Signal Processing

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Contents

1 Introduction

1.1 Literature

The theory for the lecture Signal Processing is based on the chapter *Teil A – Signale und Systeme* from the book *Signalübertragung* by Univ. Prof. Dr.-Ing. J. R. Ohm [1]. The book is available in the 12th edition as eBook in the library. It can also be downloaded as long as you are logged into the FHWS network. For the english speaking students we refer to the book *Discrete-Time Signal Processing* by Oppenheim [2].

1.2 Flipped Classroom

In the *Flipped Classroom* teaching concept, students work out the theory by themselves at home using the given literature. In the live lectures, students work together on exercises related to the theory in question. To enable you to view the theory at the appropriate lecture date during the semester beforehand, table 1 sets the prospective schedule for the upcoming semester. The lecture dates are numbered in the left column, and the associated topic is in the right column. Many topics are already known from other lectures, for example system theory. It is useful to read the corresponding chapters from

Table 1: Schedule for the upcoming semester.

No.	Topic
Continuous-Time Systems	
1	Elementary Signals, LTI Systems, Convolution
2	Similarity Theorem and Fourier Transform
Discrete-Time Signals and Systems	
3	Sampling in Time- and Frequency-Domain
4	Discrete-Time Convolution
5	Fourier Transform of Discrete Signals
6	FIR Filter Design
Filter Design	
7	Z-Transformations
8	Bilinear Transformation and IIR Filter Design
9	Partial Fraction Expansion
Correlation Functions	
10	Time-Continuous Correlation Functions
11	Correlation Functions and LTI Systems
12	Discrete-Time Correlation Functions
13	Analysis of LSI Systems
Semester Conclusion	
14 - ...	Exam Preparation, Evaluation, ...

the script before the class date. It is not necessary to understand everything directly. The goal is rather to have seen the relevant terms and formulas once, before going to the seminaristic lessons. In this class, the theory is briefly summarized, questions are clarified and deepened via examples. As a follow-up, a re-reading of the script is recommended.

1.3 eLearning

In the elearning course you will find the transcripts of the lectures, the links to the recordings of the lectures, and other teaching and class materials. Furthermore, all relevant announcements will be distributed here. It is assumed that you will

- enroll in the eLearning course Signal Processing and
- check your FHWS e-mail once a day.

1.4 Content

Chapter 2 introduces computing with continuous-time signals. In chapter 3 sampling and computing with discrete-time signals and systems is explained, especially FIR filters are explained here. Chapter 4 describes the z-transform for analysis of discrete-time IIR filters. In chapter 5 correlation functions are introduced as a tool for similarity analysis. The script ends with a summary and the collection of formulas that will be handed out as aids during the exam.

2 Continuous-Time Signals and Systems

Continuous-time elementary signals are intended to describe measured signals in compact mathematical form and to derive fundamental properties such as the relevant frequency range of the signal or system. A list of the elementary signals used with the mathematical description and a corresponding sketch can be found in table 9.

2.1 Continuous-Time Elementary Signals

The **rectangle signal** is a rectangle of width 1 and height 1:

$$\mathrm{rect}(t) = \begin{cases} 1 & |t| \leq 0.5 \\ 0 & \text{other} \end{cases} .$$

Multiplication by a square wave signal cuts out the signal in the scope of the square wave signal. The reason for this lies in the definition of the square wave signal: a multiplication by a 1 does not change the signal. A multiplication by 0 does not let the signal through, which will be illustrated by the following example:

$$\sin(2\pi f t) \cdot \mathrm{rect}(t) = \begin{cases} \sin(2\pi f t) & |t| \leq 0.5 \\ 0 & \text{other} \end{cases} .$$

So to cut out a certain signal section you only need a multiplication with the rectangle function over the corresponding signal section:

$$s(t) \cdot \mathrm{rect}\left(\frac{t - t_0}{T}\right) = \begin{cases} s(t) & t_0 - T/2 \leq t \leq t_0 + T/2 \\ 0 & \text{other} \end{cases} .$$

The **jump** simulates switch-on processes:

$$\varepsilon(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{other} \end{cases} .$$

If a voltage of $U = 5$ V is switched on at time $t = 3$ s, this could be described mathematically as follows:

$$u(t) = 5\ \mathrm{V} \cdot \varepsilon(t - 3\ \mathrm{s}) \ .$$

The **Dirac** pulse is defined as an infinitely narrow and infinitely high rectangular signal of area 1:

$$\delta(t) = \lim_{T \to 0} \frac{1}{T} \mathrm{rect}\left(\frac{t}{T}\right) \ .$$

2.2 Fourier Transform

The Fourier transform allows the calculation of the so-called spectrum. The spectrum of a signal tells us which frequencies occupy which part in the given signal. In the following,

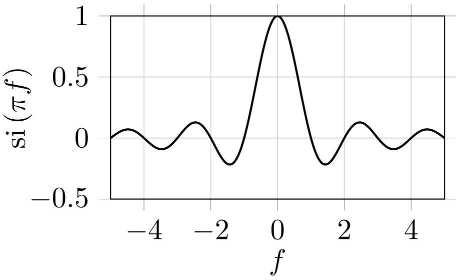

Figure 1: The si-function as a calculated spectrum of the square wave signal.

lower case is used for time signals and upper case for spectra (frequency signals). The Fourier transform is defined by:

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft} dt \ .$$

We write in abbreviated form:

$$x(t) \circ\!\!-\!\!\!-\!\!\bullet X(f) \ .$$

In the following, the Fourier transform will be clarified using the square wave signal as an example:

$$\begin{aligned}
x(t) &= \mathrm{rect}(t) \\
X(f) &= \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft} dt \\
&= \int_{-0,5}^{0,5} 1 \cdot e^{-j2\pi ft} dt \\
&= \frac{1}{-j2\pi f} \left[e^{-j2\pi ft} \right]_{-0,5}^{0,5} \\
&= \frac{1}{-j2\pi f} \left[e^{-j\pi f} - e^{j\pi f} \right] \\
&= \frac{1}{-j2\pi f} \left[-2j \sin(\pi f) \right] \\
&= \frac{\sin(\pi f)}{\pi f} \ .
\end{aligned}$$

The result of this examplary Fourier transform is written abbreviated as si-function:

$$\mathrm{si}(\pi t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & \text{other} \ . \end{cases}$$

The calculated spectrum $X(f)$ is shown in figure 1. Several things can be noted about this spectrum:

- The spectrum is defined not only for positive frequencies but also for negative frequencies. Usually, however, one interprets only the positive frequencies.

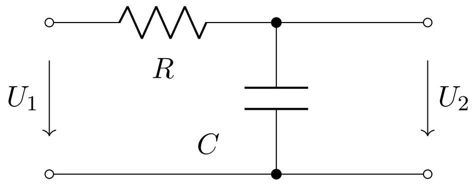

Figure 2: RC-low-pass

- For the positive frequencies, it is obviously true that the signal has more signal components at lower frequencies than at higher frequencies, since the si-function becomes smaller and smaller for large arguments. When a spectrum has the larger magnitude amplitudes at low frequencies, it is called a *low-pass signal*. If the magnitude amplitudes increase to larger amplitudes, one speaks of a *high-pass signal*. If only a certain frequency range has a large amplitude in the spectrum, it is called a band-pass signal.

2.3 Impulse Response and Transfer Function

If one excites a system with a Dirac pulse, the response of the system at the output is the so-called **impulse response**. For the RC low-pass from figure 2, the impulse response is, for example:

$$h(t) = \varepsilon(t) \cdot \frac{1}{T} \cdot e^{-t/T}.$$

The parameter $T = RC$ corresponds to the time constant of the low-pass filter. The **transfer function** is the Fourier transform of the impulse response:

$$h(t) \circ\!\!-\!\!\!-\!\!\bullet H(f) \ .$$

For the RC low-pass example above, the transfer function is:

$$\begin{aligned}
H(f) &= \int_{-\infty}^{\infty} \varepsilon(t) \cdot \frac{1}{T} \cdot e^{-t/T} \cdot e^{-j2\pi ft} dt \\
&= \int_{0}^{\infty} \frac{1}{T} \cdot e^{-t/T} \cdot e^{-j2\pi ft} dt \\
&= \frac{1}{T} \int_{0}^{\infty} e^{-(1/T + j2\pi ft)} d \\
&= \frac{1}{T} \frac{1}{-(1/T + j2\pi f)} \left[e^{-(1/T + j2\pi ft)} \right]_0^{\infty} \\
&= \frac{1}{T} \frac{1}{-(1/T + j2\pi f)} \cdot (-1) \\
&= \frac{1}{1 + j2\pi fT} \ .
\end{aligned}$$

The result agrees for $T = R \cdot C$ with the well-known voltage divider of the RC low-pass filter for AC calculations:

$$U_2 = U_1 \cdot \frac{1}{1 + j2\pi fRC} \ .$$

2.4 Cutoff Frequency

The cutoff frequency of a low-pass filter is defined as follows:

$$|H(f_g)| = \frac{1}{\sqrt{2}}\,|H(0)| \ .$$

This corresponds to the frequency at which the output signal would have only half the energy of an input signal at frequency $f = 0$ Hz. For the RC low-pass filter shown in Figure 2, the cutoff frequency can be calculated as follows:

$$\begin{aligned}
|H(f)| &= \frac{1}{\sqrt{1+(2\pi f RC)^2}} \\
|H(0)| &= 1 \\
\frac{1}{\sqrt{2}} &= \frac{1}{\sqrt{1+(2\pi f_g RC)^2}} \\
2 &= 1+(2\pi f_g RC)^2 \\
1 &= 2\pi f_g RC \\
f_g &= \frac{1}{2\pi RC} \ .
\end{aligned}$$

In the table 13 the corresponding cutoff frequencies are given for some important low-pass signals.

2.5 Continuous-Time Convolution

If a signal $x(t)$ is transmitted through a system with impulse response $h(t)$, the resulting output signal can be calculated using convolution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) d\tau \ .$$

In short form one writes for the convolution also:

$$y(t) = x(t) * h(t) \ .$$

As an example of convolution, we convolve a rectangular signal with a step:

$$\begin{aligned}
y(t) &= \mathrm{rect}(t) * \varepsilon(t) \\
&= \int_{-\infty}^{\infty} \mathrm{rect}(\tau) \cdot \varepsilon\left(t-\tau\right) d\tau \ .
\end{aligned}$$

The second factor in the integral $\varepsilon\left(t-\tau\right)$ is obviously only unequal to 0 for the range $t-\tau \geq 0$ or $t \geq \tau$. It follows that the integral becomes 0 for $t \leq -0.5$, since in this case the two functions do not overlap and one of them is always 0. For $-0.5 \leq t \leq 0.5$, one calculates the convolution integral as follows:

$$\begin{aligned}
y(t) &= \operatorname{rect}(t) * \varepsilon(t) \\
&= \int_{-\infty}^{\infty} \operatorname{rect}(\tau) \cdot \varepsilon(t-\tau)\, d\tau \\
&= \int_{-0,5}^{t} 1 \cdot 1 d\tau \\
&= [\tau]_{-0,5}^{t} \\
&= t + 0,5\ .
\end{aligned}$$

For $t > 0.5$, the integral is obviously equal to 1. From this follows the total convolution result:

$$y(t) = \begin{cases} 0 & t < -0,5 \\ t + 0,5 & -0,5 \leq t < 0,5 \\ 1 & \text{other}\ . \end{cases}$$

2.6 Calculation with the Dirac Delta Function

The Dirac delta function is the neutral element of the convolution:

$$s(t) * \delta(t) = s(t).$$

A convolution with a Dirac delta function corresponds to a weighting and a shift:

$$s(t) * a\delta(t - t_0) = as(t - t_0)\,.$$

An example of this is given in the collection of formulas. The shifted and amplified square wave signal corresponds to a square wave signal convolved with a corresponding Dirac pulse:

$$a \cdot \operatorname{rect}\left(\frac{t - t_0}{T}\right) = \operatorname{rect}\left(\frac{t}{T}\right) * a\delta(t - t_0)\,.$$

A multiplication by a Dirac pulse corresponds to cutting out the signal amplitude at the location of the Dirac (sieve property):

$$s(t) \cdot \delta(t - t_0) = s(t_0) \cdot \delta(t - t_0)\,.$$

2.7 Examples

In the following, the calculation with the elementary signals will be shown by means of selected examples. All formulas used can be taken from the formula collection.

2.7.1 Cosine with Switching Operations

The cosine is infinitely extended in the time domain and has only a single frequency in the frequency domain. To analyze infinitely extended signals, a representative section of the signal is usually analyzed. To cut out a signal, one multiplies it by a suitable window function, for example the square wave signal:

$$\begin{aligned}
\cos(2\pi F t) &\;\circ\!\!-\!\!\bullet\; \frac{1}{2}\left(\delta\left(f+F\right)+\delta\left(f-F\right)\right)\\
\mathrm{rect}(t) &\;\circ\!\!-\!\!\bullet\; \mathrm{si}\left(\pi f\right)\\
\cos(2\pi F t)\cdot\mathrm{rect}(t) &\;\circ\!\!-\!\!\bullet\; \frac{1}{2}\left(\delta\left(f+F\right)+\delta\left(f-F\right)\right)*\mathrm{si}\left(\pi f\right)\\
&= \frac{1}{2}\mathrm{si}\left(\pi\left(f+F\right)\right)+\frac{1}{2}\mathrm{si}\left(\pi\left(f-F\right)\right).
\end{aligned}$$

Before multiplication by the square wave signal, the spectrum of the cosine consisted of two Dirac pulses at frequencies $f = \pm F$. By multiplying with a square wave in the time domain, the resulting spectrum consists of si-functions at frequencies $f = \pm F$. The si-functions are significantly broader than the Dirac pulses. This effect is also called *smearing* of the spectrum, since the unique peaks of the spectrum become broader. To reduce this smearing effect, the multiplication is usually realized with signals of finite slope. An example of such a finite slope window function is the Hann window:

$$\mathrm{rect}(t)\cdot\frac{1}{2}\left(1+\cos\left(2\pi f\right)\right).$$

Figure 3 compares the magnitude spectra of the rectangular window and the Hann window. The rectangular window has the lower cutoff frequency (the narrower *main lobe*). The Hann window has the wider main lobe and thus the larger cutoff frequency. In return, it falls off much faster at higher frequencies. Usually the width of the main lobe is less critical, because the relevant signal frequency can still be read in the maximum of the main lobe. Only when two signal frequencies are close to each other, a wide main lobe of the window function prevents the detection of both frequencies: In this case, only a single frequency is analyzed. On the other hand, the fast decay of the spectrum of the window function towards high frequencies is desirable, otherwise frequencies arbitrarily far away are unnaturally increased, leading to wrong measurement results.

2.7.2 Pulse-Width Modulation

In pulse width modulation (PWM), a source voltage U_0 is switched on for a specified duration τ and then switched off for the remainder of the cycle. The cycle duration is T and $\frac{\tau}{T}$ is the duty cycle or pulse duration. PWM is used, for example, to control the speed of DC machines. The PWM signal from figure 4 can be described mathematically as follows:

$$u(t) = \mathrm{rect}\left(\frac{t}{\tau}\right) * \sum_{n=-\infty}^{\infty}\delta\left(t-nT\right).$$

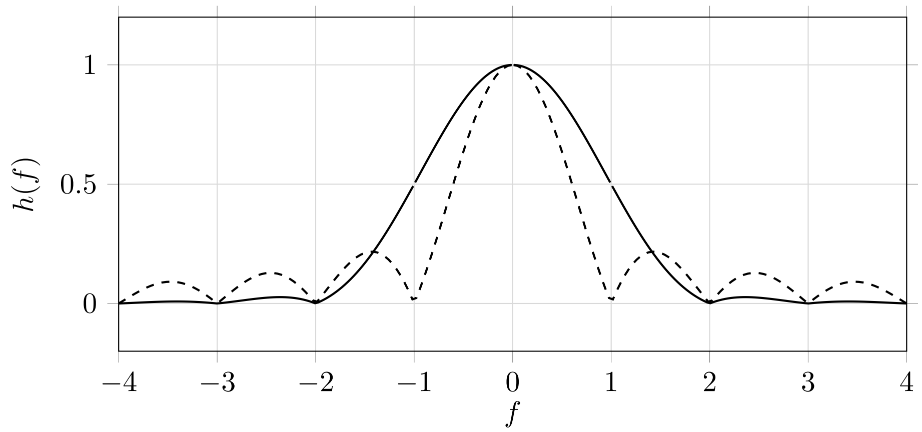

Figure 3: Comparison of the magnitude spectra of the rectangular window (dashed) and the Hann window (solid)

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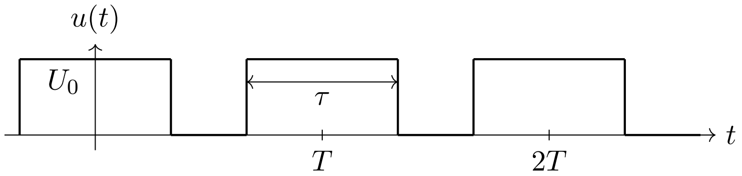

Figure 4: PWM signal in time domain.

Using the collection of formulas, one can determine the associated spectrum:

$$U(f) = \frac{\tau}{T}\mathrm{si}\left(\pi\tau f\right) \cdot \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right).$$

With the sieve property of the Dirac, the spectrum $U(f)$ can also be described as follows:

$$U(f) = \sum_{k=-\infty}^{\infty} \left(\delta\left(f - \frac{k}{T}\right) \cdot \frac{\tau}{T}\mathrm{si}\left(\pi\frac{k\tau}{T}\right)\right).$$

The spectrum $U(f)$ is visualized in figure 5.

2.7.3 Step Function of Ideal Low-Pass

From an ideal low-pass filter with cutoff frequency $f_g = 1$, the impulse response and the step response are to be determined and drawn:
The required ideal low-pass filter has the following spectrum:

$$H(f) = \mathrm{rect}\left(\frac{f}{2f_g}\right) \; .$$

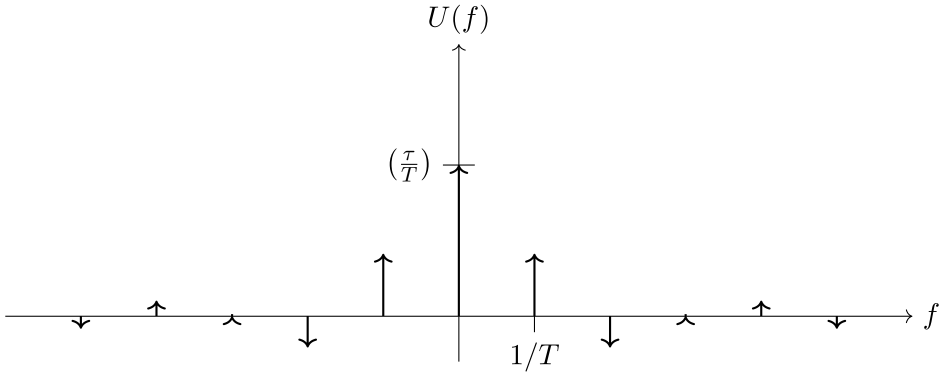

Figure 5: PWM signal in frequency domain.

Using the similarity theorem of the Fourier transform, the impulse response of the ideal low-pass filter with cutoff frequency f_g is as follows:

$$h(t) = 2f_g \mathrm{si}\left(2\pi f_g t\right) \ .$$

The step response $h_\varepsilon(t)$ is obtained by integration:

$$h_\varepsilon(t) = \varepsilon(t) * h(t) = \int_{-\infty}^{t} h(\tau) d\tau \ . \tag{1}$$

The integral over the si-function is referred to in the literature as Si. This integral is not closed-form solvable. Instead, the approximation given in the collection of formulas is used:

$$\mathrm{Si}(t) = \int_0^x \frac{\sin t}{t} dt \approx \frac{\pi}{2} - \frac{\cos x}{x + 2/\pi} - \sin\left(x\right) \cdot e^{-x} \ .$$

First, solve the integral in equation 1 for positive time points t. By substitution one obtains:

$$\begin{aligned}
\int_0^t h(x) dx &= \int_0^t 2f_g \frac{\sin\left(2\pi f_g x\right)}{2\pi f_g x} dx \\
\psi(x) &= 2\pi f_g x = \tau \\
\frac{d\tau}{dx} &= 2\pi f_g \rightarrow dx = \frac{d\tau}{2\pi f_g} \\
\int_0^t h(x) dx &= \int_{\psi(0)}^{\psi(t)} 2f_g \frac{\sin\tau}{\tau} \frac{d\tau}{2\pi f_g} \\
&= \frac{1}{\pi} \int_0^{2\pi f_g t} \frac{\sin\tau}{\tau} d\tau \\
&\approx \frac{1}{\pi} \left(\frac{\pi}{2} - \frac{\cos\left(2\pi f_g t\right)}{2\pi f_g t + 2/\pi} - \sin\left(2\pi f_g t\right) \cdot e^{-2\pi f_g t} \right) \ .
\end{aligned}$$

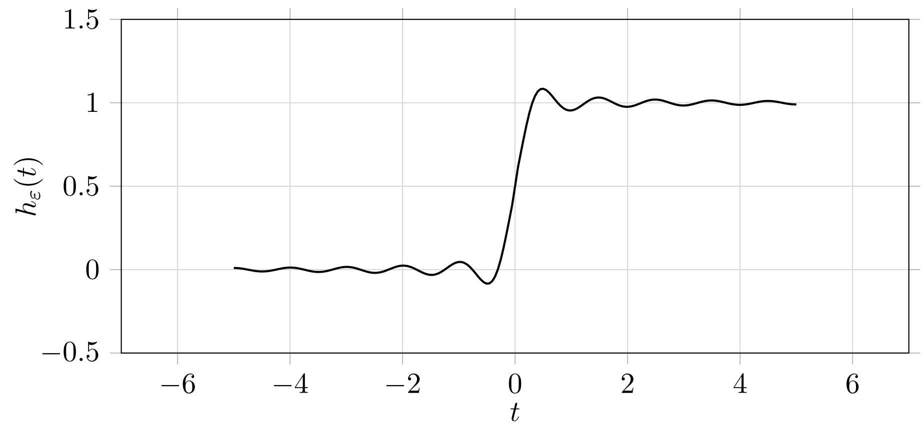

Figure 6: Step response of ideal low-pass

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It is easy to see that the integral for $t \to \infty$ tends toward the value $\frac{1}{2}$. Since the impulse response in equation 1 is symmetric, it must also hold:

$$\int_{-\infty}^{0} h(t)dt = \frac{1}{2} \ .$$

Therefore, it follows for the step response for positive time points t:

$$\begin{aligned} h_\varepsilon(t) = \int_{-\infty}^{t} h(t)dt &= \int_{-\infty}^{0} h(t)dt + \int_{0}^{t} h(t)dt \\ &\approx 1 - \frac{\cos(2\pi f_g t)}{2\pi^2 f_g t + 2} - \frac{1}{\pi}\sin(2\pi f_g t) \cdot e^{-2\pi f_g t} \ . \end{aligned}$$

The step response must be symmetric about the point $t = 0$ and $h_\varepsilon(0)$. It follows for negative t:

$$h_\varepsilon(-t) = 1 - h_\varepsilon(t) \ .$$

The step response is shown in figure 6. In the figure you can observe two things:

- The low-pass filter reduces the slope of the system response at the jump point $t = 0$, since an infinite slope would also have to have infinitely high frequencies. This can be seen in the Fourier transform of the step function.
- A switching operation at a low-pass filter inevitably leads to oscillations in the range of the switching operation.

