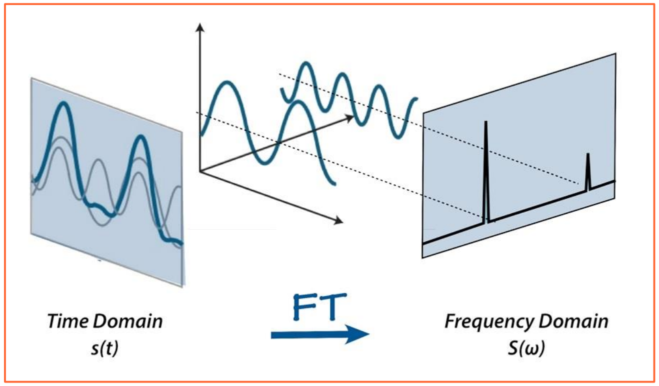
thws

System Theory

or

"If you want to achieve a certain effect,
you have to understand the **<u>cause</u>-<u>effect</u>-<u>principle!</u>**"



3rd semester, THWS

WS 23/24

Prof. Dr. Rainer Hirn

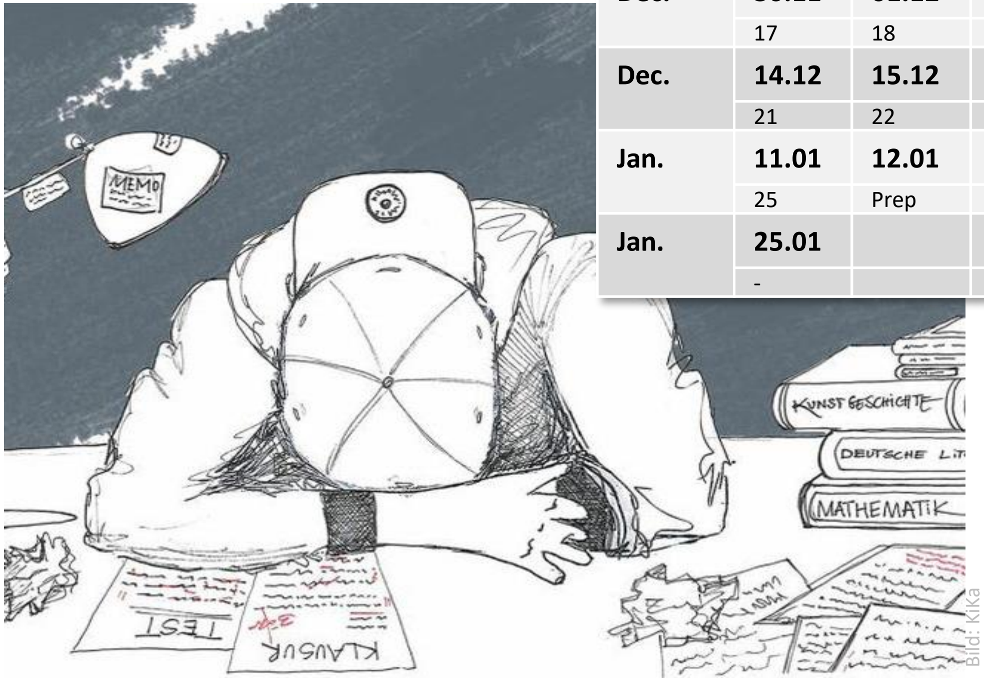
Organisation

- **Effort:** 5 ECTS / 4 SWS
- **Languages:** **1 x German** (BRO), **1 x English** (IRO)
- **Sequence:** Lectures & exercises alternately
- **Exam:** **90 Min., no auxiliary means, a formula collection will be provided**
- **E-Learning Course:** see Moodle: „Systemtheory“
- **Consultation Hours:** see profile-websites of Faculty Electrical Engineering…

Thursday (English, 10:00, 5.1.01) Thursday (German, 11:45, 9.E.24) Friday (German, 10:00, 9.E.25) Friday (English, 11:45, 9.E.26)				
Oct.	**05.10**	**06.10**	**12.10**	**13.10**
	1	2	3	4
Oct.	**19.10**	**20.10**	**26.10**	**27.10**
	5	6	7	8
Nov.	**02.11**	**03.12**	**09.11**	**10.11**
	9	10	11	12
Nov.	**16.11**	**17.11**	**23.11**	**24.11**
	13	14	15	16
Dec.	**30.11**	**01.12**	**07.12**	**08.12**
	17	18	19	20
Dec.	**14.12**	**15.12**	**21.12**	**22.12**
	21	22	23	24
Jan.	**11.01**	**12.01**	**18.01**	**19.01**
	25	Prep	Prep	?
Jan.	**25.01**			
	-			

- **Exam Preparation:**

	THWS	USA (typ.)	Systemt-Theory
Oct	-	-	-
Oct	-	Assessment 1	Test 1
		5 %	0 %
Nov	-	Assessment 2	-
		5 %	
Nov	-	**Mid Term Exam**	Test 2
		20 %	0 %
Dec	-	Assessment 3	-
		5 %	
Dec	-	Assessment 4	Test 3
		5 %	0 %
Jan	**Final Exam**	**Final Exam**	**Final Exam**
	100 %	**60 %**	**100 %**



OUTLINE

1. Basics of Signals & Systems

- **1.1 Properties of signals**
  - Signal classes
  - Test signals
  - Functional algebra
- **1.2 Properties of systems**
  - Definition of a system
  - Linearity & time invariance
  - Causality & stability

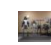
2. Analog World

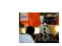
- **2.1 System descriptions in the time domain**
  - Modeling and simulation
  - State space representation
  - System description through differential equations
  - System response in the time domain (four-step method, convolution)
- **2.2 System descriptions using the Laplace transformation**
  - Laplace transform
  - Transfer function
  - Complex system structures
  - Elementary transferring links
  - Pole Zero Diagrams & Stability
  - Existence of the Laplace transform
- **2.3 System descriptions using the Fourier transformation**
  - Fourier series
  - Fourier transform
  - Comparison of Laplace & Fourier transformation
  - Spectrum and frequency response
  - Bode diagram and locus curves
  - Phase and group delay,
  - Minimum phase systems, All passes, Basics of filter design

3. Digital World

- **3.1 Basic properties of time-discrete signals**
  - Scanning and reconstruction
  - Discrete-time test signals
- **3.2 System descriptions in the time domain**
  - Difference equation
  - Discrete-time approximation of continuous-time systems
  - Solution of a difference equation in the time domain
- **3.3 System descriptions by z-Transform**
  - Z transformation
  - Transfer function of time-discrete systems
  - Pole Zero Diagram & Stability
  - Matched- and Bilinear-Transformation
  - Impulse and jump invariant transformation
  - Realization of time-discrete systems
  - FIR, IIR filters
  - Discrete-time state space representation
- **3.4 System descriptions by discrete time Fourier Transform**
  - Discrete Time Fourier Transformation (ZDFT)
  - Spectrum and frequency response of time-discrete systems
  - Bode diagram and locus curves of time-discrete systems
  - Discrete Fourier Transform (DFT)
  - Windowing and zeroPadding
  - Fast Fourier Transformation (FFT)

	Analog World	Digital World
Time Domain	1	4
Laplace Domain	2	5
Fourier Domain	3	6





1. Basics of Signals & Systems

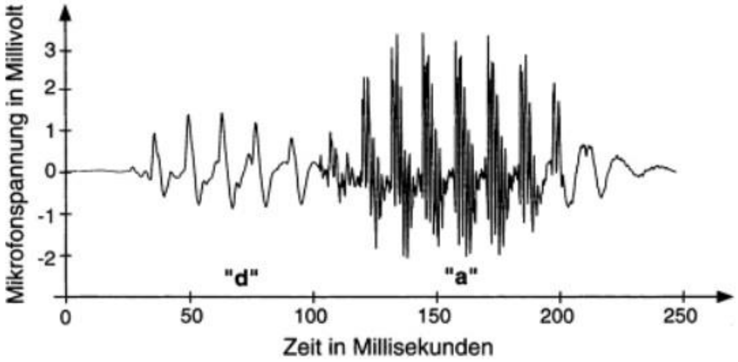
Introduction

The aim of systems theory is to describe and understand systems <u>theoretically</u> (i.e. mathematically).

