Image Analysis

Geometric Transformations and Point Operators

Linear Filters

- Average Filters
- Derivative Filters

Nonlinear Filters

- Sorting: Min, Max, Median
- Morphological Operators

Steerable Filters

- Anisotropic Averaging
- Directional Derivatives

Geometric Transformations

- Forward- and Backward Transformation
- Affine and Projective Transformations
- Interpolation

Point Operators

- Definition
- Histogram Methods

Geometric Transformations

Applications

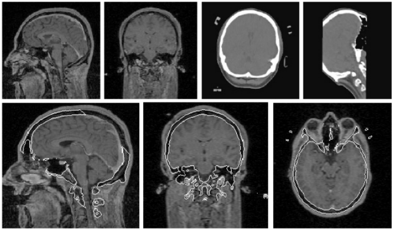
Geometric transformations affect the „Where“ of an image point and are needed in a number of image processing applications e.g.

- eliminate distortions
- image stitching
- register images

Geometric transformations can be specified in parametric form or as a vector field. The transformation involves two steps:

- transformation of coordinates
- interpolation of image values

Fusion of different object views

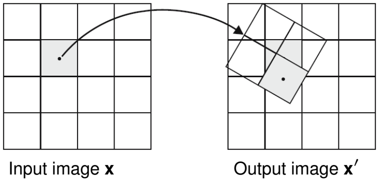


Fusion of MRT- and CT-images

Geometric Transformations

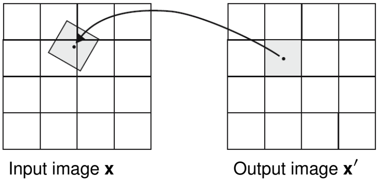
Definition

Geometric transformations define a functional relationship between the coordinates $\mathbf{x}$ and $\mathbf{x}'$ of points of two images. Here we distinguish between two types:



forward-mapping

$$\mathbf{x}' = M(\mathbf{x})$$



backward-mapping

$$\mathbf{x} = M^{-1}(\mathbf{x}')$$

Holes can occur during forward mapping, because no pixel is mapped to $\mathbf{x}'$.

Existence of the reverse mapping

If the transformation is in parametric form and the transformation rule is differentiable according to the coordinates, then the determinant of the Jacobian matrix $\mathbf{J}(\mathbf{M}(\mathbf{x}))$ can be used to determine whether an inverse transformation exists

$$\mathbf{J}(\mathbf{M}(\mathbf{x})) = \left|\frac{\partial \mathbf{M}(\mathbf{x})}{\partial \mathbf{x}}\right| = \begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{vmatrix}$$

- $|\mathbf{J}(\mathbf{x})| \neq 0 \ \forall \ \mathbf{x}$:
  A neighborhood about every $\mathbf{x}$ exists over which $\mathbf{M}^{-1}$ exists.
- $|\mathbf{J}(\mathbf{x})| = 0$: $\mathbf{M}^{-1}$ does not exist

Polynomial 2D Mappings

Many vector fields can be approximated by polynomial transformations

$$x' = \sum_{r=0}^{m} \sum_{k=0}^{m-r} a_{rk} x^r y^k \, , \quad y' = \sum_{r=0}^{m} \sum_{k=0}^{m-r} b_{rk} x^r y^k \, .$$

These transformations are always linear in the coefficients a_{rk}, b_{rk} but polynomial in the coordinates x, y.
The bilinear transformation is a simple and often used polynomial transformation

$$\begin{aligned} x' &= a_{00} + a_{10} x + a_{01} y + a_{11} xy \, , \\ y' &= b_{00} + b_{10} x + b_{01} y + b_{11} xy \, . \end{aligned}$$

At least 4 non-collinear points are needed to determine the bilinear parameters. If there are more than four points, a solution can be found e.g. via the method of least squares.

