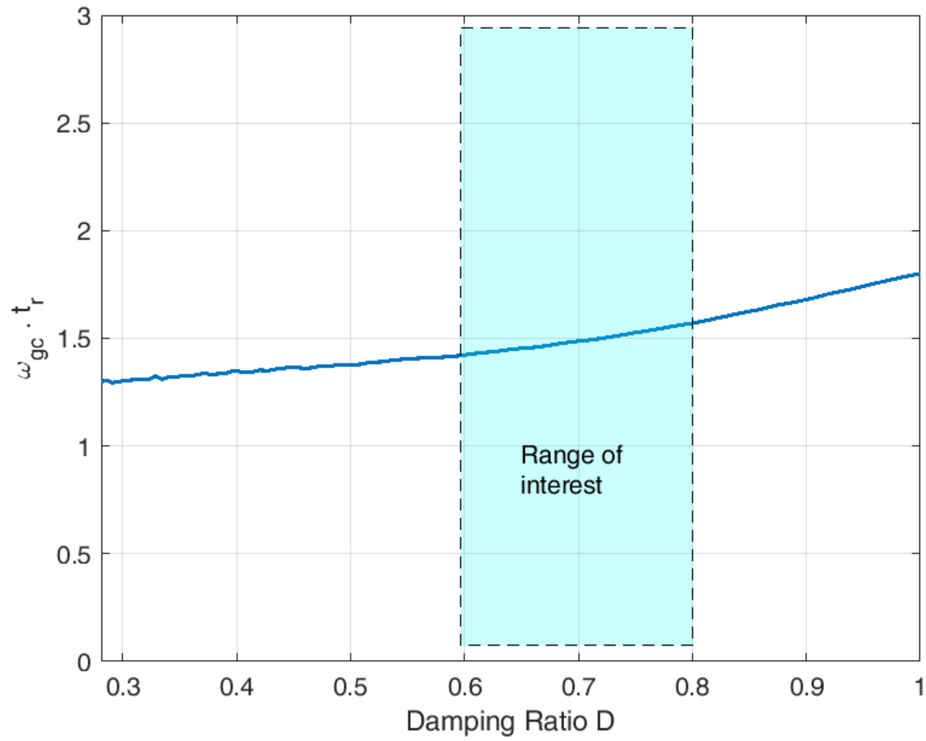


Figure 4.12: Relationship between gain crossover frequency and rise time.

The steady-state accuracy of the closed loop depends on the loop gain at low frequencies. In case of step-like changes in set-point, the following rule can be formulated for steady-state accuracy:

Steady-state accuracy: An increase in the open-loop dc-gain causes a decrease in the steady-state error. This discussion is valid only valid for constant values of set-point.

$$|G_0(0)| = A(0) \uparrow \quad \Rightarrow \qquad e_\infty \downarrow$$

Thumb rule: $e_\infty = 1/(1+A(0))$ $\Rightarrow A(0) = \frac{1}{e_\infty} - 1$

4.3.2 Controller design

The controller design involves two steps:

1. Convert the control specifications such as t_r, M_P, e_∞ etc. to frequency response characteristics like $\omega_{gc}, \phi_M, A(0)$ using the following thumb rules:

Rise time ↔ Gain crossover frequency	$t_r \cdot \omega_{gc} \approx 1.5$
Maximum overshoot ↔ Phase margin	$M_P[\%] + \phi_M[°] \approx 70$
Steady-state error ↔ $A(0)=\lvert G(j0)\rvert$	$A(0) = 1/e_\infty - 1$

2. Change the frequency response of the open loop by configuring the controller so that characteristics calculated above are realised.

■ **Example 4.1** Consider the following transfer function of a plant:

$$G_P(s) = \frac{5}{(0.1s+1)(0.5s+1)}$$

1. Design a P controller so that the steady-state control error after a step-like change in set-point is 5%.
2. Design a P controller so that the phase margin is 60°.
3. Design a minimum-order controller, which satisfies the following tracking performance specifications:
   - Steady-state control error is 0.
   - Maximum overshoot is 10%
   - Rise time is 0.25 s.
4. Solve the last problem using Matlab.

1. $G_p(s) = \dfrac{5}{(0{,}1s+1)(0{,}5s+1)}$

$G_c(s) = K_c$

Open loop: $G_o(s) = G_c(s) \cdot G_p(s) = \dfrac{5K_c}{(0{,}1s+1)(0{,}5s+1)}$

Requirement: $e_\infty = 0{,}05$

$\Rightarrow A(0) \overset{!}{=} \dfrac{1}{e_\infty} - 1 = \dfrac{1}{0{,}05} - 1 = 19$

$A(\omega) = \lvert G_o(j\omega)\rvert = \left\lvert \dfrac{5k_c}{(j0{,}1\omega+1)(j0{,}5\omega+1)} \right\rvert = \dfrac{5k_c}{\sqrt{((0{,}1\omega)^2+1)((0{,}5\omega)^2+1)}}$

■

$A(0) = \dfrac{5K_c}{1} \overset{!}{=} 19 \Rightarrow K_c = 19/5$

2. $G_p(s) = \dfrac{5}{(0{,}1s+1)(0{,}5s+1)}$ | Requirement: $\phi_M = 60^\circ$

$G_c(s) = K_c = ?$

$$G_o(s) = \frac{5K_c}{(0{,}1s+1)(0{,}5s+1)}$$

$$G_o(j\omega) = \frac{5K_c}{(j0{,}1\omega+1)(j0{,}5\omega+1)}$$

$$A(\omega) = \frac{5K_c}{\sqrt{(0{,}1\omega)^2+1}\,\sqrt{(0{,}5\omega)^2+1}}$$

$$\phi(\omega) = -\operatorname{atan}(0{,}1\omega) - \operatorname{atan}(0{,}5\omega) \;\text{- - -}$$

$$\phi_M = \phi(\omega_{gc}) + 180^\circ \Rightarrow \phi(\omega_{gc}) = 60^\circ - 180^\circ = -120^\circ$$

$$-120^\circ = -\operatorname{atan}(0{,}1\omega_{gc}) - \operatorname{atan}(0{,}5\omega_{gc})$$

$$\operatorname{atan}(0{,}1\omega_{gc}) + \operatorname{atan}(0{,}5\omega_{gc}) = 120^\circ$$

$$\tan\Big(\underbrace{\operatorname{atan}(0{,}1\omega_{gc})}_{\alpha} + \underbrace{\operatorname{atan}(0{,}5\omega_{gc})}_{\beta}\Big) = \tan(120^\circ)$$

$$\frac{0{,}1\omega_{gc} + 0{,}5\omega_{gc}}{1-(0{,}1\omega_{gc})\cdot(0{,}5\omega_{gc})} = -\sqrt{3}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\cdot\tan\beta}$$

$$\frac{0{,}6\,\omega_{gc}}{1-0{,}05\,\omega_{gc}^2} = -\sqrt{3}$$

$$0{,}6\,\omega_{gc} = -\sqrt{3} + \sqrt{3}\cdot 0{,}05\,\omega_{gc}^2$$

$$\sqrt{3}\cdot 0{,}05\,\omega_{gc}^2 - 0{,}6\,\omega_{gc} - \sqrt{3} = 0$$

$$\omega_{gc}^2 - \frac{0{,}6}{0{,}05\cdot\sqrt{3}}\,\omega_{gc} - \frac{1}{0{,}05} = 0$$

$$\omega_{gc}^2 - 4\sqrt{3}\,\omega_{gc} - 20 = 0$$

$$\omega_{gc}^2 - 4\sqrt{3}\,\omega_{gc} - 20 = 0$$

$$\omega_{gc} = \frac{4\sqrt{3}}{2} \pm \frac{\sqrt{(4\sqrt{3})^2 - 4(-20)}}{2}$$

$$= 2\sqrt{3} \pm \frac{\sqrt{48+80}}{2}$$

$$\omega_{gc} = 2\sqrt{3} \pm \frac{\sqrt{128}}{2} = \underline{\underline{9{,}12}} \quad \cancel{\quad} -2{,}19$$

$$A(\omega_{gc}) \overset{!}{=} 1$$

$$\frac{5K_c}{\sqrt{(0{,}1\cdot 9{,}12)^2+1}\cdot\sqrt{(0{,}5\cdot 9{,}12)^2+1}} = 1$$

$$K_c = \frac{\sqrt{(0{,}912^2+1)(4{,}56^2+1)}}{5} = 1{,}26$$

Ex. 4.1
Part 3:

$$G_P(s) = \frac{5}{(0.1s+1)(0.5s+1)}$$

Required: Controller transfer function $G_c(s)$
- minimum order
- $e_\infty = 0$
- $M_p = 10\%$
- $t_r = 0{,}25\,s$

Solution:

Requirements
$$\begin{cases} e_\infty = 0 \Rightarrow A(0) = \frac{1}{e_\infty} - 1 = \infty \\ M_p = 10\% \Rightarrow M_p + \phi_M \approx 70 \Rightarrow \phi_M \overset{!}{=} 70 - 10 = 60^\circ \\ t_r = 0{,}25\,s \Rightarrow \omega_{gc} \cdot t_r \approx 1{,}5 \Rightarrow \omega_{gc} = \frac{1{,}5}{t_r} = 6\ \text{rad/s} \end{cases}$$

$$A(\omega) = |G_o(j\omega)| = |G_c(j\omega) \cdot G_p(j\omega)|$$

$$A(0) \to \infty$$

$$A(\omega) = |G_c(j\omega)| \cdot \left| \frac{5}{(j0{,}1\omega+1)(j0{,}5\omega+1)} \right|$$

$$A(0) = |G_c(j0)| \cdot \frac{5}{1} \overset{!}{=} \infty$$

$$|G_c(j0)| = \infty$$

We need an integrator in our controller $G_I(s) = \frac{1}{T_I \cdot s}$
because $|G_i(j\omega)| = \frac{1}{T_I \omega}$

$$G_I(j0) = \infty$$

$$G_I(s) = \frac{1}{T_I s} \quad \Big| \text{ Has a pole at } s = 0$$

First requirement is fullfilled.

1 First proposal: we use an I-controller

$$G_c(s) = \frac{1}{T_I s}$$

$$G_0(s) = G_c(s) \cdot G_p(s) = \frac{1}{T_I s} \cdot \frac{5}{(0{,}1s+1)(0{,}5s+1)}$$

- inadequate; because we have only one adjustable parameter T_I but we have satisfy two conditions $\Rightarrow$ impossible.