- Systems theory is **<u>the</u>** indispensable theory for **<u>all</u> further subjects of modern technology and <u>far</u> beyond!**

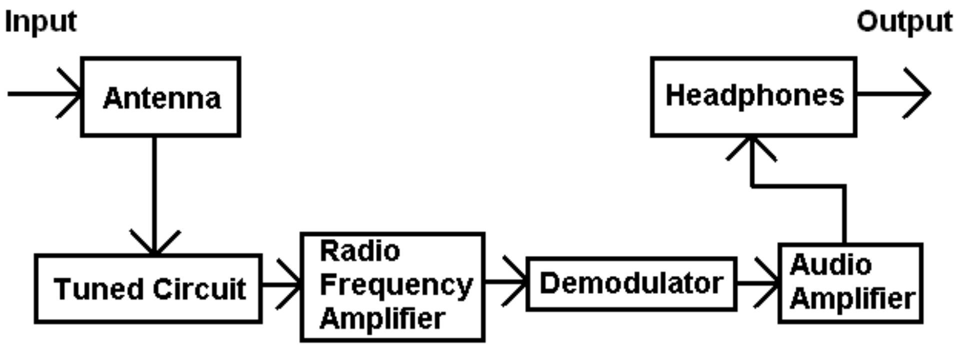
- Examples of signals and systems:

Audio-**Signal:**

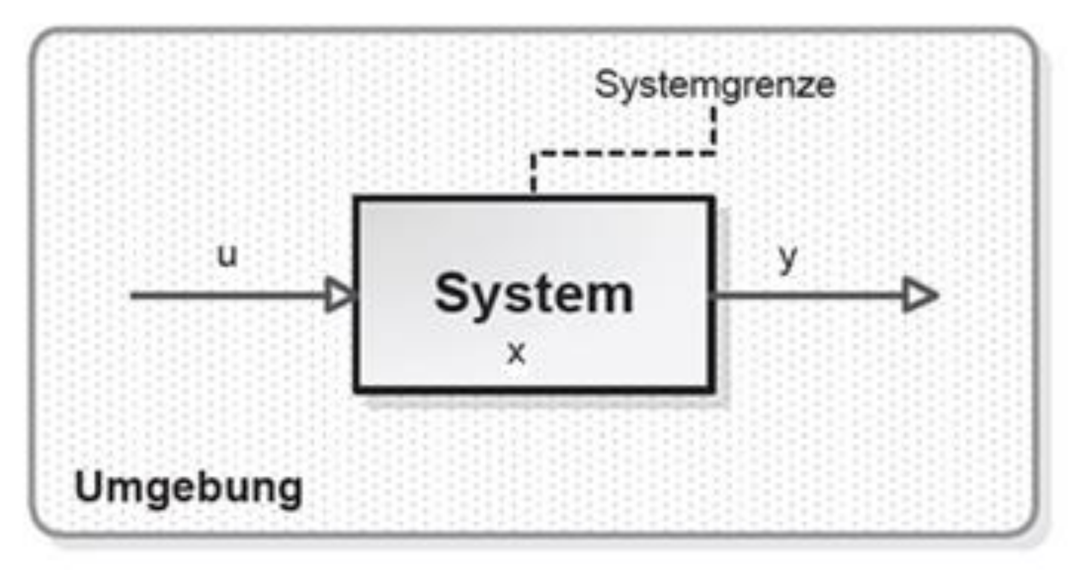


Beispiel eines zeitkontinuierlichen Signals: Sprachsignal der Silbe „da“

Radio-**System:**



If you want to create a certain effect, you must "only" understand the **cause-effect-principle**!



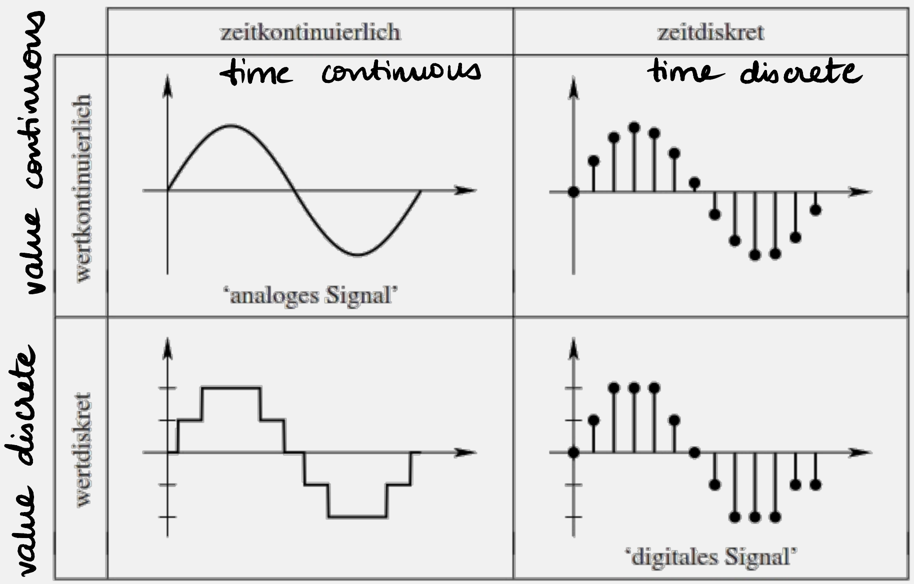
Attention: Systems theory delivers **only** mathematical descriptions, it does **not** provide concrete realization of systems!

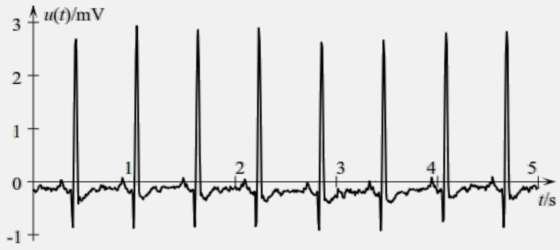
- **Advantage:**
  Not going into details helps tkeeping the **overview of "the big picture"** and allows the **transferability to completely different areas of science**!
- **Disadvantage:**
  Unfortunately a certain **lack of clarity** ...

Signal Classes

Signals:

- are arbitrary **functions** or **sequences of values** of dependent variables of one or more independent variables (e.g. time, location).
- **Four** types can be distinguished:





EKG-Verlauf als Beispiel für ein Zeitsignal

- **Continuous time signals** are described as **x(t),**
  Domain of definition $t \in \mathbb{R}$.
- **Discrete-time signals** are described as **x[k]:**
  Domain of definition $k \in \mathbb{Z}$.