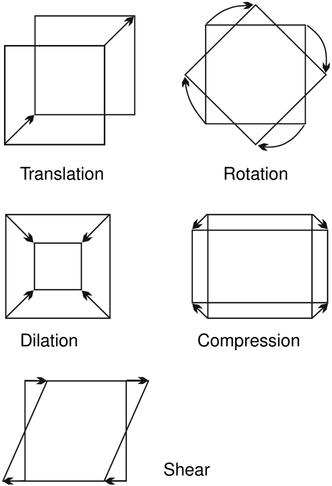
Affine 2D Mappings

The affine mapping is a linear coordinate transformation that includes the elementary transformations translation, rotation, dilation, compression and shear.

$$
\begin{aligned}
x' &= a_{00} + a_{10}x + a_{01}y\,, \\
y' &= b_{00} + b_{10}x + b_{01}y\,.
\end{aligned}
$$

Triangle $\rightarrow$ Triangle
Rectangle $\rightarrow$ Parallelogram

At least 3 non-collinear points are needed to determine the affine parameters. With more than three points a solution can be found e.g. by the method of least squares.



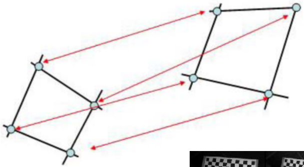
Geometric Transformations

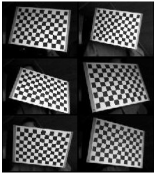
Perspective 2D Mappings

In order to describe the geometric mapping of a projected surface between two camera shots from different viewing angles, a non-linear, so-called perspective mapping is applied

$$x' = \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + 1}$$
$$y' = \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + 1}$$

Triangle $\rightarrow$ Triangle
Rectangular $\rightarrow$ General quadrangle





At least 4 non-collinear points are needed to determine the perspective parameters. With more than four points a solution can be found e.g. by the method of least squares.

Interpolation of Pixel Values

An interpolation of the sample points is obtained via a convolution with an interpolation filter h:

$$G(x, y) = \sum_{m,n} h(x - m\Delta x, y - n\Delta y) G(m\Delta x, n\Delta y)$$

The perfect interpolation function would be an infinitely extended sinc function

$$h_{ideal}(x, y) = \frac{sin(\pi x/\Delta x)}{\pi x/\Delta x} \frac{sin(\pi y/\Delta y)}{\pi y/\Delta y} .$$

Since in practice only local interpolation filters are applied to keep the computational effort low, image information is generally lost during geometric operations.

Geometric Transformations

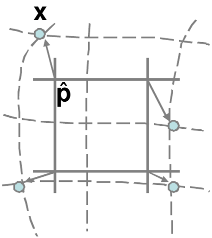
Interpolation of Pixel Values

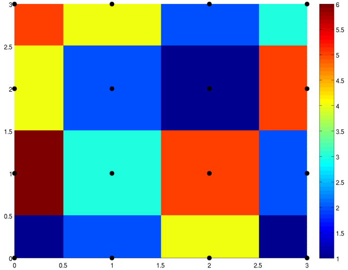
In practice, three main types of interpolation are used:

Nearest-Neighbor-Interpolation

The interpolated gray value $G(\mathbf{x})$ at the position $\mathbf{x} = (x, y)^\top$ corresponds to the closest gray value $G(\hat{\mathbf{p}})$ on the discrete pixel grid $\hat{\mathbf{p}}$, where $\mathbf{p} = (m\Delta x, n\Delta y)^\top$.

$$G(\mathbf{x}) = G(\hat{\mathbf{p}})\,, \quad \text{where} \quad \hat{\mathbf{p}} = argmin_{\mathbf{p}} \|\mathbf{p} - \mathbf{x}\|\,.$$



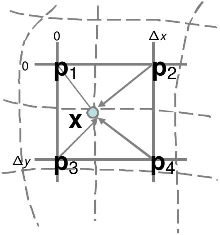


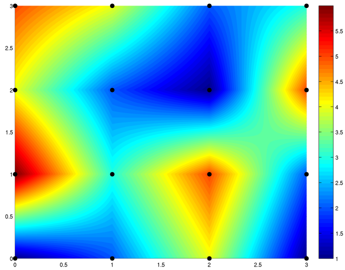
Interpolation of Pixel Values

Bilinear Interpolation

The interpolated gray value $G(\mathbf{x})$ at the position $\mathbf{x} = (x, y)^\top$ is obtained by the bilinear form

$$
\begin{aligned}
G(\mathbf{x}) &= \sum_{i=0}^{1} \sum_{j=0}^{1} a_{ij} x^i y^j \\
&= a_{00} + a_{10} \frac{x}{\Delta x} + a_{01} \frac{y}{\Delta y} + a_{11} \frac{xy}{\Delta x \Delta y} \,. \\
a_{00} &= G(\mathbf{p}_1) \,, \\
a_{10} &= G(\mathbf{p}_2) - G(\mathbf{p}_1) \,, \\
a_{01} &= G(\mathbf{p}_3) - G(\mathbf{p}_1) \,, \\
a_{11} &= G(\mathbf{p}_1) - G(\mathbf{p}_2) - G(\mathbf{p}_3) + G(\mathbf{p}_4) \,.
\end{aligned}
$$