2. Second proposal:

$$\left(\frac{K_c}{T_I} = P'\right.$$

$$\cancel{K_c \frac{1}{T_I s}}$$

PI controller: $G_c(s) = K_c \cdot \left(1 + \frac{1}{T_I s}\right) = K_c \cdot \frac{(T_I s + 1)}{T_I s}$

$$G_0(s) = G_c(s) \cdot G_p(s) = \frac{K_c \cdot (T_I s+1)}{T_I s} \cdot \frac{5}{(0{,}1s+1)(0{,}5s+1)}$$

$$G_0(j\omega) = \frac{K_c \cdot (j\omega T_I + 1)}{j\omega T_I} \cdot \frac{5}{(j0{,}1\omega+1)(j0{,}5\omega+1)}$$

$$A(\omega) = \frac{5K_c \sqrt{(\omega T_I)^2+1}}{\omega T_I \cdot \sqrt{(0{,}1\omega)^2+1} \cdot \sqrt{(0{,}5\omega)^2+1}}$$

$$\phi(\omega) = \text{atan}(\omega T_I) - 90^\circ - \text{atan}(0{,}1\omega) - \text{atan}(0{,}5\omega)$$

$$\phi_M = \phi(\omega_{gc}) + 180^\circ \quad ; \quad A(\omega_g) = 1$$

$$\phi(\omega_{gc}) = \phi_M - 180^\circ = 60^\circ - 180^\circ = -120^\circ$$

$$-120^\circ = \text{atan}(6T_I) - 90^\circ - \text{atan}(0{,}1 \cdot 6) - \text{atan}(0{,}5 \cdot 6)$$

$$\text{atan}(6T_I) = -120^\circ + 90^\circ + \text{atan}(0{,}6) + \text{atan}(3)$$

$$= -30^\circ + 31^\circ + 71{,}5^\circ = 72{,}5^\circ$$

$$T_I = \frac{1}{6} \cdot \tan(72{,}5^\circ) = \frac{1}{6} \cdot 3{,}17 = 0{,}53$$

$$A(\omega) = \frac{5K_c\sqrt{(\omega T_I)^2+1}}{\omega T_I \cdot \sqrt{(0{,}1\omega)^2+1} \cdot \sqrt{(0{,}5\omega)^2+1}}$$

$$A(\omega_{gc}) \overset{!}{=} 1$$

$$\frac{5K_c \cdot \sqrt{(3{,}17)^2+1}}{3{,}17 \cdot \sqrt{0{,}6^2+1}\,\sqrt{3^2+1}} \overset{!}{=} 1$$

$$K_c = \frac{3{,}17 \cdot \sqrt{1{,}36} \cdot \sqrt{10}}{5 \cdot \sqrt{11{,}1}} = 0{,}7$$

4.3.3 Direct cancellation of poles and zeros

This design technique allows to cancel poles and zeros of the plant with controller zeros and poles. This approach has the following advantages:

1. simplification of the further design process
2. direct removal of the poles responsible for undesirable behaviour such as excessive oscillation or slow speed of response etc.

After cancellation, the remaining parameters can be determined by using the method described in the previous section.

Important note: *Poles and zeros of the plant, which are located in the right half-plane are not allowed to be cancelled by the controller. Such cancellation would cause instability in the system due to modelling inaccuracy and actuator saturation.*

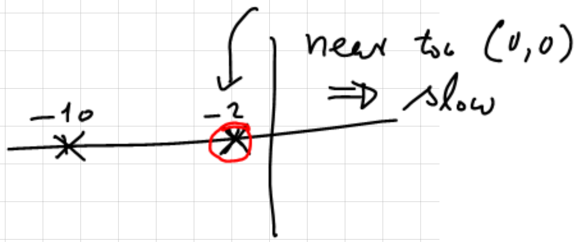
■ **Example 4.2 — Cancelling poles, which are responsible for slow response.**

$$G_P(s) = \frac{5}{(0.1s+1)(0.5s+1)}$$

Design a PI controller so that: $G_c(s) = K_c\left(1 + \frac{1}{T_I s}\right)$

1. the slower pole of the plant is compensated by a controller zero.
2. $\phi_M \overset{!}{=} 60°$

Solution: G_P Poles $s_{\infty 1} = -10$ $s_{\infty 2} = -2$



Controller:

$$G_c(s) = K_c \cdot \left(1 + \frac{1}{T_I s}\right) = \frac{K_c(T_I s + 1)}{T_I s}$$

Controller zero: $s_0 T_I + 1 = 0 \Rightarrow s_0 = -\frac{1}{T_I} \overset{!}{=} -2$

$$\Rightarrow \underline{\underline{T_I = 0{,}5}}$$

$$G_0(s) = G_c(s) \cdot G_p(s)$$

■

$$G_0(s) = \frac{K_c \cdot \cancel{(0{,}5s+1)}}{0{,}5s} \cdot \frac{5}{(0{,}1s+1)\cancel{(0{,}5s+1)}}$$

$$G_0(s) = \frac{5K_c}{0{,}5s(0{,}1s+1)} = \frac{10K_c}{s\cdot(0{,}1s+1)}$$

Using Matlab to calculate K_c for $\phi_M = 60°$

$\omega_{gc} \approx 5{,}86\ \frac{rad}{s}$ $\quad$ $K_c = 0{,}68$

■ **Example 4.3 — Cancelling poles, which are responsible for oscillation.**

$$G_P(s) = \frac{1.5}{(1+0.1s)\left(\left(\frac{s}{5}\right)^2 + 2\cdot 0.2\frac{s}{5} + 1\right)}$$

Design a PIDT$_1$-controller so that conjugate complex poles of the plant are cancelled by controller zeros, with a phase margin of 60°.

$$G_c(s) = K_c \cdot \left(1 + \frac{1}{T_I s} + \frac{T_D s}{T_V s + 1}\right)$$

$$= K_c \cdot \frac{T_I s (T_V s + 1) + T_V s + 1 + T_D s \cdot T_I s}{T_I s \cdot (T_V s + 1)}$$

$$= K_c \cdot \frac{T_I T_V s^2 + T_I s + T_V s + 1 + T_D T_I s^2}{T_I s (T_V s + 1)}$$

$$= K_c \cdot \frac{(T_D + T_V) T_I s^2 + (T_I + T_V) s + 1}{T_I s \cdot (T_V s + 1)}$$

$$G_o(s) = G_c(s) \cdot G_P(s)$$

$$= K_c \cdot \frac{\cancel{(T_D + T_V) T_I s^2 + (T_I + T_V) s + 1}}{T_I s \cdot (T_V s + 1)} \cdot \frac{1.5}{(1+0.1s)\cancel{\left(\left(\frac{s}{5}\right)^2 + 2\cdot 0.2\frac{s}{5} + 1\right)}}$$

$$G_o(s) = \frac{1{,}5 K_c}{T_I s (T_V s + 1)(0{,}1 s + 1)}$$

$$G_o(s) = \frac{1{,}5 K_c}{0{,}04 s (0{,}04 s + 1)(0{,}1 s + 1)}$$

■

$K_c = ?$ s. th. $\phi_M = 60°$

Matlab: $\Rightarrow K_c = 0{,}11$

$$(T_D + T_V) T_I = \frac{1}{5^2} \rightarrow$$

$$T_I + T_V = \frac{2 \cdot 0{,}2}{5} \rightarrow$$

Suppose $T_V = 0{,}2 \cdot \frac{1}{5} = 0{,}04$

$T_I = 0{,}08 - 0{,}04 = 0{,}04$

$$T_D = \frac{1}{25} \frac{1}{T_I} - T_V$$

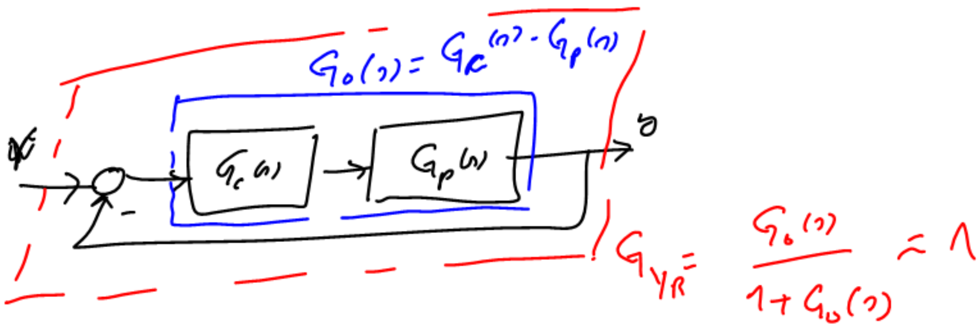
$$T_D = 0{,}04 / \cancel{0{,}08}\ 0{,}04 - T_V = \cancel{0{,}76}\ 0{,}96$$

4.3.4 Optimisation of the closed-loop magnitude response

The magnitude optimum (German: Betragsoptimum) method is widely used to optimise the current control loop for electric machines and other RL-circuits. This design technique is suitable for plants with real poles only. The objective is to calculate the parameters of PI or PID controllers in such a way that the bandwidth of the control is maximised.

$$|G_{YR}(j\omega)| \begin{cases} = 1 & \text{for} \quad \omega = 0 \\ \cong 1 & \text{for} \quad 0 < \omega \leq \omega_B \end{cases}$$

with $\omega_B \to \max$.



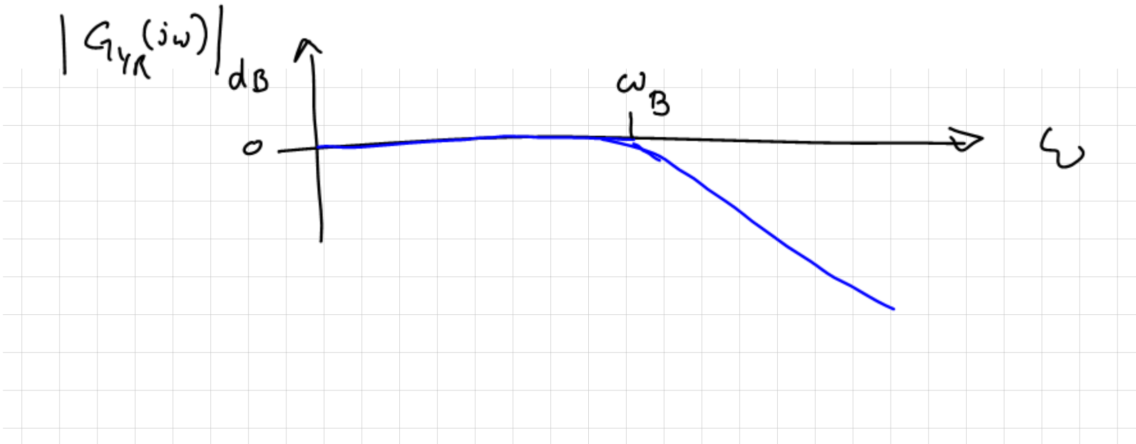


Figure 4.13: Magnitude of the closed-loop system

In order to achieve the objective it is necessary that the maximum number of derivatives of $G_{YR}(j\omega)$ become zero at $\omega = 0$. Given that a plant with real poles is described by the following transfer function

$$G_P(s) = \frac{K_P}{1 + a_1 s + \cdots + a_n s^n}$$

Parameters of a PI controller can be calculated as:

$$T_I = a_1 + \frac{a_3 - a_1 a_2}{a_1^2 - a_2}, \qquad K_C = \frac{1}{2K_P} \cdot \left(\frac{a_1(a_1^2 - a_2)}{a_1 a_2 - a_3} - 1 \right)$$

For further details of the design method please consult the relevant literature e.g. [Föllinger94, Schröder09].