Signal Classes

Determined Signals:

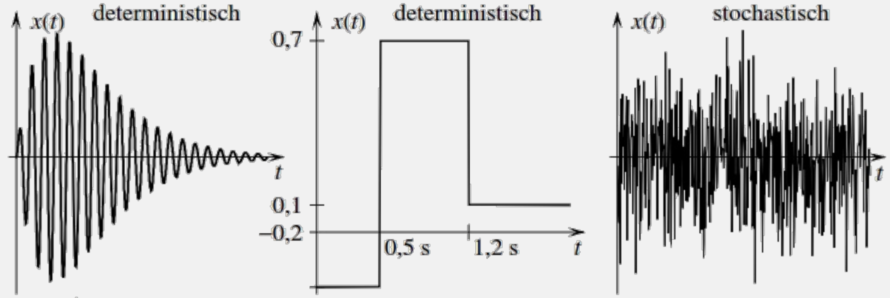
- the **signal curve is known** for all times.
  And through a math. Rule or table**exactly writable** (e.g. sin (ωt) or fixed sequence of numbers).

Random / Stochastic Signals:

stochastic : outcomes based on random probability

predictable

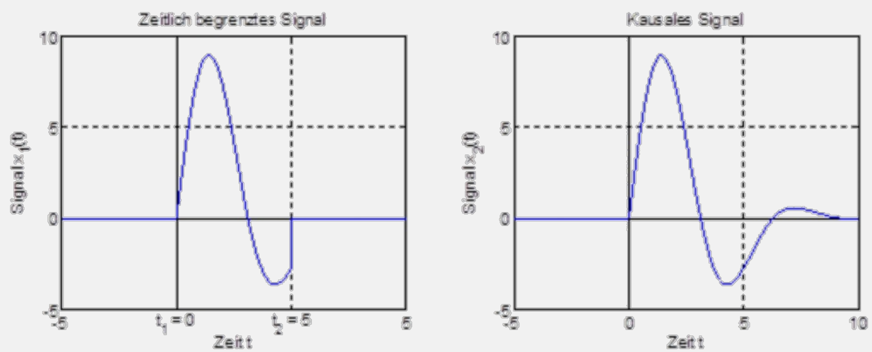
- the **signal curve is <u>not</u> known at all.**
  Only statistical signal properties, such as mean value or standard deviation (e.g. noise signals).



Limitation and Causality in signals:

- **time-limited**, if a signal lasts a finite time only, i.e. it disappears outside a finite interval:
- **causal** if the following applies to the signal: **x (t) = 0 for t <0**. otherwise it is **non-causal.**

eg sin, cos



Signal Classes (Energy & Power)

- Example of electrical resistance:

  The square of the voltage or the current is a measure for the converted power, i.e. the following applies:

$$P = UI = \frac{U^2}{R} = I^2 R \qquad P(t) = \frac{U^2(t)}{R} = R i^2(t)$$

$$P = \frac{\Delta E}{\Delta t} = \frac{dE}{dt} \Rightarrow \int P dt = \int dE = E$$

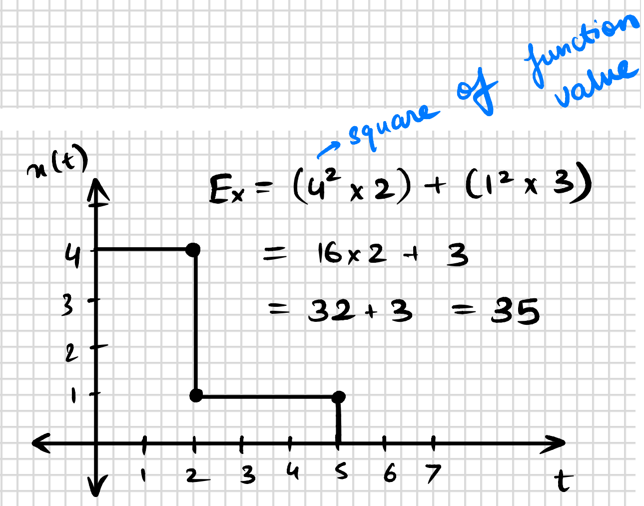
$$E = \int_{-\infty}^{\infty} P(t)\,dt = \frac{1}{R}\int_{-\infty}^{\infty} u^2(t)\,dt = R\int_{-\infty}^{\infty} i^2(t)\,dt$$

Energy & Energy Signals

In systems theory, the energy of a signal is defined (minimalistically) as:

$$\boxed{E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt}$$

- In words: The **energy** of a signal is "**the total area under its square**".
- A signal becomes **Energy signal** called if it has a finite energy, ie if: **E_x** $< \infty$.



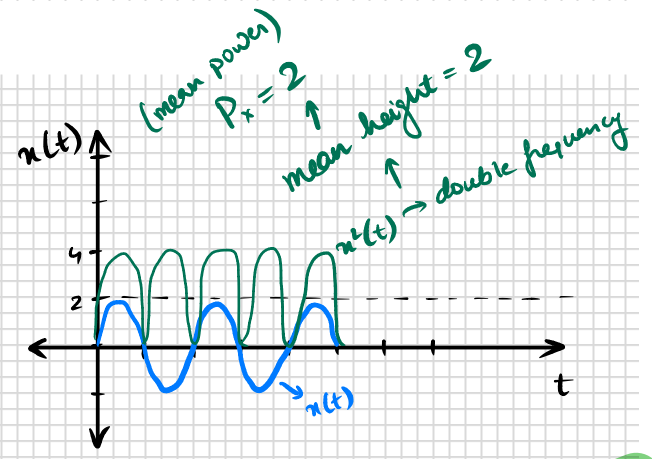
Power & Power Signals

In systems theory, the power of a signal is defined as:

$$\boxed{P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\,dt}$$

- In words: The **power** of a signal is "**the average height of its square**".
- A signal becomes **Power signal** called if it has a finite power, ie if: **P_x** $< \infty$.

The power is used to describe signals using <u>infinite</u> Energy (e.g. the sine signal) is required.

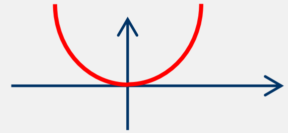


Note: The amount bars are of course only required for complex signals!

Signal Classes

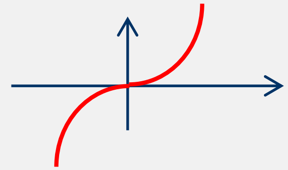
Even Signals:

- A signal is called even if: **$x(t) = x(-t)$** (= axis symmetry),



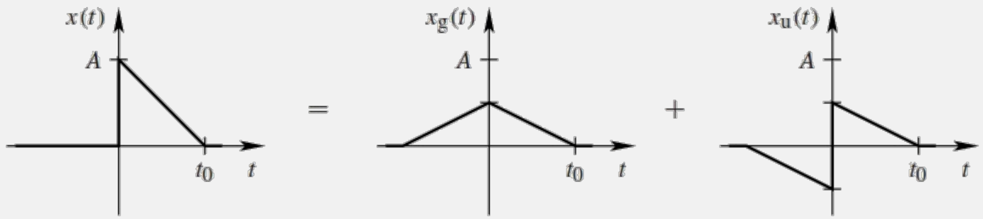
Odd Signals:

- A signal is called odd if: **$x(t) = -x(-t)$** (= point symmetry).



Even and odd parts:

- Any signal x (t) can be converted into a even part $x_G(t)$ and an odd part $x_u(t)$, the formula is:
  - **$x_G(t) = ½ [x(t) + x(-t)]$**
  - **$x_u(t) = ½ [x(t) - x(-t)]$**



Zerlegung eines Signals in seinen geraden und ungeraden Anteil

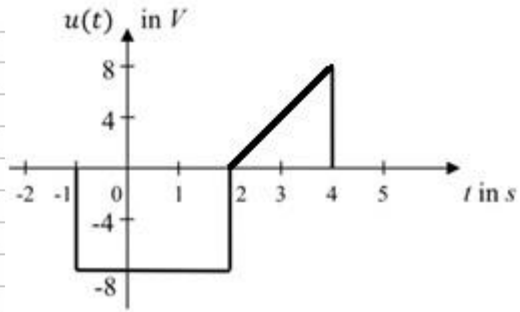
- A complex signal is called:
  - **conjugates even** if: **$x(t) = x^*(-t)$**
  - **conjugates odd** if: **$x(t) = -x^*(-t)$**.