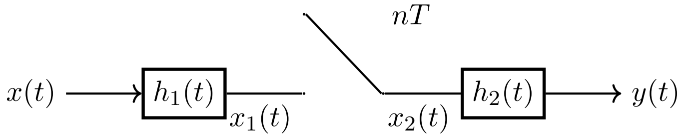

Figure 7: The ideal scanner.

Figure 8: Correlation between AD-, DA-converter, digital signal processing (DSP) and the ideal sampler from figure 7.

3 Discrete-Time Signals and Systems

The real world is analog[1]. In order to process signals of the analog world in a microprocessor, analog-to-digital conversion, digital-to-analog conversion and algotihmes for discrete-time signals are needed. This chapter will explain these areas of signal processing.

3.1 Sampling in Time- and in Frequency-Domain

The theory for this chapter is explained in detail in [1, Chapters 4.1 and 4.2].

In general, a sampler and its associated reconstruction filter is constructed as shown in figure 7: The continuous-time signal $x(t)$ is limited to the frequency range $-f_g \leq f \leq f_g$ by a low-pass filter $h_1(t)$. Subsequently, the signal $x_1(t)$ is sampled at time points nT, where T stands for the **sample duration**. The reciprocal of the sampling duration is the **sampling rate**: $r = \frac{1}{T}$. Since the signal $x_2(t)$ is not equal to 0 only at discrete time points $t = nT$, $x_2(t)$ can also be considered as a discrete-time signal $x_2(n)$. This can be manipulated and analyzed by any digital signal processing, as will be shown in the coming chapters. If one wants to generate a continuous-time signal $y(t)$ again from the digital signal $x_2(t)$, one has to filter the digital signal through a reconstruction low-pass filter. This usually has the cutoff frequency $f_{g,h_2} = \frac{1}{2T}$.

Figure 8 shows the relationship between analog-to-digital (AD) conversion, digital signal processing (DSP), and digital-to-analog (DA) conversion and the ideal sampler. The AD-converter corresponds to the low-pass $h_1(t)$ and the sampler from figure 7. The DA-converter essentially corresponds to the reconstruction low-pass $h_2(t)$.

3.1.1 The Ideal Sampler

The ideal sampler measures the current function value in a (theoretically) infinitely short instant of time and stores it in a register. This can be described mathematically as multiplication by a Dirac pulse, see also section 2.6. This happens not only once,

[1]quote by: Professor of Devices, RWTH Aachen University, 2002-2003

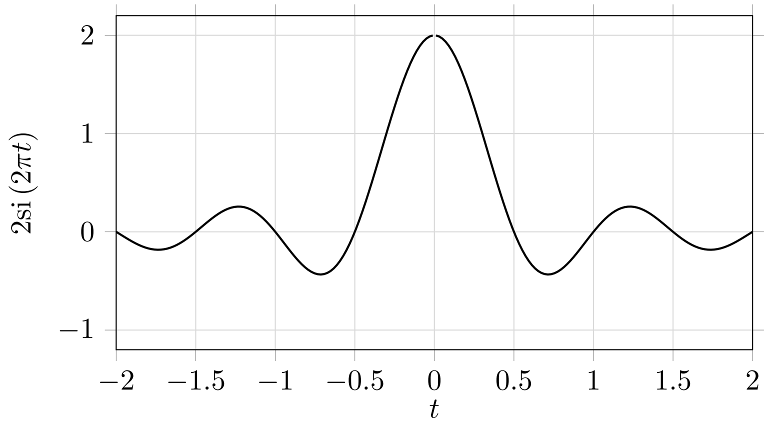

Figure 9: Example signal $x_1(t)$ before sampling.

but repeats at periodic intervals T (sampling duration). This can be represented as multiplication by an infinite sum of shifted Dirac pulses:

$$x_2(t) = x_1(t) \cdot \sum_{n=-\infty}^{\infty} \delta\left(t - nT\right) \ . \tag{2}$$

Looking at the spectrum $X_2(f)$ of the sampled signal $x_2(t)$, the product becomes a convolution. The infinite sum of Dirac pulses in the time domain becomes an infinite sum of Dirac pulses in the frequency domain after the Fourier transform (see collection of formulas):

$$\sum_{n=-\infty}^{\infty} \delta\left(t - nT\right) \circ\!\!-\!\!\bullet \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right) \ .$$

Fourier transform of the equation 2 gives the spectrum of the sampled signal:

$$X_2(f) = X_1(f) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right) \ . \tag{3}$$

A convolution with the Dirac momentum corresponds to a displacement. For this reason, it can be seen from the formula 3 that the spectrum of a sampled signal corresponds to the periodic repetition of the original spectrum.

The mathematical description of the sampling in the equations 2 and 3 will be visualized with an example: Figure 11 shows the spectrum $X_1(f) = \mathrm{rect}\left(\frac{f}{2}\right)$ of the example signal $x_1(t) = 2\mathrm{si}\left(2\pi t\right)$. By sampling with sample duration $T = \frac{1}{3}$, the spectrum is amplified by a factor of $\frac{1}{T} = 3$ and repeated at locations 3, 6, ..., as shown in figure 12. To ensure that the periodically repeated spectra do not overlap, the input signal must be a low-pass signal. The resulting condition for the sampling duration is called **sampling theorem**:

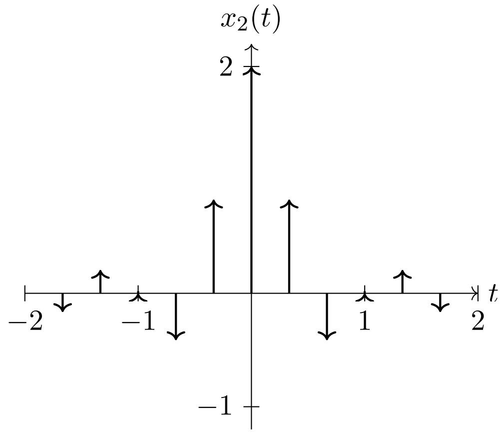

Figure 10: The signal $x_2(t)$ is formed by sampling the signal $x_1(t)$ from figure 9 with $T = \frac{1}{3}$.

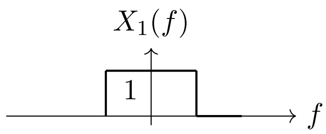

Figure 11: Spectrum $X_1(f)$ of the time signal from figure 9 before sampling.

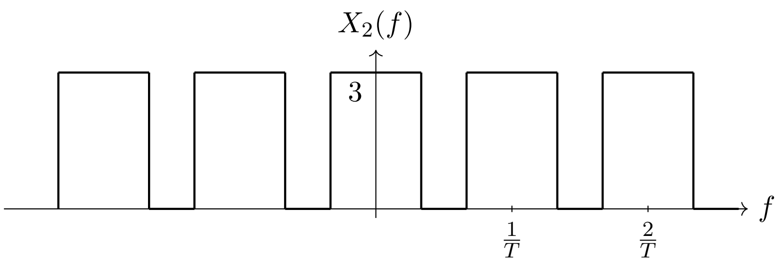

Figure 12: Spectrum $X_2(f)$ of sampled signal $x_2(t)$ from figure 10.

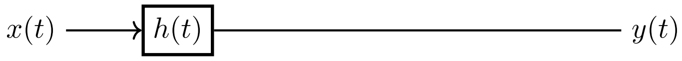

Figure 13: Reverse image of ideal scanner.

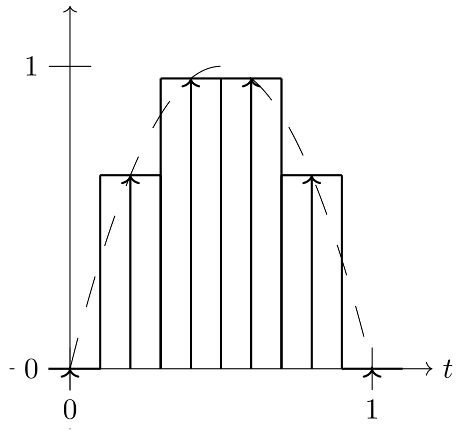

Figure 14: Sample&hold reconstruction in the time domain. The original signal $x(t)$ is drawn dashed

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$$f_g \leq \frac{1}{2T}. \tag{4}$$

If the condition from equation 4 is not satisfied, the periodic repetitions of the spectrum overlap. These distortions due to violation of the sampling theorem are called **Alias**. To ensure that the sampling theorem is satisfied for each $x(t)$, a low-pass filter $h_1(t)$ is usually connected in front of the sampler, whose cutoff frequency satisfies the equation 4. To reconstruct a continuous-time function $y(t)$ from the sampled values in $x_2(t)$, a second low-pass filter $h_2(t)$ is needed. For the input signal to be perfectly reconstructed ($x_1(t) = y(t)$), $h_2(t)$ must be an ideal low-pass:

$$h_2(t) = 2f_g T \mathrm{si}\left(2\pi f_g t\right) \circ\!\!-\!\!\!-\!\!\bullet H_2(f) = T \cdot \mathrm{rect}\left(\frac{f}{2f_g}\right) .$$

Assuming that $h_1(t)$ and $h_2(t)$ correspond to an ideal low-pass filter with cutoff frequency $f_g \leq \frac{1}{2T}$, the circuit diagram in figure 7 can be replaced by the equivalent circuit diagram in figure 13 with the ideal low-pass $h(t)$ with cutoff frequency f_g. Both systems are indistinguishable by observing their input and output signals.

3.1.2 Sample&Hold

Replacing the ideal low-pass filter on the reconstruction side with

$$h_2(t) = \mathrm{rect}\left(\frac{t}{T}\right) \tag{5}$$

one obtains a sample&hold member. In physically realizable sample&hold links, the sampled value is actually held until the next sampling time. This corresponds to a reconstruction filter

$$h_2(t) = \mathrm{rect}\left(\frac{t - T/2}{T}\right) .$$

Both filters have the same magnitude frequency response and differ only in phase. For simplicity, in the following we assume a sample&hold link according to equation 5. Assuming that the input signal $x_1(t)$ is an ideal low-pass signal with cutoff frequency f_g and that the sampling rate is $T = \frac{1}{2f_g}$, the spectrum of the sampled signal results in

$$X_2(f) = \mathrm{rect}\left(\frac{f}{2f_g}\right) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right) = \frac{1}{T} .$$

Using the ideal low-pass filter as a reconstruction filter, we get back (except for an amplitude change) the original spectrum of $X_1(f)$ at the output of the system:

$$\begin{aligned} H_2(f) &= \mathrm{rect}\left(\frac{f}{2f_g}\right) \\ Y(f) &= X_2(f) \cdot H_2(f) = \frac{1}{2f_g}\mathrm{rect}\left(\frac{f}{2f_g}\right) . \end{aligned}$$

In contrast, the sample&hold link introduces distortions:

$$\begin{aligned} h_2(t) &= \mathrm{rect}\left(\frac{t}{T}\right) = \mathrm{rect}\left(2f_gT\right) \\ H_2(f) &= \frac{1}{2f_g}\mathrm{si}\left(\pi\frac{f}{2f_g}\right) \\ Y(f) &= X_2(f) \cdot H_2(f) = \frac{1}{2f_g}\mathrm{si}\left(\pi\frac{f}{2f_g}\right) . \end{aligned}$$

These distortions are visualized in figure 15. It can be seen that, compared to the ideal reconstruction filter, the sample&hold element attenuates the signal in the passband and does not attenuate it sufficiently in the stopband.

3.1.3 Linear Interpolation

Replacing the ideal low-pass on the reconstruction side with

$$h_2(t) = \Delta\left(\frac{t}{T}\right) = \Delta\left(2f_gt\right)$$

a linear interpolation is obtained. Assuming the same configuration as for the sample&hold link, the following output signal is obtained:

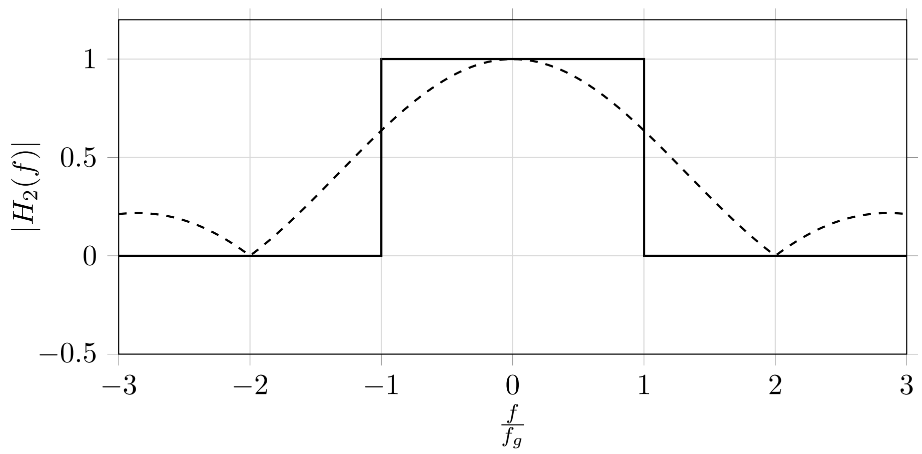


Figure 15: Comparison of the magnitude spectrum of the ideal sample (solid) and the sample&hold limb (dashed)

.

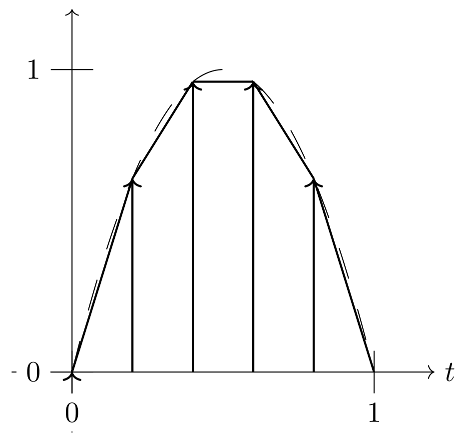


Figure 16: Linear interpolation in the time domain. The original signal $x(t)$ is drawn dashed

.

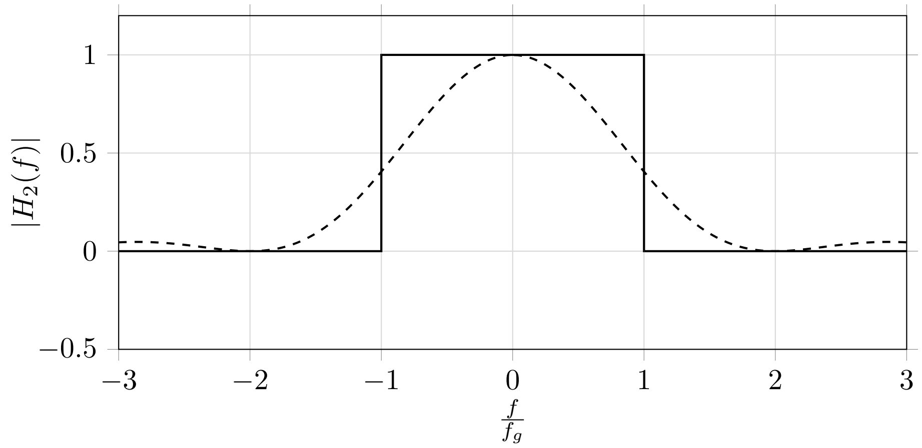


Figure 17: Comparison of the magnitude spectrum of the ideal sampler (solid) and the linear interpolator (dashed)

.

$$\begin{aligned} H_2(f) &= \frac{1}{2f_g}\mathrm{si}^2\left(\pi\frac{f}{2f_g}\right) \\ Y(f) &= X_2(f)\cdot H_2(f) = \frac{1}{2f_g}\mathrm{si}^2\left(\pi\frac{f}{2f_g}\right) \quad . \end{aligned}$$

Figure 17 again shows the resulting magnitude spectra for the ideal low-pass and linear interpolator. As with the sample&hold element, it is obvious that in the passband the signal is attenuated and in the stopband the signal is not fully attenuated. This is the frequency domain representation of the error introduced by the linear interpolation.

3.1.4 Examples of Sampling

The interaction of the three components $h_1(t)$, $h_2(t)$ and the sampler shall be explained by examples.

Example 1: Sampling without Alias Given is an ideal sampler with

$$\begin{aligned} T &= \frac{1}{4} \\ h_1(t) &= \delta(t) \\ x(t) &= \cos(3\pi t) \text{ and} \\ h_2(t) &= \mathrm{si}(4\pi t) \quad . \end{aligned}$$

The transfer function $H_2(f)$ of the reconstruction low-pass $h_2(t)$ is obtained via the similarity theorem:

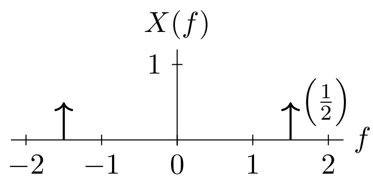


Figure 18: Sampling example 1: $X(f)$.

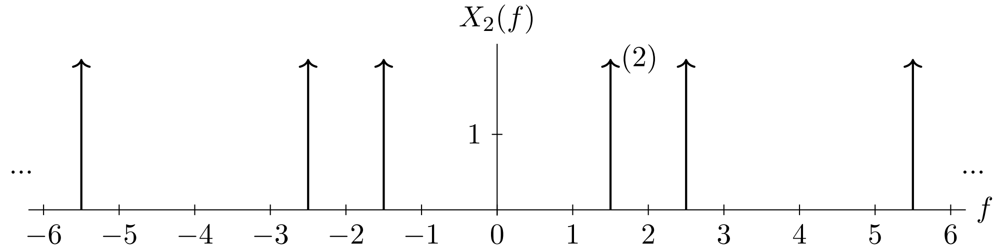


Figure 19: Sample 1: $X_2(f)$.

$$h_2(t) = \mathrm{si}\,(4\pi t) \circ\!\!-\!\!\bullet H_2(f) = \frac{1}{4}\mathrm{rect}\left(\frac{f}{4}\right) \quad . \tag{6}$$

The sketch of $X(f)$ indicating all characteristic values is shown in figure 18. The sketch of $X_2(f)$ indicating all characteristic values can be seen in figure 19. The spectrum of $X(f)$ is repeated periodically. The period here is $\frac{1}{T} = 4$. One can clearly see that by this period, the overlapping spectra do not overlap, so the sampling theorem is respected. The reconstruction low-pass from equation 6 cuts out from the spectrum $X_2(f)$ the two Dirac pulses at $f = \pm\frac{3}{2}$. Therefore $Y(f) = X(f)$ and therefore also $y(t) = x(t)$ holds.

Example 2: Sampling with Alias Given is an ideal sampler with

$$\begin{aligned} T &= \frac{1}{2} \\ h_1(t) &= \delta(t) \\ x(t) &= \cos(3\pi t) \text{ and} \\ h_2(t) &= \mathrm{si}\,(4\pi t) \ . \end{aligned}$$

The transfer function $H_2(f)$ of the reconstruction low-pass $h_2(t)$ is given in equation 6. The sketch of $X(f)$ is identical to the figure 18 from Example 1. The sketch of $X_2(f)$ showing all characteristic values is shown in figure 20. It can be seen that the sampling duration is too large or the sampling rate is chosen too small. The sampling theorem is violated because the highest frequency occurring in the signal $f_g = 1.5$ is greater than half the sampling rate $\frac{1}{2T} = 1$. Therefore the periodically repeated spectra overlap and a reconstruction by an arbitrary low-pass filter $h_2(t)$ is no longer possible. The sketch

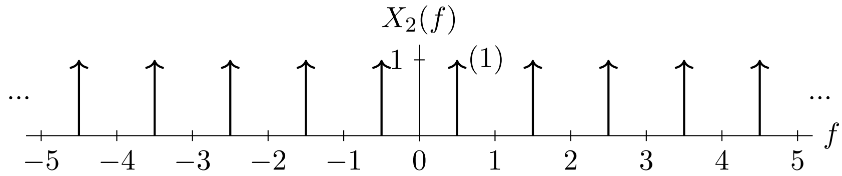


Figure 20: Sample 2: $X_2(f)$.