Special case:

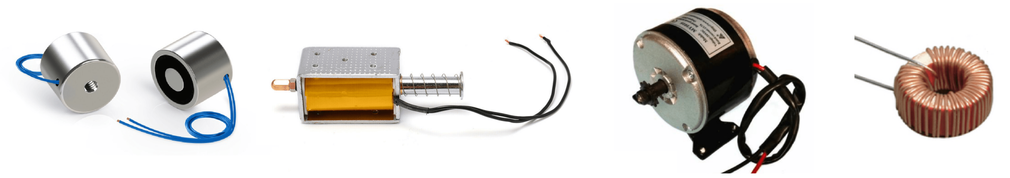
For a second-order plant with transfer function

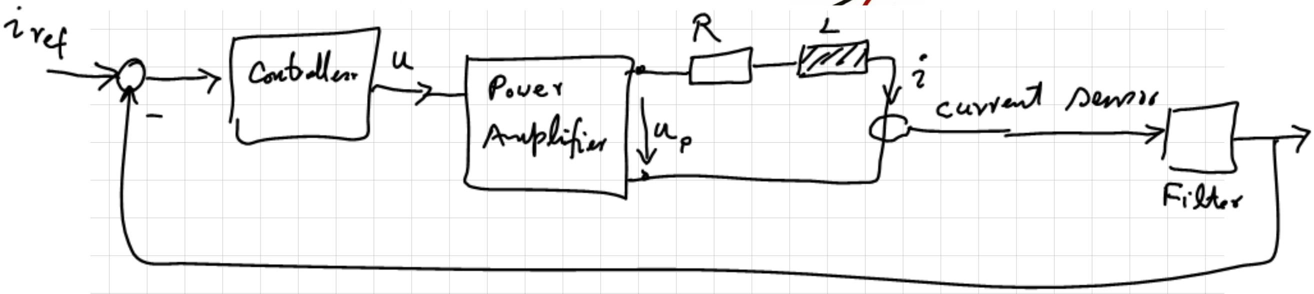
$$G_S(s) = \frac{K_S}{(1 + T_1 s)(1 + T_\sigma s)}$$

with $T_\sigma < T_1 \leq 4T_\sigma$, the controller parameters can be calculated as follows:

$T_I = T_1$ and $K_R = \frac{T_1}{2K_S T_\sigma}$

■ **Example 4.4 Current control in an RL circuit**





Assume $\frac{u_p}{u} = 1 \Rightarrow u = u_p$

Plant: $\frac{I(s)}{U(s)} = \frac{1/R}{L/R\, s + 1}$; Filter: $G_F(s) = \frac{1}{T_F\, s + 1}$

$$G_p(s) = \frac{1/R}{L/R\, s + 1} \cdot \frac{1}{T_F s + 1}$$

Assume Parameter values: $R = 2\,\Omega$ $L = 4\,mH$ | $T_F = 0{,}5\,ms$

$$G_p(s) = \frac{1/2}{\frac{4\cdot 10^{-3}}{2} s + 1} \cdot \frac{1}{0{,}5\cdot 10^{-3} s + 1}$$

$\frac{L}{R} = \frac{4\cdot 10^{-3}\,H}{2\,\Omega} = 2\cdot 10^{-3}\,s$

$$= \frac{0{,}5}{(2\cdot 10^{-3} s + 1)(0{,}5\cdot 10^{-3} s + 1)}$$

$$G_p(s) = \frac{\overset{K_p}{0{,}5}}{\underset{a_2}{10^{-6}} s^2 + \underset{a_1}{2{,}5\cdot 10^{-3}} s + 1}$$

$\underline{\underline{T = 0{,}002\, s}}$ $\underline{\underline{K_c = 4{,}25}}$

■

4.3.5 The symmetrical optimum

The symmetrical optimum was developed for the speed control of a dc motor. This method is used to design a PI controller for a plant, which already has an integrator in its model. The plant transfer function has the following form:

$$G_P(s) = \frac{K_P}{(T_1 s + 1)\,\cdot\, T_2 s}$$

For the sake of disturbance rejection, the controller should also have an integral action. Two integrators in the open loop cause a phase shift of $-180°$. But, in order to achieve a stable control system the phase margin must be positive. The controller zero at $s = -1/T_I$ could help to modify the phase response in the positive direction if the corner frequency $1/T_I$ is kept lower than the plant corner frequency $1/T_1$. By selecting the geometric mean of $1/T_I$ and $1/T_1$ as gain crossover frequency a positive phase margin can be achieved. The value of the phase margin depends on the the ratio of these corner frequencies.
For a given phase margin ϕ_M the controller parameters can be calculated as:

$$T_I = \left(\frac{1+\sin\phi_M}{\cos\phi_M}\right)^2 T_1, \qquad \omega_{gc} = \frac{1}{\sqrt{T_I \cdot T_1}}, \qquad K_C = \frac{1}{K_P}\cdot \omega_{gc} T_2.$$

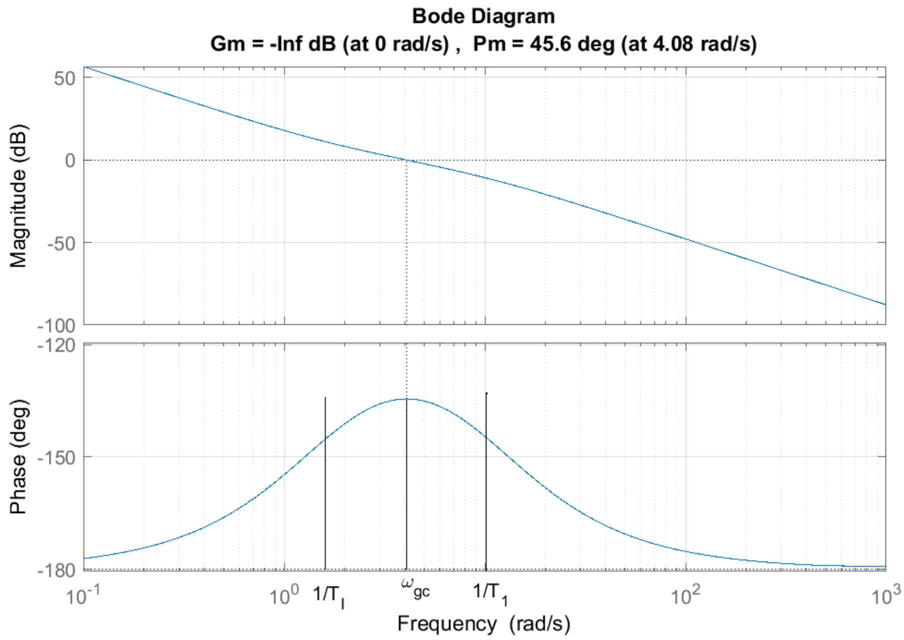
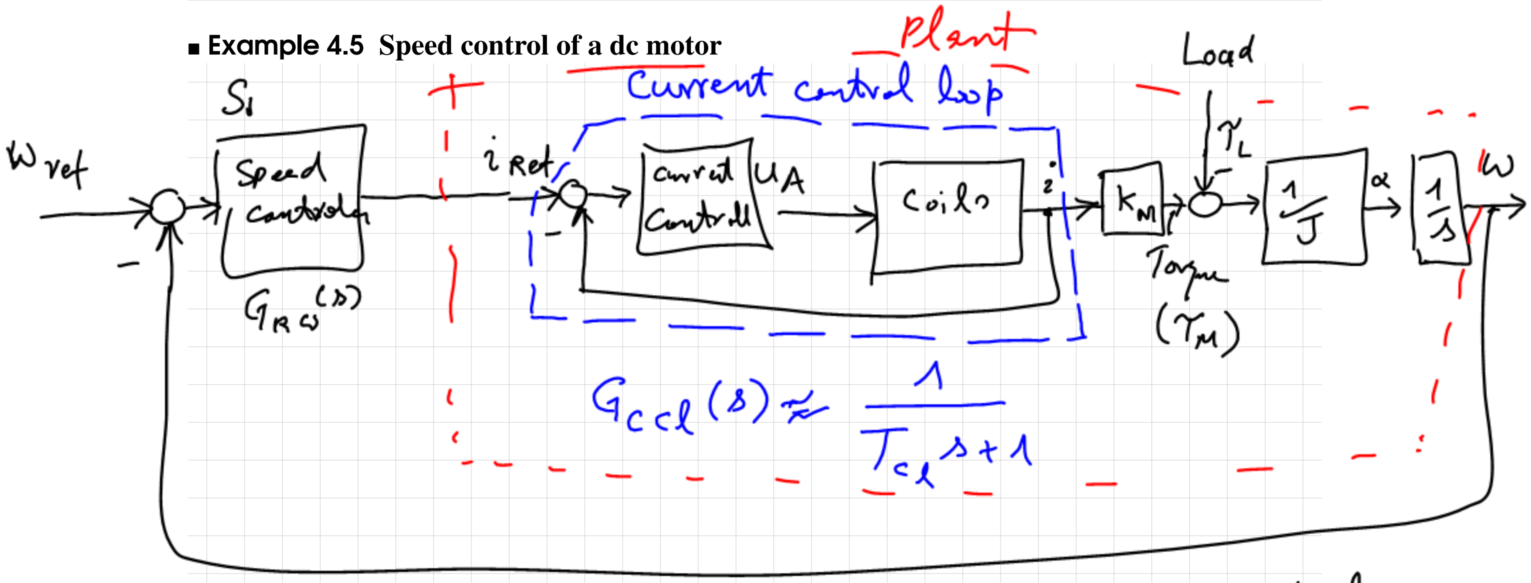


Figure 4.14: The symmetrical optimum

This method is suitable for designing a PI controller for the following cases:

1. The plant has already integral behaviour e.g. an IT1 system $G_P(s) = \frac{K_P}{(T_1 s+1)\,\cdot\, T_2 s}$
2. The plant does not have integral behaviour but one group of time constants of the plant is significantly larger than the other.

