

Low Pass Filter

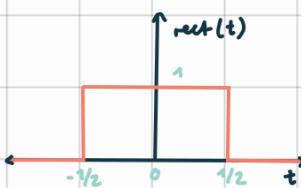
Mass reacts slowly to acceleration

How to describe the PWM signal:

Elementary Signals:

Rectangular Function

$$\text{rect}(t) = \begin{cases} 1, & \text{for } |t| \leq 0.5 \\ 0, & \text{else} \end{cases}$$



Models Switching Process

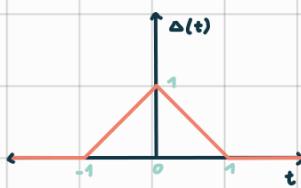
Step Function

$$\varepsilon(t) = \begin{cases} 1, & \text{for } t \geq 0 \\ 0, & \text{else} \end{cases}$$



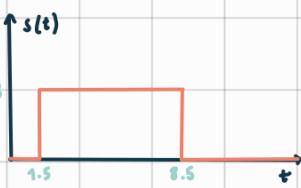
Triangular Function

$$\Delta(t) = \begin{cases} 1 - |t|, & \text{for } |t| \leq 1 \\ 0, & \text{else} \end{cases}$$



Amplitude, Shift, Expansion

$$s(t) = 3 \cdot \text{rect}\left(\frac{t-5}{7}\right)$$



$$\text{LB: } \frac{t-5}{7} = -\frac{1}{2} \Rightarrow t = 1.5, \text{ UB: } \frac{t-5}{7} = \frac{1}{2} \Rightarrow t = 8.5$$

Convolution

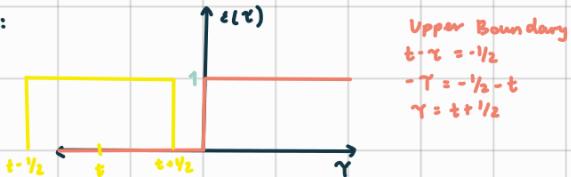
↔

$$n(t) \xrightarrow{h(t)} y(t) \quad y(t) = n(t) * h(t) = \int_{-\infty}^{\infty} n(\tau) \cdot h(t-\tau) d\tau$$

$n(t) = \varepsilon(t)$  switch on at  $t=0$     $h(t) = \text{rect}(t)$  short time integrator / low pass

$$y(t) = \int_{-\infty}^{\infty} \varepsilon(\tau) \cdot \text{rect}(t-\tau) d\tau = \int_{-\infty}^{\infty} \varepsilon(\tau) \cdot \text{rect}(t-\tau) d\tau$$

Case 1:



Upper Boundary  
 $t-\tau = -1/2$   
 $-\tau = -1/2 - t$   
 $\tau = t + 1/2$

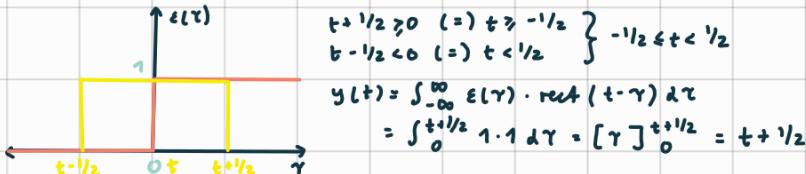
Lower Boundary  
 $t-\tau = 1/2$   
 $\tau = t - 1/2$

$t + 1/2 < 0 \Rightarrow t < -1/2 \Rightarrow y(t) = 0$  because at least 1 signal = 0  
or: Both signals don't overlap

$$y(t) = \begin{cases} 0, & \text{for } t \leq -1/2 \\ t + 1/2, & \text{else} \\ 1, & \text{for } t \geq 1/2 \end{cases}$$



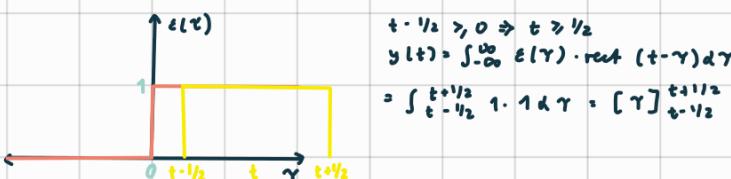
Case 2:



$$t + 1/2 \geq 0 \quad (= t \geq -1/2) \\ t - 1/2 \leq 0 \quad (= t \leq 1/2) \quad \Rightarrow -1/2 \leq t \leq 1/2$$

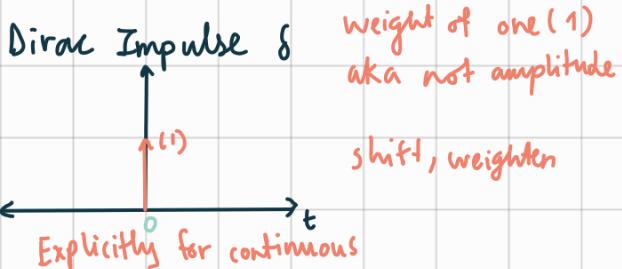
$$y(t) = \int_{-\infty}^{\infty} \varepsilon(\tau) \cdot \text{rect}(t-\tau) d\tau \\ = \int_0^{t+1/2} 1 \cdot 1 d\tau = [\tau]_0^{t+1/2} = t + 1/2$$

Case 3:



$$t - 1/2 > 0 \Rightarrow t > 1/2 \\ y(t) = \int_{-\infty}^{\infty} \varepsilon(\tau) \cdot \text{rect}(t-\tau) d\tau$$

$$= \int_{t-1/2}^{t+1/2} 1 \cdot 1 d\tau = [\tau]_{t-1/2}^{t+1/2} = t + 1/2 - t + 1/2 = 1$$



Neutral element of convolution

$$y(t) = n(t) * \delta(t) = n(t)$$

Convolution with shifted Dirac

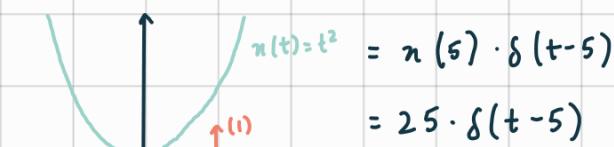
$$y(t) = n(t) * \delta(t-5) = n(t-5)$$

Ex:  $\epsilon(t) * \delta(t-5)$



Multiplication with Dirac

$$n(t) = t^2, y(t) = n(t) * \delta(t-5)$$



Used for AD conversion

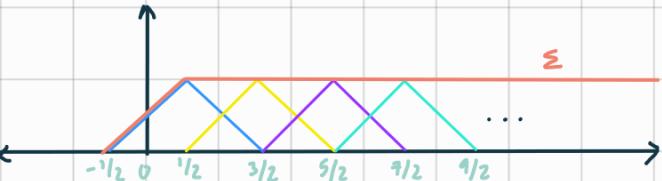
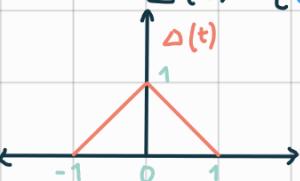
$$y(t) = h(t) * \epsilon(t)$$

$$= \text{rect}(t) * \text{rect}(t) * \sum_{n=0}^{\infty} \delta\left(t - \frac{2n+1}{2}\right)$$

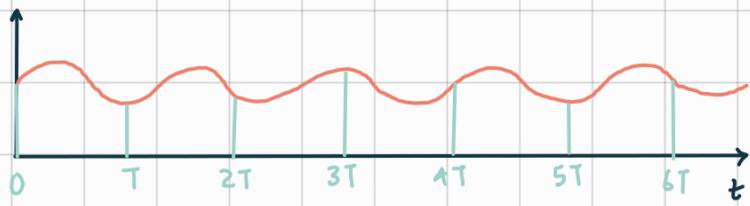
$$\text{rect}(t) * \text{rect}(t) = T \cdot \Delta\left(\frac{t}{T}\right), \text{ where } T=1$$

$$y(t) = \Delta(t) * \sum_{n=0}^{\infty} \delta\left(t - \frac{2n+1}{2}\right)$$

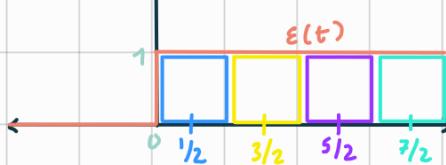
$$= \Delta(t) * [\delta(t-\frac{1}{2}) + \delta(t-\frac{3}{2}) + \delta(t-\frac{5}{2}) + \delta(t-\frac{7}{2}) \dots]$$