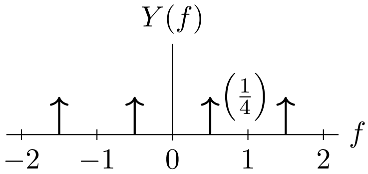


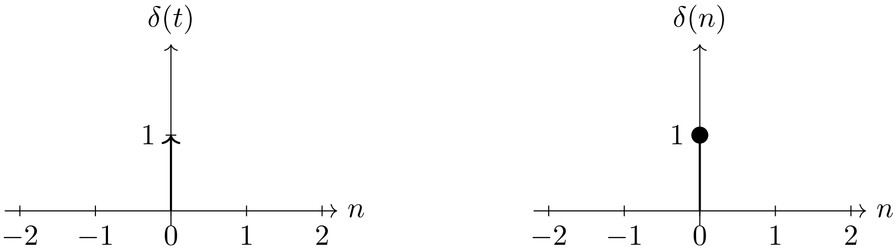
Figure 21: Sampling example 2: $Y(f)$.

of $Y(f)$ showing all characteristic values can be seen in figure 21. From this sketch, the time signal $y(t)$ can be determined:

$$y(t) = \frac{1}{2}\cos(3\pi t) + \frac{1}{2}\cos(\pi t) \quad .$$

3.2 Discrete-Time Elementary Signals

Analogous to the continuous-time elementary signals from section 2.1, one can also describe discrete-time signals using elementary signals. The most important difference here concerns the continuous-time Dirac pulse $\delta(t)$ versus the discrete-time Dirac pulse $\delta(n)$. The Dirac pulse has an infinite height and the area 1. Thus, its energy is infinite. The delta pulse is a pulse of height 1 and has a finite energy. To illustrate this difference graphically, Dirac pulses are drawn as arrows. Delta pulses, on the other hand, have a round end, as shown in figure 22. All other discrete-time elementary signals can be interpreted as sampled variants of the continuous-time elementary signals. For example,



(a) Dirac-pulse.

(b) delta-impulse

Figure 22: Different representations of the Dirac and delta function.

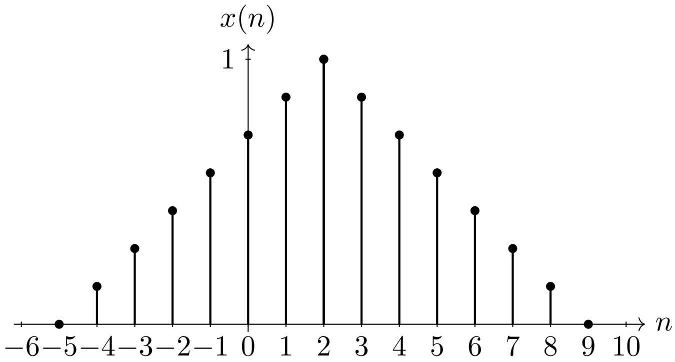


Figure 23: The signal $x(n) = \delta\left(\frac{2-n}{7}\right)$.

the discrete-time step function is a series of delta pulses starting at $n = 0$:

$$\varepsilon(n) = \sum_{m=0}^{\infty} \delta(n-m) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{other} \end{cases} .$$

As another example, draw the following signal specifying all characteristic values:

$$x(n) = \Delta\left(\frac{2-n}{7}\right) .$$

To sketch this signal we must keep in mind that the continuous-time signal $\Delta(t)$ is defined as follows:

$$\Delta(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{other} \end{cases} .$$

It follows that to sketch $x(n)$, one must determine when the argument of the triangular function $\frac{2-n}{7}$ has a magnitude less than or equal to 1:

$$\begin{aligned} \frac{2-n}{7} &\geq -1 \\ \rightarrow n &\leq 9 \\ \frac{2-n}{7} &\leq 1 \\ \rightarrow n &\geq -5 . \end{aligned}$$

It follows that the discrete signal $x(n)$ is defined as follows:

$$x(n) = \Delta\left(\frac{2-n}{7}\right) = \begin{cases} 1 - \left|\frac{2-n}{7}\right| & -5 \leq n \leq 9 \\ 0 & \text{other} \end{cases} .$$

In other words, it is the triangular signal $\Delta\left(\frac{2-t}{7}\right)$ sampled with sampling duration $T=1$. It follows that the signal $\Delta\left(\frac{2-t}{7}\right)$ is drawn as an envelope and then the amplitudes of the delta pulses are plotted, as shown in figure 23.

3.3 LTI Systems

The theory for this chapter is explained in detail in [1, Chapters 4.3.1 and 4.3.4]. The abbreviation LTI means *Linear time invariant*. For discrete-time systems, it is often referred to as *Linear shift invariant* (LSI). In the context of this lecture, LTI is used simplistically for discrete-time and continuous-time systems.

3.3.1 Linearity and Time Invariance

When a signal is transmitted through an LTI system, the output signal can be calculated using convolution. Thus, a prerequisite for using convolution is to verify that a system is linear and time invariant. How linearity and time invariance can be proved/disproved is shown below with examples.

Linear and Time Invariant A circuit with an adder, time delay element, and multiplier is linear and time invariant:

$$y(n) = \mathrm{Tr}\{x(n)\} = x(n) + 3 \cdot x(n-1) \ .$$

The proof of linearity is:

$$\begin{aligned} a \cdot \mathrm{Tr}\{x_1(n)\} + b \cdot \mathrm{Tr}\{x_2(n)\} &= a \cdot (x_1(n) + 3 \cdot x_1(n-1)) \\ &\quad +b \cdot (x_2(n) + 3 \cdot x_2(n-1)) \\ \mathrm{Tr}\{a \cdot x_1(n) + b \cdot x_2(n)\} &= (a \cdot x_1(n) + b \cdot x_2(n)) \\ &\quad +3 \cdot (a \cdot x_1(n-1) + b \cdot x_2(n-1)) \ . \end{aligned}$$

Both equations obviously give the same result.The proof for time invariance is:

$$\begin{aligned} y(n) &= \mathrm{Tr}\{x(n)\} = x(n) + 3 \cdot x(n-1) \\ y\left(n-n_0\right) &= x\left(n-n_0\right) + 3 \cdot x\left(n-1-n_0\right) \\ \mathrm{Tr}\{x\left(n-n_0\right)\} &= x\left(n-n_0\right) + 3 \cdot x\left(n-1-n_0\right) \ . \end{aligned}$$

The last two lines are obviously identical. Thus the time invariance is proved.

Non-Linear and Time-Invariant An example of a non-linear and time-invariant system is the ReLU (Rectified Linear Unit), which is often used in neural networks. It is defined by

$$y(n) = \mathrm{Tr}\{x(n)\} = \mathrm{ReLU}\,(x(n)) = \begin{cases} x(n) & x(n) \geq 0 \\ 0 & \text{other} \end{cases}$$

The counterproof against linearity is trivial for $a = 2$, $b = 3$, $x_1(n) = -1$ and $x_2(n) = 1$:

$$\begin{aligned} a \cdot \mathrm{Tr}\{x_1(n)\} + b \cdot \mathrm{Tr}\{x_2(n)\} &= 2 \cdot 0 + 3 \cdot 1 = 3 \\ \mathrm{Tr}\{a \cdot x_1(n) + b \cdot x_2(n)\} &= \mathrm{Tr}\{2 \cdot (-1) + 3 \cdot 1\} = 1 \ . \end{aligned}$$

Both lines do not give the same result. Thus, a counterexample is found and linearity is disproved. The ReLU is time invariant:

$$\begin{aligned} y(n) &= \mathrm{Tr}\{x(n)\} = \begin{cases} x(n) & x(n) \geq 0 \\ 0 & \text{other} \end{cases} \\ y\,(n - n_0) &= \begin{cases} x\,(n - n_0) & x\,(n - n_0) \geq 0 \\ 0 & \text{other} \end{cases} \\ \mathrm{Tr}\{x\,(x - x_0)\} &= \begin{cases} x\,(n - n_0) & x\,(n - n_0) \geq 0 \\ 0 & \text{other} \end{cases} \end{aligned}$$

The last two lines obviously lead to the same result in general.

Linear and Not Time-Invariant In a modulator, a signal is multiplied by a sinusoidal oscillation:

$$y(n) = \mathrm{Tr}\{x(n)\} = x(n) \cdot \cos\,(2\pi F n) \ .$$

A modulator is linear but not time-varying. Linearity can be proved as follows:

$$\begin{aligned} a \cdot \mathrm{Tr}\{x_1(n)\} + b \cdot \mathrm{Tr}\{x_2(n)\} &= a \cdot \cos\,(2\pi F n) \cdot x_1(n) + b \cdot \cos\,(2\pi F n) \cdot x_2(n) \\ \mathrm{Tr}\{a \cdot x_1(n) + b \cdot x_2(n)\} &= (a \cdot x_1(n) + b \cdot x_2(n)) \cdot \cos\,(2\pi F n) \ . \end{aligned}$$

Both lines obviously lead to the same result. The counterproof against time invariance runs as follows:

$$\begin{aligned} y(n) &= \mathrm{Tr}\{x(n)\} = x(n) \cdot \cos\,(2\pi F n) \\ y\,(n - n_0) &= x\,(n - n_0) \cdot \cos\,(2\pi F\,(n - n_0)) \\ \mathrm{Tr}\{x\,(n - n_0)\} &= x\,(n - n_0) \cdot \cos\,(2\pi F n) \ . \end{aligned}$$

Obviously, the last two lines do not generally lead to the same result.

Non-Linear and Non-Time-Invariant A simple example of a non-linear and non-time invariant mapping is phase modulation, such as that used in satellite television transmission:

$$y(n) = \mathrm{Tr}\{x(n)\} = \cos\left(2\pi\left(Fn + x(n)\right)\right)\ .$$

The counterproof against linearity is trivial for $a = 2$, $b = 3$, $x_1(n) = -1$ and $x_2(n) = 1$:

$$\begin{aligned}
a \cdot \mathrm{Tr}\{x_1(n)\} + b \cdot \mathrm{Tr}\{x_2(n)\} &= 2\cos\left(2\pi\left(Fn - 1\right)\right) + 3\cos\left(2\pi\left(Fn + 1\right)\right) \\
&= 5\cos\left(2\pi\left(Fn\right)\right) \\
\mathrm{Tr}\{a \cdot x_1(n) + b \cdot x_2(n)\} &= \cos\left(2\pi\left(Fn + 2 \cdot (-1) + 3 \cdot 1\right)\right) \\
&= \cos\left(2\pi\left(Fn\right)\right)\ .
\end{aligned}$$

The counterevidence for time invariance is:

$$\begin{aligned}
y(n) &= \mathrm{Tr}\{x(n)\} = \cos\left(2\pi\left(Fn + x(n)\right)\right) \\
y\left(n - n_0\right) &= \cos\left(2\pi\left(F\left(n - n_0\right) + x\left(n - n_0\right)\right)\right) \\
\mathrm{Tr}\{x\left(n - n_0\right)\} &= \cos\left(2\pi\left(Fn + x\left(n - n_0\right)\right)\right)\ .
\end{aligned}$$

The last two lines are obviously not equal, disproving time invariance.

3.3.2 FIR and IIR

If a filter has an impulse response of finite length, this is called *Finite Impulse Response* or abbreviated **FIR**. In contrast, if the impulse response is infinite in length, it is called an *Infinite Impulse Response* filter or also abbreviated **IIR**. Examples would be:

$$\begin{aligned}
h_1(n) &= \mathrm{rect}\left(\frac{n}{7}\right) \text{ FIR filter} \\
h_2(n) &= \varepsilon(n) \text{ IIR filter.}
\end{aligned}$$

3.3.3 Discrete-Time Convolution

If a signal $x(n)$ is transmitted through a filter with impulse response $h(n)$ one calculates the output signal $y(n)$ of the filter via discrete-time convolution:

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m) \cdot h(n - m)\ . \tag{7}$$

The formula 7 means that any impulse of $x(n)$ at location n_0 with amplitude a causes the shifted and amplified impulse response as a reaction in $y(n)$:

$$x(n) = a \cdot \delta\left(n - n_0\right) \rightarrow y(n) = a \cdot h\left(n - n_0\right)\ .$$

If $x(n)$ consists of two impulses the shifted and amplified impulse responses must be added:

$$x(n) = a_0 \cdot \delta\left(n - n_0\right) + a_1 \cdot \delta\left(n - n_1\right) \rightarrow y(n) = a_0 \cdot h\left(n - n_0\right) + a_1 \cdot h\left(n - n_1\right) \ .$$

Generalizing $x(n)$ to any number of pulses, the formula for convolution follows:

$$\begin{aligned} x(n) &= \sum_{m=-\infty}^{\infty} a_m \cdot \delta\left(n - m\right) \\ y(n) &= \sum_{m=-\infty}^{\infty} a_m \cdot h\left(n - m\right), \end{aligned}$$

where a_m is the amplitude of $x(m)$.

Example 1

$$\begin{aligned} x(n) &= \delta\left(n-2\right) - \delta\left(n-5\right) \\ h(n) &= \varepsilon(n) \\ y(n) &= \left(\delta\left(n-2\right) - \delta\left(n-5\right)\right) * \varepsilon(n) \\ &= \varepsilon\left(n-2\right) - \varepsilon\left(n-5\right) \\ &= \delta\left(n-2\right) + \delta\left(n-3\right) + \delta\left(n-4\right) \ . \end{aligned}$$

Example2

$$\begin{aligned} x(n) &= \delta\left(n\right) + \frac{1}{2}\delta\left(n-1\right) \\ h(n) &= \sum_{k=0}^{\infty} \frac{1}{2^k}\delta\left(n-k\right) \ . \end{aligned}$$

For $h(n)$ we can derive the following recursion:

$$\begin{aligned} h(n) &= \sum_{k=0}^{\infty} \frac{1}{2^k}\delta\left(n-k\right) \\ &= \delta(n) + \frac{1}{2}\delta\left(n-1\right) + \frac{1}{4}\delta\left(n-2\right) + \frac{1}{8}\delta\left(n-4\right) + ... \\ &= \delta(n) + \frac{1}{2}\left(\delta\left(n-1\right) + \frac{1}{2}\delta\left(n-2\right) + \frac{1}{4}\delta\left(n-4\right) + ...\right) \\ &= \delta(n) + \frac{1}{2}h(n-1) \ . \end{aligned}$$

It follows for $y(n)$:

$$
\begin{aligned}
y(n) &= h(n) * \left(\delta\left(n\right) + \frac{1}{2}\delta\left(n-1\right)\right) \\
&= h(n) + \frac{1}{2}h(n-1) \\
&= \delta(n) + \frac{1}{2}h(n-1) + \frac{1}{2}h(n-1) \\
&= \delta(n) + h(n-1) \\
&= \delta(n) + \sum_{k=0}^{\infty} \frac{1}{2^k}\delta\left(n-k-1\right) \ .
\end{aligned}
$$

3.3.4 Causality

For **causal** discrete-time signals:

$$x(n) = 0, \text{ for } n < 0 \ .$$

For **anticausal** signals holds:

$$x(n) = 0, \text{ for } n > 0 \ .$$

If both conditions are not satisfied, we speak of **non-causal** signals. It follows that a signal is either causal, anticausal or noncausal. In continuous-time, often only causal signals and systems are considered, since only these are physically realizable. In the discrete-time, this is not true. For example, consider the following convolution with a causal system.

$$
\begin{aligned}
x(n) &= \varepsilon(n) + \varepsilon(n-1) \\
h(n) &= \delta\left(n\right) - 2\delta\left(n-1\right) + \delta\left(n-2\right) \\
y(n) &= x(n) * h(n) = \delta\left(n\right) - \delta\left(n-2\right)
\end{aligned}
$$

it is noticeable that the high-pass $h(n)$ filters out the edge of the input signal and filters away the DC component. Before convolution, the midpoint of the jump of $x(n)$ is at $n = 0$. If the high-pass $h(n)$ detects signal changes, one would expect the largest magnitude amplitudes in $y(n)$ at the jump point $n = 0$. However, the largest amplitudes in magnitude are symmetrical around $n = 1$. Thus, the detection of the signal edge by the high-pass filter is time-delayed due to the causality of the filter. On the other hand, if we convolve with the non-causal version of the filter, we get the following result:

$$
\begin{aligned}
x(n) &= \varepsilon(n) + \varepsilon(n-1) \\
h(n) &= \delta\left(n+1\right) - 2\delta\left(n\right) + \delta\left(n-1\right) \\
y(n) &= x(n) * h(n) = \delta\left(n+1\right) - \delta\left(n-1\right) \ .
\end{aligned}
$$

Thus, the largest amplitudes of $y(n)$ in terms of magnitude are around $n = 0$, so there is no delay due to the convolution. From this simple example, we can see that non-causal digital filtering means nothing more than shifting the filter relative to the input signal. Thus, to apply the filter in this specific example, one only needs to buffer the input signal for the duration of one sample so that the filter can *look* into the future. It follows that noncausal filters in discrete time only cause additional latency in the system but are technically easy to implement.

3.4 Fourier Transform of Discrete-Time Signals

The theory for this chapter is explained in detail in [1, Chapter 4.3.5]. The Fourier transform of discrete-time signals can be determined using three formulas. First, the sampled values can be transformed directly:

$$X_a(f) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j2\pi fnT} \ .$$

This option is especially recommended (not very computationally expensive) when $x(n)$ has only a few samples not equal to 0. The second possibility is to determine the Fourier spectrum of the continuous-time signal and then periodically repeat the spectrum:

$$X_a(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(f - \frac{k}{T}\right) \ .$$

This option is recommended when $X(f)$ is bandlimited and no alias occurs, so the periodic spectra do not overlap in the spectral domain. The third possibility is the z-transform and subsequent substitution: $z \rightarrow e^{-j2\pi fT}$. This third option is explained in detail in the context of the z-transform in 4. As a simple example, the Fourier transform of $h(n) = \delta\left(n-1\right) + 2\delta\left(n-2\right) + \delta\left(n-3\right)$ is determined using the first method. Let $T = 1$ hold:

$$\begin{aligned}
H(f) &= e^{-j2\pi f\cdot 1} + 2e^{-j2\pi f\cdot 2} + e^{-j2\pi f\cdot 3} && (8)\\
&= e^{-j4\pi f} \cdot \left(e^{j2\pi f} + 2 + e^{-j2\pi f}\right) && (9)\\
&= e^{-j4\pi f} \cdot \left(\cos\left(2\pi f\right) + j\sin\left(2\pi f\right) + 2 + \cos\left(2\pi f\right) - j\sin\left(2\pi f\right)\right) && (10)\\
&= e^{-j4\pi f} \cdot \left(2\cos\left(2\pi f\right) + 2\right) \ . && (11)
\end{aligned}$$

Split into real and imaginary parts, it follows:

$$\begin{aligned}
H(f) &= \cos\left(4\pi f\right) \cdot \left(2\cos\left(2\pi f\right) + 2\right) - j\sin\left(4\pi f\right) \cdot \left(2\cos\left(2\pi f\right) + 2\right)\\
\mathrm{Re}\{H(f)\} &= \cos\left(4\pi f\right) \cdot \left(2\cos\left(2\pi f\right) + 2\right)\\
\mathrm{Im}\{H(f)\} &= -\sin\left(4\pi f\right) \cdot \left(2\cos\left(2\pi f\right) + 2\right) \ .
\end{aligned}$$

3.4.1 Linear Phase System

The phase of a transfer function can be interpreted as a delay:

$$x(n-n_0) \circ\!\!-\!\!\bullet X(f) \cdot e^{-j2\pi f n_0 T} \ .$$

It can be concluded that a linear phase system ($\varphi(f) = c \cdot f$) delays each frequency of the input signal equally. For this reason, linear phase is an important property of transmission systems. One can express the transfer function from equation 11 in magnitude:

$$|H(f)| = 2\cos(2\pi f) + 2$$

and phase:

$$\begin{aligned}
\varphi(f) &= \arctan\left(\frac{-\sin(4\pi f)\cdot(2\cos(2\pi f)+2)}{\cos(4\pi f)\cdot(2\cos(2\pi f)+2)}\right) \\
&= \arctan(-\tan(4\pi f)) \\
&= -4\pi f \ .
\end{aligned}$$

From the linear phase criterion $\varphi(f) = c \cdot f$, it can be concluded that above impulse response/transfer function has a linear phase with $c = -4\pi$. In general, any signal that is axisymmetric about a time n_0 has also a linear phase.

3.5 Discrete Fourier Transform

Assuming a periodic signal with period N, instead of using the Fourier transform of discrete-time signals from section 3.4, one can also use the Discrete Fourier Transform (DFT). It exists implemented in almost every software as *Fast Fourier Transform* (FFT). The transformation specification is:

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi nk/K}, 0 \leq k < K \ .$$

The index k corresponds here to a physical frequency of

$$\begin{aligned}
f &= k \cdot \delta f \\
\delta f &= \frac{1}{KT} \ .
\end{aligned}$$

Here Δf is the frequency resolution of the DFT and T is the sampling duration of the time-discrete signal $x(n)$. The parameter K can be used to control the frequency resolution of the DFT. If $K > N$ is valid, one speaks also of *zero-padding*, since in this case, the time signal $x(n)$ is artificially extended by $K - N$ zeros before the transformation. The inverse discrete Fourier transform is defined by:

$$x(n) = \frac{1}{K} \sum_{k=0}^{K-1} X(k) \cdot e^{j2\pi nk/K}, 0 \leq n < N \ .$$

For a real time signal $x(n)$, the following symmetry condition holds for the spectrum calculated by DFT:

$$X(k) = X^*(K - k) \ .$$

Because of this symmetry relation, it is common in signal processing to *discard* the second half of the spectrum after DFT, since this second half does not contain any new information. The redundancy of the second half of the spectrum is analogous to the statement of the sampling theorem: in a discrete-time signal $x(n)$, only frequencies in the range $-\frac{1}{2T} \leq f \leq \frac{1}{2T}$ may be interpreted. The second symmetric half of the spectrum starts at $k = K/2$:

$$\begin{aligned} f &= k \cdot \delta f \\ . &= \frac{K}{2} \cdot \frac{1}{KT} \\ &= \frac{1}{2T} \ . \end{aligned}$$

It follows that the redundancy of the second half of the spectrum is analogous to the sampling theorem.