■ **Example 4.5** **Speed control of a dc motor**



Transfer function of the plant for speed control.

$$G_p(s) = G_{ccl}(s)\cdot K_M \cdot \frac{1}{J}\cdot\frac{1}{s} = \frac{K_M}{(T_{cl}s+1)\cdot J\cdot s}$$

$\Rightarrow$ an IT_1 system

Suppose: No disturbance. (tracking control)

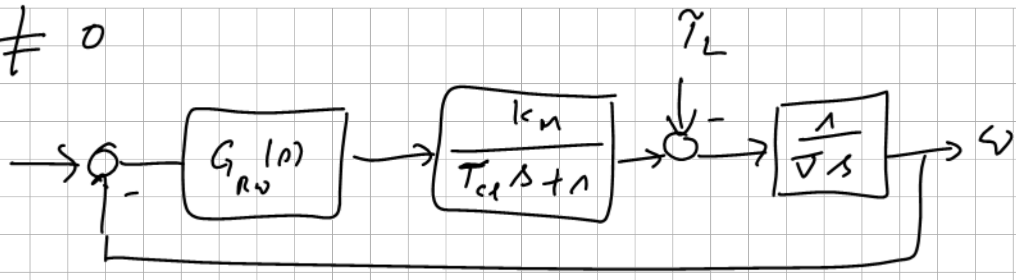
■ $$\frac{\omega(s)}{\omega_{ref}(s)} = \frac{G_{R\omega}(s)\cdot G_p(s)}{1+G_{R\omega}(s)\cdot G_p(s)}$$

Steady-state gain of the control loop.

$$K_0 = \frac{G_{R\omega}(0)\cdot \overset{\infty}{G_p(0)}}{1+G_{R\omega}(0)\cdot \underset{\infty}{G_p(0)}} = \frac{G_{R\omega}(0)}{\underbrace{1/G_p(0)}_{0} + G_{R\omega}(0)} = \frac{G_{R\omega}(0)}{G_{R\omega}(0)} = 1$$

$\Rightarrow$ Steady-state accuracy is achieved even if $G_{R\omega}(s) = K_R$ (P-controller)

ii) $\tilde{T}_L \neq 0$



$$\frac{\omega(s)}{\tilde{T}_L(s)} = \frac{-\frac{1}{Js}}{1 + G_{R\omega}(s) \cdot \frac{k_M}{T_{cl}s+1} \cdot \frac{1}{Js}}$$

$$= \frac{-1}{Js + k_M \cdot G_{R\omega}(s) \cdot \frac{1}{T_{cl}s+1}}$$

Steady state gain

$$K_{ss} = \frac{-1}{0 + k_M \cdot G_{R\omega}(0) \cdot \frac{1}{0+1}}$$

$$= -\frac{1}{k_M \cdot G_{R\omega}(0)} \overset{!}{=} 0$$

disturbance.

$\Rightarrow G_{R\omega}(0) \to \infty \Rightarrow$ Speed controller must have an integrator.

■ **Example 4.6** Please design a PI controller for the following plant using symmetrical optimum method. The phase margin should be 37°.

$$G_P(s) = \frac{\overset{K_P}{2}}{(\underset{T_1}{0.3}s+1)\cdot \underset{T_2}{1.5}s}$$

a) Graphical method with Matlab

$$G_c(s) = K_c \cdot \left(1 + \frac{1}{T_I s}\right)$$

$$\phi_M = 45^\circ$$

$$G_c(s) = 1 \cdot \frac{(s+0{,}56)}{s}$$

$$T_I = \frac{1}{0{,}56} = 1{,}79$$

$$K_c = 1$$

b) With the help of the formulae

$$T_I = \left(\frac{1+\sin\phi_M}{\cos\phi_M}\right)^2 T_1, \qquad \omega_{gc} = \frac{1}{\sqrt{T_I \cdot T_1}}, \qquad K_C = \frac{1}{K_P}\cdot\omega_{gc}T_2.$$

$$T_I = \left(\frac{1+\sin 45^\circ}{\cos 45^\circ}\right)^2 \cdot 0{,}3 = \left(\frac{1{,}707}{0{,}707}\right)^2 \cdot 0{,}3 = 1{,}77\ s$$

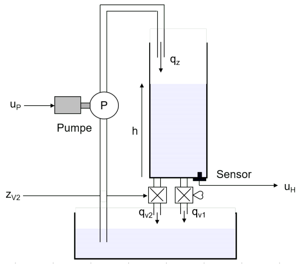
$$\omega_{gc} = \frac{1}{\sqrt{1{,}77\cdot 0{,}3}} = \overset{1{,}37}{\cancel{2{,}14}}\ rad/s$$

$$K_c = \frac{1}{2}\cdot \overset{1{,}37}{\cancel{2{,}14}} \cdot 1{,}5 = \cancel{1{,}6} = 1{,}02$$

■

■ **Example 4.7** A plant with a small and a large time constant

Please design a PI controller for the following plant using symmetrical optimum method. The phase margin should be 45°.



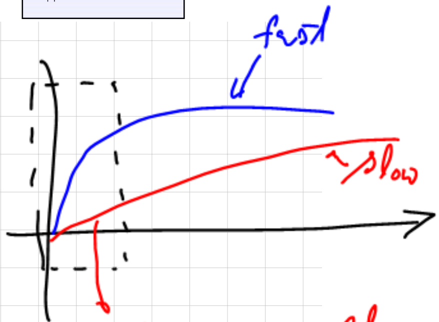
$$G_P(s) = \frac{\overbrace{10}^{K_P}}{(\underbrace{0.1}_{T_1}s+1)\cdot(\underbrace{5}_{T_2}s+1)}$$

$$T_I = \left(\frac{1+\sin\phi_M}{\cos\phi_M}\right)^2 T_1, \qquad \omega_{gc} = \frac{1}{\sqrt{T_I\cdot T_1}}, \qquad K_C = \frac{1}{K_P}\cdot\omega_{gc}T_2.$$

$$T_i = \left(\frac{1+\sin 45}{\cos 45}\right)^2\cdot 0{,}1 = \left(\frac{1{,}7}{0{,}7}\right)^2\cdot 0{,}1 = 0{,}590$$

$$\omega_{gc} = \frac{1}{\sqrt{0{,}1\cdot 0{,}59}} = 4{,}1 \text{ rad/s}$$

$$K_c = \frac{1}{10}\cdot 4{,}1\cdot 5 = 2{,}05$$



In this small time window it looks like an integrate

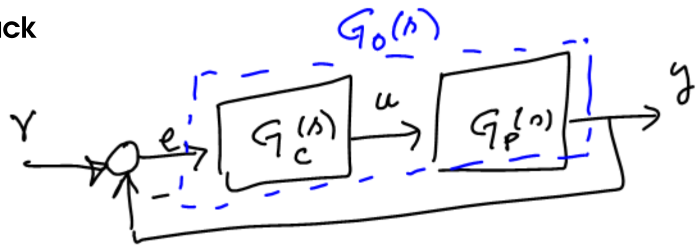
→ Symmetrical optimum.

■

4.4 Pole-placement method

Important properties of a dynamic system like speed of response, oscillation behaviour, stability etc. depend on the location of its poles. Poles can be moved to better locations in order to improve dynamic behaviour of a system. The relationship between system properties and pole locations can be used to translate control specification into location of desired poles. In a second step, the controller parameters can be calculated to place the closed-loop poles directly at desired locations.

4.4.1 Influencing pole locations using feedback



Controller:

$$G_C(s) = \frac{B_C(s)}{A_C(s)}$$

Plant:

$$G_P(s) = \frac{B_P(s)}{A_P(s)}$$

The open loop:

$$G_0(s) = \frac{B_C(s) \cdot B_P(s)}{A_C(s) \cdot A_P(s)}$$

The closed loop:

$$G_{YR}(s) = \frac{B_C(s) \cdot B_P(s)}{A_C(s) \cdot A_P(s) + B_C(s) \cdot B_P(s)}$$

The closed-loop poles i.e. the roots of the denominator polynomial can be influenced by the controller polynomials $A_C(s)$ and $B_C(s)$. Calculation of controller parameters for a given set of closed-loop poles is explained with the help of the following example.

■ **Example 4.8** Consider an unstable plant with the following transfer function:

$$G_P(s) = \frac{1}{0.5s - 0.5}$$

The task is to design a PI controller for this plant so that the closed-loop poles move to -2 and -3.

$$G_C(s) = K_C \left(1 + \frac{1}{T_I s}\right) = \frac{K_C \left(T_I s + 1\right)}{T_I s}$$

Solution:

$$G_P(s) = \frac{1}{0.5s - 0.5}, \qquad G_C(s) = K_C\left(1 + \frac{1}{T_I s}\right)$$

Determine K_c & T_I so that closed-loop system has poles at -2 & -3

Closed-loop transfer function:

$$G_{YR}(s) = \frac{G_c(s) \cdot G_p(s)}{1 + G_c(s) \cdot G_p(s)} = \frac{K_c\left(1+\frac{1}{T_I s}\right) \cdot \frac{1}{0{,}5s - 0{,}5}}{1 + K_c \cdot \left(1 + \frac{1}{T_I s}\right) \cdot \frac{1}{0{,}5s - 0{,}5}}$$

$$= \frac{K_c(T_I s + 1)}{T_I s(0{,}5s - 0{,}5) + K_c(T_I s + 1)}$$

$$= \frac{K_c(T_I s + 1)}{0{,}5 T_I s^2 - 0{,}5 T_I s + K_c T_I s + K_c}$$

$$= \frac{K_c(T_I s + 1)}{0{,}5 T_I s^2 + (K_c - 0{,}5) T_I s + K_c}$$

$$= \frac{K_c \cdot (T_I s + 1)}{0{,}5 T_I \left(s^2 + \frac{(K_c - 0{,}5)}{0{,}5} s + \frac{K_c}{0{,}5 T_I}\right)}$$

$$P(s) = (s+2)(s+3) = 1 s^2 + 5s + 6$$

$$4 s^2 + 20 s + 24$$

Comparing the coefficients:

$$\frac{K_c - 0{,}5}{0{,}5} = 5 \Rightarrow K_c = 5 \cdot 0{,}5 + 0{,}5 = 3$$

$$\frac{K_c}{0{,}5 T_I} = 6 \Rightarrow T_I = \frac{K_c}{0{,}5 \cdot 6} = \frac{3}{3} = 1\, s.$$

■

4.4.2 How to select closed-loop poles?

By appropriate selection of closed-loop poles, the transient response of the control system can be optimized. The following considerations are helpful in this task:

1. For the stability of the control loop it is necessary that all poles are located in the left half-plane (LHP).
2. In order to achieve a good speed of response, the natural frequency i.e. the distance between poles and the origin (0,0) of the s-plane must be large enough.
3. For better oscillation behaviour, the damping ratio of the poles should be sufficiently large.
4. In order to avoid unnecessary noise amplification and problems related to actuator saturation, the natural frequency should not be made too large.