Sampling



Old Example



$$\begin{aligned} \epsilon(t) &= \text{rect}(t-\frac{1}{2}) + \text{rect}(t-\frac{3}{2}) + \text{rect}(t-\frac{5}{2}) + \text{rect}(t-\frac{7}{2}) \dots \\ &= \text{rect}(t) * \delta(t-\frac{1}{2}) + \text{rect}(t) * \delta(t-\frac{3}{2}) + \text{rect}(t) * \delta(t-\frac{5}{2}) \dots \end{aligned}$$

Distributive Law of Convolution

$$\begin{aligned} \epsilon(t) &= \text{rect}(t) * [\delta(t-\frac{1}{2}) + \delta(t-\frac{3}{2}) + \delta(t-\frac{5}{2}) \dots] \\ &= \text{rect}(t) * \sum_{n=0}^{\infty} \delta(t - \frac{2n+1}{2}) \end{aligned}$$

$$y(t) = h(t) * n(t), \quad h(t) = \text{rect}(t)$$



$$n(t) = \text{rect}(t) * \delta(t-\frac{1}{2}) - \text{rect}(t) * \delta(t-\frac{3}{2}) \dots$$

$$= \text{rect}(t) * \sum_{n=0}^{\infty} \delta(t - \frac{2n+1}{2}) (-1)^n$$

$$y(t) = \text{rect}(t) * \text{rect}(t) * \sum_{n=0}^{\infty} \delta(t - \frac{2n+1}{2}) (-1)^n$$

$$= \Delta(t) * \sum_{n=0}^{\infty} (\frac{2n+1}{2}) (-1)^n$$



$$n=0 : \Delta(t) \delta(t-\frac{1}{2}) (-1)^0, \quad n=1 : \Delta(t) \delta(t-\frac{3}{2}) (-1)^1$$

$$n=2 : \Delta(t) \delta(t-\frac{5}{2}) (-1)^2, \quad n=3 : \Delta(t) \delta(t-\frac{7}{2}) (-1)^3$$

## Similarity Theorem of the Fourier Transform

$$\text{Ex: } n(t) = 2 \cdot \text{si}(3\pi t) \quad | \quad \text{si}(nt) = \frac{\sin(nt)}{\pi t}, \text{ si}(0) = 1$$



Sketch the spectrum by magnitude and phase:

$$\text{Magnitude: } |X(f)| = \sqrt{\text{Re}(X(f))^2 + \text{Im}(X(f))^2} = \frac{2}{3} \text{ rect}\left(\frac{f}{2}\right)$$

$$\text{Phase: } \varphi(f) = \arctan \frac{\text{Im}(X(f))}{\text{Re}(X(f))} = 0$$

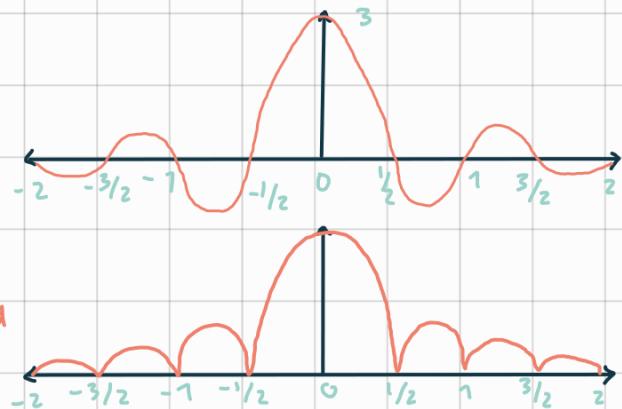
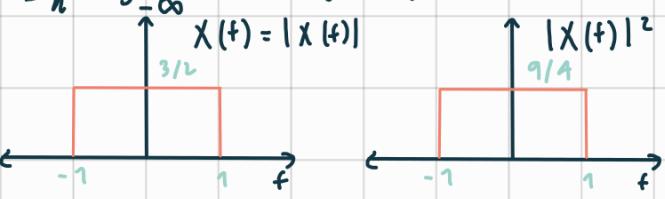
## Parseval Theorem

$$E_n = \int_{-\infty}^{\infty} |n(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

The energy of a signal can be calculated in time and image domain

$$\text{Ex: } n(t) = 3 \text{ si}(2\pi t) \rightarrow \frac{3}{2} \text{ rect}\left(\frac{f}{2}\right) = X(f)$$

$$E_n = \int_{-\infty}^{\infty} |3 \text{ si}(2\pi t)|^2 dt \quad \text{Integration too complicated}$$

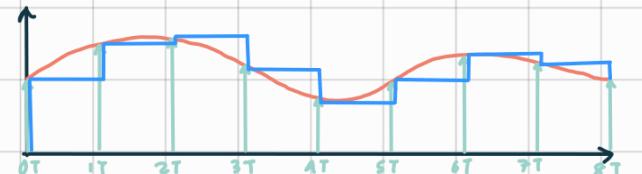


$$E_n = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} \frac{9}{4} \text{ rect}\left(\frac{f}{2}\right) df = \int_{-1}^1 \frac{9}{4} df = \frac{9}{2}$$

## PDF: Power Density Function

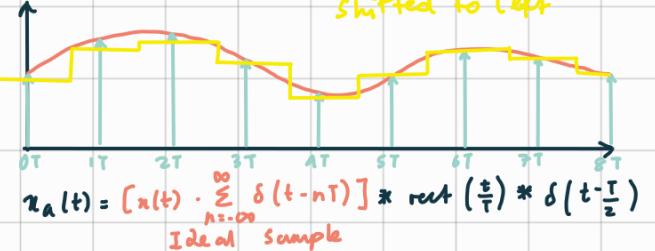
Non-ideal Sample: Sample and Hold

Real Sample and Hold:



$$x_a(t) = [x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT)] * \text{rect}\left(\frac{t}{T}\right) * \delta\left(t - \frac{T}{2}\right)$$

Ideal Sample and Hold:



$$x_a(t) = [x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT)] * \text{rect}\left(\frac{t}{T}\right) * \delta\left(t - \frac{T}{2}\right)$$

Ideal Sample

## Alternative Representation in Frequency Domain

$$X_a(f) = X(f) * \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{T}) \quad \xrightarrow{t_0} a \cdot e^{-2\pi i f t_0}$$

$$X_a(f) = 2 \cdot e^{-2\pi i f \cdot 0} + 1,5 \cdot e^{-2\pi i f \cdot T} + 1,75 \cdot e^{-2\pi i f \cdot 2T} + \dots$$

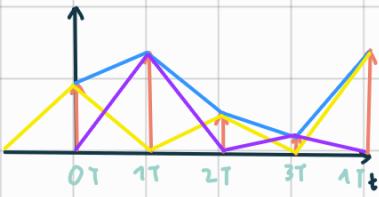
$$\Rightarrow X_a(f) = \sum_{n=-\infty}^{\infty} x_a(nT) \cdot e^{-2\pi i f \cdot nT}$$

## Reconstruction (Interpolation)



- We get a continuous signal via the envelope function
- Envelope functions are smooth
- Low Pass Filters form envelope functions

## Linear Interpolation



## Linear Interpolated Envelope Ideal Interpolation

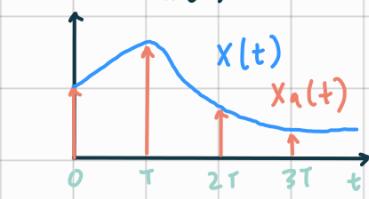
$$y(t) = n_a(t) * \Delta \left( \frac{t}{T} \right)$$



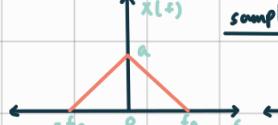
$y(t) = n_a(t) * \sum_{k=-\infty}^{\infty} si(\pi \frac{t-kT}{T})$   
Sum of all si functions leads to the envelope

## Frequency Domain

M155  
Domain:  $X(t) \xrightarrow{\text{Sampling}} x_a(t) = X(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT)$



$\bullet X_a(f) = X(f) * \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{T})$

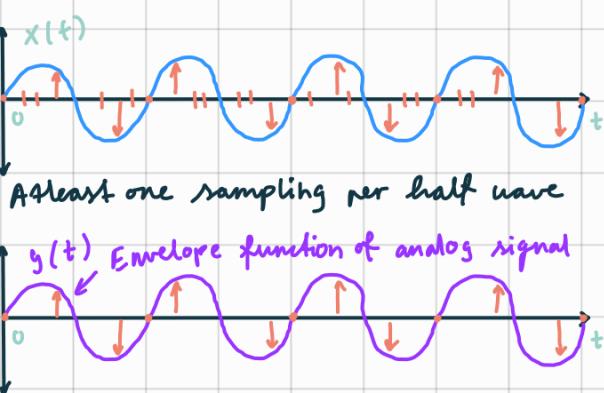


So that the spectra don't overlap  
 $\frac{1}{T} - f_g > f_g \Rightarrow 2f_g \leq \frac{1}{T}$