3.6 Inverse System

If one can characterize transmission over an LTI system by convolution, it may be of interest to determine the input signal to the system from the measured output signal (given a known impulse response):

$$\begin{aligned} y(n) &= x(n) * h(n) \\ x(n) &= y(n) * h^{-1}(n) = x(n) * h(n) * h^{-1} \ . \end{aligned}$$

Since the delta function is the neutral element of the (discrete-time) convolution, it follows from the last equation that the convolution of the system with the associated inverse system must yield the delta function:

$$\delta(n) = h(n) * h^{-1}(n) \ . \tag{12}$$

3.6.1 Inverse System in Time-Domain

Due to noise processes in system identification, the determination of the input signal is generally not exact, but only an estimate. The procedure will be explained with an example. The system is an accumulator:

$$h(n) = \varepsilon(n) \ .$$

Convolution of $h(n)$ with the inverse system $h^{-1}(n)$ should again yield the delta function. To form the inverse system $h^{-1}(n)$, the principle of convolution is applied iteratively. One must add up shifted and amplified instances of the impulse response $h(n)$ so that the result is the delta impulse $\delta(n)$. Using the accumulator as a concrete example, this means that $h(n)$ has a delta pulse at location $n = 0$. All following delta pulses must be removed. This can be done by subtracting the accumulator delayed by one:

$$\begin{aligned} \delta(n) &= h(n) - h(n-1) \\ &= h(n) * (\delta(n) - \delta(n-1)) \\ \rightarrow h^{-1}(n) &= \delta(n) - \delta(n-1) \ . \end{aligned}$$

The principle will be illustrated by a second example: In the following $h(n)$ shall consist of two delta pulses:

$$h(n) = 2\delta(n) + 2\delta(n-1) \ .$$

Again, shifted and amplified (damped) instants of $h(n)$ are to be added up to obtain the delta function as a result. The first instance is certainly

$$\frac{1}{2}h(n) = \delta(n) + \delta(n-1) \ .$$

Since the result is to be the delta function, the second function $\delta(n-1)$ must be removed by another instance of $h(n)$:

$$\frac{1}{2}h(n) - \frac{1}{2}h(n-1) = \delta(n) - \delta(n-2) \ .$$

Thus the result of the convolution on the first two digits $0 \leq n \leq 1$ is correct. Again a delta pulse interferes, this time at $n = 2$:

$$\frac{1}{2}h(n) - \frac{1}{2}h(n-1) + \frac{1}{2}h(n-2) = \delta(n) + \delta(n-3) \ .$$

From the perpetual compensation of the remaining delta impulse, a remaining delta impulse is generated again. From this it can be concluded that the summation must never end:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{2} (-1)^k \, h(n) &= \delta(n) \\ h^{-1}(n) &= \sum_{k=0}^{\infty} \frac{1}{2} (-1)^k \, \delta(n-k) \ . \end{aligned}$$

Thus, the inverse filter of $h(n)$ is not stable.

3.6.2 Inverse System in Frequency-Domain

Transforming the equation 12 into the frequency domain we get:

$$\begin{aligned} 1 &= H(f) \cdot H^{-1}(f) \\ H^{-1}(f) &= \frac{1}{H(f)} \ . \end{aligned}$$

Thus, provided the transfer function has no zeros, the inverse system can be determined by taking the inverse of the transfer function $H(f)$.

3.7 FIR Filter Design

The theory for this chapter is explained in detail in [1, chapters 5.1 to 5.3]. In the context of this lecture, a distinction is made between low-passes (allow low frequencies to pass), band-passes (allow a specific frequency band to pass), and high-passes (allow high frequencies to pass). In the literature, one can find other configurations such as band-stop, which, for example, blocks a specific frequency band.
As an introduction to this topic, the general effect of low-pass and high-pass will be discussed using a simple example: The simple low-pass from figure 24 is defined over:

$$h_{\mathrm{TP}}(n) = \frac{1}{4}\delta(n+1) + \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n-1) \circ\!\!-\!\!\bullet H_{\mathrm{TP}}(f) = \frac{1}{2} + \frac{1}{2}\cos(2\pi f) \ .$$

The corresponding high-pass from figure 25 is:

$$h_{\mathrm{HP}}(n) = \frac{1}{4}\delta(n+1) - \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n-1) \circ\!\!-\!\!\bullet H_{\mathrm{TP}}(f) = \frac{1}{2} - \frac{1}{2}\cos(2\pi f) \ .$$

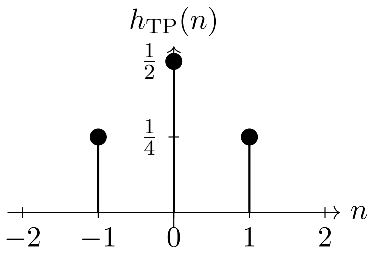
A simple signal model for an edge or signal jump is the jump function $x(n) = \varepsilon(n)$, visualized in figure 26. Filtering this edge with a low-pass filter will smooth the edge:

$$y(n) = x(n) * h_{\mathrm{TP}}(n) = \frac{1}{4}\delta(n+1) + \frac{3}{4}\delta(n) + \varepsilon(n-1) \ .$$

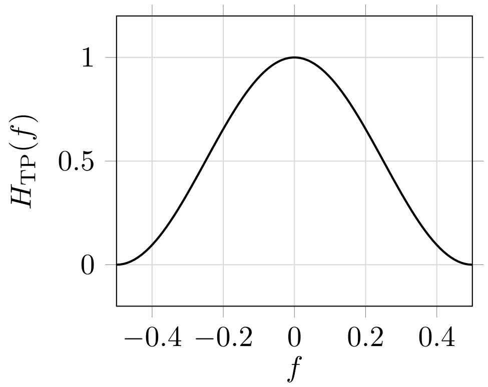
In general, the smoothing properties with respect to edges are used to attenuate abrupt signal changes, for example as noise reduction. The result of smoothing is visualized in figure 27. Filtering this edge with a high-pass filters out constant signal amplitudes. So the high-pass is an edge detector/jump detector:

$$y(n) = x(n) * h_{\mathrm{HP}}(n) = \frac{1}{4}\delta(n+1) - \frac{1}{4}\delta(n) \ .$$

The result of the filtering is visualized in figure 28.

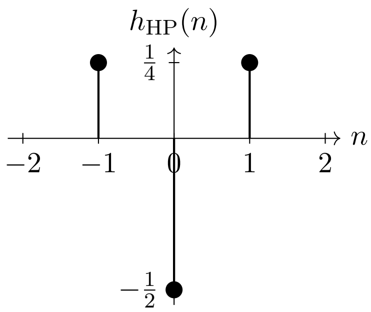


(a) Display in time domain.

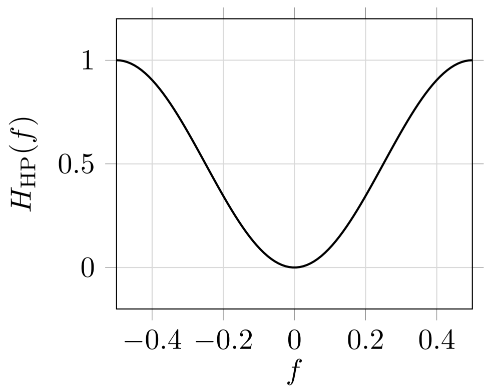


(b) Representation in frequency-domain for $-0.5 \leq f \leq 0.5$.

Figure 24: Example low-pass.



(a) Display in time domain



(b) Representation in frequency domain for $-0.5 \leq f \leq 0.5$

Figure 25: Example high pass

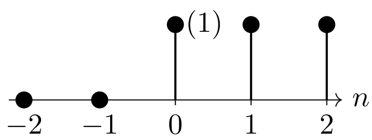


Figure 26: Input signal $x(n) = \varepsilon(n)$.

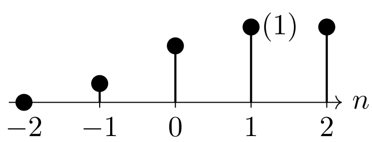


Figure 27: $y(n) = x(n) * h_{\mathrm{TP}}(n)$.

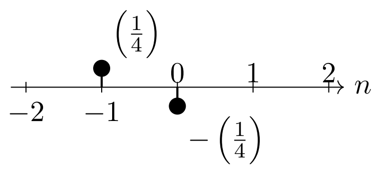


Figure 28: $y(n) = x(n) * h_{\mathrm{HP}}(n)$.

3.7.1 The-Time-Bandwidth-Product

From the similarity theorem of the Fourier transform

$$x(bt) \circ\!\!-\!\!\bullet \frac{1}{|b|} X\left(\frac{f}{b}\right), b \neq 0 \tag{13}$$

follows: The narrower a signal is in the time-domain, the wider it is in the frequency-domain. The wider a signal is in the time-domain, the narrower it is in the frequency-domain. This can be easily illustrated by the extreme case of the square wave pulse and the ideal low-pass:

$$\begin{aligned} \mathrm{rect}(t) \quad &\circ\!\!-\!\!\bullet \quad \mathrm{si}\,(\pi f) \\ \mathrm{si}(\pi t) \quad &\circ\!\!-\!\!\bullet \quad \mathrm{rect}(f)\ . \end{aligned}$$

The rectangular pulse has an impulse response with finite expansion $(-0.5 \geq t \geq 0.5)$ and has an infinitely expanded spectrum. The ideal low-pass filter has an infinitely extended impulse response and a spectrum that is limited to the frequency range $-0.5 \leq f \leq 0.5$. In general, the similarity theorem leads to the statement: The steeper the edges of a filter are in the frequency domain, the longer the filter must be in the time domain. Longer filters lead to larger average signal delays and to higher computational effort. For this reason, a compromise must be made in FIR filter design between time resolution, and thus delay and computational cost, on the one hand, and frequency resolution on the other.
The spectral bandwidth f_Δ is usually defined by the cutoff frequency f_g: $f_\Delta = 2f_g$. Analogously, if one defines the temporal expansion t_Δ as the range in which the magnitude of the impulse response is larger than $\frac{1}{\sqrt{2}} \cdot |h_{\max}|$, it follows from equation 13 that the time-bandwidth product

$$f_\delta \cdot t_\delta = \mathrm{const}$$

must result in a constant. This equation is also called the uncertainty principle of communications engineering [1, chapter 5.2].

3.7.2 The Ideal Low-Pass

The ideal low-pass filter with cutoff frequency f_g passes all signals with frequencies $f \leq f_g$ unchanged and blocks all signal components with frequencies $f > f_g$. From this,

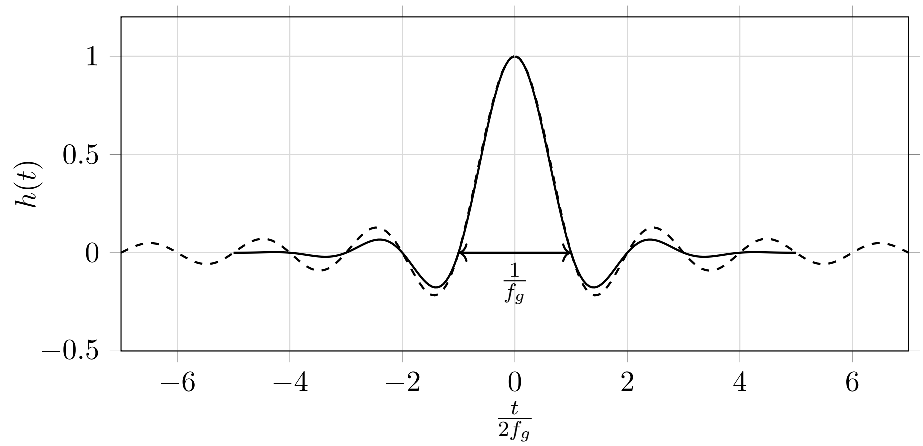


Figure 29: Ideal low-pass (dashed) and time-limited lowpass (solid), the reciprocal of the cutoff frequency is the distance between the first positive and the first negative zero of the lowpass

.

one can derive the transfer function of the ideal low-pass filter:

$$H_{\mathrm{TP}}(f) = \begin{cases} 1 & |f| \leq f_g \\ 0 & \text{other} \end{cases} = \mathrm{rect}\left(\frac{f}{2f_g}\right). \tag{14}$$

The associated impulse response is:

$$2f_g \mathrm{si}\left(2\pi f_g t\right) \circ\!\!-\!\!\!-\!\!\bullet \mathrm{rect}\left(\frac{f}{2f_g}\right).$$

It has its zeros at $t = \frac{n}{2f_g}$. The ideal low-pass is non-causal and infinitely extended, which prevents its realization as a circuit. However, it will be shown below that it can be suitably approximated.

3.7.3 Discrete-Time Low-Pass as FIR Filter

In the simplest case, the temporal expansion can be limited by a multiplication with a square wave signal of width T_0. To reduce spectral smearing by the window function, a Hann window is used instead, as shown in figure 29:

$$h_{\mathrm{TP}}(t) = 2f_g \mathrm{si}\left(2\pi f_g t\right) \cdot \mathrm{rect}\left(\frac{t}{T_0}\right) \cdot \frac{1}{2}\left(1 + \cos\left(2\pi \frac{t}{T_0}\right)\right).$$

For the filter to be realized, a time shift of $t_0 = \frac{T_0}{2}$ must ensure causality:

$$h_{\mathrm{TP}}(t) = \left(f_g \mathrm{si}\left(2\pi f_g t\right) \cdot \mathrm{rect}\left(\frac{t}{T_0}\right) \cdot \left(1 + \cos\left(2\pi \frac{t}{T_0}\right)\right)\right) * \delta\left(t - \frac{T_0}{2}\right).$$

Table 2: Samples of a discrete-time low-pass filter with a cutoff frequency of $f_g = 16$ kHz and limiting the impulse response to $0 \leq n < 16$ at a sampling rate of $\frac{1}{T} = 48$ kHz.

n	$h(n)$
0	0
1	$171,54$
2	$-534,6$
3	0
4	$2259,04$
5	$-4116,63$
6	0
7	$26209,54$
8	$26209,54$
9	0
10	$-4116,63$
11	$2259,04$
12	0
13	$-534,6$
14	$171,54$
15	0

Then the low-pass is sampled with sample duration T and the discrete-time low-pass is obtained as FIR filter:

$$h_{\mathrm{TP}}(n) = f_g \mathrm{si}\left(2\pi f_g T\left(n - n_0\right)\right) \cdot \left(1 + \cos\left(2\pi \frac{T}{T_0}\left(n - n_0\right)\right)\right)$$

with $n_0 = \frac{T_0}{2T}$ and $0 \leq n < N$. This corresponds to a filter length of $N = \frac{T_0}{T}$. The filter has its maximum at n_0. It follows that the average delay of the input signals is approximately equal to n_0. The samples for a filter with $N = 16$, $f_g = 16$ kHz and $\frac{1}{T} = 48$ kHz are listed in table 2.

3.7.4 Discrete-Time Bandpass as FIR Filter

To turn an ideal low-pass filter with cutoff frequency f_g into an ideal band-pass filter of bandwidth $f_\delta = 2f_g$ and center frequency f_c, shift the transfer function to the center frequency.

A shift to the center frequency f_c corresponds to a convolution with the Dirac pulse:

$$H_{\mathrm{BP}}(f) = H_{\mathrm{TP}}(f) * \frac{1}{2}\left(\delta\left(f + f_c\right) + \delta\left(f - f_c\right)\right)$$

with $H_{\mathrm{TP}}(f)$ as introduced in equation 14. The impulse response of the ideal low-pass filter is obtained analogously:

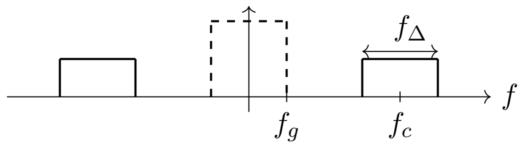


Figure 30: Ideal low-pass with cutoff frequency f_g (dashed) and ideal band-pass with center frequency f_c and bandwidth f_Δ (solid).

$$h_{\mathrm{BP}}(t) = h_{\mathrm{TP}}(t) \cdot \cos\left(2\pi f_c t\right) \ . \tag{15}$$

It follows that the discrete-time bandpass has the following impulse response:

$$h_{\mathrm{BP}}(n) = \frac{f_\Delta}{2} \mathrm{si}\left(\pi f_\Delta T\left(n - n_0\right)\right) \cdot \left(1 + \cos\left(2\pi \frac{T}{T_0}\left(n - n_0\right)\right) \cdot \cos\left(2\pi f_c T\left(n - n_0\right)\right)\right) \ . \tag{16}$$

3.7.5 Discrete-Time High-Pass as FIR Filter

The discrete-time high-pass filter with cutoff frequency f_{HP} is formed in two steps. First, a low-pass filter with cutoff frequency

$$f_{g,\mathrm{TP}} = \frac{1}{2T} - f_{g,\mathrm{HP}}$$

is formed. Second, this low-pass is shifted to the center frequency $f_c = \frac{1}{2T}$. Multiplying by the cosine from equation 15 to shift to half the sampling frequency simplifies accordingly to:

$$\cos\left(2\pi \frac{1}{2T} T n\right) = \cos\left(\pi\left(n - n_0\right)\right) \ .$$

Thus, for the discrete-time high-pass filter, it follows:

$$h_{\mathrm{HP}}(n) = f_{g,\mathrm{TP}} \mathrm{si}\left(2\pi f_{g,\mathrm{TP}} T\left(n - n_0\right)\right) \cdot \left(1 + \cos\left(2\pi \frac{T}{T_0}\left(n - n_0\right)\right) \cdot \cos\left(\pi\left(n - n_0\right)\right)\right) \ .$$

The discrete-time low-pass, band-pass, and high-pass filters presented in this section are all linear-phase because they are symmetric FIR filters.

4 Z-Transform

The theory for this chapter is explained in detail in [1, Chapter 4.3.9]. The Laplace-transform allows frequency analysis and stability analysis for continuous-time systems. The z-transform allows the same for discrete-time systems. It is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT) \cdot z^{-n} \text{ with } z = e^{\sigma T + j2\pi fT}. \tag{17}$$

All z for which the sum of equation 17 converges ($|X(z)| < \infty$) form the **convergence region**.

4.1 Relation Z-Transform and Fourier Transform

In this chapter, procedures are presented to determine the z-transform of signals. Provided that the unit circle belongs to the convergence region of the z-transform, the associated Fourier transform can be determined by the following substitution:

$$\rightarrow e^{j2\pi fT} \quad . \tag{18}$$

T here corresponds to the sampling duration. For example, according to the collection of formulas, it is valid:

$$x(n) = \left(\frac{2}{3}\right)^n \varepsilon(n) \circ\!\!-\!\!\bullet X(z) = \frac{1}{1 - \frac{2}{3}z^{-1}}$$

with the convergence range $|z| > \left|\frac{2}{3}\right|$. For the unit circle $|z| = 1$ holds. Thus the unit circle belongs to the convergence domain. So, by substitution from equation 18, the Fourier transform can be formed:

$$X(f) = \frac{1}{1 - \frac{2}{3}e^{-j2\pi fT}} \quad .$$

4.2 Difference Equations and their Z-Transforms

In the time-domain, discrete-time circuits can be described by their difference equations:

$$y(n) = \sum_{k=0}^{K} b_k \cdot x(n-k) + \sum_{l=1}^{L} a_l \cdot y(n-l).$$

The associated z-transform can be easily determined: One replaces each time delay by a multiplication by z^{-1}:

$$Y(z) = \sum_{k=0}^{K} b_k \cdot X(z) \cdot z^{-k} + \sum_{l=1}^{L} a_l \cdot Y(z) \cdot z^{-l}.$$

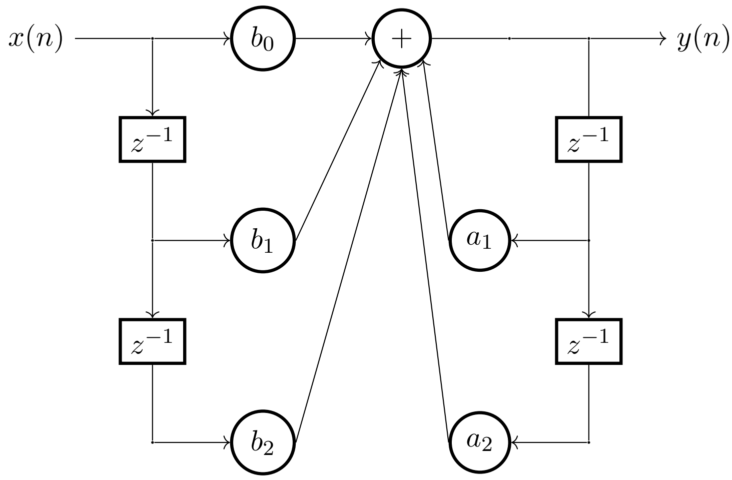


Figure 31: A circuit with multipliers, adders and time delay elements to realize a difference equation.

From this, the transfer function can be determined by transforming:

$$Y(z) - \sum_{l=1}^{L} a_l \cdot Y(z) \cdot z^{-l} = \sum_{k=0}^{K} b_k \cdot X(z) \cdot z^{-k} \tag{19}$$

$$.Y(z) \cdot \left(1 - \sum_{l=1}^{L} a_l \cdot z^{-l}\right) = X(z) \cdot \left(\sum_{k=0}^{K} b_k \cdot z^{-k}\right) \tag{20}$$

$$\rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{K} b_k \cdot z^{-k}}{1 - \sum_{l=1}^{L} a_l \cdot z^{-l}} \ . \tag{21}$$

If all a_l are zero, $H(z)$ is an FIR filter. If only one a_l is nonzero, then $H(z)$ is an IIR filter. In figure 31, the circuit fitting to the difference equation

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + a_1 y(n-1) + a_2 y(n-2)$$

is represented. Its z-transform is:

$$Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) \ .$$

The corresponding transfer function in the z-domain is then

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} \ .$$

4.3 Stability

For a discrete-time circuit with multipliers, adders, and delay elements, such as in figure 31, the stability of the system is of particular interest in circuit analysis. In an unstable

n	$x(n)$	$x_1(n)$	$y(n)$
-1	0	0	0
0	$0,01$	$0,01$	0
1	$0,01$	$0,03$	$0,02$
2	$0,01$	$0,07$	$0,06$
3	0.01	0.15	0.14
4	0.01	0.31	0.30
$\vdots$	$\vdots$	$\vdots$	

Table 3: Example waveform for unstable system.

system, for certain input signals, the output signal may have infinite energy, which may damage subsequent systems. For this reason, the statement of stability, or the constraint under which conditions a system remains stable, is an important characteristic of a system.