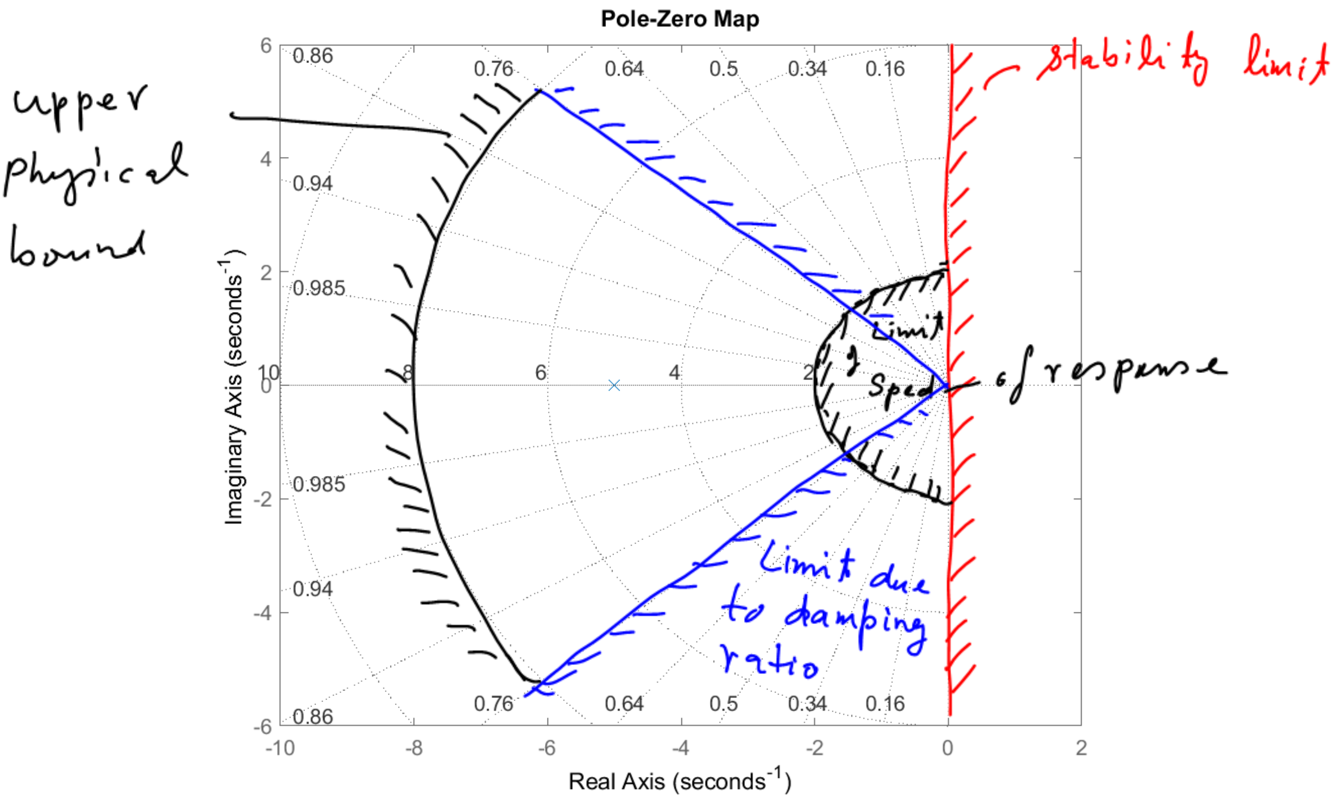


Figure 4.15: Selection of closed-loop poles

4.4.3 Demerits of the method

1. Number of freely placeable poles depends on the number of adjustable controller parameters. For example, in case of a PI controller with two parameters only two poles can be freely placed.
2. This method places the poles of the closed loop at desired positions, but does not consider its zeros. The design process can generate controller zeros at undesired location, which could deteriorate the control performance. In the previous example, the controller zero is located at $s = -1$, which causes an overshoot of 25%. Whereas, the control loop without this zero would be an over-damped system without any overshoot (see Figure 4.16).
3. This method is not applicable if the plant has a dominant transport delay.

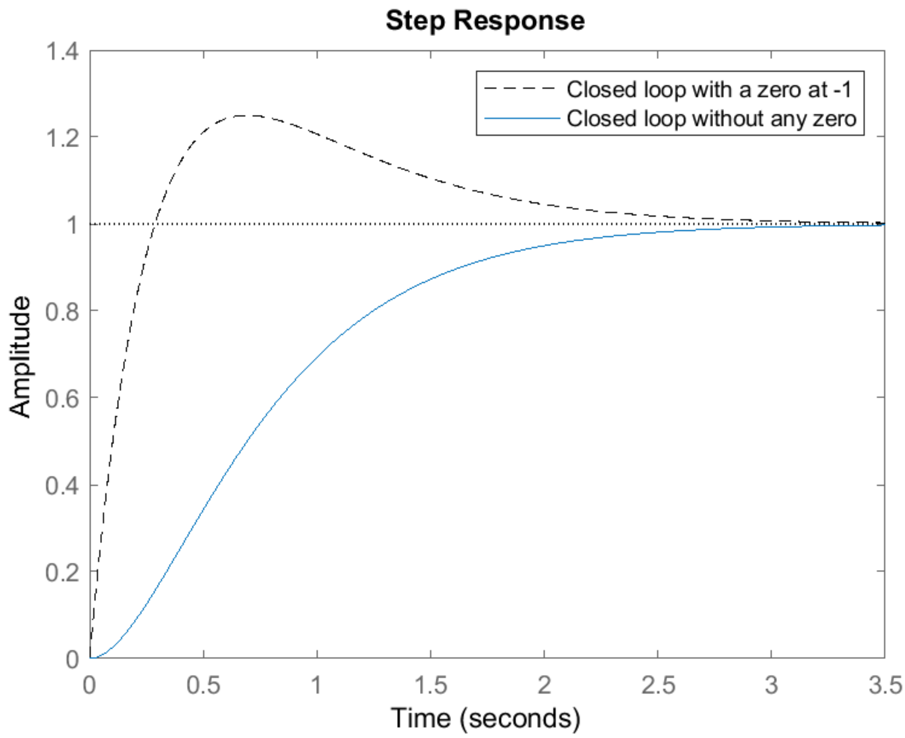
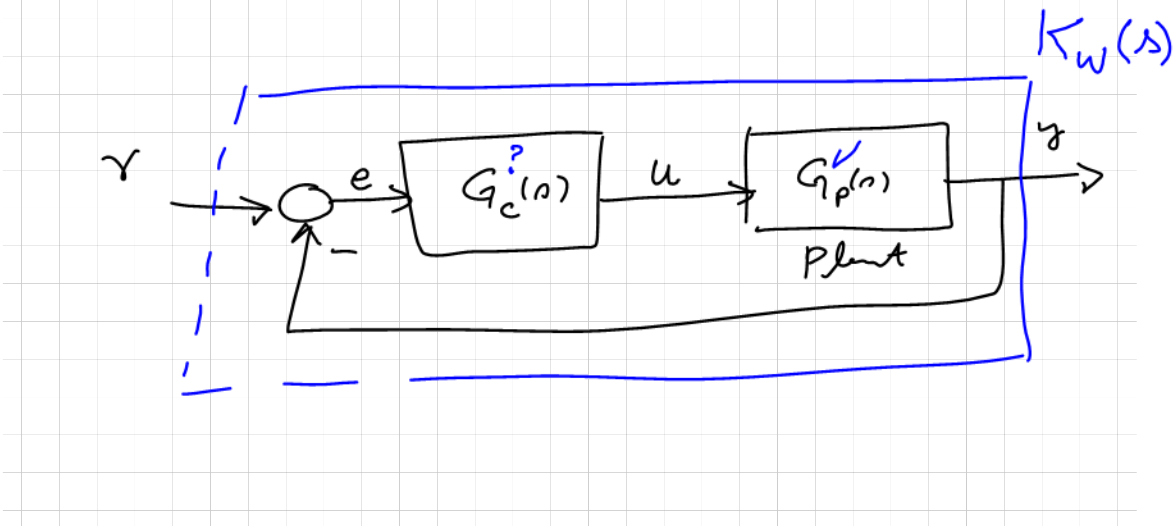


Figure 4.16: Closed-loop step response with two real poles

4.5 Desired transfer function approach (compensation method)

4.5.1 Basic idea (set-point tracking)

The basic idea of this design method is to express the control specifications in form of a transfer function $K_W(s)$. Consequently, the controller $G_C(s)$ is designed so that the closed-loop system has the specified transfer function.



The closed-loop transfer function of a standard feedback loop can be given as:

$$G_{YR}(s) = \frac{G_P(s)G_C(s)}{1+G_P(s)G_C(s)}$$

with

$$G_P(s) = \frac{B(s)}{A(s)} = \frac{b_0+b_1s+\cdots+b_ms^m}{a_0+a_1s+\cdots+s^n}, \quad n \geq m$$

The desired transfer function $K_W(s)$ of the closed-loop is defined as

$$K_W(s) = \frac{\alpha(s)}{\beta(s)} = \frac{\alpha_0+\alpha_1s+\cdots+\alpha_\nu s^\nu}{\beta_0+\beta_1s+\cdots+s^\mu}, \quad \mu > \nu$$

Comparing both transfer functions, we get

$$\frac{G_P(s)G_C(s)}{1+G_P(s)G_C(s)} \stackrel{!}{=} K_W(s)$$

Solving this equation for $G_C(s)$ results in

$$G_C(s) = \frac{K_W(s)}{1-K_W(s)} \cdot \frac{1}{G_P(s)}$$

$$G_C(s) = \frac{\alpha(s)}{\beta(s)-\alpha(s)} \cdot \frac{A(s)}{B(s)}$$

The controller can be calculated by using above expression if the plant transfer function $G_P(s)$ is known and the desired tracking model $K_W(s)$ is already specified. Now the question is how to select the transfer function $K_W(s)$. The following discussion can help to answer this question.