	$x(t)$	$y(t)$	$x(t) \pm y(t)$	$x(t) \cdot y(t)\,,\; x(t)/y(t)$	$x(y(t))$
even	gerade	gerade	gerade	gerade	gerade
	gerade	ungerade	—	ungerade	gerade
odd	ungerade	gerade	—	ungerade	gerade
	ungerade	ungerade	ungerade	gerade	ungerade

Symmetrieeigenschaften bei der Verknüpfung von Funktionen

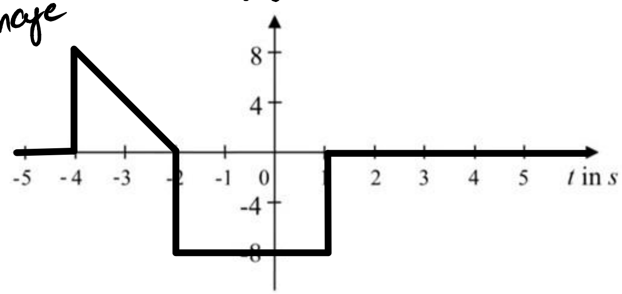
Ü1.1 Signal Classes

a) Draw the version u(-t) mirrored to the signal u(t), as well as also its even and odd part $u_G(t)$ and $u_U(t)$.

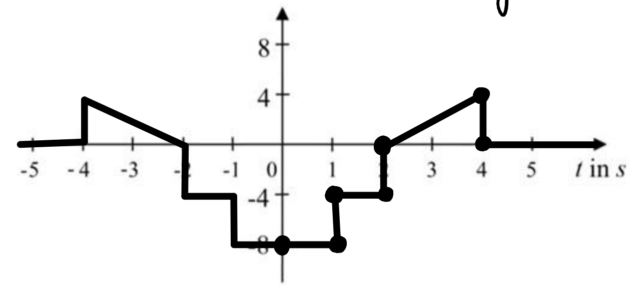


mirror image

u(−t)

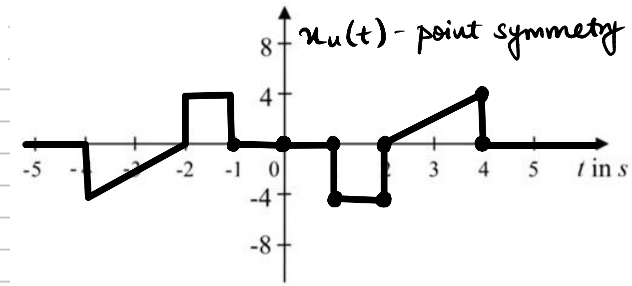


$x_G(t)$ – line Symmetric

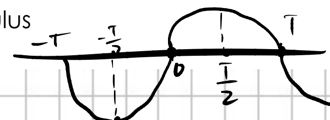


- $x_G(t) = ½ [x(t) + x(−t)]$
- $x_u(t) = ½ [x(t) - x(−t)]$

$x_u(t)$ – point symmetry



b) Calculate the power of a sinusoidal signal sin (ωt) using the integral calculus



$$x(t) = \sin(\omega t)$$

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\,dt$$

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sin^2(\omega t)\,dt = \frac{1}{T_P}\int_0^{T_P}\left(\frac{1}{2} - \frac{\cos 2\omega t}{2}\right)dt$$

time period

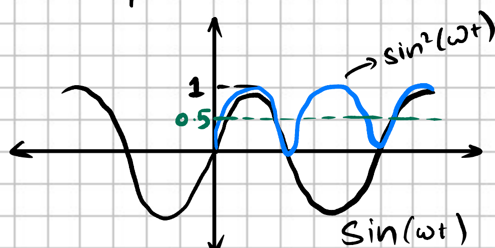
$$\cos 2x = 1 - 2\sin^2 x$$

$$\sin^2 x = \frac{1}{2} - \frac{\cos 2x}{2}$$

$$= \frac{1}{T_P}\left(\frac{t}{2} - \frac{\sin 2\omega t}{2\cdot 2\omega}\right)\Bigg|_0^{T_P} = \frac{1}{T_P}\left(\frac{T_P}{2} - \frac{\sin 2\omega T_P}{4\omega}\right)$$

$$= \frac{1}{2} - \underbrace{\frac{\sin 2\omega T_P}{4\omega}}_{0 \text{ for one } T_P} = \frac{1}{2} = \boxed{0.5}$$

↳ mean of (height)^2



Step function:

- ε **(t)** = 1, $t \geq 0$
  ε **(t)** = 0 , $t < 0$
  Other terms in the literature is also $\boldsymbol{\sigma}$ **(t)**.

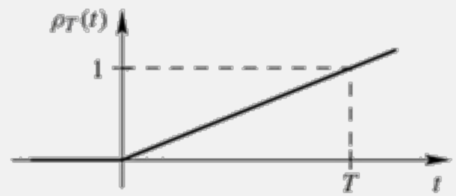
$\varepsilon(t)$

Sprungfunktion $\varepsilon(t)$

Ramp function:

- ρ **(t)** = $t \cdot \varepsilon$ (t)

$\rho(t)$



Rampenfunktion $\rho_T(t)$

if not mentioned, height = width = 1

Rectangle and triangle function:

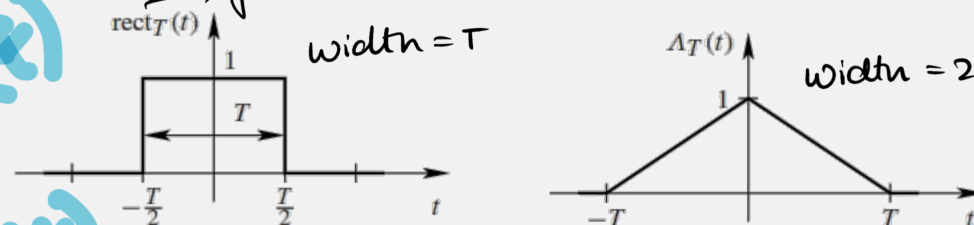
- **rect$_T$(t)** = 1, $-T/2 < t < T/2$
  rect$_T$(t) = 0 , otherwise
- Λ_T**[t]** = $1 - |t/T|$, $|t| < T$
  = 0, $|t| > T$
- For **T = 1** the basic function are obtained **rect(t) and Λ(t)**.

rect(t)

$\Lambda_T(t)$

width = T

width = 2T

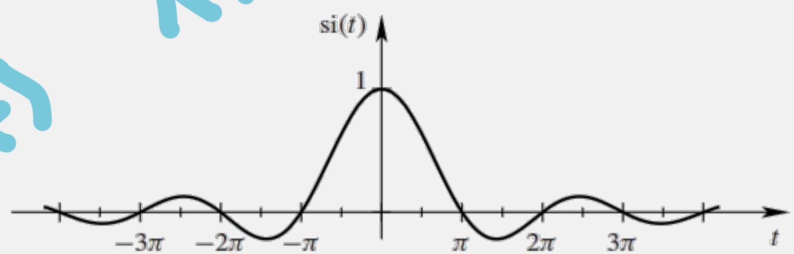


Rechteckimpuls $\mathrm{rect}_T(t)$ und Dreieckimpuls $\Lambda_T(t)$

si-function:

- **si (t) = sin (t)/t**

si(t)