## Sampling Theorem (Nyquist Theorem)

$f_g \leq \frac{1}{2T}$  with  $f_s = \frac{1}{T}$   
sample rate  
 $f_g \leq \frac{1}{2} f_s \mid 2f_g \leq f_s$

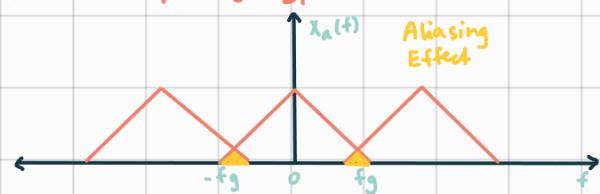
## Sampling Theorem in Time Domain Representation



At least one sampling per half wave

$y(t)$  Envelope function of analog signal

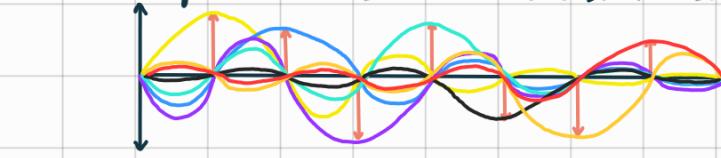
Anti-Example  $f_g > \frac{1}{2T}$ :



## Violation of Sampling Theorem

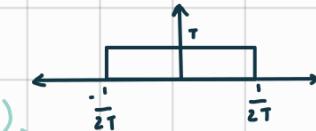


## New Example: Reconstruction via si-Function

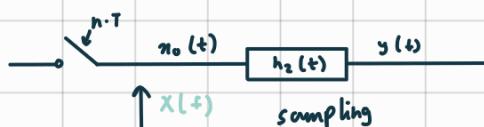


Sum of all shifted si-functions  $\rightarrow$  signal envelope

We know:  $si(\pi \frac{t}{T}) \xrightarrow{\circ} T \cdot rect(T \cdot f)$

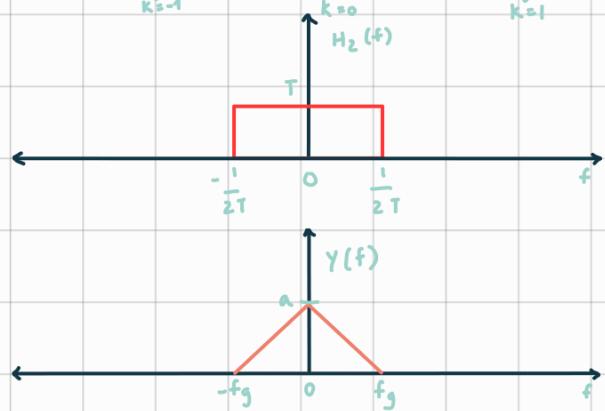


## Reconstruction in frequency-domain via ideal interpolation



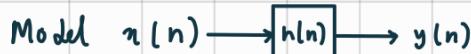
## Multiplication

Sampling Theorem  
 $f_g \leq \frac{1}{2T}$



## Discrete Convolution

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m) \cdot h(n-m)$$



$\approx LS\ I \rightarrow$  Linear Shift Invariant



Example:  $x(n) = \delta(n) + \frac{1}{2} \delta(n-1)$

$$h(n) = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \cdot \delta(n-k) = \left(\frac{1}{2}\right)^n \cdot \varepsilon(n) \quad \text{Time unit step}$$

a) Draw  $h(n)$  with specification of all characteristic values



b) Is  $h(n)$  stable according to BIBO?  
Bounded Input Bounded Output

$$\text{To prove: } \sum_{n=-\infty}^{n=\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \cdot \varepsilon(n) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

$2 < \infty$ , Hence Proved,  $h(n)$  is stable

c) Is  $h(n)$  causal?

Yes, because  $h(n) = 0, n < 0$

Anti-causal:  $h(n) = 0, n > 0$

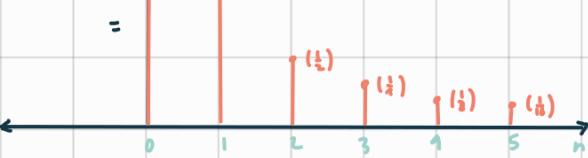
Non-causal:  $h(n) \neq 0, n > 0$  and  $n < 0$

d) Determine  $y(n) = x(n) * h(n) = h(n) * x(n)$

$$y(n) = \left[ \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \cdot \delta(n-k) \right] * \left[ \delta(n) + \frac{1}{2} \delta(n-1) \right]$$

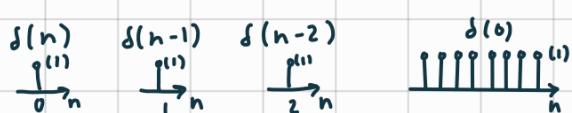
$$= h(n) + \frac{1}{2} h(n-1)$$

$$=$$



$$= \delta(n) + \left(\frac{1}{2}\right)^{n-1} \cdot \varepsilon(n-1) = \delta(n) + h(n-1)$$

$$= \varepsilon(n) - \left(\frac{1}{2}\right)^{n-2} \varepsilon(n-2)$$



Example:

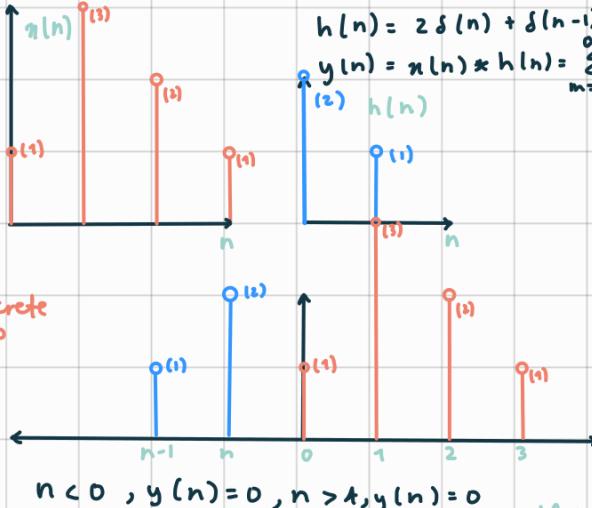
$$x(n) = \delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)$$

$$n = [1 \quad 3 \quad 2 \quad 1]$$

$$h(n) = 2\delta(n) + \delta(n-1)$$

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m) \cdot h(n-m)$$

shift + flip



$$n < 0, y(n) = 0, n > 4, y(n) = 0$$

$$n = 0, y(0) = 1 \cdot 2 = 2$$

$$n = 1, y(1) = 1 \cdot 1 + 2 \cdot 3 = 7$$

$$n = 2, y(2) = 1 \cdot 3 + 2 \cdot 2 = 7$$

$$n = 3, y(3) = 1 \cdot 2 + 2 \cdot 1 = 4$$

$$n = 4, y(4) = 1 \cdot 1 = 1$$

$$y(n) = 2 \cdot \delta(n) + 7 \delta(n-1) + 7 \delta(n-2) + 4 \delta(n-3) + \delta(n-4)$$



Convolution via shift & summation

$$y(n) = x(n) * h(n)$$

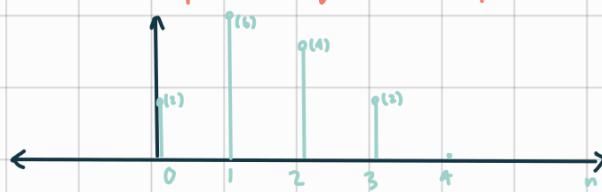
$$= x(n) * [2\delta(n) + \delta(n-1)]$$

Distributive law

$$= x(n) * 2\delta(n) + x(n) * \delta(n-1)$$

=  $2x(n) + x(n-1)$  Convolution with

Delta Impulse equals shift in time



This method is much faster if  $x(n)$  or  $h(n)$  have few values ≠ 0

## Non-causal Filter

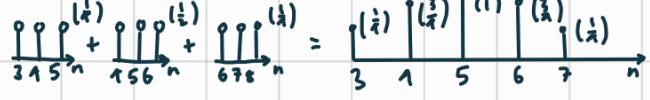
$$n(n) = \delta(n-3) + \delta(n-4) + \delta(n-5)$$