■ **Example 4.9**

$$G_P(s) = \frac{5}{s^2+0.5s+2} \;;$$

Required: $G_c(s) = ?$

compensation controller with

$$K_w(s) = \frac{4}{s^2+4s+4} \quad \checkmark$$

$$G_c(s) = \frac{K_w(s)}{1-K_w(s)} \cdot \frac{1}{G_p(s)}$$

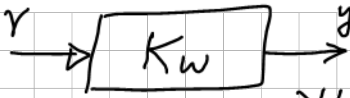
$$= \frac{\frac{4}{s^2+4s+4}}{1-\frac{4}{s^2+4s+4}} \cdot \frac{s^2+0{,}5s+2}{5}$$

■

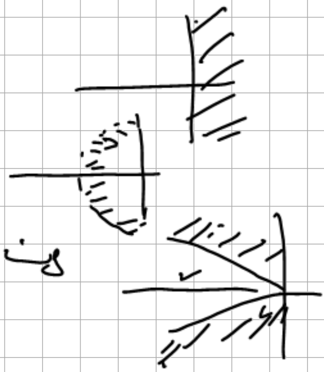
$$= \frac{4}{s^2+4s+4-4} \cdot \frac{s^2+0{,}5s+2}{5}$$

$$G_c(s) = \frac{4\,(s^2+0{,}5s+2)}{5\,s\,(s+4)}$$

How to select $K_w(s)$?



$$K_w(s) = \frac{Y(s)}{R(s)}$$



1. $K_w(s)$ must be stable.
2. $K_w(s)$ " " fast
3. " have good damping
4. $e_\infty = 0 \Rightarrow K_w(0) = 1$
5. $G_c(s)$ must be proper $\Rightarrow \mu - \nu \geq n - m$

(For a minimum-order controller $\mu - \nu = n - m$)

Example: Given $G_p(s) = \dfrac{2}{s+0,3}$

Required: $K_w(s) = ?$

$G_c(s) = ?$

Suggestion 1: $K_w(s) = \dfrac{1 \cdot 6}{(s+2)(s+3)} = \dfrac{6}{s^2+5s+6}$ α β

$$G_c = \frac{6}{s^2+5s+6-6} \cdot \frac{s+0,3}{2}$$

$$G_c(s) = \frac{3(s+0,3)}{s(s+5)}$$

<u>Example 2:</u> $G_p(s) = \dfrac{s+1}{s^2+0,5s+2}$

$$K_w(s) = \frac{2}{s+2}$$

$$G_c(s) = \frac{2}{s+2-2} \cdot \frac{s^2+0,5s+2}{s+1} = \frac{2(s^2+0,5s+2)}{s \cdot (s+1)}$$

Example 3: $G_p(s) = \dfrac{1}{s^2+0,5s+2}$

$$K_w(s) = \frac{2}{s+2}$$

$$G_c(s) = \frac{2}{s+2-2} \cdot \frac{s^2+0,5s+2}{1} = \frac{2(s^2+0,5s+2)}{s} \quad !$$

$\Rightarrow G_c$ is not proper $\Leftarrow$ deg(Numerator) > deg(Denom.)

$\Rightarrow G_c$ cannot be implemented.

$K_w(s) = \dfrac{2}{s+2}$ is not suitable for this plant !

Example: $G_p(s) = \dfrac{s-1}{(s+1)(s+0{,}5)}$ | Plant has one zero in RHP

$m = 1 \quad \& \quad n = 2 \qquad n - m = 1$

$$K_w(s) = \frac{\omega_0}{s+\omega_0} \qquad (\omega_0 = 2)$$

$$K_w(s) = \frac{2}{s+2}$$

$$G_c(s) = \frac{2}{s+2-2} \cdot \frac{(s+1)(s+0{,}5)}{s-1}$$

$$G_c(s) = \frac{2(s+1)(s+0{,}5)}{s\,(s-1)}$$

This cancellation is not allowed.

$K_w(s)$ is wrong!

Correction:

$$G_p(s) = \frac{\overbrace{(s-1)}^{B^+}\cdot 1}{(s+1)(0{,}5+s)}$$

$$K_w(s) = \frac{\overbrace{(s-1)}^{B^+}\cdot\overbrace{\boxed{-4}}^{\alpha^*}}{(s+2)^2}$$

$$G_c(s) = \frac{\cancel{(s-1)}(-4)}{s^2+4s+4+4s-4} \cdot \frac{(s+1)(s+0{,}5)}{\cancel{(s-1)}}$$

$$= \frac{-4\cdot(s+1)(s+0{,}5)}{s\,(s+8)}$$

Example 2: $G_p(s) = \dfrac{2}{(s+0{,}5)(s-2)}$ $\quad m=0$, $n=2$ $\quad n-m=2$

$$K_w(s) = \frac{4}{(s+2)^2} = \frac{4}{s^2+4s+4}$$

$$G_c = \frac{4}{s^2+4s} \cdot \frac{(s+0{,}5)(s-2)}{2}$$

$$G_c(s) = \frac{2(s+0{,}5)(s-2)}{s(s+4)}$$

This cancellation is not allowed!

$K_w(s)$ is wrong!

Ex. $$G_p(s) = \frac{2}{\underbrace{(s-1)}_{A^+} \cdot 1}$$

$$K_w(s) = \frac{\alpha(s)}{\beta(s)} \qquad \gamma(s) = \beta(s) - \alpha(s) = \underbrace{\tilde{\gamma}(s) \cdot (s-1)}$$

Let us try $\mu = 1$

$$K_w(s) = \frac{\omega_0}{s+\omega_0}$$

$$\gamma(s) = s + \omega_0 - \omega_0 = s$$

impossible.

$\mu = 2$; $\nu = 1$

$$K_w(s) = \frac{\alpha_1 s + 1}{\beta_2 s^2 + \beta_1 s + 1}$$

$$\gamma(s) = \beta_2 s^2 + \beta_1 s + 1 - \alpha_1 s - 1 = \beta_2 s^2 + (\beta_1 - \alpha_1) s$$

$$= \beta_2 s \left(s + \underbrace{\left(\frac{\beta_1 - \alpha_1}{\beta_2}\right)}_{-1} \right)$$

$$\frac{\beta_1 - \alpha_1}{\beta_2} = -1 \Rightarrow \underline{\underline{\alpha_1 = \beta_1 + \beta_2}}$$

$$K_w(s) = \frac{\alpha_1 s + 1}{\left(\frac{s}{2} + 1\right)^2} = \frac{\alpha_1 s + 1}{\underbrace{0{,}25}_{\beta_2} s^2 + \underbrace{\ }_{\beta_1} s + 1}$$

$$K_w(s) = \frac{1{,}25 s + 1}{0{,}25 s^2 + s + 1}$$

$$G_c(s) = \frac{1{,}25 s + 1}{0{,}25 s^2 + s + 1 - 1{,}25 s - 1} \cdot \frac{s-1}{2}$$

$$= \frac{(1{,}25 s + 1)(s-1)}{(0{,}25 s^2 - 0{,}25 s) \quad 2} = \frac{(1{,}25 s + 1)\cancel{(s-1)}}{0{,}25 s \cancel{(s-1)} \cdot 2}$$

$$= \frac{1{,}25 s + 1}{0{,}5 s}$$

Plant has some zeros in right half-plane (RHP)

The compensation controller contains an inverse of the plant model. That means it cancels the poles and zeros of the plant directly. Such a cancellation is not a problem as long as the poles and zeros of the plant are located in the left half plane. Cancelling RHP zeros would lead to instability in a practical implementation of the control. This cancellation must be avoided. To avoid direct cancellation, the RHP zeros of the plant must also be are defined as zero of the closed loop model $K_W(s)$. For that purpose the numerator polynomial of the plant $B(s)$ is written in form of two factors:

$$G_p(s) = \frac{B(s)}{A(s)} = \frac{B^+(s)B^-(s)}{A(s)} \quad \text{RHP}$$

$$B(s) = B^+(s)B^-(s).$$

The polynomial $B^-(s)$ has all its roots in the left half plane, whereas, the roots of the polynomial $B^+(s)$ are located in the right half plane. Now the numerator $\alpha(s)$ of $K_W(s)$ can be written as:

$$G_c(s) = \frac{\alpha(s) \; B^+\cdot\alpha^*}{\beta(s)-\alpha(s)} \cdot \frac{A(s)}{\boxed{\cancel{B^+(s)}}\, B^-(s)}$$

$$\alpha(s) = B^+(s)\alpha^*(s)$$

The controller has the following transfer function

$$G_R(s) = \frac{\alpha^*(s)}{\beta(s)-\alpha(s)} \cdot \frac{A(s)}{B^-(s)}$$

Plant has some poles in right half-plane (RHP)

In this case the denominator polynomial $A(s)$ can be factorised in RHP and LHP parts $A(s) = A^+(s)A^-(s)$. The part $A^+(s)$ is not allowed to be cancelled by controller zeros. This is avoided if polynomial $\gamma(s) = \beta(s) - \alpha(s)$ contains $A^+(s)$ as its factor.

$$G_p(s) = \frac{B(s)}{\underbrace{A^+(s)}_{\text{RHP}}\, A^-(s)}$$

$$\beta(s) - \alpha(s) = \gamma(s) = A^+(s)\gamma^*(s)$$

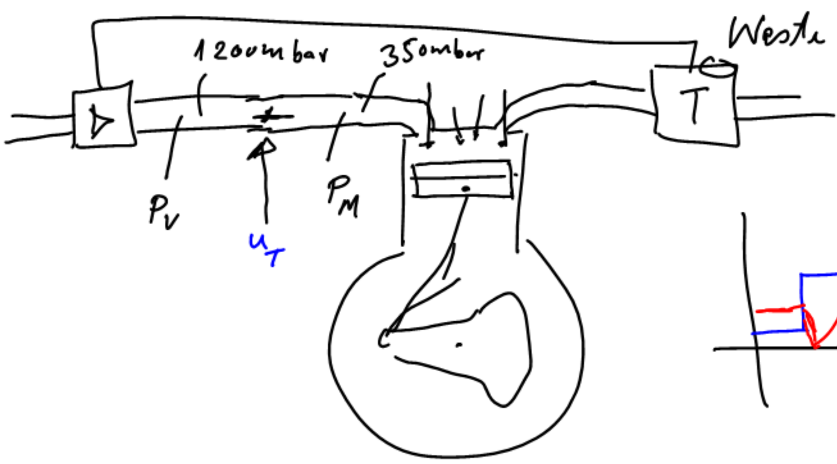
The resulting controller transfer function

$$G_c(s) = \frac{\alpha(s)}{\underbrace{\beta(s)-\alpha(s)}_{\gamma(s) = \cancel{A^+(s)}\cdot\gamma^*(s)}} \cdot \frac{\cancel{A^+(s)}\cdot A^-(s)}{B(s)}$$

$$G_R(s) = \frac{\alpha(s)}{\gamma'(s)} \cdot \frac{A^-(s)}{B(s)}$$

does not cancel RHP poles of the plant directly.

> ***Important note:*** Poles and zeros of the plant, which are located in the right half-plane are not allowed to be cancelled by the controller. Such cancellation would cause instability in the system due to modelling inaccuracy and actuator saturation.