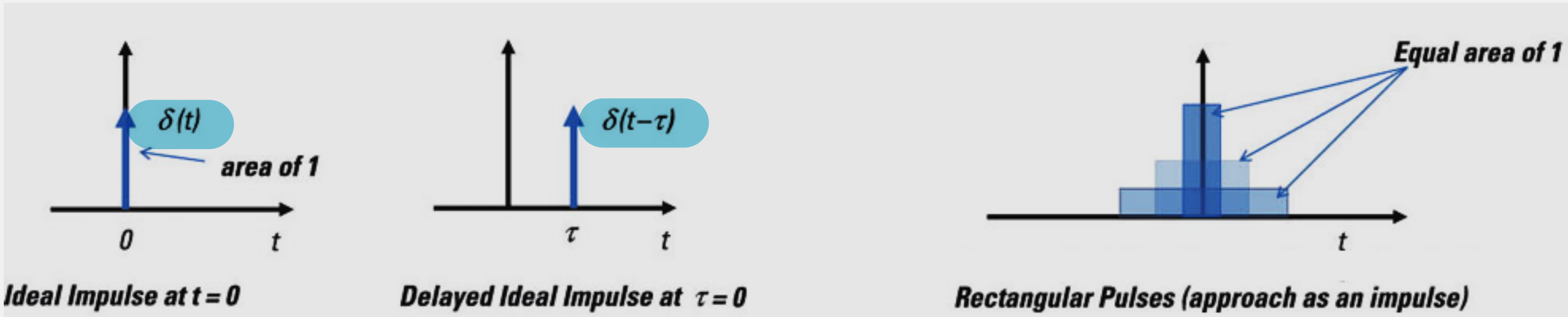
si-Funktion

Dirac Impulse:

- It is an infinitely high, infinitely short signal of **area one**,
- It is not a function, but a so-called **distribution.**
  It could be seen as an an infinitely small rectangle with area one…
- The Dirac pulse is causal and a power signal!

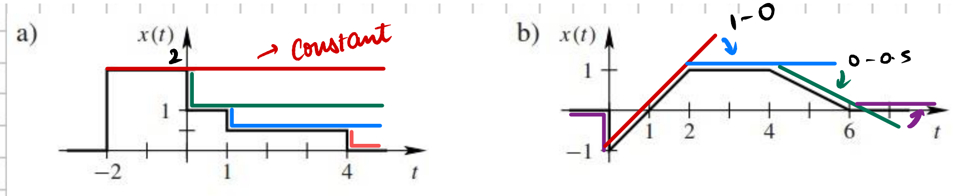
Properties of the Dirac Impulse:

$$\int_{-\infty}^{\infty}\delta(t)dt=1 \qquad \int_{-\infty}^{t}\delta(\tau)d\tau=\varepsilon(t) \qquad \delta(t)=\frac{d\varepsilon(t)}{dt} \qquad \delta(at)=\frac{1}{|a|}\delta(t) \qquad \int_{-\infty}^{\infty}\delta(t)\cdot\Phi(t)dt=\Phi(0)$$



Ex.1.2 Functional algebra

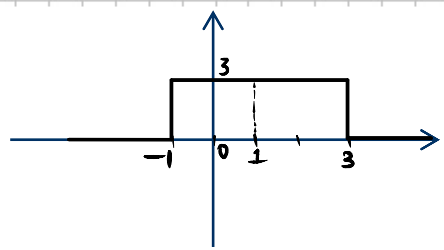
a) Display the following signals using jump and ramp functions:



a) $x(t) = 2\varepsilon(t+2) - \varepsilon(t) - 0.5\,\varepsilon(t-1) - 0.5\,\varepsilon(t-4)$

b) $x(t) = -\varepsilon(t) + 1.5\rho(t) - \rho(t-2) - 0.5\rho(t-4) + 0.5\rho(t-6)$

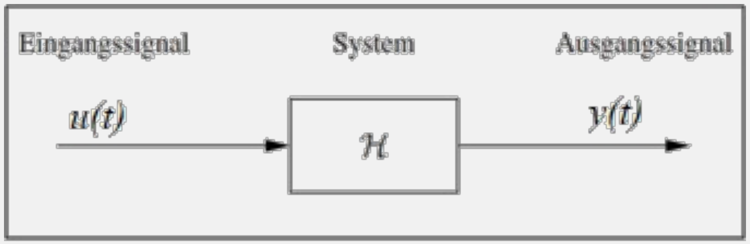
b) Sketch the signal: $x(t) = \underbrace{3}_{\text{height}}\,\mathrm{rect}_{\underbrace{4th}_{\text{width}}}(\underbrace{t-1}_{\text{center}})$



Definition System

System:

- A system is a delimited functional unit that interacrs with its **environsment** **<u>only</u>** through **input** and **output signals.**.



- <u>Example: Direct setting of the DE of an RC element:</u>

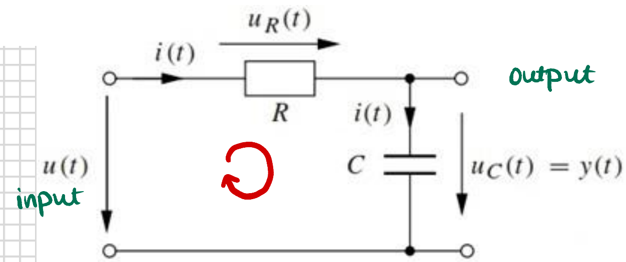
① $R = \frac{U_R}{i}$ ② $i = C\frac{du_c}{dt}$

③ $U_R + U_C = u$

$$\frac{dU_c}{dt} = \frac{i}{c} = \frac{U_R}{RC} = \frac{U - U_c}{RC}$$

$$\Rightarrow U = \frac{RC\, dU_c}{dt} + U_c$$

$$\Rightarrow U(t) = RC\frac{dU_c}{dt} + u_c$$



$$\underbrace{U_R + U_c = U(t)}$$

Linearity and Time Invariance

Linear system:

For a linear system, the answer to a linear combination of Input signals is equal to the corresponding linear combination of the individual system responses (**Superposition principle**).

$$f\{a_1 \cdot u_1(t) + a_2 \cdot u_2(t)\} = a_1 \cdot f\{u_1(t)\} + a_2 \cdot f\{u_2(t)\}$$

- Short: "Input and output are proportional to each other".

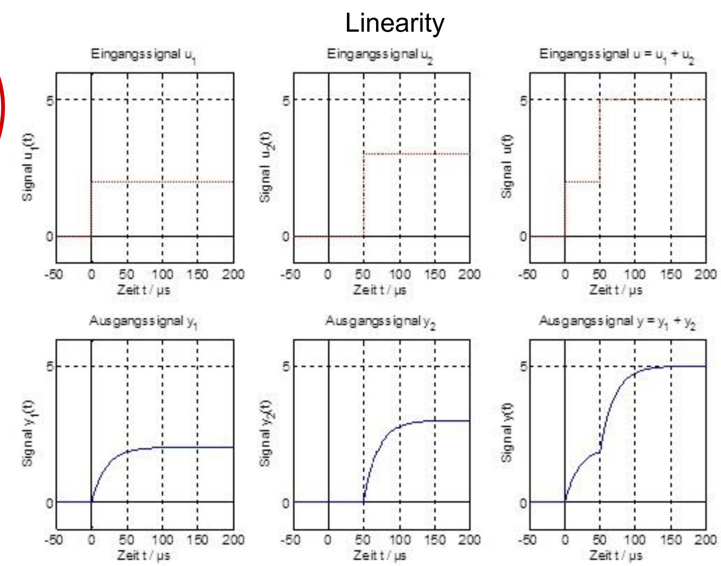
how to tell if a system is linear?