Low pass filter, causal:

$$h(n) = \frac{1}{4} \delta(n) + \frac{1}{2} \delta(n-1) + \frac{1}{4} \delta(n-2)$$

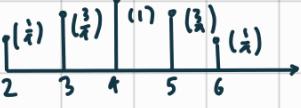
$$y(n) = n(n) * h(n)$$

$$= n(n) * \left[ \frac{1}{4} \delta(n) + \frac{1}{2} \delta(n-1) + \frac{1}{4} \delta(n-2) \right]$$



Center at 5 not 4, which is why:

$$\text{Non-causal } h(n) = \frac{1}{4} \delta(n+1) + \frac{1}{2} \delta(n) + \frac{1}{4} \delta(n-1)$$



## Fourier Transformation for Time Discrete Signals

$$\text{Exercise: } X(k) = \sum_{n=0}^{\infty} n(n) \cdot \underbrace{\cos\left(\frac{\pi}{N}(n+\frac{1}{2})k\right)}_{\text{Basis function } h_k(n)} \quad \begin{array}{l} \text{Discrete Cosine} \\ \text{Transformation} \\ \text{DCT} \end{array}$$

a) Derive 3 Filters for  $0 \leq k < N$ ,  $N=3$ ,  $0 \leq n \leq N$

$$k=0, n=0: \cos\left(\frac{\pi}{3}(0+\frac{1}{2}) \cdot 0\right) = 1$$

$$n=1: \cos\left(\frac{\pi}{3}(1+\frac{1}{2}) \cdot 0\right) = 1$$

$$n=2: \cos\left(\frac{\pi}{3}(2+\frac{1}{2}) \cdot 0\right) = 1$$

$$k=1, n=0: \cos\left(\frac{\pi}{3}(0+\frac{1}{2}) \cdot 1\right) = \frac{\sqrt{3}}{2}$$

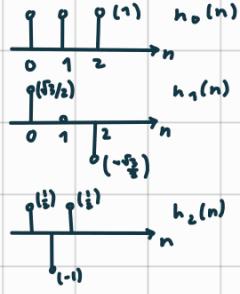
$$n=1: \cos\left(\frac{\pi}{3}(1+\frac{1}{2}) \cdot 1\right) = 0$$

$$n=2: \cos\left(\frac{\pi}{3}(2+\frac{1}{2}) \cdot 1\right) = -\frac{\sqrt{3}}{2}$$

$$k=2, n=0: \cos\left(\frac{\pi}{3}(0+\frac{1}{2}) \cdot 2\right) = \frac{1}{2}$$

$$n=1: \cos\left(\frac{\pi}{3}(1+\frac{1}{2}) \cdot 2\right) = -1$$

$$n=2: \cos\left(\frac{\pi}{3}(2+\frac{1}{2}) \cdot 2\right) = \frac{1}{2}$$



## Cut off frequency

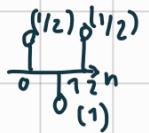
$$\sqrt{3} \sin(2\pi f T) = \frac{f_3}{2} \pi/4$$

$$f_{c1} = \frac{1}{2\pi T} \arcsin\left(\frac{1}{\sqrt{2}}\right)$$

$$f_{c1} = \frac{\pi}{2\pi \cdot 1T} = \frac{1}{8T}$$

$$f_{c2} = f_1 - f_{c1}$$

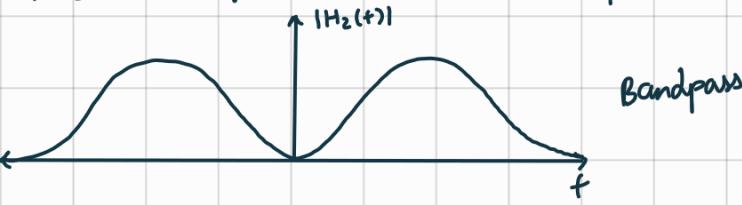
$$h_2(n) = \frac{1}{2} \delta(n) - \delta(n-1) + \frac{1}{2} \delta(n-2)$$



$$H_2(n) = \frac{1}{2} e^{-j2\pi f_0 T} - e^{-j2\pi f T} + \frac{1}{2} e^{-j2\pi f \cdot 2T}$$

$$= e^{-j2\pi f T} \left( \frac{1}{2} e^{j2\pi f T} - 1 + \frac{1}{2} e^{-j2\pi f T} \right)$$

$$|H_2(n)| = \left| \frac{1}{2} \cdot 2 \cdot \cos(2\pi f T) - 1 \right|$$



## Problems:

- Steep slopes (infinitely high)

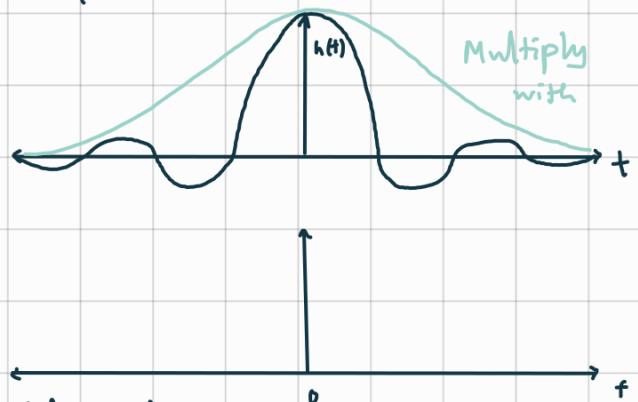
→ infinite circuit effort

- Impulse response is infinite

- Not causal

## Solutions:

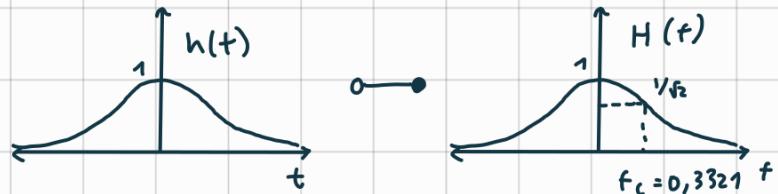
- Limit the response in time typically with a window function (ex: Hann, Hamming, Kaiser etc.)



Time limited impulse response has an infinite spectrum

## Example Exercise:

$$h(t) = e^{-\pi t^2} \rightarrow H(f) = e^{-\pi f^2}$$



- a) Determine  $b$  ( $b > 0$ ) so that  $f_c = 1\text{kHz}$

$$\text{Similarity theorem: } x(bt) \xrightarrow{} \frac{1}{|b|} X\left(\frac{f}{b}\right)$$

$$n(t) = e^{-\pi t^2} \rightarrow n(bt) = e^{-\pi (bt)^2}$$

$$\frac{1}{|b|} e^{-\pi \left(\frac{f}{b}\right)^2}$$

$$\text{We know: } h(f_c) = e^{-\pi \cdot 0,3321^2} = \frac{1}{\sqrt{2}}$$

$$e^{-\pi \left(\frac{1\text{kHz}}{b}\right)^2} = \frac{1}{\sqrt{2}} \Rightarrow \frac{1\text{kHz}}{b} = 0,3321 \Rightarrow b = 3010\text{Hz}$$

$$\Rightarrow h_{LP}(t) = 3010 \cdot e^{-\pi (3010\text{Hz} \cdot t)^2}$$

neglect Hz

LP ≈ Low Pass

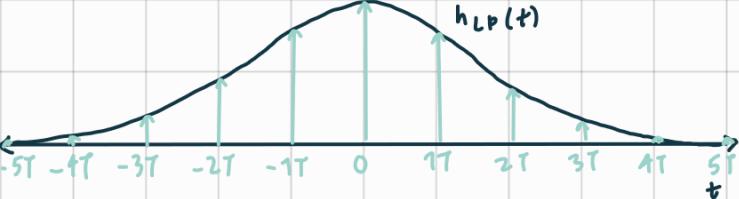
$$H_{LP}(f) = 1 \cdot e^{-\pi \left(\frac{f}{3010\text{Hz}}\right)^2}$$

$$\sin(\pi t) \xrightarrow{} \text{rect}(t)$$

$$\text{Similarity theorem: } x(bt) \xrightarrow{} \frac{1}{|b|} \cdot X\left(\frac{f}{b}\right)$$

$$\text{here } b = 2 \cdot f_c$$

b) Determine  $h(n)$  for  $T = \frac{1}{16\text{kHz}}$  and  $|n| \leq 5$



$$n=0: h(0) = 3010$$

$$n=1, n=-1: h(1) = 3010 \cdot e^{-\pi(3010\text{Hz} \cdot \frac{1}{16\text{kHz}})^2}$$

$$= 2693 = h(-1)$$

$$n=2, n=-2: h(2) = h(-2) = 1929$$

$$n=3, n=-3: h(3) = h(-3) = 1106$$

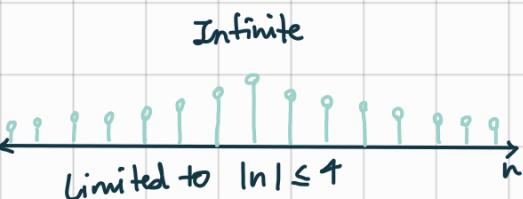
$$n=4, n=-4: h(4) = h(-4) = 508$$

$$n=5, n=-5: h(5) = h(-5) = 185$$

c) Sketch of  $h(n)$  for  $|n| \leq 5$



d) Determine the maximum deviation / error between the infinite impulse and the response limit to  $|n| \leq 1$