4.3.1 Stability in the Time-Domain

In an unstable system, there is a possibility that the system's response will build up in response to an input signal. A practical example of this is the feedback between a microphone and the connected loudspeaker: the microphone measures an arbitrary quiet background noise. This background noise is emitted amplified by the loudspeaker. This loudspeaker signal is picked up by the microphone and reproduced again amplified by the loudspeaker. The system builds up and an unpleasant characteristic whistle is generated. Its signal energy is limited only by the power of the amplifier/speaker. A simple mathematical model for this can be derived as follows. Let the background noise be the input signal $x(n)$, the microphone signal be the auxiliary variable $x_1(n)$, and the loudspeaker signal be $y(n)$. The loudspeaker reproduces the microphone signal with a slight delay after amplification:

$$y(n) = 2 \cdot x_1(n-1).$$

The gain with a factor of 2 is arbitrarily chosen here. The microphone measures the background noise $x(n)$ with an exemplary amplitude of $\frac{1}{100}$ together with the loudspeaker signal. This joint measurement can be considered as an addition:

$$x_1(n) = y(n) + x(n) = y(n) + \frac{1}{100}.$$

In the following, we assume that the microphone is turned on at time $n = 0$. This results in the time history from table 3. It is easy to see that the signal amplitudes in the system keep building up. In theory, these values can become arbitrarily large. In practice, the AD- and DA-converters limit the value ranges. The corresponding difference equation for this system is obtained after a short consideration:

$$\begin{aligned} y(n) &= 2x_1(n-1) \\ x_1(n) &= x(n) + y(n) \\ \rightarrow y(n) &= 2x(n-1) + 2y(n-1) \end{aligned}$$

The associated z-transform is:

$$Y(z) = 2z^{-1} \cdot X(z) + 2z^{-1} \cdot Y(z)\ .$$

From this follows the transfer function for the system:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2z^{-1}}{1-2z^{-1}}\ .$$

Then, according to the table, the associated impulse response is 20

$$h(n) = 2\delta(n-1) * (2^n \varepsilon(n)) = 2\left(2^{n-1}\varepsilon(n-1)\right) = 2^n \varepsilon(n-1)\ . \tag{22}$$

If the impulse response is known, one can check the stability of a system using the Bounded Input Bounded Output (BIBO) criterion: For a difference equation to be stable, the summation of the impulse response must be less than infinity.

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty.$$

Obviously, the summation over the impulse response from equation (22) tends to infinity. Thus, the system is unstable.

4.3.2 Pole-Zero Diagram

In the pole-zero diagram, the pole subsections are shown as $\times$ and the zeros as $\circ$. Furthermore, the specification of the gain H_0 is necessary to fully describe the signal or the system. Let a signal $x(n)$ be given:

$$x(n) = 5 \cdot 0,7^n \cos\left(2\pi\frac{n}{6}\right)\varepsilon(n).$$

According to the collection of formulas, the associated z-transform is:

$$\begin{aligned} X(z) &= 5\frac{1-0,7\cos\left(2\pi\frac{1}{6}\right)z^{-1}}{1-1,4\cos\left(2\pi\frac{1}{6}\right)z^{-1}+0,49z^{-2}} \\ &= 5\frac{1-0,35z^{-1}}{1-0,7z^{-1}+0,49z^{-2}} \end{aligned}$$

To create the pole-zero diagram, the fraction must be expanded so that only positive powers of z occur. Furthermore, the prefactors before the highest power of z in the

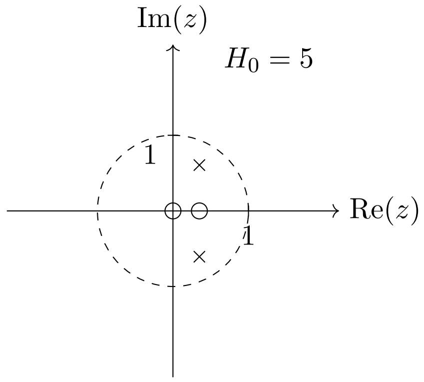


Figure 32: Example pole-zero diagram, the unit circle is drawn for illustration only.

numerator and denominator must equal 1 so that the gain H_0 can be determined as the prefactor before the fractional rational function:

$$H(z) \cdot \frac{z^2}{z^2} = 5\frac{z^2 - 0,35z}{z^2 - 0,7z + 0,49} = 5\frac{1 \cdot z^2 - 0,35z}{1 \cdot z^2 - 0,7z + 0,49} \ . \tag{23}$$

Thus, the gain H_0 is equal to the coefficient b_0 from the formula 21. The next step is to determine the zeros from the numerator polynomial and the poles from the denominator polynomial. It results in:

$$\begin{aligned} z_{N,1} &= 0 \\ z_{N,2} &= 0,35 \\ z_{P,1} &= 0,35 + j0,606 \\ z_{P,2} &= 0,35 - j0,606 \ . \end{aligned}$$

This gives the pole-zero diagram in figure 32. Multiple pole and zero points, as in the example transfer function $H(z) = 2\frac{z^3}{(z-0,5)^2}$, are plotted as shown in figure 33.

4.3.3 Stability in Z-Domain

The stability criterion in the z-domain is: all pole locations of the z-transform must lie within the unit circle. Similarly, the unit circle must belong to the convergence region of the z-transform. Since FIR filters can have only single or multiple pole positions at the location $z = 0$, FIR filters are always stable.

4.4 Inverse Filter

As we have seen in the previous sections, the associated circuit can be inferred from a difference equation. Likewise, the associated z-transform can be determined. The difference equation and thus its z-transform describe a filter. If one wants to undo the filtering by this difference equation, as shown in figure 34, one must simply redefine the

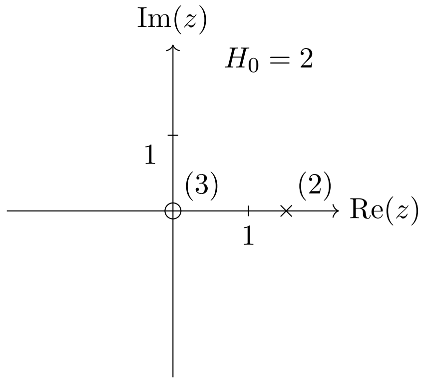


Figure 33: Visualization of multiple poles and zeros for the transfer function $H(z) = 2\frac{z^3}{(z-0,5)^2}$.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) \longrightarrow \boxed{h^{-1}(n)} \longrightarrow \tilde{x}(n)$$

Figure 34: Filtering by a discrete-time system, described by its z-transform $h(n) \circ\!\!-\!\!\bullet H(z)$, can be undone by swapping pole and zero points, so that ideally $\tilde{x}(n) = x(n)$ holds

.

pole points to zero points and the zero points to pole points in the z-transform. This will be illustrated by the example of the z-transform from equation (23):

$$\begin{aligned}
H(z) = \frac{Y(z)}{X(z)} &= 5\frac{z^2 - 0,35z}{z^2 - 0,7z + 0,49} = 5\frac{1 \cdot z^2 - 0,35z}{1 \cdot z^2 - 0,7z + 0,49} \\
&= 5\frac{z \cdot (z - 0,35)}{(z - 0,35 + j0,606) \cdot (z - 0,35 + j0,606)} \\
H^{-1}(z) &= \frac{1}{5}\frac{(z - 0,35 + j0,606) \cdot (z - 0,35 + j0,606)}{z \cdot (z - 0,35)} \ .
\end{aligned}$$

Since from the original transfer function $H(z)$ all the zeros are located in the unit circle, all the poles of $H^{-1}(z)$ are also located in the unit circle and the inverse filter is also stable. The difference equation of the inverse filter can be determined as follows:

$$\begin{aligned}
H^{-1}(z) &= \frac{1}{5}\frac{z^2 - 0,7z + 0,49}{z^2 - 0,35z} \\
\frac{\tilde{X}(z)}{Y(z)} &= \frac{1}{5}\frac{1 - 0,7z^{-1} + 0,49z^{-2}}{1 - 0,35z^{-1}} \\
5 \cdot Y(z) \cdot \left(1 - 0,35z^{-1}\right) &= \tilde{X}(z)\left(1 - 0,7z^{-1} + 0,49z^{-2}\right) \\
\tilde{x}(n) &= 5y(n) - 1.75y(n-1) + 0.7\tilde{x}(n-1) - 0.49\tilde{x}(n-2) \ .
\end{aligned}$$

In the given example, the filtering is undone by $H(z)$ without error, so $\tilde{x}(n) = x(n)$.

4.5 Bilinear Transformation

The bilinear transformation transforms filters and signals from the Laplace-domain to the z-domain and vice versa.

4.5.1 Differential Equations in Laplace-Domain

The output voltage $u_2(t)$ of the RC low-pass filter from figure 2 can be described by the following equations:

$$\begin{aligned} u_2(t) &= \frac{Q(t)}{C} \\ .u_1(t) &= R \cdot i(t) + u_2(t) \\ &= R \cdot \frac{dQ(t)}{dt} + u_2(t) \\ &= RC \cdot \frac{du_2(t)}{dt} + u_2(t)\ . \end{aligned}$$

Equations in which variables (here $u_2(t)$) occur as (multiple) derivatives or (multiple) integrals are called differential equations. Differential equations can be expressed in simple form in the Laplace domain, since the Laplace transform turns derivatives into multiplication by s and integrals into division by s. For example, the behavior of the RC low-pass filter in the Laplace-domain can be described as follows:

$$\begin{aligned} U_1(s) &= RC \cdot sU_2(s) + U_2(s) \\ H(s) &= = \frac{U_2(s)}{U_1(s)} = \frac{1}{1 + sRC} \end{aligned}$$

with $s = \sigma + j2\pi f$. This is ultimately a voltage divider with the following component impedances:

$$\begin{aligned} \text{resistor} \quad & Z = R \\ \text{coil} \quad & Z = sL \\ \text{capacitor} \quad & Z = \frac{1}{sC}\ . \end{aligned}$$

To simulate differential equations by discrete-time circuits must be transformed from the Laplace-plane to the z-plane. Equation 17 provides the following exact mapping:

$$s = \frac{1}{T} \log z.$$

The bilinear transformation approximates this exact mapping from the Laplace plane to the z-plane by the first member of the Taylor series of the natural logarithm:

$$s = \frac{2}{T}\left(\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \dots\right) \tag{24}$$

$$s \approx \frac{2}{T}\frac{z-1}{z+1}. \tag{25}$$

Using the RC low-pass as an example, this leads to the following discrete-time transfer function:

$$H(z) \approx \frac{1}{1+\frac{2}{T}\frac{z-1}{z+1}RC} = \frac{T\cdot(z+1)}{z\,(T+2RC)+(T-2RC)}.$$

4.5.2 Frequency Warping

Since the Taylor series from equation 25 is evolved around the point $z=1$, the mapping of the frequency axis is exact for $z=1$. This corresponds to the frequency $f=0$ Hz. The further the frequency deviates from 0 Hz, the more erroneous the mapping 25. For the frequency mapping to be accurate at a given frequency f_0, equation (25) must be modified as follows:

$$s \approx c\cdot\frac{z-1}{z+1}\ .$$

The derivation of the constant c goes as follows: The complex frequency s of the Laplace transform is defined by

$$s = \sigma + j2\pi f\ .$$

From this it follows for the frequency f of the Fourier transform:

$$\begin{aligned} f &= \frac{1}{2\pi}\mathfrak{Im}\,(s) && (26)\\ &= \frac{1}{2\pi}\mathfrak{Im}\left(\frac{1}{T}\log z\right) && (27)\\ &\approx \frac{1}{2\pi}\mathfrak{Im}\left(c\cdot\frac{z-1}{z+1}\right)\ . && (28)\end{aligned}$$

To obtain the frequencies f in discrete time, set $\sigma=0$ and obtain for the complex variable z:

$$z = e^{j2\pi fT}\ .$$

Substituting this into equation 28 we get the following formula:

$$f \approx \frac{1}{2\pi}\mathfrak{Im}\left(c\cdot\frac{e^{j2\pi fT}-1}{e^{j2\pi fT}+1}\right)\ .$$

To determine the constant c it is required in the following that the equation must be exactly satisfied for a frequency f_0:

$$\begin{aligned} f_0 &= \frac{1}{2\pi}\mathfrak{Im}\left(c\cdot\frac{e^{j2\pi f_0T}-1}{e^{j2\pi f_0T}+1}\right)\\ &= \frac{1}{2\pi}\mathfrak{Im}\left(c\cdot\frac{e^{j\pi f_0T}-e^{-j\pi f_0T}}{e^{j\pi f_0T}+e^{-j\pi f_0T}}\right)\\ &= \frac{1}{2\pi}\mathfrak{Im}\left(c\cdot\frac{2j\sin(\pi f_0T)}{2\cos(\pi f_0T)}\right)\\ &= \frac{1}{2\pi}c\cdot\frac{2\sin(\pi f_0T)}{2\cos(\pi f_0T)}\\ &= \frac{1}{2\pi}c\cdot\tan(\pi f_0T)\ .\end{aligned}$$

It follows for the constant c:

$$c = \frac{2\pi f_0}{\tan(\pi f_0 T)} \ . \tag{29}$$

This non-linear distortion of the frequency axis is called frequency warping. With an RC low-pass filter, the correct mapping of frequencies in the pass-band and stop-band is not that important. The most important thing is that the cutoff frequency is mapped as accurately as possible. The cutoff frequency of the RC low-pass is:

$$f_g = \frac{1}{2\pi RC}$$

It follows for the factor c from equation 29:

$$\begin{aligned} c &= \frac{2\pi \cdot \frac{1}{2\pi RC}}{\tan\left(\pi \frac{1}{2\pi RC} T\right)} \\ &= \frac{1}{RC \tan\left(\frac{T}{2RC}\right)} \end{aligned}$$

For a high-pass filter, analogous to a low-pass filter, the cutoff frequency would be chosen as the reference frequency f_0. For a band-pass, the center frequency is chosen as the reference frequency f_0.

4.6 Determining the output signal of an IIR filter

One determines the output signal of an FIR filter with the discrete-time convolution in the time domain or with the Fourier transform of discrete-time signals in the frequency-domain. With the partial fraction decomposition and the tables for the z-transformation (20), the output signal of an IIR filter can be determined. For this the following steps are necessary:

- z-transformation of the input signal and the difference equation of the circuit,
- multiplication of both z-transforms to form the z-transform of the output signal: $Y(z) = H(z) \cdot X(z)$,
- partial fraction decomposition of $Y(z)$ and
- back transformation into the time domain: $y(n) \circ\!\!-\!\!\bullet Y(z)$.

These steps are explained below with an example. Let the input signal be

$$x(n) = (-0,5)^n \varepsilon(n) - 2(-0,5)^{n-1}\varepsilon(n-1)$$

and the difference equation of the IIR filter is

$$y(n) = x(n+1) - 4x(n) + 3x(n-1) + 0,5y(n-1) \ .$$

4.6.1 Z-Transform

The z-transforms are correspondingly:

$$\begin{aligned} X(z) &= \frac{1-2z^{-1}}{1+0.5z^{-1}} = \frac{z-2}{z+0.5} \\ H(z) &= \frac{z-4+3z^{-1}}{1-0,5z^{-1}} = \frac{(z-1)\cdot(z-3)}{z-0,5}. \end{aligned}$$

4.6.2 Multiplication

The z-transform of the output signal is:

$$Y(z) = X(z)\cdot H(z) = \frac{z^3-6z^2+11z-6}{z^2-0,25}.$$

4.6.3 Partial Fraction Decomposition

If the numerator degree is greater than the denominator degree, a polynomial division is first performed:

$$\left(z^3-6z^2+11z-6\right) : \left(z^2-0,25\right) = z-6+\frac{11,25z-7,5}{z^2-0,25}.$$

Next, the partial fraction decomposition is applied to the division remainder. First, the denominator is decomposed into its terms. Then, the division remainder is equated to an addition of fractions. Each of these fractions has one degree less in the numerator polynomial than in the denominator polynomial:

$$\frac{11,25z-7,5}{z^2-0,25} = \frac{A}{z-0,5} + \frac{B}{z+0,5}\ .$$

By multiplying by the entire denominator polynomial, an equation is set up to determine the coefficients A and B of the numerator polynomials of the right-hand side of the system of equations:

$$11.25z-7.5 = (z+0.5)\cdot A + (z-0.5)\cdot B\ . \tag{30}$$

If one inserts arbitrary values for z, one obtains an equation. Choosing as many values for z as one has coefficients in the equation (30) produces a system of equations whose solution provides the values for the coefficients. In principle, the choice of values for z is arbitrary. It is convenient to use the pole places to keep the system of equations as simple as possible. Alternatively, one can also use small natural numbers for z to obtain equations as simple as possible. In the following the pole places of $Y(z)$ are used for z to determine the coefficients A and B. Substituting the pole location $z=-0.5$ into the equation (30) yields the following equation:

$$\begin{aligned} 11.25\cdot(-0.5)-7.5 &= -B \\ .13,125 &= B \end{aligned}$$

Substituting the pole location $z = 0.5$ into the equation (30) yields the following equation:

$$\begin{aligned} 11.25 \cdot 0.5 - 7.5 &= A \\ -1,875 &= A \end{aligned}$$

This gives the following partial fraction decomposition:

$$\frac{z^3 - 6z^2 + 11z - 6}{z^2 - 0,25} = z - 6 - \frac{1,875}{z - 0,5} + \frac{13,125}{z + 0,5}. \tag{31}$$

4.6.4 Backtransformation

Dividing the additive terms of the result of the partial fraction decomposition in equation (31) again by the highest powers of z:

$$z - 6 - \frac{1,875z^{-1}}{1 - 0,5z^{-1}} + \frac{13,125z^{-1}}{1 + 0,5z^{-1}}$$

Then convert the result to time signals using the table 20:

$$\begin{aligned} \delta(n+1) &\quad \circ\!\!-\!\!\bullet \quad z \\ \delta(n) &\quad \circ\!\!-\!\!\bullet \quad 1 \\ 0,5^n \varepsilon(n) &\quad \circ\!\!-\!\!\bullet \quad \frac{1}{1 - 0,5z^{-1}} \\ (-0,5)^n \varepsilon(n) &\quad \circ\!\!-\!\!\bullet \quad \frac{1}{1 + 0,5z^{-1}}. \end{aligned}$$

This gives the time signal at the output:

$$y(n) = \delta(n+1) - 6\delta(n) - 1,875 \cdot 0,5^{n-1}\varepsilon(n-1) + 13,125 \cdot (-0,5)^{n-1}\varepsilon(n-1).$$

4.6.5 Partial Fraction Decomposition with Complex Conjugate Poles

In the context of this lecture, we always assume difference equations with real-valued coefficients. In this case, complex-valued pole places always appear as complex conjugate pairs. A complex conjugate pole pair ($z_P = \mathfrak{Re}(z_P) + j\mathfrak{Im}(z_P)$ and $z_P^* = \mathfrak{Re}(z_P) - j\mathfrak{Im}(z_P)$) always leads to a denominator polynomial of degree 2 with real-valued coefficients:

$$\begin{aligned} Y(z) &= \frac{1}{(z + z_P) \cdot (z + z_P^*)} \\ &= \frac{1}{(z + \mathfrak{Re}(z_P) + j\mathfrak{Im}(z_P)) \cdot (z + \mathfrak{Re}(z_P) - j\mathfrak{Im}(z_P))} \\ &= \frac{1}{z^2 + 2\mathfrak{Re}(z_P)\, z + \mathfrak{Re}(z_P)^2 + \mathfrak{Im}(z_P)^2}. \end{aligned}$$

The following is an example of a partial fraction decomposition with a complex conjugate pole pair:

$$Y(z) = \frac{12z+5}{\left(z^2+z+\frac{5}{16}\right)\cdot\left(z-\frac{1}{4}\right)}.$$

For the partial fraction decomposition of $Y(z)$, the following holds for each term: the numerator polynomial always has one polynomial degree less than the denominator polynomial:

$$Y(z) = \frac{12z+5}{\left(z^2+z+\frac{5}{16}\right)\cdot\left(z-\frac{1}{4}\right)} = \frac{Az+B}{z^2+z+\frac{5}{16}} + \frac{C}{z-\frac{1}{4}}.$$

Again, multiply by the denominator polynomial:

$$12z+5 = (Az+B)\cdot\left(z-\frac{1}{4}\right) + C\cdot\left(z^2+z+\frac{5}{16}\right).$$

Substituting any three values for z gives a system of equations that can be solved for the unknowns A, B, and C:

$$\begin{aligned}
z=0 &\quad:\quad 5 = B\cdot\frac{-1}{4} + C\cdot\frac{5}{16} \\
z=1 &\quad:\quad 17 = (A+B)\cdot\frac{3}{4} + C\cdot\frac{37}{16} \\
z=-1 &\quad:\quad -7 = (-A+B)\cdot\frac{-5}{4} + C\cdot\frac{5}{16}
\end{aligned}$$