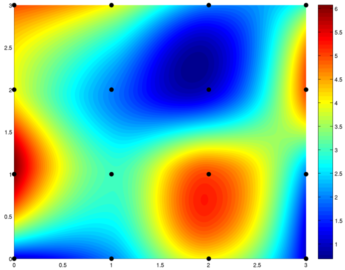
Interpolation of Pixel Values

Bicubic Interpolation

The interpolated gray value $G(\mathbf{x})$ at the position $\mathbf{x} = (x, y)^\top$ is obtained via the bicubic form

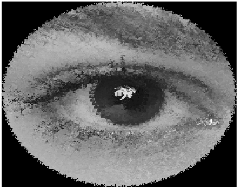
$$G(\mathbf{x}) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} x^i y^j \, .$$

The 16 coefficients a_{ij} are given by the brightness values $G(\mathbf{p}_i)$, where $i = 1 \dots 4$, and their derivatives $G_x(\mathbf{p}_i)$, $G_y(\mathbf{p}_i)$ and $G_{xy}(\mathbf{p}_i)$.

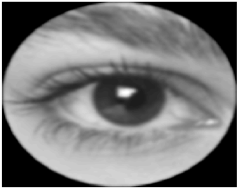


Interpolation of Pixel Values - Examples

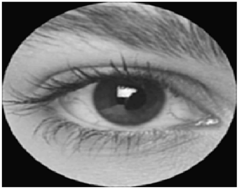
36-times Rotation about 10°



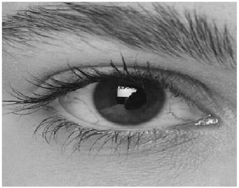
Nearest-Neighbor NN



Bilinear BL



Bicubic BK



Original

- NN: False high wavenumbers are generated, → many artifacts.
- BL: High wavenumbers are attenuated, → strong smoothing.
- BK: Slight smoothing, few artifacts.

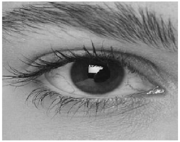
Point Operators

Applications

Point operators influence the „What“ of an image point and modify the gray values of individual image points only depending on the gray value itself and possibly on the position of the pixel. Point operators are needed for the following applications:

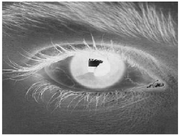
- correction and optimization of lighting,
- detection of underflow and overflow,
- contrast enhancement and stretching,
- image averaging in multi-image shots,
- correction of inhomogeneous illumination,
- radiometric calibration.

Original



Threshold

Inverse



Point Operators

Definition

There are both homogeneous (locationally invariant) and inhomogeneous (locationally variant) point operators. They are generally not reversible $\rightarrow$ Information is lost.

Homogeneous Point Operators

Each gray value $G(\mathbf{p})$ is mapped to itself via the function $\mathcal{F}$:

$$G'(\mathbf{p}) = \mathcal{F}\Big(G(\mathbf{p})\Big) \, .$$

Inhomogeneous Point Operators

Each gray value $G(\mathbf{p})$ is mapped to itself via a function $\mathcal{F}$ that depends on the location $\mathbf{p}$:

$$G'(\mathbf{p}) = \mathcal{F}\Big(G(\mathbf{p}), \mathbf{p}\Big) \, .$$

Random Variables - Probability Density Functions

The gray value G of an image at a certain pixel is a measure for the measured irradiance E. Since the measurement process is subject to statistical fluctuations, the irradiance or the gray value of each pixel is assumed to be a random variable with a certain continuous $p(x = E)$ or discrete $p(X = G)$ probability density function (pdf). Each pdf satisfies two conditions:

$$\begin{aligned} p(X) &\geq 0\,, \quad \forall\, X = G \in \mathbb{N}\,, \\ \sum_{X=1}^{Q} p(X) &= 1\,. \end{aligned}$$

$$\begin{aligned} p(x) &\geq 0\,, \quad \forall\, x = E \in \mathbb{R}\,, \\ \int_{-\infty}^{\infty} p(x)dx &= 1\,. \end{aligned}$$

Q equals quantization steps.