Maximum error: 185 at  $|n|=5$

e) Sketch the block diagramm for a causal realization of  $h(n)$

with multiplication, delay, addition



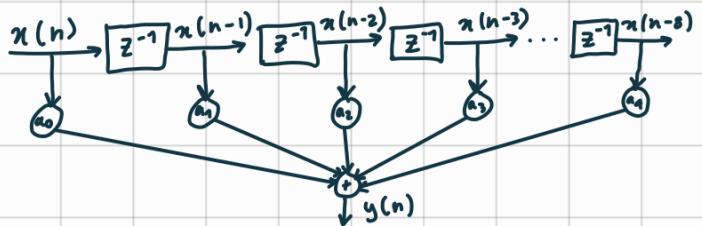
$$h(n) = [a_0 \ a_1 \ a_2 \ \dots \ a_8]$$

$$a_0 = a_8 = 508, a_1 = a_7 = 1106$$

$$a_2 = a_6 = 1929, a_3 = a_5 = 2693$$

$$a_4 = 3010$$

$$x(n) \rightarrow [h(n)] \rightarrow y(n)$$



Time discrete convolution

$$y(n) = h(n) * x(n) = \sum_{m=-\infty}^{\infty} h(m) x(n-m)$$

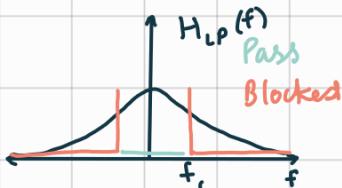
here m:  $a_0 \dots a_8$

$$\Rightarrow y(n) = \sum_{m=0}^8 h(m) \cdot x(n-m)$$

$$= h(0) \cdot x(n-0) + h(1) \cdot x(n-1) + \dots + h(8) \cdot x(n-8)$$

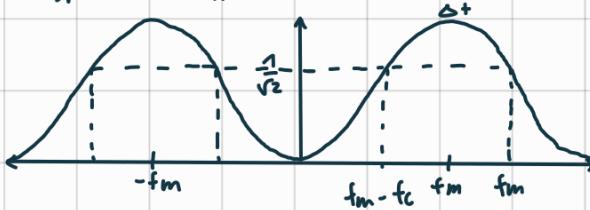
f) Bandpass filter with center frequency

$f_m = 4\text{kHz}$ . Derive  $H_{BP}(f)$ ,  $h_{BP}(t)$  and the bandwidth  $\Delta f$



$$H_{BP}(f) = H_{TP}(f) * \frac{1}{2} [\delta(f - f_m) + \delta(f + f_m)]$$

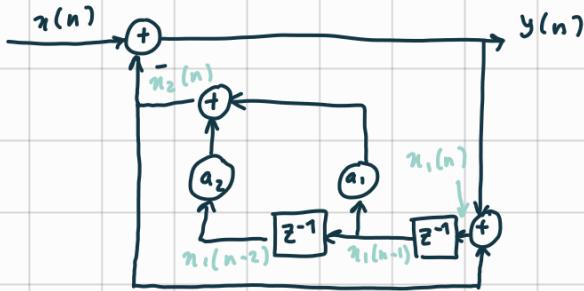
$$h_{BP}(t) = h_{TP}(t) \cdot \cos(2\pi f_m t)$$



$$\Delta f = 2f_c = 2\text{kHz}$$

## Z-Transform

Predictor of 2nd order in "closed loop" config.



a) Determine the z-transform of the transfer function

- Determine intermediate signals after all adder

$$x_1(n) = y(n) + x_2(n) \rightarrow Y(z) + X_2(z) \dots (1)$$

$$x_2(n) = a_2 \cdot x_1(n-2) + a_1 \cdot x_1(n-1) \rightarrow X_2(z) = a_2 \cdot z^{-2} \cdot X(z) + a_1 \cdot z^{-1} \cdot X(z) \dots (2)$$

$$y(n) = x(n) - x_2(n) \rightarrow X(z) - X_2(z) \dots (3)$$

Z-Transform:  $a \cdot X(n-n_0) \rightarrow a \cdot z^{-n_0} \cdot X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{X(z) - X_2(z)}{X(z)} = 1 - a_2 \cdot z^{-2} + a_1 \cdot z^{-1}$$

b) Determine the difference equation  $y(n) = \dots$

$$\begin{aligned} Y(z) &= X(z) \cdot (1 - a_2 z^{-2} - a_1 z^{-1}) \\ &= X(z) - a_2 z^{-2} X(z) - a_1 z^{-1} X(z) \end{aligned}$$

y(n) = x(n) - a\_2 x(n-2) - a\_1 x(n-1)

Assume:  $H(z) = \frac{Y(z)}{X(z)} = \frac{1 - a_1 z^{-1} - a_2 z^{-2}}{1 - b_1 z^{-1}}$

$$Y(z) (1 - b_1 z^{-1}) = X(z) (1 - a_1 z^{-1} - a_2 z^{-2})$$

y(n) - b\_1 y(n-1) = x(n) - a\_1 x(n-1) - a\_2 x(n-2) + b\_1 y(n-1)

c) Now set  $a_1 = 1$  and  $a_2 = -\frac{1}{2}$

Draw the pole-zero-diagram (PN Diagram)

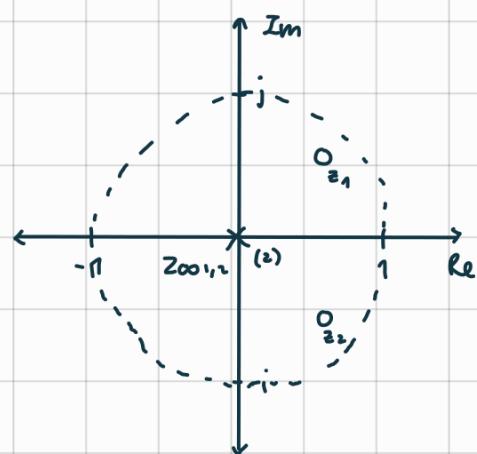
$$\begin{aligned} H(z) &= (1 - a_1 z^{-1} - a_2 z^{-2}) \cdot \frac{z^2}{z^2} \\ &= \frac{z^2 - a_1 z - a_2}{z^2} = \frac{z^2 - z + \frac{1}{2}}{z^2} \end{aligned}$$

Pole:  $z_{\infty 1,2} = 0$

Zeroes:  $z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm i}{2}$

$$\frac{z^2 - z + \frac{1}{2}}{z^2} = 1 \cdot \frac{(z - 1/2 - 1/2i)(z - 1/2 + 1/2i)}{z \cdot z}$$

Amplification factor:  $H_0 = 1$



Example for  $H_0$

$$H(z) = \frac{3z^2 - 2}{z^2}$$

Poles: 0  
Zeros:  $z = \pm \sqrt{\frac{2}{3}}$   
 $H(z) = \frac{(z - \sqrt{\frac{2}{3}})(z + \sqrt{\frac{2}{3}})}{z^2} \cdot H_0 \quad H_0 = \frac{3}{2}$



d) Is the system stable?