Solving the system of equations we get the following values for the coefficients:

$$\begin{aligned}
A &= -12,8 \\
B &= -4 \\
C &= 12,8
\end{aligned}$$

This gives the following partial fraction decomposition:

$$\begin{aligned}
Y(z) &= \frac{12z+5}{\left(z^2+z+\frac{5}{16}\right)\cdot\left(z-\frac{1}{4}\right)} \\
&= \frac{-12,8z-4}{z^2+z+\frac{5}{16}} + \frac{12,8}{z-\frac{1}{4}} \\
&= \frac{-12,8z^{-1}-4z^{-2}}{1+z^{-1}+\frac{5}{16}z^{-2}} + \frac{12,8z^{-1}}{1-\frac{1}{4}z^{-1}} \\
&= z^{-1}\cdot\frac{-12,8-4z^{-1}}{1+z^{-1}+\frac{5}{16}z^{-2}} + 12,8z^{-1}\frac{1}{1-\frac{1}{4}z^{-1}}.
\end{aligned}$$

From the table (20) we obtain the following time signal:

$$\begin{aligned}
y(n) &= -12,8\delta\left(n-1\right) * \left(\sqrt{\frac{5}{16}}^{n} \cdot \cos\left(2\pi \cdot 0,426 \cdot n\right) \cdot \varepsilon(n)\right) \\
&\quad -4\delta\left(n-1\right) * \left(\sqrt{\frac{5}{16}}^{n} \cdot \sin\left(2\pi \cdot \left(-0,176\right) \cdot n\right) \cdot \varepsilon(n)\right) \\
&\quad +12,8\delta\left(n-1\right) * \left(\frac{1}{4^{n}}\varepsilon(n)\right) \\
&= -12,8 \cdot \sqrt{\frac{5}{16}}^{n-1} \cdot \cos\left(2\pi \cdot 0,426 \cdot (n-1)\right) \cdot \varepsilon(n-1) \\
&\quad +4 \cdot \sqrt{\frac{5}{16}}^{n-1} \cdot \sin\left(2\pi \cdot 0,176 \cdot (n-1)\right) \cdot \varepsilon(n-1) \\
&\quad +12,8 \cdot \frac{1}{4^{n-1}}\varepsilon(n-1)\ .
\end{aligned}$$

4.6.6 Partial Fraction Decomposition with Multiple Poles

If the z-transform has multiple pole places, one term per pole place degree is introduced for each multiple pole place. With each term the pole degree increases, as shown in the following example:

$$Y(z) = \frac{7z+5}{\left(z+1\right)^{2} \cdot \left(z+2\right)} = \frac{A}{z+1} + \frac{B}{\left(z+1\right)^{2}} + \frac{C}{z+2}.$$

The resulting system of equations is solved as usual: multiply by the entire denominator and substitute arbitrary values for z to determine the unknown coefficients A, B, and C:

$$\begin{aligned}
7z+5 &= \left(z+1\right)\left(z+2\right)A + \left(z+2\right) + \left(z+1\right)^{2}C \\
z=0 &: \quad 5 = 2A + 2B + C \\
z=-1 &: \quad -2 = B \\
z=-2 &: \quad -9 = C \\
&\rightarrow \quad A = 9 \\
Y(z) &= \frac{9}{z+1} - \frac{2}{\left(z+1\right)^{2}} - \frac{9}{z+2} \\
&= \frac{9z^{-1}}{1+z^{-1}} - \frac{2z^{-2}}{\left(1+z^{-1}\right)^{2}} - \frac{9z^{-1}}{1+2z^{-1}}
\end{aligned}$$

From this, the time domain signal can be determined as usual:

$$y(n) = \Big(\left(9+2(n-1)\right)(-1)^{n-1} - 9(-2)^{n-1}\Big) \cdot \varepsilon(n-1)$$

If the degree of the pole is higher than 2, the associated time-domain signal can no longer be solved using the table 20 from the collection of formulas. In this case, the theorem *derivation of image function* helps:

$$n \cdot x(n) \circ\!\!-\!\!\bullet - z \cdot \frac{dX(z)}{dz} \quad . \tag{32}$$

If one wants to find the time domain representation of a third degree pole, one derives the image function of a second degree pole:

$$\begin{aligned} \frac{d}{dz}\frac{bz^{-1}}{\left(1-bz^{-1}\right)^2} &= \frac{-bz^{-2}\cdot\left(1-bz^{-1}\right)^2 - bz^{-1}\cdot 2bz^{-2}\left(1-bz^{-1}\right)}{\left(1-bz^{-1}\right)^4} \\ &= \frac{-bz^{-2}+b^2z^{-3}-2b^2z^{-3}}{\left(1-bz^{-1}\right)^3} \\ &= \frac{-bz^{-2}-b^2z^{-3}}{\left(1-bz^{-1}\right)^3} \quad . \end{aligned}$$

Applying to this derivative the theorem from equation 32, we obtain:

$$\begin{aligned} nb^n\varepsilon(n) \quad &\circ\!\!-\!\!\bullet \quad \frac{bz^{-1}}{\left(1-bz^{-1}\right)^2} \\ \to n^2b^n\varepsilon(n) \quad &\circ\!\!-\!\!\bullet \quad -z\cdot\frac{-bz^{-2}-b^2z^{-3}}{\left(1-bz^{-1}\right)^3} \\ &= \quad \frac{bz^{-1}+b^2z^{-2}}{\left(1-bz^{-1}\right)^3} \quad . \end{aligned}$$

The determination of the time-domain representation of fourth- or higher-degree poles follows the same iterative scheme.

4.7 Exercises

4.7.1 Task 1: Inverse Z-Transform

For the following z-transforms, determine the corresponding signals in the time domain.

- $\frac{z^2+z+1}{1+\frac{1}{3}z^{-1}}$
- $\frac{z+1}{z^2-1}$
- $\frac{1}{1-6z^{-1}+9z^{-2}}$
- $\frac{1}{1-3z^{-1}+4z^{-2}}$

4.7.2 Task 2: Sketch with all Characteristic Values given

- $x(n) = (0,7)^n \cdot \varepsilon(n)$
- $x(n) = n\,(0,5)^n \cdot \varepsilon(n)$
- $x(n) = \delta\,(n+1) * (n\,(0,5)^n \cdot \varepsilon(n))$

5 Correlation Functions

Correlation functions give a measure of the similarity between two signals. For the purpose of this lecture, only the similarities between two power signals will be considered. Power signals and their correlation functions are not part of the lecture. An energy signal has a finite signal energy:

$$E_s = \int_{-\infty}^{\infty} s^2(t)dt < \infty \ .$$

Similarly, power signals are defined by:

$$E_s = \int_{-\infty}^{\infty} s^2(t)dt \to \infty \ .$$

5.1 Continuous-Time Correlation Functions

The theory for this chapter is explained in detail in [1, Chapters 6.1 to 6.4].

5.1.1 Cross-Correlation Function

The cross-correlation function between two signals $s(t)$ and $g(t)$ is defined as follows:

$$\varphi_{sg}^E(\tau) = \int_{-\infty}^{\infty} s(t) \cdot g(t+\tau)dt = s(-\tau) * g(\tau) \ .$$

The larger the function value of the correlation function, the greater the similarity between $s(t)$ and $g(t)$. As an example, determine the correlation function between $s(t) = \mathrm{rect}(t)$ and $g(t) = \mathrm{rect}(t-3)$:

$$\begin{aligned}
s(t) &= \mathrm{rect}(t) = s(-t) \\
g(t) &= \mathrm{rect}(t-3) = \mathrm{rect}(t) * \delta(t-3) \\
\varphi_{sg}^E(\tau) &= s(-t) * g(t) = \mathrm{rect}(\tau) * \mathrm{rect}(\tau) * \delta(\tau - 3) \\
&= \Delta(\tau) * \delta(\tau - 3) \\
&= \Delta(\tau - 3) \ .
\end{aligned}$$

The cross-correlation function $\varphi_{sg}^E(\tau)$ has a maximum at $t_0 = 3$. This means that both signals $s(t)$ and $g(t)$ are maximally similar when $s(t)$ is shifted by $t_0 = +3$ and when $g(t)$ is shifted by $t_0 = -3$, respectively. To make the similarities between different pairs of signals comparable, there is the normalized correlation coefficient. This corresponds to the cross-correlation function normalized by the signal energies:

$$p_{sg}^E(\tau) = \frac{\varphi_{sg}^E(\tau)}{\sqrt{E_s \cdot E_g}}, -1 \leq p_{sg}^E(\tau) \leq 1 \ .$$

The normalized correlation coefficient has the following meaning:

- $p_{sg}^E(\tau_0) = 1$ means that the two signals are identical except for a positive amplitude change when g is shifted to the left by τ_0,

- $p_{sg}^E(\tau_0) = -1$ means that the two signals are identical except for a negative amplitude change when g is shifted to the left by τ_0, and

- $p_{sg}^E(0) = 0$ means that the two signals are **orthogonal**.

The two cross-correlation functions between two signals have the following symmetry properties:

$$\varphi_{sg}^E(-\tau) = \varphi_{gs}^E(\tau) \ .$$

The Fourier transform of the cross-correlation function is defined as follows:

$$\begin{aligned} \varphi_{sg}^E(\tau) &= s(-\tau) * g(\tau) \\ \circ\!\!-\!\!\bullet\, \Phi_{sg}^E(f) &= S^*(f) \cdot G(f) = \int_{-\infty}^{\infty} \varphi_{sg}^E(\tau) \cdot e^{-j2\pi f\tau} d\tau \ . \end{aligned}$$

Again, as an example, consider the cross-correlation function between $s(t) = \mathrm{rect}(t)$ and $g(t) = \mathrm{rect}(t-3)$:

$$\begin{aligned} S(f) &= \mathrm{si}\,(\pi f) = S^*(f) \\ G(f) &= \mathrm{si}\,(\pi f) \cdot e^{-j2\pi 3f} \\ \circ\!\!-\!\!\bullet\, \Phi_{sg}^E(f) &= S^*(f) \cdot G(f) = \mathrm{si}\,(\pi f)^2 \cdot e^{-j6\pi f} \\ \Phi_{gs}^E(f) &= G^*(f) \cdot S(f) = \mathrm{si}\,(\pi f)^2 \cdot e^{j6\pi f} \end{aligned}$$

5.1.2 Orthogonality

Two signals $s(t)$ and $g(t)$ are orthogonal if the cross-correlation function becomes 0 at the point $\tau = 0$:

$$\varphi_{sg}^E(0) = \int_{-\infty}^{\infty} s(t) \cdot g(t) dt = 0 \ . \tag{33}$$

From this we can derive two simple cases where orthogonality must be given:

- if $s(t)$ and $g(t)$ do not overlap in the time domain, for example $s(t) = \mathrm{rect}(t-0.5)$ and $g(t) = \mathrm{rect}(t+0.5)$ or

- if $s(t)$ is an even signal and $g(t)$ is an odd signal (or vice versa). Also in this case, the integral in equation 33 must become 0. An example of this would be $s(t) = \Delta(t)$ and $g(t) = \mathrm{rect}(t-0.5) - \mathrm{rect}(t+0.5)$.

Equivalently, orthogonality can also be formulated in the frequency domain:

$$\varphi_{sg}^E(0) = \int_{-\infty}^{\infty} S^*(f) \cdot G(f) df = 0 \ . \tag{34}$$

Again, if the two spectra of the two signals do not overlap, then when integrating in equation 34, one of the two spectra is always equal to 0 and thus the entire integral is

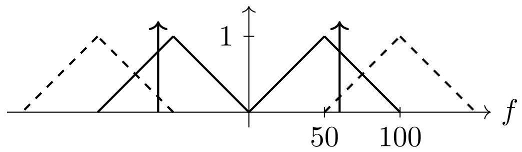


Figure 35: Signal analysis with overlapping (non-orthogonal) filters. It is assumed that the signal to be analyzed is a cosine oscillation at $F = 60$ Hz.

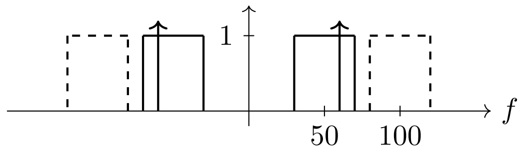


Figure 36: Signal analysis with non-overlapping (orthogonal) filters. It is assumed that the signal to be analyzed is a cosine oscillation at $F = 60$ Hz.

equal to 0. An example of this would be two band-pass signals:

$$\begin{aligned} s(t) &= \mathrm{si}\,(\pi t) \cdot \cos 2\pi 100 t \\ \circ\!\!-\!\!\!-\!\!\bullet S(f) &= \mathrm{rect}\,(f) * \frac{1}{2}\left(\delta\,(f+100) + \delta\,(f-100)\right) \\ g(t) &= \mathrm{si}\,(\pi t) \cdot \cos 2\pi 50 t \\ \circ\!\!-\!\!\!-\!\!\bullet G(f) &= \mathrm{rect}\,(f) * \frac{1}{2}\left(\delta\,(f+50) + \delta\,(f-50)\right)\ . \end{aligned}$$

Both spectra do not overlap. Thus both signals are orthogonal.

Applications for Orthogonality The orthogonality condition is important whenever signals are to be analyzed. As a simple example, we will measure how much signal energy around $f_0 = 50$ Hz and how much signal energy around $f_1 = 100$ Hz are contained in a measurement signal. It is assumed that a cosine oscillation with $F = 60$ Hz is present in the measurement signal. If two overlapping filters are used for the measurement as in figure 35, for the given measurement signal both filters will have an output signal other than 0. However, if one uses non-overlapping (orthogonal) filters, only one of the two filters will output a non-zero signal (or none) for any measured signal. Thus, a simple signal detector would detect either a signal in the frequency band around f_0 or a signal in the frequency band around f_1.
On the other hand, it was shown in section 3.7.1 that steep-edged filters are not trivial to realize. For this reason, filters are generally used for signal analysis which, on the one hand, satisfy the orthogonality condition as best as possible and, on the other hand, cause as little realization effort as possible, for example, by steep filter edges.

5.1.3 Autocorrelation Function

If one compares a signal with itself using the correlation function, one obtains the so-called autocorrelation function:

$$\varphi_{ss}^{E}(\tau) = \int_{-\infty}^{\infty} s(t) \cdot s(t+\tau) dt = s(-\tau) * s(\tau) \ .$$

The autocorrelation function has the following important properties:

- the maximum is always at $\tau = 0$: $\varphi_{ss}^{E}(0) \geq \left|\varphi_{ss}^{E}(\tau)\right|$,
- the autocorrelation function is axisymmetric: $\varphi_{ss}^{E}(-\tau) = \varphi_{ss}^{E}(\tau)$,
- if $s(t)$ is time bounded, $\varphi_{ss}^{E}(\tau)$ is also time bounded and has at most twice the width of $s(t)$, and
- its spectrum is purely real-valued and larger 0.

5.1.4 Energy Density Spectrum

The Fourier transform of the autocorrelation function is called the energy density spectrum:

$$|S(f)|^2 = \Phi_{ss}^{E}(f) = \int_{-\infty}^{\infty} \varphi_{ss}^{E}(\tau) \cdot e^{-j2\pi f\tau} d\tau \ .$$

The energy density spectrum indicates how much signal energy is active in a given frequency range. Accordingly, the total signal energy is defined by:

$$E_s = \int_{-\infty}^{\infty} \Phi_{ss}^{E}(f) df$$

The energy density spectrum and the autocorrelation function shall be explained by an example:

$$\begin{aligned}
s(t) &= e^{-\pi t^2} \\
S(f) &= e^{-\pi f^2} \\
|S(f)|^2 &= \left(e^{-\pi f^2}\right)^2 = e^{-2\pi f^2} \\
\Phi_{ss}^{E}(f) &= |S(f)|^2 = e^{-\pi\left(\sqrt{2}\cdot f\right)^2} \\
\varphi_{ss}^{E}(\tau) &= \frac{1}{\sqrt{2}} e^{-\pi t^2/2}
\end{aligned}$$

From the autocorrelation function, the energy of $s(t)$ can be read:

$$E_s = \varphi_{ss}^{E}(0) = \frac{1}{\sqrt{2}} \ .$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Figure 37: A continuous-time LTI system with input signal $x(t)$, impulse response $h(t)$, and output signal $y(t)$.

For example, if one wants to calculate what signal energy of $s(t)$ passes through an ideal low-pass filter with cutoff frequency $f_g = 0.1$, one must solve the following integral:

$$\begin{aligned}
\int_{-0,1}^{0,1} \Phi_{ss}^E(f)df &= \int_{-0,1}^{0,1} e^{-\pi\left(\sqrt{2}\cdot f\right)^2} df \\
&= \sqrt{2\pi\sigma^2} \int_{-0,1}^{0,1} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-f^2/\left(2\sigma^2\right)} df \text{ with } \sigma^2 = \frac{1}{4\pi} \\
&= \frac{1}{\sqrt{2}} \int_{-0,1}^{0,1} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-f^2/\left(2\sigma^2\right)} df \\
&= \frac{1}{\sqrt{2}} \left(\int_{-\infty}^{0,1} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-f^2/\left(2\sigma^2\right)} df - \int_{-\infty}^{-0,1} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-f^2/\left(2\sigma^2\right)} df \right) \\
&= \frac{1}{2\sqrt{2}} \left(\operatorname{erfc}\left(\frac{-0,1}{\sqrt{2\sigma^2}} \right) - \operatorname{erfc}\left(\frac{0,1}{\sqrt{2\sigma^2}} \right) \right) \\
&= \frac{1}{2\sqrt{2}} \left(\operatorname{erfc}\left(-0,1\sqrt{2\pi} \right) - \operatorname{erfc}\left(0,1\sqrt{2\pi} \right) \right) \\
&= \frac{1}{2\sqrt{2}} \left(2 - \operatorname{erfc}\left(0,1\sqrt{2\pi} \right) - \operatorname{erfc}\left(0,1\sqrt{2\pi} \right) \right) \\
&= \frac{1}{\sqrt{2}} \left(1 - \operatorname{erfc}\left(0,1\sqrt{2\pi} \right) \right) \\
&\approx \frac{1}{\sqrt{2}} \left(1 - \frac{e^{-(0,1)^2 2\pi}}{\sqrt{\pi}\cdot 0,1 \cdot \sqrt{2\pi} + e^{-0,6\cdot 0,1\cdot\sqrt{2\pi}}} \right) \\
&= 0,20\ .
\end{aligned}$$

5.2 Correlation Functions and LTI Systems

The theory for this chapter is explained in detail in [1, Chapter 6.5]. In the following, we assume an LTI system as in figure 37. The autocorrelation function of the output signal is defined by:

$$\begin{aligned}
\varphi_{yy}^E(\tau) &= y(-\tau) * y(\tau) \\
\text{.with } y(t) &= x(t) * h(t) \\
\text{gives } \varphi_{yy}^E(\tau) &= (x(-\tau) * h(-\tau)) * (x(\tau) * h(\tau)) \\
&= x(-\tau) * h(-\tau) * x(\tau) * h(\tau) \\
&= x(-\tau) * x(\tau) * h(-\tau) * h(\tau) \\
\rightarrow \varphi_{yy}^E(\tau) &= \varphi_{xx}^E(\tau) * \varphi_{hh}^E(\tau)
\end{aligned}$$

The last equation is called Wiener-Lee relation and defines the relation of the autocorrelation functions for an LTI system. By Fourier transform, the Wiener-Lee relation is obtained in the spectral domain:

$$\Phi^E_{yy}(f) = \Phi^E_{xx}(f) \cdot \Phi^E_{hh}(f) \ .$$

The cross-correlation function between the input and output signals in an LTI system can be derived in the same form:

$$\begin{aligned} \varphi^E_{xy}(\tau) &= x(-\tau) * y(\tau) \\ \text{.with } y(t) &= x(t) * h(t) \\ \text{gives } \varphi^E_{xy}(\tau) &= x(-\tau) * (x(\tau) * h(\tau)) \\ &= x(-\tau) * x(\tau) * h(\tau) \\ \varphi^E_{xy}(\tau) &= \varphi^E_{xx}(\tau) * h(\tau) \ . \end{aligned}$$

Similarly, the Fourier transform of the cross-correlation function between the input and output signals in an LTI system is:

$$\Phi^E_{xy}(f) = \Phi^E_{xx}(f) \cdot H(f) \ .$$

5.3 Discrete-Time Correlation Functions

The theory for this chapter is explained in detail in [1, Chapter 6.7].
For discrete-time signals, the integrals must be replaced by sums. For example, energy and power signals are defined as follows:

$$\text{energy signal: } E_s = \sum_{-\infty}^{\infty} s^2(n) < \infty$$

$$\text{power signal: } E_s = \sum_{-\infty}^{\infty} s^2(n) \to \infty \ .$$

The other formulas for calculating correlation functions for discrete-time signals can be found in the collection of formulas.

5.3.1 Example of a Discrete-Time Cross-Correlation Function

To illustrate the discrete-time cross-correlation function, consider the following example:

$$\begin{aligned} x(n) &= \delta(n-1) + 2\delta(n-2) \\ y(n) &= \delta(n+1) + \delta(n) + \delta(n-1) \\ \varphi^E_{xy}(m) &= x(-m) * y(m) = \sum_{n=-\infty}^{\infty} x(n) \cdot y(n+m) \\ \to \varphi^E_{xy}(0) &= 1 \\ \to \varphi^E_{xy}(-1) &= 1 \cdot 1 + 2 \cdot 1 = 3 \\ \to \varphi^E_{xy}(-2) &= 1 \cdot 1 + 2 \cdot 1 = 3 \\ \to \varphi^E_{xy}(-3) &= 1 \cdot 2 = 2 \ . \end{aligned}$$

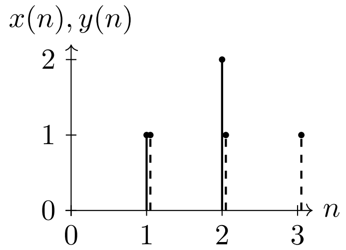


Figure 38: Visualization of the computation of the discrete-time cross-correlation function $\varphi^E_{xy}(-2)$. For better visualization $y(n-2)$ is drawn dashed and slightly offset.