Linear DGL: Nonlinear DGL:

$$a_2\ddot{y} + a_1\dot{y} + a_0 y = b_1\dot{u} + b_0 u$$

$$a_2\sin(\ddot{y}) + a_1\dot{y}^2 + a_0 y_0 = \cdots$$

operations on derivatives ⇒ non linear



Time invariant system: → same output, just delayed

A system reacts to a delayed input signal with a correspondingly delayed output signal, the system is:

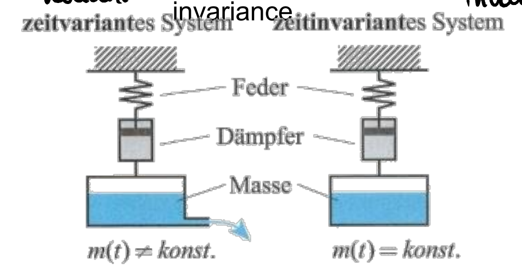
$$y(t - t_0) = f\{u(t - t_0)\}.$$

- Short: "The parameter the DE must be constant".

Time invariance

time variant

time invariant



mass changes ⇒ output changes

output const w.r.t time

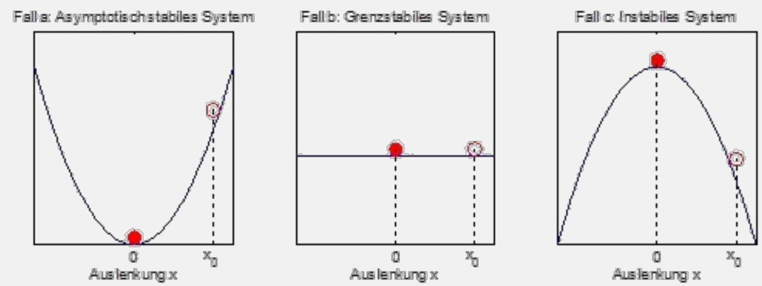
LTI system:

- An LTI system is both **linear and time invariant** (**L**.inear **T**ime-**I**nvariant system).

Causal system:

- „In a causal system, the effect occurs at the earliest at time of the cause, but never before!"
  Physically meaningful and feasible systems are because of the **Cause and effect principle** alwas causal!

Stable system:



- Does a system respond to a limited input signal with a limited output signal, the system is said to be stable. (**BIBO stability**, engl. 'Bounded Input - Bounded Output').

System with / without memory:

- A system whose output depends not only on the current input but also on previous inputs is called a system with Memory, otherwise without.

System with multiple inputs / outputs:

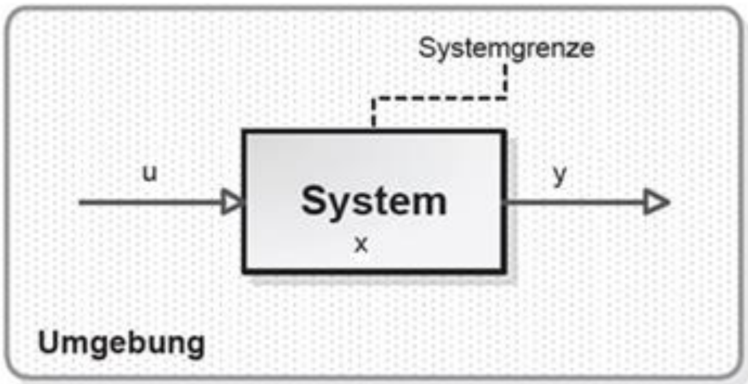
- **SISO** (English Single Input Single Output), **MIMO** (Multiple Input Multiple Output).

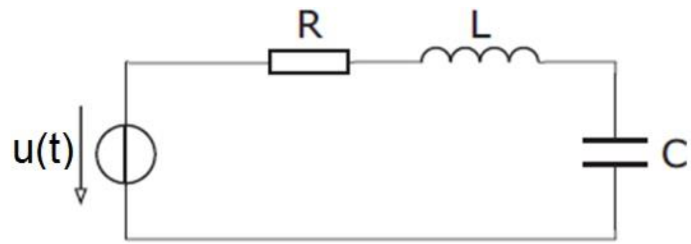
Without memory:

With memory:

2.1 System Descriptions in the Time Domain

Signals of Modeling:





System inputs u

- are the variables that **excite/stimujlate** the system from the **environment** through the system boundary, they **cannot be directly influenced** by the actual model!

System outputs y

- are the model variables of interest, they are **provided by the model** to the environment.

System states x

- **States x** describe the **content** of **storage elements**.

Recipe for Modeling & Simulation

1) Define interface

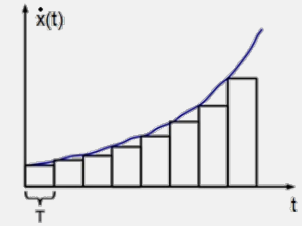
Definition of all inputs u, state variables x and outputs y of the system.

2) Define coordinate system

Introduction of clear coordinates, identification of all forces and moments or currents and voltages.

3) Write down balance equations

Creation of all element and topology equations (e.g. Kirchhoff, Newton, ...).



4) Reshape balance equations

Transform the above equations in such a way that the derivatives of all state variables can be explicitly calculated (i.e. stand alone on one side): $\boxed{\dot{x}_i = f(x, u, p)}$

5) Derive solution

Draw or create the simulation model.

Example: Modeling & simulation of the RC element:

The given input variable u is u(t), the output variable y is the voltage $u_c(t)$. → for capacitor : U_c

for inductor : i_L

① input $u = u(t)$ output $y = U_c(t)$ internal state $x = u_c(t)$

②

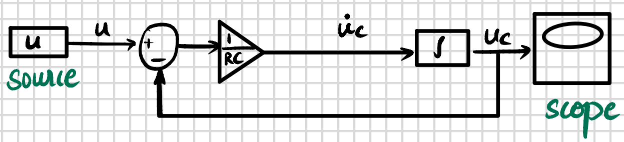
③ $U_R = iR$ $(Q = CU_c)$ $i = C \cdot \frac{dU_c}{dt}$

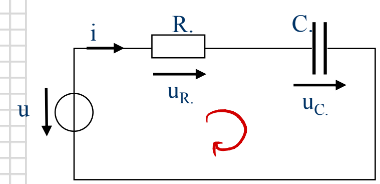
KVL : $U = U_R + U_c$

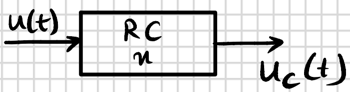
④ $\frac{U_R}{R} = C\frac{dU_c}{dt}$

$\Rightarrow \frac{dU_c}{dt} = \frac{U_R}{RC} = \frac{U - U_c}{RC}$

$\Rightarrow \dot{x} = \frac{dU_c}{dt} = \dot{u}_c = \frac{U - U_c}{RC}$



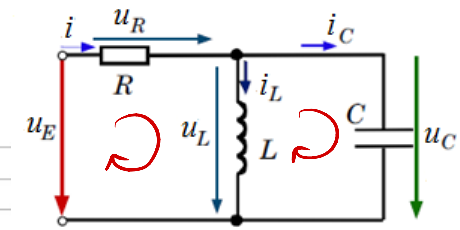




Ex.2.1 Modeling & Simulation (Parallel Resonant Circuit)

A variable voltage $u_E(t)$ is connected to a parallel resonant circuit at time t = 0.

a) We are looking for a simulation model for the variation of the voltage across the capacitor over time.