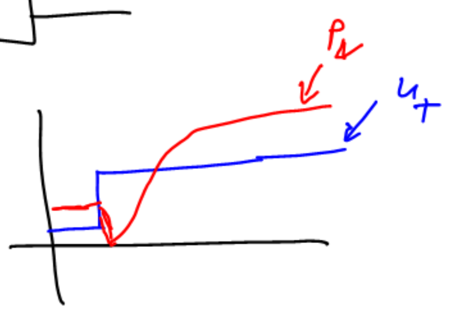


Table 4.3: Transfer function and step response of $K_W(s)$ in binomial form

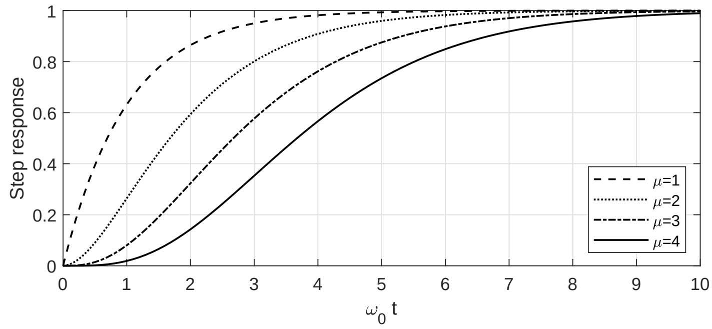
μ	$K_W(s)$	Step response
1	$\frac{\omega_0}{s+\omega_0}$	
2	$\frac{\omega_0^2}{s^2+2\omega_0 s+\omega_0^2}$	
3	$\frac{\omega_0^3}{s^3+3\omega_0 s^2+3\omega_0^2 s+\omega_0^3}$	
4	$\frac{\omega_0^4}{s^4+4\omega_0 s^3+6\omega_0^2 s^2+4\omega_0^3 s+\omega_0^4}$	

Table 4.4: Transfer function and step response of $K_W(s)$ in Butterworth form

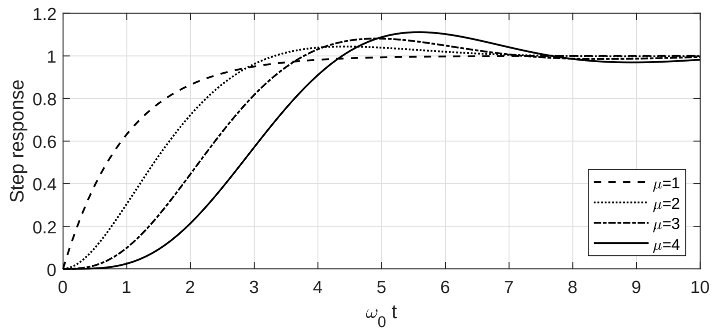
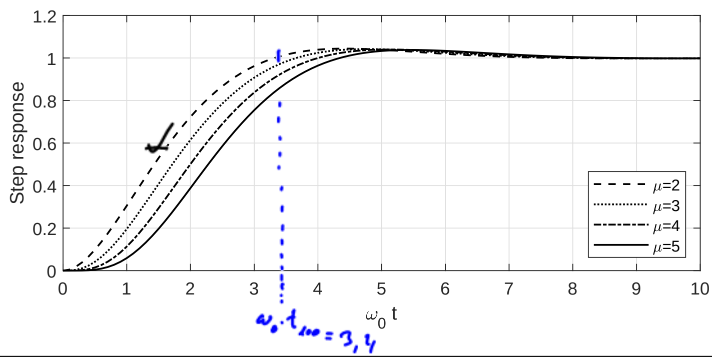
μ	$K_W(s)$	Step response
1	$\frac{\omega_0}{s+\omega_0}$	
2	$\frac{\omega_0^2}{s^2+1.4\omega_0 s+\omega_0^2}$	
3	$\frac{\omega_0^3}{s^3+2\omega_0 s^2+2\omega_0^2 s+\omega_0^3}$	
4	$\frac{\omega_0^4}{s^4+2.6\omega_0 s^3+3.4\omega_0^2 s^2+2.6\omega_0^3 s+\omega_0^4}$	

Table 4.5: Transfer function and step response of $K_W(s)$ in Weber form with $D = 0.7$

μ	$K_W(s)$	Step response
2	$\frac{\omega_0^2}{s^2+1.4\omega_0 s+\omega_0^2}$ ✓	
3	$\frac{\omega_0^3}{(s^2+1.4\omega_0 s+\omega_0^2)(0.28s+\omega_0)}$	
4	$\frac{\omega_0^4}{(s^2+1.4\omega_0 s+\omega_0^2)(0.28s+\omega_0)^2}$	
5	$\frac{\omega_0^5}{(s^2+1.4\omega_0 s+\omega_0^2)(0.28s+\omega_0)^3}$	

4.6 Actuator saturation

4.6.1 Integrator-Windup Problem

4.6.2 Anti-Windup Schemes

Example:

$$G_P(s) = \frac{1,5}{s\,(0,1s+1)}$$

$$G_C(s) = K_C \cdot \left(1 + \frac{1}{T_I s}\right)$$

} as in exp. 4

Controller design: Symmetrical optimum with $\Phi_M = 45°$

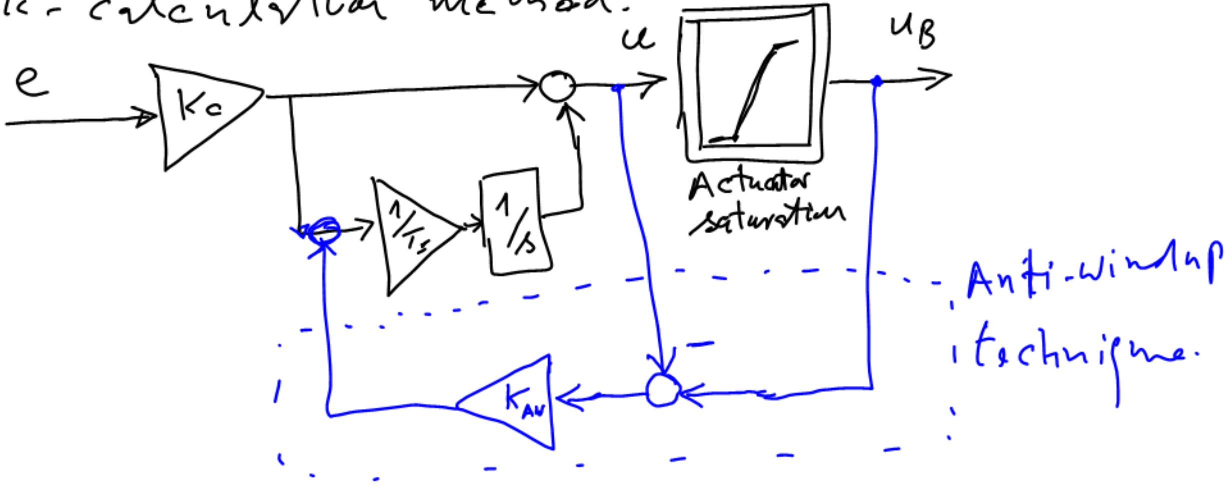
$\Rightarrow$ $T_I = 0,58\,s$ $\quad K_C = 2,86$

$\Rightarrow$ Integrator windup problem due to actuator saturation.

Anti-Windup techniques:

1. Limited intergrator.

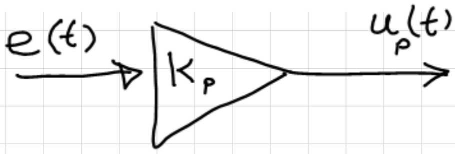
2. Back-calculation method.



later

4.1.2 Implementation as an electronic circuit

P controller



$$u_p(t) = K_p \cdot e(t)$$

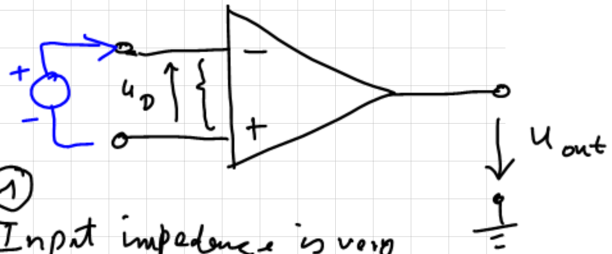
Node ①:

$$i_1 + i_2 = 0$$

$$\frac{e}{R_1} + \frac{u_p}{R_2} = 0$$

$$u_p = \underbrace{-\frac{R_2}{R_1}}_{K_p} e$$

Operational amplifier:



① Input impedance is very high.

$Z_{in} \to \infty \Rightarrow i_{in} = 0$

② Gain is very high

$$\frac{U_{out}}{U_D} \to \infty \Rightarrow u_D = 0$$

⇒ virtual short circuit.

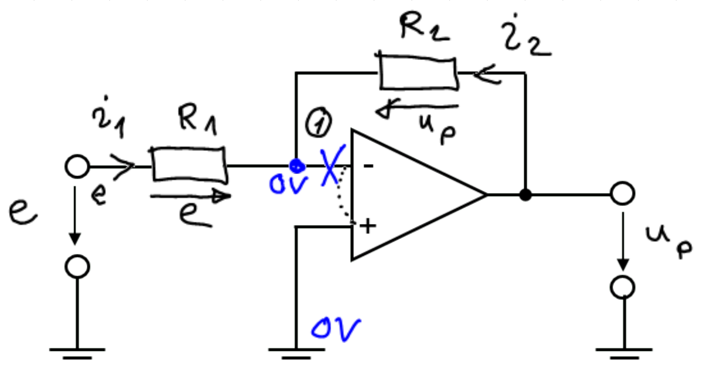
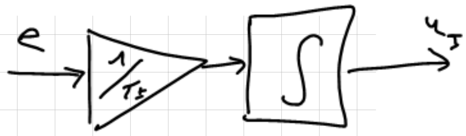


Figure 4.5: Implementation of the P controller as an analogue circuit

I controller

$$u_I(t) = \frac{1}{T_I}\int e\,dt$$



Node equation:

$$i_1 + i_2 = 0$$

$$\frac{e}{R_I} + C_I \cdot \frac{du_I}{dt} = 0$$

$$\frac{du_I}{dt} = -\frac{1}{R_I C_I}\, e(t)$$

$$u_I = -\frac{1}{\underbrace{R_I C_I}_{T_I}}\int e(t)\,dt$$

Capacitor:

$$u_c = \frac{1}{C}\int i_c\,dt$$

$$i_c = C\,\frac{du_c}{dt}$$

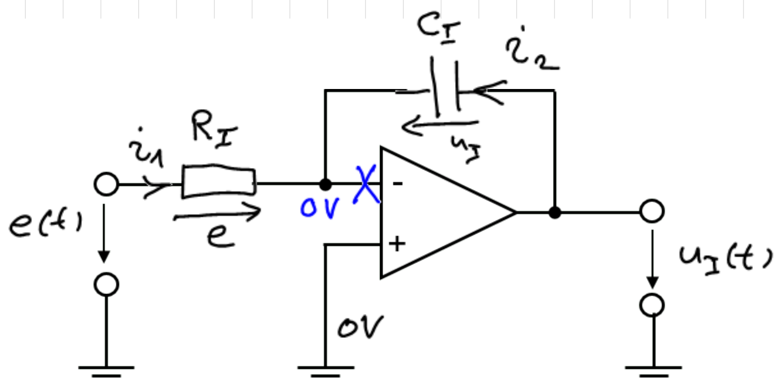


Figure 4.6: Implementation of the I controller as an analogue circuit

D controller

$$u_D(t) = T_D \frac{de}{dt}$$

Node equation:

$$i_1 + i_2 = 0$$

$$C_D \cdot \frac{de}{dt} + \frac{u_D}{R_D} = 0$$

$$u_D = -\underbrace{R_D C_D}_{T_D} \cdot \frac{de}{dt}$$

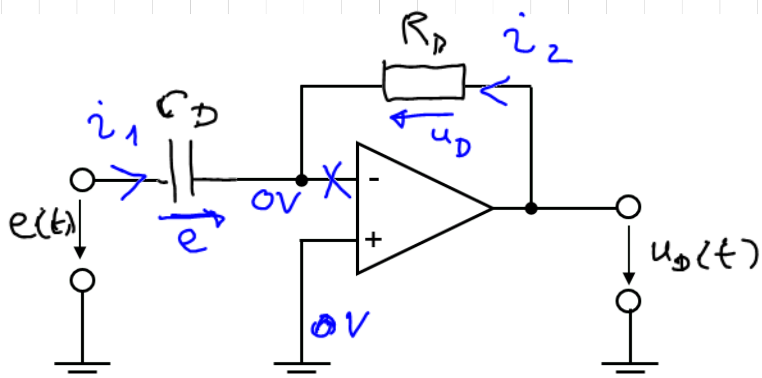


Figure 4.7: Implementation of the D or DT1 controller as an analogue circuit

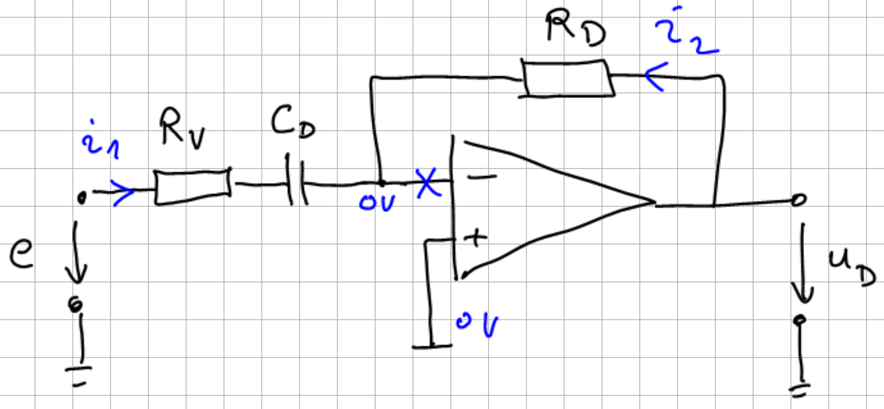
Problem: if $e(t)$ is changed like a step funct ⇒ $i_1 \to \infty$, $i_2 \to \infty$

$u_D \to \infty$

Which are not realisable.

Solution: Limit i_1 !

DT1 - Controller.



Node equation:

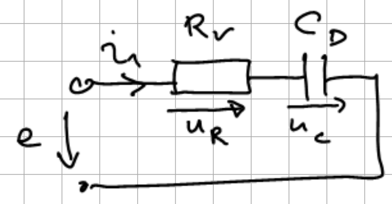
$$i_1 + i_2 = 0$$

$$C_D \frac{de}{dt} - R_V C_D \frac{di_1}{dt} + \frac{u_D}{R_D} = 0$$

$$C_D \frac{de}{dt} + \frac{R_V C_D}{R_D} \frac{du_D}{dt} + \frac{u_D}{R_D} = 0$$

$$R_D C_D \frac{de}{dt} + R_V C_D \frac{du_D}{dt} + u_D = 0$$

$$\boxed{R_V C_D \frac{du_D}{dt} + u_D = - R_D C_D \frac{de}{dt}}$$



$$e = u_R + u_C$$

$$e = i_1 R_V + \frac{1}{C_D} \int i_1 \, dt$$

$$\frac{de}{dt} = R_V \frac{di_1}{dt} + \frac{1}{C_D} i_1$$

$$i_1 = C_D \frac{de}{dt} - R_V C_D \frac{di_1}{dt}$$

$$i_1 = - i_2$$

$$i_1 = - \frac{u_D}{R_D}$$

$$\frac{di_1}{dt} = - \frac{1}{R_D} \cdot \frac{du_D}{dt}$$

$$R_V C_D s \cdot U_D(s) + U_D(s) = - R_D C_D s E(s)$$

$$\frac{U_D(s)}{E(s)} = \underbrace{- \frac{\overbrace{R_D C_D}^{T_D} s}{\underbrace{R_V C_D}_{T_V} s + 1}}_{DT1}$$

PIDT1 controller

$$U(s) = K_c \left(1 + \frac{1}{T_I s} + \frac{T_D s}{T_V s + 1}\right) \cdot E(s)$$

$$= -K_c \left(\underbrace{-1}_{P} + \underbrace{\frac{-1}{T_I s}}_{I} + \underbrace{\frac{-T_D s}{T_V s + 1}}_{DT1} \right) E(s)$$

$$= -\underbrace{K_c}_{\frac{R_K}{R_1}} \left(\underbrace{-E(s)}_{x_1} + \underbrace{\frac{-E(s)}{T_I s}}_{x_2} + \underbrace{\frac{-T_D s E(s)}{T_V s + 1}}_{x_3} \right)$$

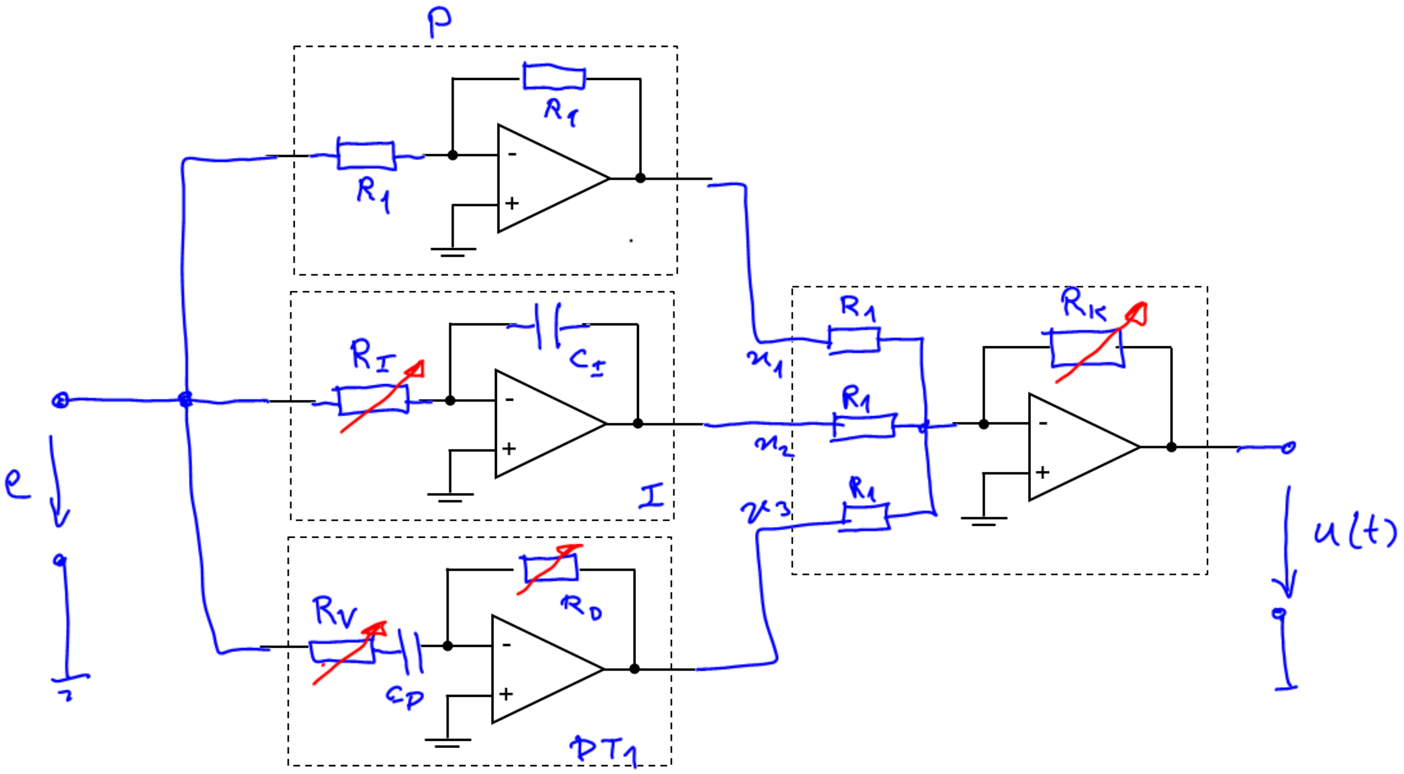
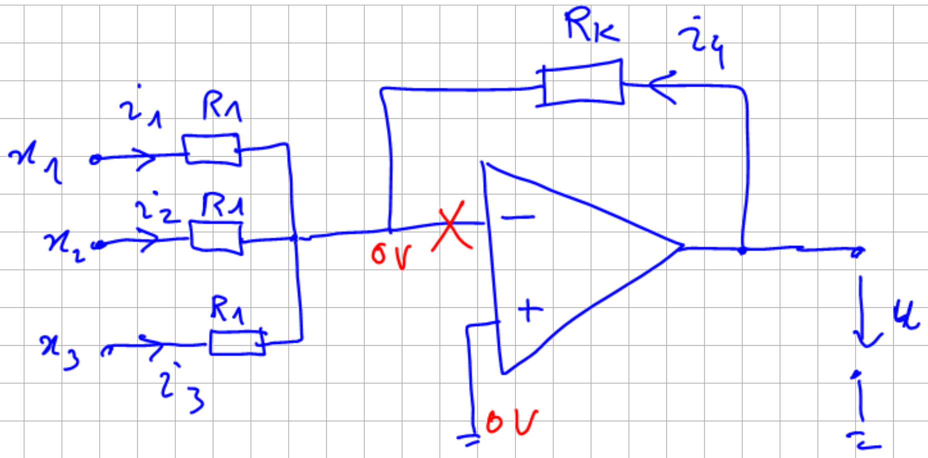


Figure 4.8: Implementation of the PIDT1 controller as an analogue circuit

Parameters: $K_c = \frac{R_K}{R_1}$; $T_I = R_I \cdot C_I$

$T_D = R_D C_D$; $T_V = R_V C_D$

$R_1 = \text{const} =$



$$i_1 + i_2 + i_3 + i_4 = 0$$

$$\frac{x_1}{R_1} + \frac{x_2}{R_1} + \frac{x_3}{R_1} + \frac{u}{R_K} = 0$$

$$u = -\frac{R_K}{R_1}\left(x_1 + x_2 + x_3\right)$$