The calculation of the cross-correlation function for $m = -2$ is also visualized in figure 38. $y(n)$ is shifted by $m = -2$. Then all *overlapping* amplitudes are multiplied and added up.
In summary, the cross-correlation function $\varphi^E_{xy}(m)$ can be written as follows:

$$\varphi^E_{xy}(m) = 2\delta(m+3) + 3\delta(m+2) + 3\delta(m+1) + \delta(m) \ .$$

Due to the symmetry properties of the cross-correlation function, the cross-correlation function $\varphi^E_{yx}(m)$ is calculated to:

$$\varphi^E_{yx}(m) = \varphi^E_{xy}(-m) = \delta(m) + 3\delta(m-1) + 3\delta(m-2) + 2\delta(m-3) \ .$$

5.3.2 Example of a Discrete-Time Autocorrelation Function

The autocorrelation function for $x(n)$ is calculated as follows:

$$\begin{aligned}\varphi^E_{xx}(m) &= x(-m) * x(m) = \sum_{n=-\infty}^{\infty} x(n) \cdot x(n+m) \\ &= 2\delta(m+1) + 5\delta(m) + 2\delta(m-1) \ .\end{aligned}$$

The associated energy density spectrum is:

$$\begin{aligned}\Phi^E_{xx}(f) &= 2e^{j2\pi fT} + 5 + 2e^{-j2\pi fT} \\ &= 4\cos(2\pi fT) + 5 \ .\end{aligned}$$

5.4 System Analysis

The following assumes an LTI system as shown in figure 40. If the input and output signals are known, the impulse response and transfer function can be estimated.

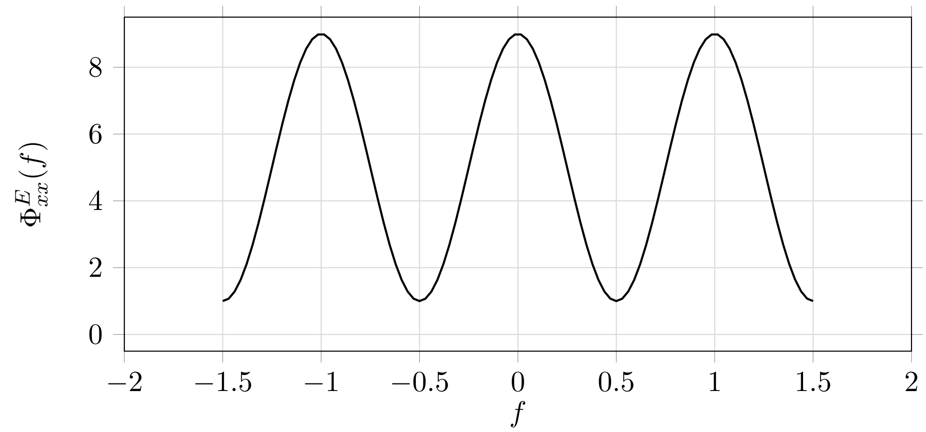


Figure 39: Autocorrelation function $\Phi_{xx}^{E}(f) = 4\cos(2\pi f T) + 5$ for $T = 1$.

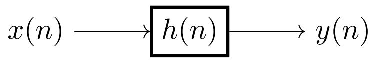


Figure 40: An LTI system with input signal $x(n)$, impulse response $h(n)$, and output signal $y(n)$.

5.4.1 Delay and Attenuation

As a first simple estimate of an LTI system, it can be approximated as an attenuator and delay element. Examples of such systems would be cables that attenuate the input signal due to losses and delay the output signal relative to the input signal due to the speed of light. Since, in general, no system is lossless and/or delayless, the model of an attenuator and delay element for any system is an initial estimate of the system properties. A delay and an attenuation can be described by the following convolution/impulse response:

$$y(n) = h(n) * x(n) = (a \cdot \delta(n - n_0)) * x(n) = a \cdot x(n - n_0).$$

The cross correlation function between the input signal and the output signal is then:

$$\begin{aligned}
\varphi_{xy}^{E}(m) &= x(-m) * y(m) \\
&= a \cdot (x(-m) * x(m) * \delta(m - n_0)) \\
&= a \cdot \delta(m - n_0) * \varphi_{xx}^{E}(m) \\
&= a \cdot \varphi_{xx}^{E}(m - n_0).
\end{aligned}$$

Since the autocorrelation function $\varphi_{xx}^{E}(m)$ has a unique maximum at $m = 0$, the position of the maximum of the cross-correlation function φ_{xy}^{E} corresponds to the delay of the system. The damping factor a can be determined by the following formula:

$$a = \frac{\varphi_{xy}^{E}(n_0)}{\varphi_{xx}^{E}(0)}.$$

This uniquely determines the LTI system with impulse response $h(n) = a \cdot \delta(n - n_0)$.

5.4.2 Least Squares Estimation of an Impulse Response

In what follows, we assume that the impulse response of the system from figure 40 can be approximated by a causal FIR filter of length K:

$$h(n) = \sum_{k=0}^{K-1} b_k \cdot \delta\left(n-k\right)$$

The formula of the discrete convolution then gives:

$$y(n) \approx \sum_{k=0}^{K-1} h(k) \cdot x(n-k).$$

A simple approach to estimate the values of the FIR filter $h(n)$ is to minimize the squared error e:

$$e = \sum_n \left(\sum_{k=0}^{K-1} h(k) \cdot x(n-k) - y(n) \right)^2 .$$

For minimization, the derivative of the filter is formed and set to zero:

$$\begin{aligned}
\frac{en}{dh(k)} &= \sum_n 2 \cdot \left(\sum_{m=0}^{K-1} h(m) \cdot x(n-m) - y(n) \right) \cdot x(n-k) \\
&= 2 \sum_n \left(\sum_{m=0}^{K-1} h(m) \cdot x(n-m) \right) \cdot x(n-k) - 2 \sum_n y(n) \cdot x(n-k) \\
&= 2 \sum_{m=0}^{K-1} h(m) \left(\sum_n x(n-m) \cdot x(n-k) \right) - 2 \sum_n y(n) \cdot x(n-k) \\
&= 2 \sum_{m=0}^{K-1} h(m) \varphi_{xx}^E(-m-k) - 2\varphi_{yx}(-k) \\
&= 2 \sum_{m=0}^{K-1} h(m) \varphi_{xx}^E(m+k) - 2\varphi_{xy}(k) \\
&\overset{!}{=} 0
\end{aligned}$$

From this follows an equation for each coefficient of the FIR filter:

$$\sum_{m=0}^{K-1} h(m) \cdot \varphi_{xx}^E(m+k) = \varphi_{xy}^E(k).$$

All K equations form the following linear system of equations:

$$\begin{pmatrix} \varphi_{xx}^E(0) & \varphi_{xx}^E(1) & ... & \varphi_{xx}^E(K-1) \\ \varphi_{xx}^E(1) & \varphi_{xx}^E(2) & ... & \varphi_{xx}^E(K) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{xx}^E(K-1) & \varphi_{xx}^E(K) & ... & \varphi_{xx}^E(2K-2) \end{pmatrix} \cdot \begin{pmatrix} h(0) \\ h(1) \\ \vdots \, h(K-1) \end{pmatrix} = \begin{pmatrix} \varphi_{xy}^E(0) \\ \cdot\varphi_{xy}^E(1) \\ \vdots \, \varphi_{xy}^E(K-1) \end{pmatrix} . \tag{35}$$

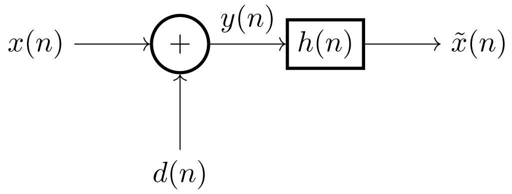


Figure 41: System model for the Wiener filter.

Solving the system of equations 35 provides an estimate of the FIR filter coefficients $h(n)$. A subsequent Fourier transform for discrete-time signals provides the associated transfer function $H(f)$.

5.5 Wiener Filter

The Wiener filter starts from the system model in figure 41: The input signal $x(n)$ is perturbed by noise $d(n)$. The disturbed signal $y(n) = x(n)+d(n)$ is recorded by a sensor. After the recording, a filter $h(n)$ is to be found such that $x(n) \approx \tilde{x}(n)$. Assuming that the noise $d(n)$ and the useful signal $x(n)$ are uncorrelated (i.e., have no linear dependencies), holds:

$$\Phi_{yy}^{E}(f) = \Phi_{xx}^{E}(f) + |D(f)|^2 \ .$$

For this case, the optimal filter can be derived as follows:

$$H(f) = \frac{\Phi_{xx}^{E}(f)}{\Phi_{xx}^{E}(f) + |D(f)|^2} \ . \tag{36}$$

Since noise signals are usually not energy signals, $|D(f)|^2$ is used instead of $\Phi_{dd}^{E}(f)$ in the formula 36. A detailed distinction between energy and power signals and their impact on the theory of correlation functions is beyond the scope of this lecture. For the interested reader, please refer to [1, Chapter 7]. The formula 36 represents the optimal solution for additive noise, as shown in figure 41. The only problem with the Wiener filter is the estimation of the parameters $\Phi_{xx}^{E}(f)$ and $|D(f)|^2$.

6 Summary

In the short time available, this script cannot cover the entire spectrum of signal processing routines. Rather, it is intended to review and reinforce certain aspects of the system theory lecture around the central concept of LTI systems. In addition, emphasis will be placed on aspects of filter design and system analysis. If a signal is to be filtered with a linear-phase filter and computational complexity is not a problem, FIR filters are the tool of choice. A design specification is presented in section 3.7 for low-pass, band-pass, and high-pass filters. If computational complexity is an issue and suitable filters are known in the Laplace domain, a method to transform IIR filters from the Laplace domain to the z-domain is presented in section 4.5. In the z-domain, a realization of the IIR filter as a difference equation in a DSP is possible. To analyze signals in general, the notion of orthogonality is introduced in section 5.1.2. An analysis method of LTI systems based on their input and output signals is presented in section 5.4. The idea behind the algorithms presented is: measure signals at the input and output of the LTI system, estimate the impulse response from them, and use them to determine the transfer function of the system.

If the transfer function is to be determined from a given system and thus the system is to be characterized, a method for FIR filters is presented in the section 3.4 using the Fourier transform for discrete-time signals. An analysis of IIR filters is performed in discrete-time via the difference equation, which is presented in 4.3.

For further reading on all the topics presented, we refer to [1] as throughout the script.

7 Solutions to the Exercises

7.1 Z-Transformation

7.1.1 Task 1

a

$$\frac{z^2+2z+1}{1+\frac{1}{3}z^{-1}} = \left(z^2+2z+1\right)\cdot\frac{1}{1-\left(-\frac{1}{3}\right)z^{-1}}$$

Multiplication in the z-domain becomes a convolution in the time-domain. The numerator becomes a sequence of delta pulses that simply realize time shifts. The reverse transformation of the denominator is looked up in the collection of formulas:

$$\left(\delta\left(n+2\right)+2\delta\left(n+1\right)+\delta\left(n\right)\right)*\left(\left(-\frac{1}{3}\right)^n\varepsilon(n)\right)$$

Resolving the convolution, we get the following solution:

$$\left(-\frac{1}{3}\right)^{n+2}\varepsilon\left(n+2\right)+2\left(-\frac{1}{3}\right)^{n+1}\varepsilon\left(n+1\right)+\left(-\frac{1}{3}\right)^{n}\varepsilon\left(n\right)$$

b) Before backtransforming, it is always worth checking whether terms can be truncated:

$$\frac{z+1}{z^2-1}=\frac{z+1}{(z+1)\cdot(z-1)}=\frac{1}{z-1}$$

Then, all positive powers of the denominator must be converted to negative powers so that the solutions of the collection of formulas can be used. For this one expands the fraction suitably:

$$\frac{1}{z-1}\cdot\frac{z^{-1}}{z^{-1}}=\frac{z^{-1}}{1-z^{-1}}$$

The subsequent back transformation is again done using the collection of formulas:

$$\delta\left(n-1\right)\cdot\left(1^n\cdot\varepsilon(n)\right)=\varepsilon\left(n-1\right)$$

Here $1^n=1$ and the convolution with the delta function shifts the solution one unit to the right.

c) The denominator of $\frac{1}{1-6z^{-1}+9z^{-2}}$ corresponds to a binomial formula:

$$\frac{1}{1-6z^{-1}+9z^{-2}}=\frac{1}{\left(1-3z^{-1}\right)^2}\ .$$

By suitable transformation, the z-transform is obtained from the table in the collection of formulas:

$$\frac{1}{\left(1-3z^{-1}\right)^2}=\frac{1}{3}z\frac{3z^{-1}}{\left(1-3z^{-1}\right)^2}\ .$$

The transformation to the time-domain goes as follows:

$$\frac{1}{3}\delta\left(n+1\right)*\left(n\cdot 3^n\cdot\varepsilon(n)\right)=\frac{1}{3}\cdot\left(n+1\right)\cdot 3^{(n+1)}\cdot\varepsilon\left(n+1\right)\ .$$

d) The denominator of $\frac{1}{1-3z^{-1}+4z^{-2}}$ does not correspond to any binomial formula. By suitable transformation one obtains the z-transform from the table of the collection of formulas:

$$\frac{1}{1-3z^{-1}+4z^{-2}} = \frac{1}{b\sin\left(2\pi F\right)}z \cdot \frac{b\sin\left(2\pi F\right)\cdot z^{-1}}{1-2b\sin\left(2\pi F\right)\cdot z^{-1}+b^2z^{-2}} \ .$$

In the denominator, proceed sequentially as follows:

$$\begin{aligned} b^2 = 4 \quad &\rightarrow \quad b = 2 \\ 2b\sin\left(2\pi F\right) = 3 \quad &\rightarrow \quad \sin\left(2\pi F\right) = 0,75 \\ &\rightarrow \quad F = \frac{1}{2\pi}\arcsin(0,75) = 0,135 \ . \end{aligned}$$

From this follows the z-transform:

$$\frac{2}{3}z \cdot \frac{b\sin\left(2\pi F\right)\cdot z^{-1}}{1-2b\sin\left(2\pi F\right)\cdot z^{-1}+b^2z^{-2}}$$

and the time domain representation:

$$\frac{2}{3}\delta\left(n+1\right) * \left(2^n\cdot\sin\left(2\pi 0,135n\right)\cdot\varepsilon\left(n\right)\right) \ .$$

If one still carries out the convolution with the delta impulse one obtains:

$$\frac{2}{3}2^{n+1}\cdot\sin\left(2\pi 0,135(n+1)\right)\cdot\varepsilon\left(n+1\right) \ .$$

7.1.2 Task 2

a) The sketch can be seen in figure 42. The corresponding function values are in table 4.

Table 4: Value table of function $x(n) = (0,7)^n \cdot \varepsilon(n)$.

n	x(n)
-2	0
-1	0
0	1
1	0.7
2	0.49
3	0.343
4	0.240
5	0.168

b) The sketch can be seen in figure 43. The corresponding function values are in table 5.

c) The sketch can be seen in figure 44. The corresponding function values are in table 6.

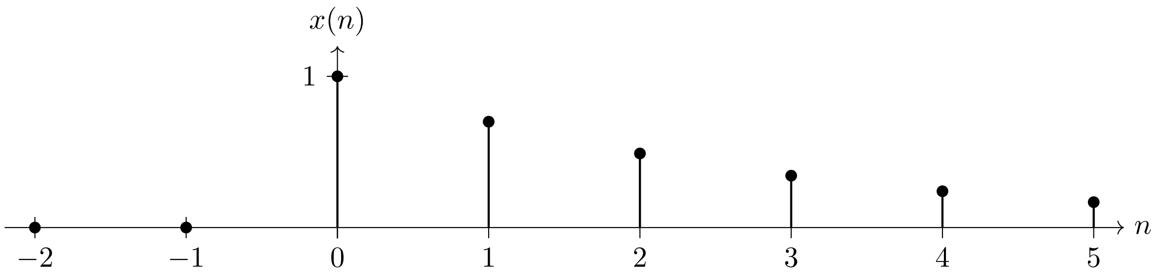


Figure 42: Solution to task 2 a).

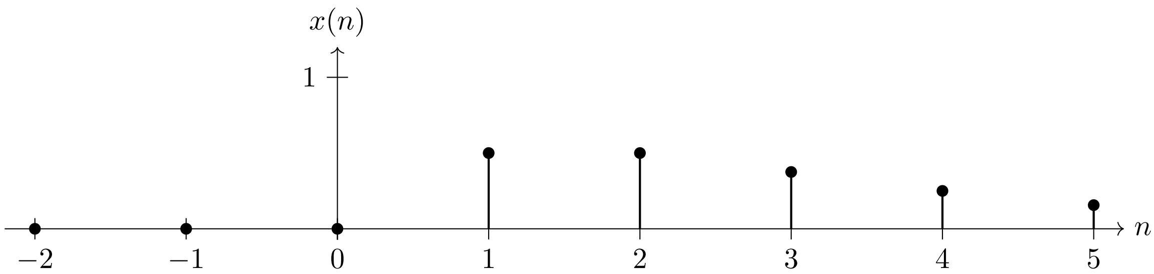


Figure 43: Solution to task 2 b).

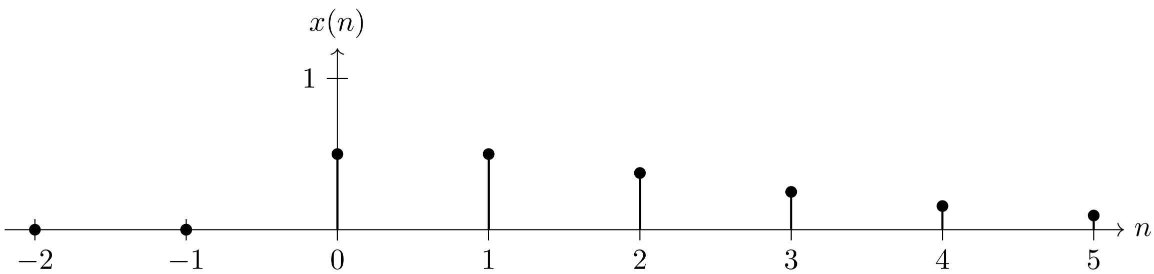


Figure 44: Solution to task 2 c).

Table 5: Value table of function $x(n) = n\,(0,5)^n \cdot \varepsilon(n)$.

n	x(n)
-2	0
-1	0
0	0
1	0.5
2	0.5
3	0.375
4	0.250
5	0,156

Table 6: Value table of function $x(n) = \delta\,(n+1) * (n\,(0,5)^n \cdot \varepsilon(n))$.

n	x(n)
-2	0
-1	0
0	0,5
1	0.5
2	0.375
3	0.250
4	0.156
5	0,094

8 Formulary

Table 7: Common Theorems

Euler's theorem :	e^{jx}	$=$	$\cos x + j\sin x$
	$e^{jx} - e^{-jx}$	$=$	$2j\sin x$
	$e^{jx} + e^{-jx}$	$=$	$2\cos x$
Geometric series	$\sum_{n=0}^{\infty} c \cdot a^n$	$=$	$\frac{c}{1-a}$, für $\lvert a\rvert < 1$
n-th partial sum	$\sum_{n=0}^{N} c \cdot a^n$	$=$	$c\frac{1-a^{N+1}}{1-a}$
Sine and Cosine	1	$=$	$\cos^2 x + \sin^2 x$
	$\cos^2 x$	$=$	$\frac{1}{2}\left(1+\cos(2x)\right)$
	$\sin(x+y)$	$=$	$\sin x\cos y + \sin y\cos x$
	$\cos(x+y)$	$=$	$\cos x\cos y - \sin x\sin y$
	$\sin x + \sin y$	$=$	$2 \cdot \sin\frac{x+y}{2} \cdot \cos\frac{x-y}{2}$
	$\cos x + \cos y$	$=$	$2 \cdot \cos\frac{x+y}{2} \cdot \cos\frac{x-y}{2}$

Table 8: Antiderivatives

$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt$	$\approx$	$\frac{\pi}{2} - \frac{\cos x}{x+2/\pi} - \sin(x) \cdot e^{-x}$
with absolute error $< 0,04$ for $x \geq 0$		
$y = \mathrm{erfc}(x)$	$\approx$	$\frac{e^{-x^2}}{\sqrt{\pi}x + e^{-0,6x}}$
with relative error $< 0,02$ for $x \geq 0$		
$\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\tau-\mu)^2}{2\sigma^2}} d\tau$	$=$	$\frac{1}{2}\mathrm{erfc}\left(\frac{\mu-t}{\sqrt{2\sigma^2}}\right)$
Symmetry properties: $\mathrm{erfc}(-x)$	$=$	$2 - \mathrm{erfc}(x)$
Inverse function: $x = \mathrm{erfc}^{-1}(y)$	$\approx$	$1,1 \cdot \sqrt{-\log y} - 0,43 \cdot (-\log y)^{\frac{1}{6}}$
with absolute error $< 0,04$ for $0 \leq x \leq 6$		