Yes, because all poles are within the unit circle

Example unit-step

Marginally stable



Stable: Because the amplitude doesn't increase

Non-stable: Because the amplitude never becomes 0

Example  $h(n)$  for poles outside of unit circle

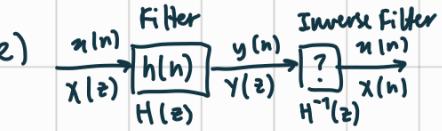


High pitch noise from mic is an unstable system

Example  $h(n)$  for poles inside of unit circle



BIBO:  $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$



$$Y(z) = X(z) \cdot H(z) \Rightarrow X(z) = \frac{1}{H(z)} \cdot Y(z) = H^{-1}(z) \cdot Y(z)$$

Determine the inverse filter  $H^{-1}(z)$

$$H(z) = \frac{z^2 - z + 1/2}{z^2} \Rightarrow H^{-1}(z) = \frac{z^2}{z^2 - z + 1/2}$$

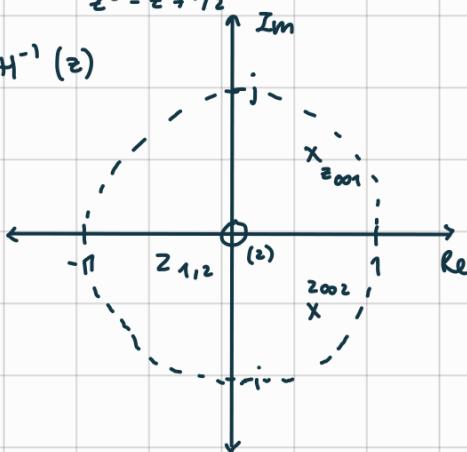
f) Pole-Zero-Diagram of  $H^{-1}(z)$

Poles become zeroes

Zeroes become poles

$$H_0 = 1$$

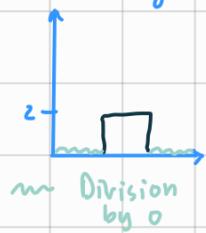
$H^{-1}(z)$  is stable



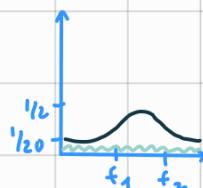
Example of a non-invertible system



Ideal



Division by 0



Noise will be multiplied by 20

Real

Amplification of Noise  
Solution:  
Apply window function

## Relation between Z transform and the Fourier Transform

$$Z \rightarrow e^{sT} \quad s = \sigma + j\omega = \sigma + j2\pi f$$

$$Z \rightarrow e^{j2\pi f T} \quad T: \text{sample rate}$$

$$H(z) = \frac{z^2 - z + 1/2}{z^2} = 1 - z^{-1} + \frac{1}{2} z^{-2}$$

$$H(f) = 1 - e^{-j2\pi f T} + \frac{1}{2} e^{-j4\pi f T}$$

## Bilinear Transform and Frequency Warping (Z-transform)

Laplace Transform:

$$\text{---} \boxed{R} \quad V(s) = R \cdot I(s)$$

$$\text{---} \boxed{L} \quad V(s) = s \cdot L \cdot I(s)$$

$$\text{---} \boxed{C} \quad V(s) = \frac{1}{sC} \cdot I(s)$$

Steps of the Bilinear Transform:

1) Description of the circuit via Laplace-Transform

2) Substitute  $s \approx c \frac{z-1}{z+1}$  with  $c = \frac{2\pi f_0}{\tan(\pi f_0 T)}$

$c = \dots$

Frequency Warping

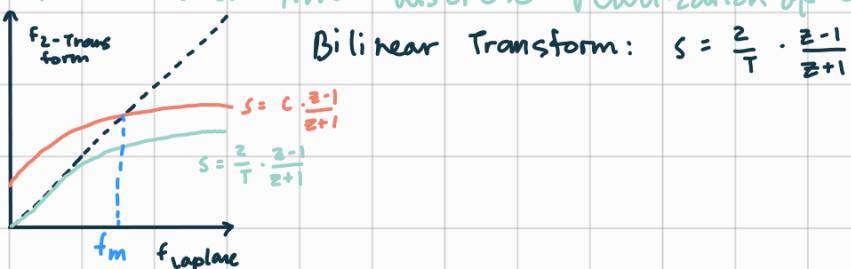
T: Sampling Time

$f_0$ : Frequency at which the approximation is precise:

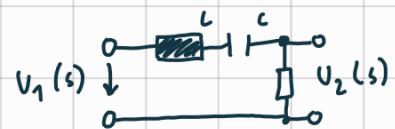
Low / High Pass Filter: Cut off frequency

Bandpass: Center Frequency

⇒ Allows a time-discrete realization of a system



Example: Series resonant circuit



Transfer function:

$$H(s) = \frac{U_2(s)}{U_1(s)} = \frac{R}{R + sL + \frac{1}{sC}}$$

Resonance frequency  $f_0 = ?$

Definition of  $f_0$ :  $H(s)$  is completely real

$$sL + \frac{1}{sC} = 0 \Rightarrow sL = -\frac{1}{sC} \Rightarrow s^2LC = -1 \Rightarrow s^2 = -\frac{1}{LC}$$

$$s = \sigma + j\omega = \sigma + j2\pi f_0$$

$$\sigma = 0$$

$$\Rightarrow (j2\pi f_0)^2 = -\frac{1}{LC} \Rightarrow 4\pi^2 f_0^2 = \frac{1}{LC} \Rightarrow f_0 = \frac{1}{2\pi\sqrt{LC}}$$

Determine a time-discrete realization of the filter by using the bilinear transform.

$$T = \frac{1}{1000} \text{ s}, R = 100 \Omega, L = 100 \text{ mH}, C = 100 \mu\text{F}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = 50.3 \text{ Hz}$$

$$\zeta = \frac{2\pi f_0}{\tan(\pi f_0 T)} = 1983 \text{ Hz}$$

$$H(s) = \frac{R}{R + sL + \frac{1}{sC}} \cdot \frac{sC}{sC} = \frac{sRC}{sRC + s^2LC + 1}$$

Bilinear Transform:  $s \approx \zeta \cdot \frac{z-1}{z+1}$

$$H(z) = \frac{\zeta \cdot \frac{z-1}{z+1} \cdot RC}{\zeta \frac{z-1}{z+1} RC + \zeta^2 \frac{(z-1)^2}{(z+1)^2} LC + 1} \cdot \frac{(z+1)^2}{(z+1)^2} = \frac{\zeta \cdot (z-1)(z+1) \cdot RC}{\zeta \cdot (z-1)(z+1) RC + \zeta^2 (z-1)^2 LC + (z+1)^2}$$

$$= \frac{\zeta \cdot RC \cdot (z^2 - 1)}{\zeta \cdot RC (z^2 - 1) + \zeta^2 LC (z^2 - 2z + 1) + (z^2 + 2z + 1)}$$

$$= \frac{b_1 z^2 + b_2 z + b_3}{a_1 z^2 + a_2 z + a_3}$$

$$b_1 = \zeta RC = 1983 \text{ Hz} \cdot 100 \Omega \cdot 100 \mu\text{F} = 19,83 \quad \text{Unit Check: } \frac{1}{s} \cdot \frac{V}{A} \cdot \frac{As}{V} = 1$$

$$b_2 = 0$$

$$b_3 = -\zeta RC = -19,83$$

$$a_1 = \zeta RC + \zeta^2 LC + 1 = 60,15$$

$$a_2 = -2\zeta^2 LC - 2 = -76,65$$

$$a_3 = -\zeta RC + \zeta^2 LC + 1 = 20,49$$

$$\Rightarrow H(z) = \frac{19,83 \cdot z^2 + 0 \cdot z - 19,83}{60,15 \cdot z^2 - 76,65z + 20,49}$$

ex: Look at PN Diagram

Difference between Equation and Block Diagram?

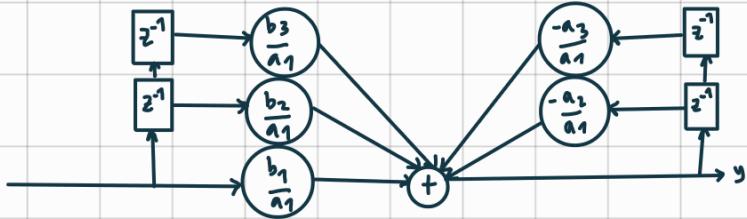
$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_1 z^2 + b_2 z + b_3}{a_1 z^2 + a_2 z + a_3} \cdot \frac{z^{-2}}{z^{-2}} = \frac{b_1 + b_2 z^{-1} + b_3 z^{-2}}{a_1 + b_2 z^{-1} + b_3 z^{-2}}$$

$$y(z) \cdot (a_1 + a_2 z^{-1} + a_3 z^{-2}) = x(z) (b_1 + b_2 z^{-1} + b_3 z^{-2})$$

$$a_1 \cdot y(n) + a_2 \cdot y(n-1) + a_3 \cdot y(n-2) = b_1 \cdot x(n) + b_2 \cdot x(n-1) + b_3 \cdot x(n-2)$$

Time Discrete Form

$$\Rightarrow y(n) = \frac{b_1}{a_1} x(n) + \frac{b_2}{a_1} x(n-1) + \frac{b_3}{a_1} x(n-2) - \frac{a_2}{a_1} y(n-1) - \frac{a_3}{a_1} y(n-2)$$