① $u = u_E(t)$ $\quad y = u_c(t)$ $\quad x_1 = i_L(t)$ $\quad x_2 = u_c(t)$

② labelling

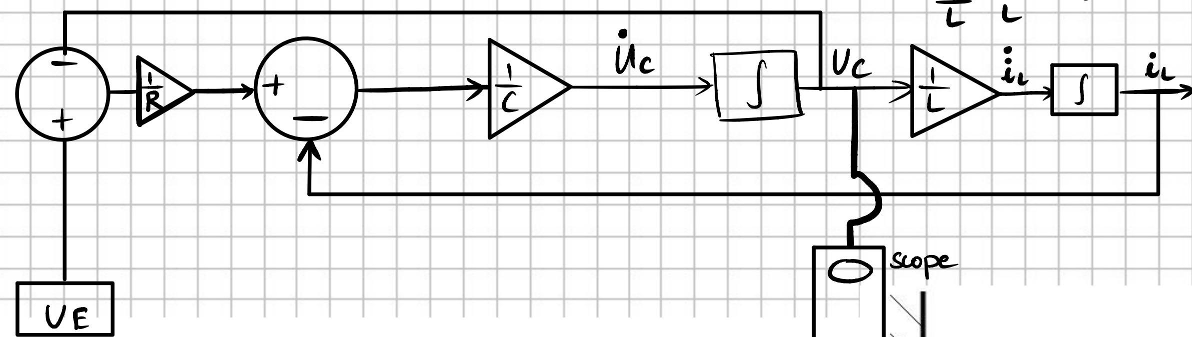
③ $U_R = iR$ $\quad U_L = L\dfrac{di_L}{dt}$ $\quad i_c = C\dfrac{du_c}{dt}$

$U_R + U_L = U_E$ $\quad U_c = U_L$ $\quad i = i_L + i_c$ $\quad U_R = U_E - U_L = U_E - U_c$

$$\frac{U_R}{R} = i_L + C\frac{du_c}{dt}$$

$$\Rightarrow \frac{du_c}{dt} = \frac{\frac{U_R}{R} - i_L}{C} = \frac{\frac{U_E - U_c}{R} - i_L}{C}$$

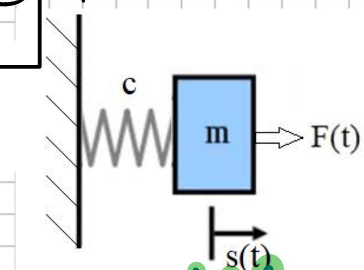
$$\frac{U_c}{L} = \frac{U_L}{L} = \dot{i}_L$$



Spring damper system *

A variable force F(t) acts on a mass m moving only horizontally. The spring c is ideal and massless.

a) We are looking for a model for the course of the location s (t).



① $u = F(t)$ $\quad y = s(t)$ $\quad x_1 = v(t)$ $\quad x_2 = s(t)$

②

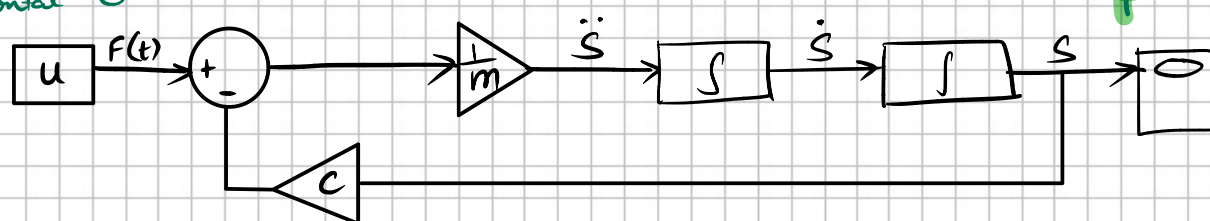
③ $F = cs$ $\quad F = ma = m\ddot{s}$

$F - cs - ma = 0$ $\quad \Sigma F_{horizontal} = 0$

$m\ddot{s} = F - cs$

$$\ddot{s} = \frac{F - cs}{m}$$

practice DE



State Space Representation of LTI-Systems

- For fast computer input, the standardized **state space representation** is often used.
  All **State variables** become one **vector** summarized, the system parameters are thus too **four matrices or vectors**.

State vector:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}}_{\text{System-Matrix}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}}_{\text{Zustd.-Vektor}} + \underbrace{\begin{bmatrix} b_{11} & \dots & b_{1m} \\ b_{21} & \dots & b_{2m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nm} \end{bmatrix}}_{\text{Eingangs-Matrix}} \cdot \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}}_{\text{Eing. Gr.-Vektor}}$$

Equation of state:

$$\boxed{\dot{x} = A \cdot x + B \cdot u}$$

Corresponds to the equations on the right with the restriction to linear systems: $\boxed{\dot{x}_i = f(x,u,p)}$

Output equation:

$$\boxed{y = C \cdot x + D \cdot u}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_r(t) \end{bmatrix} = \underbrace{\begin{bmatrix} c_{11} & \dots & c_{1n} \\ c_{21} & \dots & c_{2n} \\ \vdots & \ddots & \vdots \\ c_{r1} & \dots & c_{rn} \end{bmatrix}}_{\text{Ausgangs-Matrix}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}}_{\text{Zustd.-Vektor}} + \underbrace{\begin{bmatrix} d_{11} & \dots & d_{1m} \\ d_{21} & \dots & d_{2m} \\ \vdots & \ddots & \vdots \\ d_{r1} & \dots & d_{rm} \end{bmatrix}}_{\text{Durchgangs-Matrix}} \cdot \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}}_{\text{Eing. Gr.-Vektor}}$$

Corresponds to the equations on the right with the restriction to linear systems: $\boxed{\dot{x}_i = f(x,u,p)}$

Ex.2.3 State space representation (parallel resonant circuit)

a) Show the parallel oscillating circuit from the previous exercise in the state space representation.

$$\frac{dU_C}{dt} = \frac{\frac{U_R}{R} - i_L}{C} = \frac{\frac{U_E - U_C}{R} - i_L}{C}$$

$$x_1 = i_L(t) \qquad x_2 = U_C(t)$$

$$\dot{x}_1 = \dot{i}_L = \frac{U_L}{L} = \frac{U_C}{L} \qquad \dot{x}_2 = \dot{U}_C$$

$$u = U_E(t) \qquad y = U_C(t)$$

$$U_R = iR \qquad U_L = L\frac{di_L}{dt} \qquad i_C = C\frac{dU_C}{dt} \qquad U_R + U_L = U_E \qquad U_C = U_L \qquad i = i_L + i_C$$

$$\frac{U_E}{CR} - \frac{U_C}{RC} - \frac{i_L}{C}$$

State space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1/L \\ -1/C & -1/RC \end{bmatrix} \begin{bmatrix} \overset{i_L(t)}{x_1} \\ \underset{U_C(t)}{x_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/RC \end{bmatrix} \overset{U_E(t)}{u}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \overset{i_L}{x_1} \\ \underset{U_C}{x_2} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \underset{U_E(t)}{u}$$

State space representation (spring-mass damper)

a) Represent the spring-mass oscillator from the previous exercise in the state space representation.

$$u = F(t) \qquad y = s(t) \qquad x_1 = v = \dot{s}(t) \qquad x_2 = s(t)$$

$$\dot{x}_1 = \dot{v}(t) = \ddot{s}(t) = \frac{F - cs}{m} \qquad \dot{x}_2 = \dot{s}(t) = v(t)$$

State space form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -c/m \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \end{bmatrix} u \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

Description by a Single DE (of higher order)

Mathematical theorem:

- A DE-system of N first-order equations can always be transformed into **a single DE of Nth order!**

$$\left.\begin{aligned} \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}b_{11}\\ b_{22}\end{bmatrix}\cdot u \\ y &= \begin{bmatrix}c_1 & c_2\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix} + d\cdot u \end{aligned}\right\} \Rightarrow \quad a_2\ddot{y} + a_1\dot{y} + a_0 y = b_2\ddot{u} + b_1\dot{u} + b_0 u$$

Result: A single linear DE with constants coefficients

- the **order og differential equation is e.g. N = 2** and corresponds the so-called **system order.**
  On the left is the output variable y and its derivatives, on the right the input variable u and its derivatives.
  Information about the internal states x_i get lost!