Table 9: Time-continuous Elementary Signals

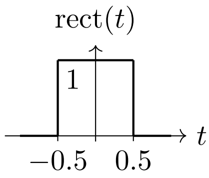
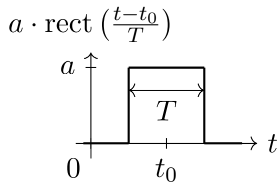
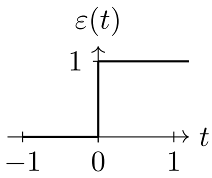
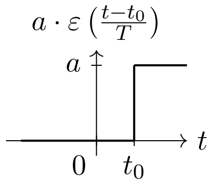
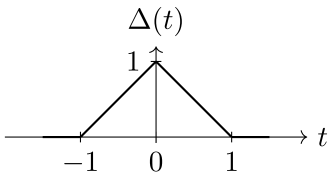
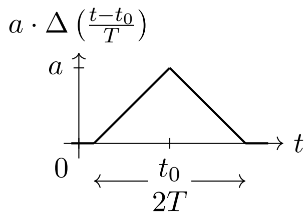
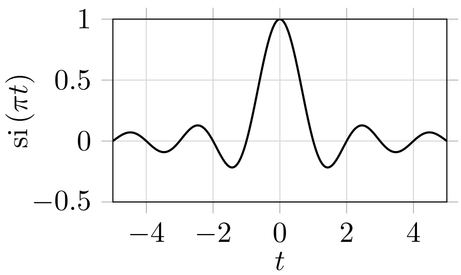
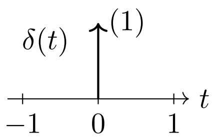
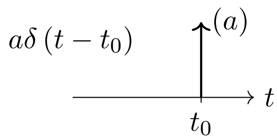
Rectangle	$\mathrm{rect}(t) = \begin{cases} 1 & \lvert t\rvert \le 0,5 \\ 0 & \text{other} \end{cases}$		
Step	$\varepsilon(t) = \begin{cases} 1 & t \ge 0 \\ 0 & \text{other} \end{cases}$		
Triangle	$\Delta(t) = \begin{cases} 1 - \lvert t\rvert & \lvert t\rvert \le 1 \\ 0 & \text{other} \end{cases}$		
si	$\mathrm{si}(\pi t) = \begin{cases} 1 & t = 0 \\ \frac{\sin(\pi t)}{\pi t} & \text{other} \end{cases}$		
Signum function	$\mathrm{sign}(t) = \begin{cases} 1 & t \ge 0 \\ -1 & \text{other} \end{cases} = 2\varepsilon(t) - 1$		
Dirac function	$\delta(t) = \lim\limits_{T \to 0} \frac{1}{T}\mathrm{rect}\left(\frac{t}{T}\right)$		

Table 10: Convolution Theorems

Linearity	$\begin{aligned} a \cdot y_1(t) + b \cdot y_2(t) &= a \cdot \mathrm{Tr}\{x_1(t)\} + b \cdot \mathrm{Tr}\{x_2(t)\} \\ &= \mathrm{Tr}\{a \cdot x_1(t) + b \cdot x_2(t)\} \end{aligned}$
Time invariance	$y(t - t_0) = \mathrm{Tr}\{x(t - t_0)\}$
Convolution integral	$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t - \tau) d\tau = x(t) * h(t)$
Convolution with the Dirac function	$s(t) * \delta(t - t_0) = s(t - t_0)$
Multiplication with the Dirac function	$s(t) \cdot \delta(t - t_0) = s(t - t_0) \cdot \delta(t - t_0)$
Stretching of the Dirac function	$\delta(bt) = \frac{1}{\lvert b \rvert} \delta(t)$, for $b \neq 0$
Commutative law of convolution	$y(t) = x(t) * h(t) = h(t) * x(t)$
Associative law of convolution	$\begin{aligned} y(t) &= x_1(t) * x_2(t) * x_3(t) \\ &= (x_1(t) * x_2(t)) * x_3(t) \\ &= x_1(t) * (x_2(t) * x_3(t)) \end{aligned}$
Distributive law of convolution	$\begin{aligned} y(t) &= (x_1(t) + x_2(t)) * x_3(t) \\ &= x_1(t) * x_3(t) + x_2(t) * x_3(t) \end{aligned}$
Causal:	$s(t) = 0$, for $t < 0$
Anti-causal:	$s(t) = 0$, for $t > 0$
Non-causal:	$s(t) \neq 0$, for $t > 0$ and $t < 0$
Stability	$\int_{-\infty}^{\infty} \lvert h(t) \rvert dt \leq B, B \in \mathbb{R}, B < \infty$

Table 11: Examples for Convolution

$\mathrm{rect}\left(\frac{t}{T}\right) * \mathrm{rect}\left(\frac{t}{T}\right)$	$=$	$T \cdot \Delta\left(\frac{t}{T}\right)$
$\mathrm{rect}\left(\frac{t}{T}\right) * T\Delta\left(\frac{t}{T}\right)$	$=$	$\begin{cases} \frac{1}{2}t^2 + \frac{3T}{2}t + \frac{9}{8}T^2 & -\frac{3T}{2} \le t < -\frac{T}{2} \\ -t^2 + \frac{3}{4}T^2 & -\frac{T}{2} \le t < \frac{T}{2} \\ \frac{1}{2}t^2 - \frac{3T}{2}t + \frac{9}{8}T^2 & \frac{T}{2} \le t < \frac{3T}{2} \\ 0 & \text{other} \end{cases}$
$\mathrm{si}\left(\pi t\right) * \mathrm{si}\left(\pi t\right)$	$=$	$\mathrm{si}\left(\pi t\right)$
$\delta\left(t - t_1\right) * \delta\left(t - t_2\right)$	$=$	$\delta\left(t - t_1 - t_2\right)$
$s(t) * \varepsilon(t)$	$=$	$\int_{-\infty}^{t} s(\tau) d\tau$
$s(t) * \mathrm{rect}\left(t\right)$	$=$	$\int_{t-0,5}^{t+0,5} s(\tau) d\tau$
$e^{-\pi t^2} * e^{-\pi t^2}$	$=$	$\frac{1}{\sqrt{2}} e^{-\pi t^2/2}$
$\frac{1}{T}\varepsilon(t)e^{-t/T} * \frac{1}{T}\varepsilon(t)e^{-t/T}$	$=$	$\frac{1}{T^2}\varepsilon(t) \cdot t \cdot e^{-t/T}$

Table 12: Fourier-Transformation Theorems

Fourier-Transformation	$x(t) \circ\!\!-\!\!\bullet X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft} dt$
Inverse Fourier-transformation	$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{j2\pi ft} dt$
Convolution and multiplication	$x(t) * y(t) \circ\!\!-\!\!\bullet X(f) \cdot Y(f)$ $x(t) \cdot y(t) \circ\!\!-\!\!\bullet X(f) * Y(f)$
Superposition	$a \cdot x_1(t) + b \cdot x_2(t) \circ\!\!-\!\!\bullet a \cdot X_1(f) + b \cdot X_2(f)$
Similarity	$x(bt) \circ\!\!-\!\!\bullet \frac{1}{\lvert b \rvert} X\left(\frac{f}{b}\right), b \neq 0$
Differentiation	$\frac{d}{dt} x(t) \circ\!\!-\!\!\bullet X(f) \cdot j2\pi f$
Integration	$\int_{-\infty}^{t} x(\tau) d\tau \circ\!\!-\!\!\bullet \frac{X(f)}{j2\pi f} + X(0) \cdot \delta(f)$
Time shift	$x(t - t_0) \circ\!\!-\!\!\bullet X(f) \cdot e^{-j2\pi f t_0}$
Frequency shift	$x(t) \cdot e^{j2\pi Ft} \circ\!\!-\!\!\bullet X(f - F)$
Surface	$\int_{-\infty}^{\infty} x(t) dt \circ\!\!-\!\!\bullet X(0)$ $x(0) \circ\!\!-\!\!\bullet \int_{-\infty}^{\infty} X(f) df$
Parsevals theorem	$E_x = \int_{-\infty}^{\infty} \lvert x(t) \rvert^2 dt = \int_{-\infty}^{\infty} \lvert X(f) \rvert^2 df$
Symmetries	$X(t) \circ\!\!-\!\!\bullet x(-f)$ $x(-t) \circ\!\!-\!\!\bullet X(-f)$
Even and odd	$x(t) = x_u(t) + x_g(t) \circ\!\!-\!\!\bullet X(f) = \mathrm{Re}\{X(f)\} + j\mathrm{Im}\{X(f)\}$ $x_g(t) = \frac{1}{2}(x(t) + x^*(-t)) \circ\!\!-\!\!\bullet \mathrm{Re}\{X(f)\}$ $x_u(t) = \frac{1}{2}(x(t) - x^*(-t)) \circ\!\!-\!\!\bullet j\mathrm{Im}\{X(f)\}$
Magnitude of a spectrum	$\lvert X(f) \rvert = \sqrt{\mathrm{Re}\{X(f)\}^2 + \mathrm{Im}\{X(f)\}^2}$
Phase of a spectrum	$\varphi(f) = \arctan\left(\frac{\mathrm{Im}\{X(f)\}}{\mathrm{Re}\{X(f)\}}\right) + k \cdot \pi, k = \begin{cases} 1 & \mathrm{Re}\{X(f)\} < 0 \\ 0 & \text{other} \end{cases}$
Phase delay	$t_p(f) = -\frac{\varphi(f)}{2\pi}$
Group delay	$t_g(f) = -\frac{1}{2\pi} \frac{d\varphi(f)}{df}$
Linear phase system	$\varphi(f) = c \cdot f$

Table 13: Examples for the Fourier-Transformation (f_g corresponds to the cutoff frequency for low-pass filters)

$\mathrm{rect}(t)$	$\circ\!\!-\!\!\bullet$	$\mathrm{si}\,(\pi f)$	$f_g = 0,4430$
$\mathrm{si}(\pi t)$	$\circ\!\!-\!\!\bullet$	$\mathrm{rect}(f)$	$f_g = 0,5$
$\Delta(t)$	$\circ\!\!-\!\!\bullet$	$\mathrm{si}^2\,(\pi f)$	$f_g = 0,3189$
$\delta(t)$	$\circ\!\!-\!\!\bullet$	1	
1	$\circ\!\!-\!\!\bullet$	$\delta(f)$	
$\sum_{n=-\infty}^{\infty} \delta\,(t-nT)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(f-\frac{k}{T}\right)$	
$e^{-\pi t^2}$	$\circ\!\!-\!\!\bullet$	$e^{-\pi f^2}$	$f_g = 0,3321$
$\frac{1}{T}\varepsilon(t)e^{-t/T}$	$\circ\!\!-\!\!\bullet$	$\frac{1}{1+j2\pi fT}$	$f_g = \frac{1}{2\pi T}$
$\frac{1}{2T}e^{-\lvert t\rvert/T}$	$\circ\!\!-\!\!\bullet$	$\frac{1}{1+(2\pi fT)^2}$	$f_g = \frac{\sqrt{\sqrt{2}-1}}{2\pi T} = \frac{0{,}6436}{2\pi T}$
$\frac{1}{2T}\mathrm{sign}(t)e^{-\lvert t\rvert/T}$	$\circ\!\!-\!\!\bullet$	$-j\frac{2\pi fT}{1+(2\pi fT)^2}$	
$\varepsilon(t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)$	
$\mathrm{sign}(t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{j\pi f}$	
$\cos\,(2\pi Ft)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{2}\left(\delta(f+F)+\delta(f-F)\right)$	
$\sin\,(2\pi Ft)$	$\circ\!\!-\!\!\bullet$	$\frac{j}{2}\left(\delta(f+F)-\delta(f-F)\right)$	
$\mathrm{rect}(t)\cdot\frac{1}{2}\left(1+\cos\,(2\pi f)\right)$	$\circ\!\!-\!\!\bullet$	$\frac{\mathrm{si}(\pi f)}{2(1-f^2)}$	$f_g = 0,7203$

Table 14: Theorems for Sampling

Sampling in time-domain	$x_a(t) = x(t)\cdot\sum_{n=-\infty}^{\infty}\delta\,(t-nT)$
Sampling in frequency-domain	$X_a(f) = X(f) * \frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$
Sampling rate	$R = \frac{1}{T}$
Sampling theorem	$\frac{1}{T} \geq 2f_g$

Table 15: Discrete-Time Elementary Signals

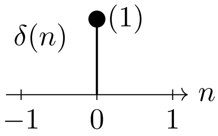
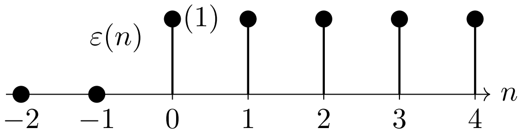
Delta impulse	$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{other} \end{cases}$	
Step	$\varepsilon(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{other} \end{cases}$	

Table 16: Theorems for Time-Discrete, Linear Signals and Systems

Linearity	$a \cdot y_1(n) + b \cdot y_2(n) = a \cdot \mathrm{Tr}\{x_1(n)\} + b \cdot \mathrm{Tr}\{x_2(n)\} = \mathrm{Tr}\{a \cdot x_1(n) + b \cdot x_2(n)\}$
Time invariance	$y\,(n - n_0) = \mathrm{Tr}\{x\,(n - n_0)\}$
Discrete convolution	$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m) \cdot h(n-m)$
Fourier-transformation	$x_a(t) = \sum_{n=-\infty}^{\infty} x\,(nT) \cdot \delta\,(t - nT)$ $\circ\!\!-\!\!\bullet X_a(f) = \sum_{n=-\infty}^{\infty} x\,(nT) \cdot e^{-j2\pi fnT} = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(f - \frac{k}{T}\right)$
Inverse Fourier-transformation	$x(n) = T \cdot \int_{-\frac{1}{2T}}^{\frac{1}{2T}} X_a(f) \cdot e^{j2\pi fnT} df$
Convolution and multiplication	$y(n) = x(n) * h(n) \circ\!\!-\!\!\bullet Y(f) = X(f) \cdot H(f)$
Causality	$s(n) = 0$, for $n < 0$
Anti causality	$s(n) = 0$, for $n > 0$
non causality	$s(n) \neq 0$, for $n > 0$ and $n < 0$
Convolution with the Delta-impulse	$s(n) * \delta\,(n - n_0) = s\,(n - n_0)$

Table 17: Theorems for the Discrete Fourier Transform (DFT)

Discrete Fourier-transformation	$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi nk/K}$, $0 \leq k < K$
Inverse discrete Fourier-transformation	$x(n) = \frac{1}{K} \sum_{k=0}^{K-1} X(k) \cdot e^{j2\pi nk/K}$, $0 \leq n < N$
Symmetry	$X(k) = X^*(K - k)$
Frequency resolution	$\Delta f = \frac{1}{KT}$ with the sampling time T and the transformation len

Table 18: Examples of the Fourier Transformation of Time-Discrete Signals

Time-Domain		Frequency-Domain
$\delta(n-n_0)$	$\circ\!\!-\!\!\bullet$	$e^{-j2\pi f n_0 T}$
$\sum_{k=0}^{\infty} a^k \delta(n-k) = a^n \varepsilon(n)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{1-a\cdot e^{-j2\pi fT}}$, for $\lvert a\rvert < 1$
$\varepsilon(n)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{1-e^{-j2\pi fT}} + \frac{1}{2}\sum_{k=-\infty}^{\infty} \delta\left(f-\frac{k}{T}\right)$
$\cos(2\pi Fn)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{2}\sum_{l=-\infty}^{\infty}\left(\delta\left(f-F-\frac{l}{T}\right)+\delta\left(f+F-\frac{l}{T}\right)\right)$
$\sin(2\pi Fn)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{2j}\sum_{l=-\infty}^{\infty}\left(\delta\left(f-F-\frac{l}{T}\right)-\delta\left(f+F-\frac{l}{T}\right)\right)$
$\delta(n+n_0)+\delta(n-n_0)$	$\circ\!\!-\!\!\bullet$	$2\cos(2\pi f n_0 T)$
$\delta(n+n_0)-\delta(n-n_0)$	$\circ\!\!-\!\!\bullet$	$2j\sin(2\pi f n_0 T)$

Table 19: Theorems for the z-transformation

z-transformation	$X(z) = \sum_{n=-\infty}^{\infty} x(nT)\cdot z^{-n}$, mit $z = e^{\sigma T + j2\pi fT}$
Stability in time-domain	$\sum_{-\infty}^{\infty} \lvert s(n)\rvert < \infty$
Stability in z-domain	All poles of the z-transform must lie within the unit circle.
Attenuation theorem	$a^{-n}\cdot x(n) \circ\!\!-\!\!\bullet X(a\cdot z)$
Derivation of the image function	$n\cdot x(n) \circ\!\!-\!\!\bullet -z\cdot\frac{dX(z)}{dz}$
Bilinear transform	$z \approx \frac{1+sT/2}{1-sT/2}$ $s \approx \frac{2}{T}\frac{z-1}{z+1}$
Frequency warping for f_0:	$s \approx c\frac{z-1}{z+1}$ $c = 2\pi f_0 \cot(2\pi f_0 T/2) = \frac{2\pi f_0}{\tan(\pi f_0 T)}$

Table 20: Examples for the z-Transformation

Time-Domain		z-transform	convergence area
$\delta\left(n-n_0\right)$	$\circ\!\!-\!\!\bullet$	z^{-n_0}	all z
$b^n\varepsilon(n)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{1-bz^{-1}}$	$\lvert z\rvert > \lvert b\rvert$
$nb^n\varepsilon(n)$	$\circ\!\!-\!\!\bullet$	$\frac{bz^{-1}}{(1-bz^{-1})^2}$	$\lvert z\rvert > \lvert b\rvert$
$b^n \cdot \cos\left(2\pi Fn\right) \cdot \varepsilon(n)$	$\circ\!\!-\!\!\bullet$	$\frac{1-b\cos(2\pi F)\cdot z^{-1}}{1-2b\cos(2\pi F)\cdot z^{-1}+b^2z^{-2}}$	$\lvert z\rvert > \lvert b\rvert$
$b^n \cdot \sin\left(2\pi Fn\right) \cdot \varepsilon(n)$	$\circ\!\!-\!\!\bullet$	$\frac{b\sin(2\pi F)\cdot z^{-1}}{1-2b\sin(2\pi F)\cdot z^{-1}+b^2z^{-2}}$	$\lvert z\rvert > \lvert b\rvert$
$c_1 \cdot b^n \cdot \cos\left(2\pi F_1 n\right) \cdot \varepsilon(n)$ $+c_2 \cdot b^n \cdot \sin\left(2\pi F_2 n\right) \cdot \varepsilon(n)$	$\circ\!\!-\!\!\bullet$	$\frac{A+Bz^{-1}}{1-2Cz^{-1}+Dz^{-2}}$	$\lvert z\rvert > b$

$$b=\sqrt{D}, F_1=\frac{1}{2\pi}\arccos\frac{C}{b}, F_2=\frac{1}{2\pi}\arcsin\frac{C}{b}, c_1=A, c_2=\frac{B}{C}+c_1$$

Table 21: Theorems on the Correlation Functions

Energy signal continuous in time	$E_s = \int_{-\infty}^{\infty} s^2(t)dt < \infty$
Energy signal discrete in time	$E_s = \sum_{n=-\infty}^{\infty} s^2(n) < \infty$
Power signal continuous in timeh	$E_s = \int_{-\infty}^{\infty} s^2(t)dt \to \infty$
Power signal discrete in time	$E_s = \sum_{n=-\infty}^{\infty} s^2(n) \to \infty$
Normalized correlation coefficient	$p_{sg}^E(\tau) = \frac{\int_{-\infty}^{\infty} s(t)\cdot g(t+\tau)dt}{\sqrt{E_s\cdot E_g}}$
Cross-correlation function	$\varphi_{sg}^E(\tau) = \int_{-\infty}^{\infty} s(t)\cdot g(t+\tau)dt = s(-\tau) * g(\tau)$ $\varphi_{sg}^E(m) = \sum_{n=-\infty}^{\infty} s(n)\cdot g(n+m) = s(-m) * g(m)$
Auto-correlation function	$\varphi_{ss}^E(\tau) = \int_{-\infty}^{\infty} s(t)\cdot s(t+\tau) = s(-\tau) * s(\tau)$ $\varphi_{ss}^E(m) = \sum_{n=-\infty}^{\infty} s(n)\cdot s(n+m) = s(-m) * s(m)$
Symmetries	$\varphi_{sg}^E(\tau) = \varphi_{gs}^E(-\tau)$ und $\varphi_{ss}^E(\tau) = \varphi_{ss}^E(-\tau)$ $\varphi_{sg}^E(m) = \varphi_{gs}^E(-m)$ und $\varphi_{ss}^E(m) = \varphi_{ss}^E(-m)$
Orthogonality	$\varphi_{sg}^E(0) = \int_{-\infty}^{\infty} s(t)\cdot g(t)dt = 0$ $\varphi_{sg}^E(0) = \sum_{n=-\infty}^{\infty} s(n)\cdot g(n) = 0$ $\varphi_{sg}^E(0) \int_{-\infty}^{\infty} S^*(f)\cdot G(f)df = 0$
Fourier-transform	$\Phi_{sg}^E(f) = S^*(f)\cdot G(f)$ $\Phi_{ss}^E(f) = S^*(f)\cdot S(f) = \|S(f)\|^2$ $\Phi_{ss}^E(f) = \int_{-\infty}^{\infty} \varphi_{ss}^E(\tau)e^{-j2\pi ft}d\tau$
Unique maximum of the auto-correlation function corresponds to the signal energy	$E_s = \int_{-\infty}^{\infty} s^2(t)dt = \varphi_{ss}^E(0) \geq \left\|\varphi_{ss}^E(\tau)\right\|$
	$E_s = \sum_{-\infty}^{\infty} s^2(n) = \varphi_{ss}^E(0) \geq \left\|\varphi_{ss}^E(m)\right\|$ $E_s = \int_{-\infty}^{\infty} \Phi_{ss}^E(f)df$
Transmission via an LTI-System: $g(t) = s(t) * h(t)$	
Wiener-Lee-relation	$\varphi_{gg}^E(\tau) = \varphi_{ss}^E(\tau) * \varphi_{hh}^E(\tau)$ $\varphi_{gg}^E(m) = \varphi_{ss}^E(m) * \varphi_{hh}^E(m)$ $\Phi_{gg}^E(f) = \Phi_{ss}^E(f)\cdot \Phi_{hh}^E(f)$
Cross-correlation between $s(t)$ and $g(t)$	$\varphi_{sg}^E(\tau) = \varphi_{ss}^E(\tau) * h(\tau)$
	$\varphi_{sg}^E(m) = \varphi_{ss}^E(m) * h(m)$ $\Phi_{sg}^E(f) = \Phi_{ss}^E(f)\cdot H(f)$
Wiener filter	$H(f) = \frac{\Phi_{xx}^E(f)}{\Phi_{xx}^E(f)+\|D(f)\|^2}$

References

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