Example: Discharge Process

MISSING

MISSING (PARTIAL FRACTION DECOMPOSITION)

## SIGNAL CORRELATION (CHAPTER 5)

Signals with finite and infinite energy

Parseval theorem:

Energy of a signal can be calculated in time and frequency domain:

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty} |s(f)|^2 df$$

electric power:  $P = V \cdot I = \frac{V^2}{R} = I^2 R$

energy:  $E = \int P dt = \int \frac{1}{R} V^2 dt$

$$R = 1 \Omega$$



$E_s < \infty \rightarrow$  Energy Signal

- $s(t) = \text{rect}(t)$

- Piano

- Discharging Processes

$E_s \rightarrow \infty \rightarrow$  Power Signal

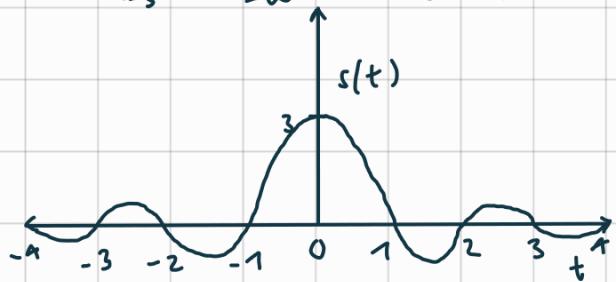
- $s(t) = \sin(2\pi f t)$

- PWM

- Power Plug 230V

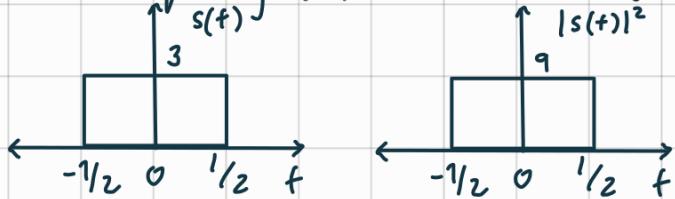
Example: Determine the energy of  $s(t) = 3 \sin(\pi t)$

$$E_s = \int_{-\infty}^{\infty} |3 \cdot \sin(\pi t)|^2 dt$$



$$\sin(\pi t) = \frac{\sin(\pi t)}{\pi t}$$

→ Frequency domain  $3 \cdot \sin(\pi t) \xrightarrow{} 3 \text{rect}(f)$



$$\text{Parseval: } E_s = \int_{-\infty}^{\infty} |s(f)|^2 df = 9$$

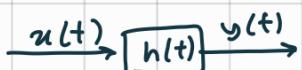
Surface / Area under the Signal

Cross-correlation for  $E_s < \infty$

$$\varphi_{sg}^E(\tau) = \int_{-\infty}^{\infty} s(\tau) \cdot g(t-\tau) dt = s(-\tau) * g(\tau)$$

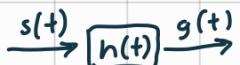
- Measure the similarity between two signals  $s(t)$  and  $g(t)$
- Find  $g(t)$  in  $s(t)$
- How much we have to shift  $g(t)$ , so that the similarity to  $s(t)$  is maximum?

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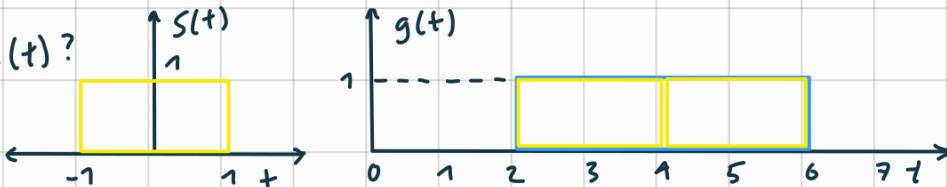
$\varphi_{sg}^E(\tau)$  is able to automatically determine the time-delay and attenuation between  $u(t)$  and  $y(t)$ .

$$\text{Example: } s(t) = \text{rect}\left(\frac{t}{2}\right)$$



$$g(t) = \text{rect}\left(\frac{t-1}{\pi}\right)$$

Which delay is caused by  $h(t)$ ?



$$\begin{aligned}
 \varphi_{sg}^E(\gamma) &= s(-\gamma) * g(\gamma) \\
 &= s(\gamma) * g(\gamma) \quad (\text{Because of symmetry}) \\
 &= \text{rect}\left(\frac{\gamma}{2}\right) * \text{rect}\left(\frac{\gamma}{2}\right) * [\delta(\gamma-3) + \delta(\gamma-5)] \\
 &= 2 \cdot \Delta\left(\frac{\pi}{2}\right) * [\delta(\gamma-3) + \delta(\gamma-5)] \quad (\text{Formula Sheet Table 11})
 \end{aligned}$$



Note  $\varphi_{sgs}^E(\gamma) = \varphi_{gs}^E(-\gamma)$  due to symmetry



$\varphi_{gs}^E(\gamma)$  has its maximum within  $3 \leq \gamma \leq 5$

When we shift  $s(t)$  by to  $= 3 \dots 5$  towards the right hand side then both signals have maximum similarity.

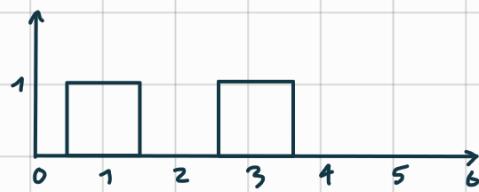
### Autocorrelation

$$\varphi_{ss}^E(\gamma) = \int_{-\infty}^{\infty} s(\tau) \cdot s(\tau + \gamma) d\tau = s(-\gamma) * s(\gamma)$$

How similar is a signal  $s(t)$  to itself when we shift it by  $\gamma$ ? (Used to find period)

Example :

$$s(t) = \text{rect}(t) * [\delta(t-1) + \delta(t-3)]$$



$$\begin{aligned}
 \varphi_{ss}^E(\gamma) &= \text{rect}(-\gamma) * [\delta(-\gamma-1) + \delta(-\gamma-3)] * \text{rect}(\gamma) * [\delta(\gamma-1) + \delta(\gamma-3)] \\
 &= \text{rect}(\gamma) * [\delta(\gamma+1) + \delta(\gamma+3)] * \text{rect}(\gamma) * [\delta(\gamma-1) + \delta(\gamma-3)] \\
 &= \text{rect}(\gamma) * \text{rect}(\gamma) * [\delta(\gamma+1) * (\delta(\gamma-1) + \delta(\gamma-3)) + \delta(\gamma+3) * (\delta(\gamma-1) + \delta(\gamma-3))] \\
 &= \Delta(\gamma) * [\delta(\gamma) + \delta(\gamma-2) + \delta(\gamma+2) + \delta(0)] \\
 &= \Delta(\gamma) * [\delta(\gamma+2) + 2\delta(\gamma) + \delta(\gamma-2)]
 \end{aligned}$$



## Correlation functions - Fourier Transform

$$\text{CCF: } \Psi_{sg}^E(\tau) = s(-\tau) * g(\tau)$$

$$\Phi_{sg}^E(f) = S^*(f) \cdot G(f)$$

$$\text{ACF: } \Psi_{ss}^E(\tau) = s(-\tau) * s(\tau)$$

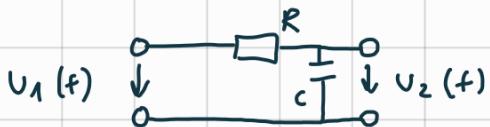
↓

$$\Phi_{ss}^E(f) = S^*(f) \cdot S(f) = |S(f)|^2$$

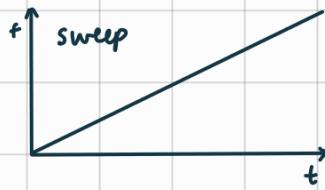
- If the energy is finite: energy density (can find?)
- For energy infinite: power density spectrum (can find?)