Technical University of Applied Sciences Würzburg-Schweinfurt

Random Variables - Histogram

If one assumes that the random variable X is independent of the image location, then the pdf can be approximated by a histogram. A histogram corresponds to the frequency distribution of the variables e.g. the gray values in an image

$$p(X) \approx h_N(X) = N_X / N \,,$$

where N_X corresponds to the number of the respective variable X e.g. of the respective gray value G in the image. It holds:

$$\lim_{N \to \infty} h_N(X) = p(X) \,.$$

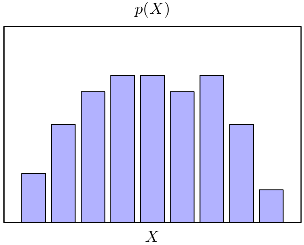
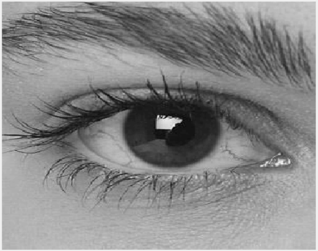


Image Statistics

Example - Histogram of an Image



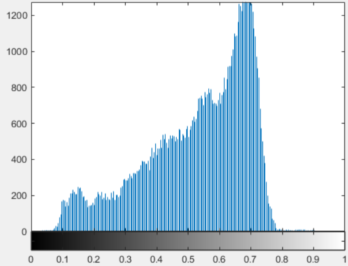


Image Statistics

Example - Histogram of image parts

Histograms with 256 bins

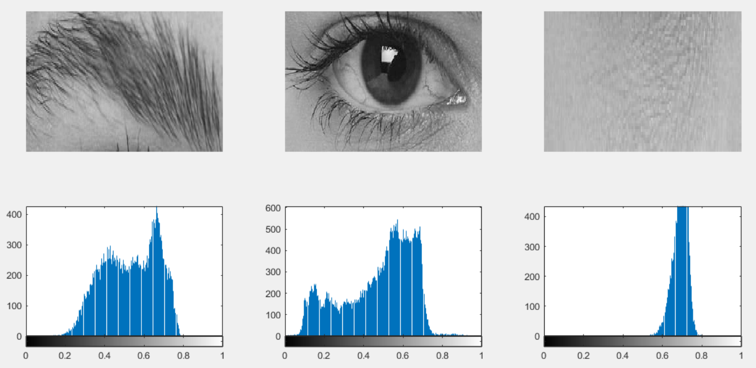
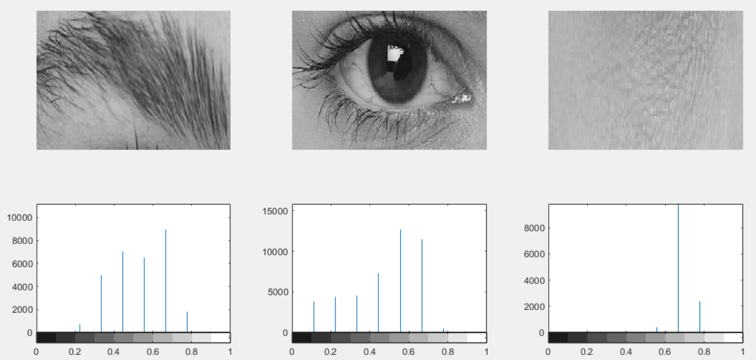


Image Statistics

Example - Histogram of image parts

Histograms with 10 Bins

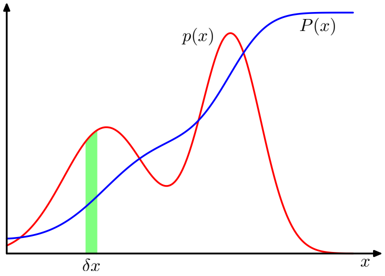


Random Variables - Cumulative Distribution Function

The probability $P(x)$ that e.g. the illuminance E lies in the interval of $]-\infty; E]$ is called cumulative distribution function and corresponds to the antiderivative of the pdf. The distribution function increases monotonically from 0 to 1 and it holds: $P'(x) = dP(x)/dx = p(x)$.

