Control Systems 1 Problem Solving Guide

Problem 3 (Solving **O**rdinary **D**ifferential **E**quations)

General Approach

Given: an ODE

Sought: Transfer function: $G_{YU} = \frac{Y(s)}{U(s)}$

Procedure:

1) Laplace transform the provided equation
2) Rearrange the equation such that $Y(s)$ and $U(s)$ are isolated
3) $G_{YU} = \frac{Y(s)}{U(s)}$

Example:

$$\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} + 5y(t) = 5u(t)$$

...Laplace transform…

$$s^2 \cdot Y(s) - s \cdot f(0) - \dot{f}(0) - 2s \cdot Y(s) + 2 \cdot f(0) + 5 \cdot Y(s) = 5 \cdot U(s)$$

...Rearranging…

$$Y(s) \cdot (s^2 - 2s + 5) = 5 \cdot U(s)$$

$$G_{YU} = \frac{Y(s)}{U(s)} = \frac{5}{s^2-2s+5}$$

Finding Steady-state gain

Given: G_{YU}

Sought: K_∞

In general: $K_\infty = \lim_{s \to 0} G_{YU}$

Example:

$$G_{YU} = \frac{Y(s)}{U(s)} = \frac{5}{s^2-2s+5}$$

$$K_\infty = \lim_{s \to 0} \frac{5}{s^2-2s+5} = 1$$

Finding Poles and Zeros

Given: G_{YU}
Sought: Poles and zeros

Procedure:

1) To find zeros: set the numerator of G_{YU} to 0. Then, solve for each value of s.
2) To find poles: set the denominator of G_{YU} to 0. Then, solve for each value of s.

Example:

$G_{YU} = \frac{Y(s)}{U(s)} = \frac{5}{s^2-2s+5}$

Zeros: $5 = 0$;; There are no zeros.
Poles: $s^2 - 2s + 5 = 0$;; Find the roots to get $(s + 1 + j2)(s + 1 - j2) = 0$

Laplace to real-time conversion

Given: $G_{YU} = \frac{Y(s)}{U(s)}$
Sought: $y(t)$

Generally, we will end up with something like the following scenario: $Y(s) = \frac{Numerator}{(s+a)(s+b)^2}$

This is generally *unsolvable*. We need to separate all the brackets in the denominator like so:

$$Y(s) = \frac{A}{s+a} + \frac{B}{(s+b)^2} + \frac{C}{(s+b)^2}$$

In order to force our $Y(s)$ to align with the equation above this line, we need to somehow reorganize the equation, such that we can find the values of A, B, C.

In this template, the approach is to get:

$$Y(s) = \frac{A \cdot (s+b)^2 + B \cdot (s+a)(s+b) + C \cdot (s+a)}{(s+a)(s+b)^2}$$

Lastly, just compare this equation with the actual values of $Y(s)$. This way **A, B and C** can be found.

Finally, use the inverse laplace table to convert $Y(s)$ to $y(t)$.

Problem 7 (Determining $y(t)$ based on $G(s)$ and $u(t)$)

General Approach

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{1+2s}$$

7.1 Please determine the expression for the output signal $y(t)$, if the following signal is applied as input and all the transients have been decayed:

I. $u(t) = 3\text{ V}\sin(\omega t)$; mit $\omega = 0.5$ [rad/s]

II. $u(t) = 3\text{ V}\sin(\omega t)$; mit $\omega = 2$ [rad/s]

III. $u(t) = 5\text{V} + 3\text{ V}\sin(\omega t)$; mit ω=1000 [rad/s]

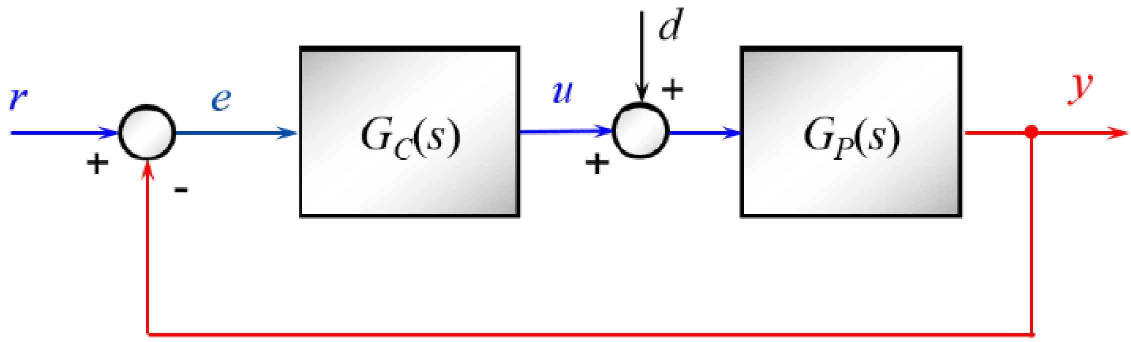
7.2 At which value of the frequency ω of the sinusoidal input are the amplitudes of the input and output signals same?

1) Find $G(j\omega)$
2) FInd $A_{dB}(\omega) = |G(j\omega)|$
3) Find $\varphi(\omega) = Arg(G(j\omega))$
4) Solve for $y(t)$ using general equation $y(t) = \overline{A} \cdot A_{dB} \cdot sin(\omega t + \varphi(\omega))$

   E.g. Case I

   $$\overline{A} = 3V;\ A_{dB} = 7.07;\ \omega = 0.5;\ \varphi(\omega) = -45°$$

Determining closed/open loop TF and finding controller constants



General Approach

Given: $G_P(s)$ and $G_C(s)$ {in terms of unknown constants}
Sought: $G_C(s)$ {with fully defined constants}

In general, the end goal is to find a fully defined $G_C(s)$.
Sought variables for a:

- P controller: K_c
- PI controller: K_c & T_I
- PID controller: K_C, T_I & T_D

You may also be given the Controller equation in terms of unknown variables. e.g.
$G_c(s) = \frac{b_1 s + b_0}{s + a_0}$

Procedure:

1) Find $G_0(s) = G_P(s) \cdot G_C(s)$
2) Find $G_{YR}(s) = \frac{Y(s)}{R(s)} = \frac{G_0}{1+G_0}$

3) Based on other given conditions, such as:
    a) $A(0)$
    b) φ_M
        i) $\varphi_M = 180° + \varphi(\omega)$
    c) e_∞
        i) Use ***given*** thumb rules in this case, or derived equation from previous part of the problem. (ref. 17.1)
    d) ω_{gc}
        i) If ω_{gc} is given, skip to step 6.
    e) For PT2 systems, ζ and/or ω_n
    f) Poles of the plant
        i) If poles are complex, reverse engineer the quadratic equation that has the given poles. $(x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2a})$
        ii) Otherwise, if the poles are real:
            (1) E.g: poles: -2 , -6 ----> (s+2)(s+6) = 0
4) Find $G_0(j\omega)$
5) FInd $A_{dB}(\omega) = |G_0(j\omega)|$
6) Set $A_{dB}(\omega_{gc}) = 1$ and find ω_{gc}
7) Find $\varphi(\omega_{gc}) = \ ?$ (based on what the problem asks)

Determining $G_C(s)$ using K_W

Given: $G_P(s)$ & K_W {either as a graph or an equation table}

Sought: $G_C(s)$

The key equation needed to solve this type of problem is:

$$G_C(s) = \frac{K_W}{1-K_W} \cdot \frac{1}{G_P(s)}$$

If K_W is already provided, then you just need to plug it into the equation above and done.

Otherwise, we need to determine what K_W is. To do that, use the following equation:

$$\mu \geq n - m + v$$

Symbol	Meaning
μ	Order of closed loop denominator
v	Order of closed loop numerator
n	Order of open loop denominator (G_P)
m	Order of open loop numerator (G_P)

***Note**: in most cases v is 0, because we don't have any "inner" feedback loops in the main system.*

If the problem specifies that we need to find a minimum order G_C, then use the equation

$$\mu \ = \ n - m + v$$

Once the order of μ is found, cross-check against given table to find the general form of K_w and solve using ω_0.

If ω_0 is not known, we need to find it from a given graph, and we should also be given what the desired rise time of the function is.

$$\omega_o = \frac{t_{found}}{t_{given}}$$

Lastly, plug in the value of ω_0 into the K_W equation, and done.