Example: RL low pass (signal model of a cable)

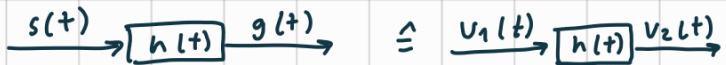


- $s(t)$  has constant spectrum:  $s(f) = 1$ 
  - Dirac Impulse
  - White noise
  - Sweep (frequency)



$$U_2(f) = U_1(f) \cdot \frac{\frac{1}{j\omega C}}{R + j\omega C} = U_1(f) \cdot \frac{1}{1 + j\omega RC}$$

$$H(f) = \frac{U_2(f)}{U_1(f)} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j2\pi f RC}$$



$$\hat{=} \quad u_1(t) \rightarrow [h(t)] \rightarrow v_2(t)$$

$$G(f) = H(f) \cdot S(f)$$

$$G(f) = \frac{1}{1 + j 2\pi f R C}$$

Determine the Energy Spectral Density of  $g(f)$ :

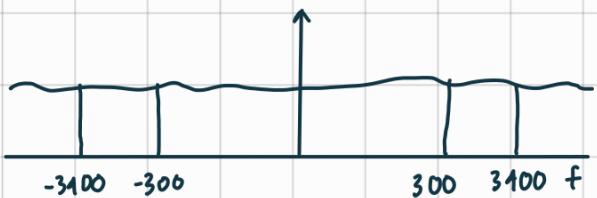
$$|h(f)|^2 = \frac{1}{(1^2 + (2\pi f R C)^2)^2} = \frac{1}{1 + (2\pi f R C)^2}$$

How much energy do we have at the output?

$$\text{Parseval: } E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

Energy within 300 Hz and 3400 Hz (ISDN)

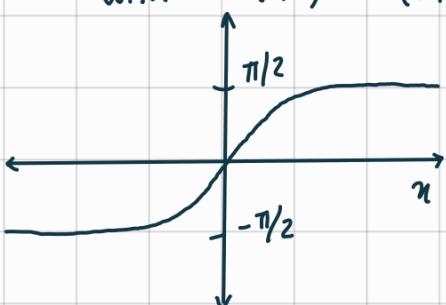
$$\begin{aligned} E &= \int_{300 \text{ Hz}}^{3400 \text{ Hz}} |h(f)|^2 df + \int_{-3400 \text{ Hz}}^{-300 \text{ Hz}} |h(f)|^2 df \\ &= 2 \cdot \int_{300 \text{ Hz}}^{3400 \text{ Hz}} |h(f)|^2 df \end{aligned}$$



e.g. Bronstein (Famous Book)

$$\text{Here: } E_g = \int_{-\infty}^{\infty} \frac{1}{1 + (2\pi f R C)^2} df = \left[ \frac{2}{\sqrt{\Delta}} \cdot \arctan \left( \frac{2\pi f}{\sqrt{\Delta}} \right) \right]_{-\infty}^{\infty}$$

$$\text{with } \Delta = \Delta a, \quad a = (2\pi R C)^2$$



$$\Rightarrow E_g = \frac{2}{\sqrt{\Delta}} \cdot \frac{\pi}{2} - \frac{2}{\sqrt{\Delta}} \left( -\frac{\pi}{2} \right) = 2 \cdot \frac{2\pi}{\sqrt{\Delta} \cdot 2} = \frac{2\pi}{1 \cdot \sqrt{4\pi^2 R^2 C^2}} = \frac{1}{2RC}$$

Correlation Functions and LTI Systems



$$h(t) * s(t)$$

$$\text{CCF: } \Psi_{sg}^E(\tau) = s(-\tau) * g(\tau) \\ = s(-\tau) * h(\tau) * s(\tau) \\ = s(-\tau) * s(\tau) * h(\tau) \\ = \Psi_{ss}^E(\tau) * h(\tau)$$

↓

$$\phi_{sg}^E(f) = |s(f)|^2 \cdot H(f)$$

$$\text{ACF: } \Psi_{gg}^E(\tau) = g(-\tau) * g(\tau) \\ = h(-\tau) * s(-\tau) * h(\tau) * s(\tau) \\ = h(-\tau) * h(\tau) * s(-\tau) * s(\tau)$$

$$\Psi_{gg}^E(\tau) = \Psi_{hh}^E(\tau) * \Psi_{ss}^E(\tau) \quad \text{Wiener-Khintchin-Theorem}$$

↓

/Wiener-Lee

$$\phi_{hh}^E(f) = \phi_{HH}^E(f) \cdot \phi_{ss}^E(f)$$

$$|h(f)|^2 = |H(f)|^2 \cdot |s(f)|^2$$

### Symmetry of the Correlation Functions

$$\text{ACF: } \Psi_{gg}^E(\tau) = \Psi_{gg}^E(-\tau)$$

$$\phi_{gh}^E(f) = \phi_{gh}^E(-f)$$

$$\text{CCF: } \Psi_{gs}^E(\tau) = \Psi_{sg}^E(-\tau)$$

$$\phi_{gss}^E(\tau) = \phi_{ss}^E(f) \cdot H(-f)$$

### Orthogonality:

Are the signals similar to each other?

### 2 Prototype signal:

$s_1(t)$ : intact bearing of motor

$s_2(t)$ : defect bearing

MISSING

Time Domain:  $\varphi_{sg}^E(0) = 0$

$$\int_{-\infty}^{\infty} s(t) \cdot g(t) dt = 0$$

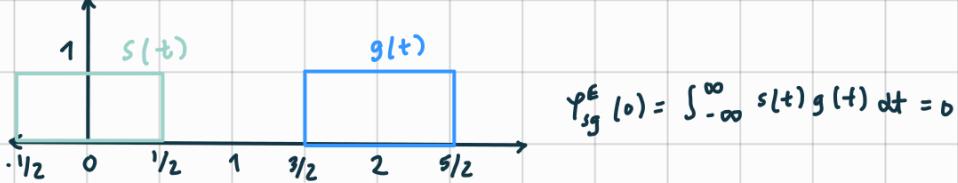
Frequency Domain:  $\varphi_{sg}^E(0) = \int_{-\infty}^{\infty} s^*(f) \cdot G(f) df = 0$

Test of Orthogonality:

- Test of Overlap (If it doesn't overlap in either  $t \rightarrow f$ , no need to check other directly orthogonal)
- Use symmetry of the signals

Example 1:

$$s(t) = \text{rect}(t), \quad g(t) = \text{rect}(t - 2)$$



$\Rightarrow$  Orthogonal because both signals don't overlap in time domain

With real functions you have symmetry after Fourier

With complex you can do anything

Example 2:  $s(t) = \text{si}(\pi t), \quad g(t) = \text{si}(\pi t) \cdot \cos(2\pi \cdot 3t)$

$$\cos(2\pi ft) = \frac{1}{2} [\delta(f+3) + \delta(f-3)]$$

MISSING

$$s(t) = \text{rect}(t), \quad G(f) = \text{rect}(f) * \frac{1}{2} [\delta(f+3) + \delta(f-3)]$$

But is doing whatever a bat does

$$\varphi_{sg}^E(\tau=0) = \int_{-\infty}^{\infty} s(t) \cdot g(t) dt = \int_{-\infty}^{\infty} s^*(f) \cdot G(f) df$$

$\Rightarrow$  At least one of the spectra is always 0

$\Rightarrow$  Orthogonal because no overlap in frequency domain

Example 3:  $s(t) = \text{rect}(t) \cdot \sin(2\pi t)$ ,  $g(t) = \text{rect}(t) \cdot \cos(2\pi t)$

MISSING

- Overlap in time-domain
- Overlap in frequency-domain

$$\int_{-\infty}^{\infty} s(t) \cdot g(t) dt$$

↑  
odd signal/  
function

Odd function:  $s(t) = -s(-t)$

Even function:  $g(t) = g(-t)$

Odd · Even = odd

$$\Rightarrow \int_{-\infty}^{\infty} \text{"odd fct"} dt = 0$$

MISSING

$\Rightarrow s(t) \& g(t)$  are orthogonal

If one is completely real and the other is completely real then they don't overlap

MISSING

Example 1:  $s(t) = \Delta(t)$ ,  $g(t) = \frac{1}{2} \cdot e^{-|t|}$

MISSING

$$\varphi_{sg}^E(0) = \int_{-\infty}^{\infty} s(t) g(t) dt$$

↑ ↑  
even even

$$s(t) > 0, g(t) > 0$$

Both signals are completely positive and thus the integral over their surface can't cancel out  $\Rightarrow$  not orthogonal

### Time Discrete Correlation Functions

$$\int \dots dt \rightarrow \Sigma$$

$$\varphi_{xy}^E(m) = x(-m) * y(m) = \sum_{n=-\infty}^{\infty} x(n) \cdot y(n-m)$$

Orthogonality of Time Discrete Functions:

$$\begin{aligned} \varphi_{xy}^E(0) &= 0 = \sum_{n=-\infty}^{\infty} x(n) \cdot y(n+0) && \text{Time Domain} \\ &= \int_{-\infty}^{\infty} X(f)^* \cdot Y(f) df && \text{frequency Do} \end{aligned}$$