$$P(X) = \sum_{G=1}^{X} p(G), \quad \forall\, G \in \mathbb{N}.$$

$$P(x) = \int_{-\infty}^{x} p(E)dE, \quad \forall\, E \in \mathbb{R}.$$



Point Operators

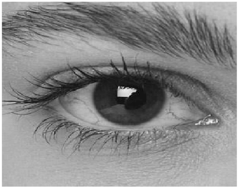
Homogeneous Operators - Examples

Contrast Enhancement

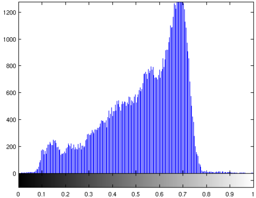
Improving the visual impression of an image by mapping the grayscale of a limited gray value range of an image to the full contrast range. This does not increase the image quality!

Example: Interval [0.08; 0.8] is mapped to [0; 1].

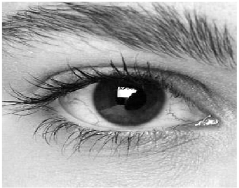
Original



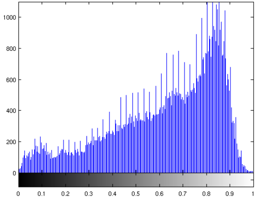
Histogram original



Enhanced image



Histogram enhanced



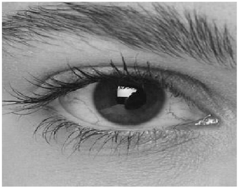
Point Operators

Homogeneous Operators - **Examples**

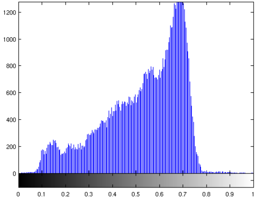
Contrast Adjustment

Improving the visual impression of an image by mapping the grayscale of a limited gray value range of an image to a larger/smaller contrast range. This does not increase the image quality!

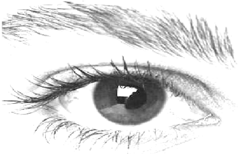
Original



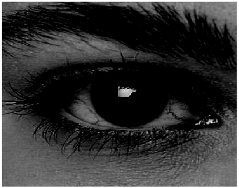
Histogram original



Dark regions enhanced [0; 0.5]



Bright regions enhanced [0.5; 1]



Point Operators

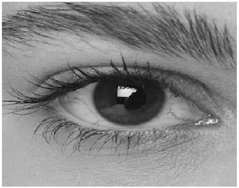
Homogeneous Operators - Examples

Histogram Equalization

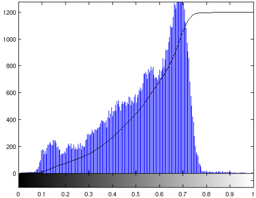
Also serves to improve contrast in gray-scale images. For this purpose, the gray scale is stretched in areas with frequent gray values and compressed in areas with less frequent gray values. The transformation corresponds simply to the distribution function $P(G)$ of the gray scale image.

$$G'(\mathbf{p}) = P(G(\mathbf{p})) \approx \sum_{n=0}^{G(\mathbf{p})} h_N(G(\mathbf{p}))\,.$$

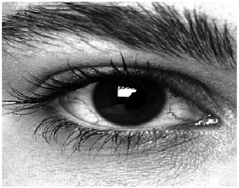
Original



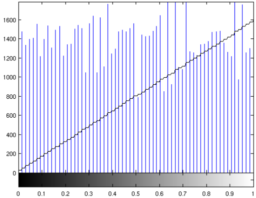
Histogram original



Equalized image



Equally distributed histogram



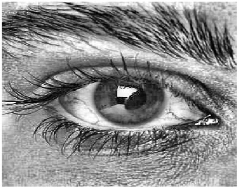
Point Operators

Inhomogeneous Operators

In contrast to homogeneous operators, where the transformation can be stored in a lookup table, for inhomogeneous operators it must usually be calculated for each pixel. Examples of inhomogeneous operators are:

- Background Subtraction,
- Temporal Averaging,
- Adaptive Histogram Equalization,
- etc.

Adaptive Histogram Equalization



Histogram Equalization