Example system of 2nd order:

$$\dot{x}_1 = 2x_1 + x_2 \qquad y = x_1$$

$$\dot{x}_2 = x_1 + u$$

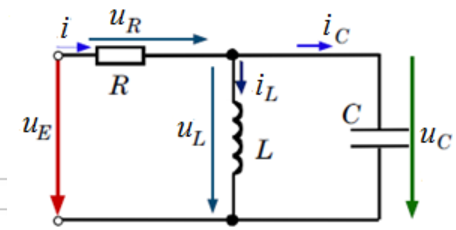
Target: single DE of 2nd order as function of y & u only

$$x_2 = \dot{x}_1 - 2x_1 = \dot{y} - 2y$$

$$\Rightarrow \dot{x}_2 = \ddot{y} - 2\dot{y}$$

$$\Rightarrow \ddot{y} - 2\dot{y} = y + u \Rightarrow \ddot{y} - 2\dot{y} - y = u$$

Ex.2.4 Converting a DE system into a single DE of a higher order



a) Describe the input-output behavior of the parallel resonant circuit with input u_E and exit u_C by a only DGL 2nd order.

$$\dot{x} = \begin{pmatrix}\dot{x}_1 \\ \dot{x}_2\end{pmatrix} = \begin{bmatrix}0 & 1/L \\ -1/C & -1/RC\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ 1/RC\end{bmatrix}u$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0\end{bmatrix}u$$

$$\dot{x}_1 = \frac{x_2}{L} = \frac{U_c}{L} = \dot{i}_L$$

$$\dot{x}_2 = -\frac{x_1}{C} - \frac{x_2}{RC} + \frac{u}{RC} = \frac{1}{C}\left(\frac{U_E - U_c}{R} - i_L\right) = \dot{U}_c$$

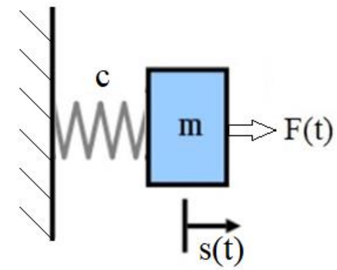
$$y = x_2 = U_c$$

$$\ddot{U}_c = \frac{1}{C}\left(\frac{\dot{U}_E - \dot{U}_c}{R} - \dot{i}_L\right) = \frac{1}{C}\left(\frac{\dot{U}_E - \dot{U}_c}{R} - \frac{U_c}{L}\right)$$

$$\Rightarrow \ddot{U}_c = \frac{\dot{U}_E}{RC} - \frac{\dot{U}_c}{RC} - \frac{U_c}{LC} \Rightarrow LC\ddot{U}_c + \frac{L}{R}\dot{U}_c + U_c = \frac{L}{R}\dot{U}_E$$

Input-output description by a single DE (higher order)

a) Describe the input-output behavior of the spring-mass oscillator using a single DGL.



$u = F(t)$ $\quad y = s(t)$ $\quad x_1 = v = \dot{s}(t)$ $\quad x_2 = s(t)$

$$\dot{x}_1 = \dot{v}(t) = \ddot{s}(t) = \frac{F - cs}{m} \qquad \dot{x}_2 = \dot{s}(t) = v(t)$$

$$\ddot{s} = \dot{v}$$

$$\ddot{s}(t) = \frac{F}{m} - \frac{cs}{m} \Rightarrow \boxed{m\ddot{s} + cs = F}$$

Four-step method*:

- Calculation of the general homogeneous solutions
- Calculation of a particulate solution
- Superposition of homogeneous and particulate solution
- Determination of the constants via initial conditions

Step and impulse responses are often used to characterize systems. With LTI systems, they can generally be calculated in the time domain using the four-step method, which we do not want to pursue any further here:

Transient Response (homogeneous solution):

- of the form of one or more e-functions

Steady State Response (particulate solution):

- on the form of the stimulus.

Example RC element: $\boxed{\mathrm{DGL:}\quad \mathrm{RC}\,\dot{\mathrm{y}} + \mathrm{y} = \mathrm{u} \qquad \mathrm{T} = \mathrm{RC}}$

A) Jump-like excitation:

$$u(t) = \varepsilon(t)$$

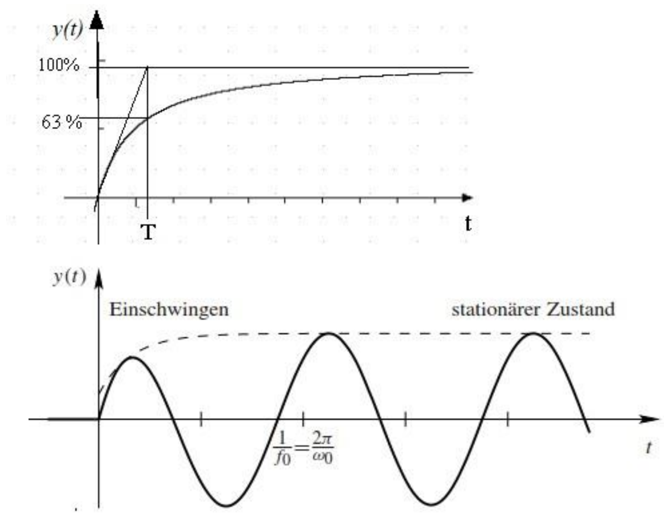
$$y(t) = [\underbrace{-e^{-t/T}}_{Einschwinganteil} + \underbrace{1}_{station\ddot{a}rer Anteil}]\cdot \varepsilon(t)$$

B) Oscillatory excitation:

$$u(t) = \cos\left(\omega t\right)\cdot \varepsilon(t)$$

$$y(t) = [\underbrace{-e^{-t/T}}_{Einschwinganteil} + \underbrace{\frac{\cos(\omega t + \phi_0)}{\cos(\phi_0)}}_{station\ddot{a}rer Anteil}]\cdot \varepsilon(t)$$

$$mit\ \phi_0 = -\arctan(\omega T)$$



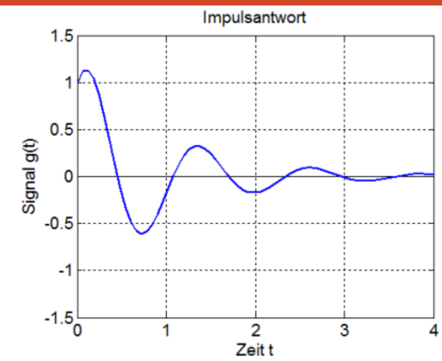
Einschwingvorgang und stationärer Zustand am Beispiel RC-Glied

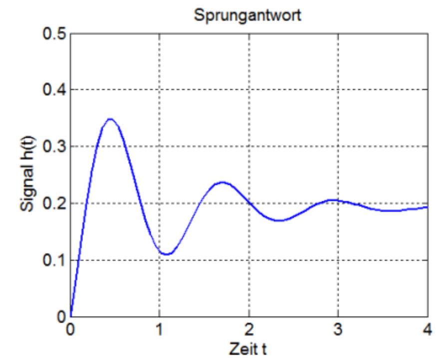
Impulse response g(t):

- Called the response of a system to a pulse **δ (t)** at the input.

Step response h(t):

- Called the response of a system to a **jump ε(t)** at the input.





Convolution:

- It calculates the output signal y(t) of an LTI system for a given impulse response g(t) **for any input signal u(t)!**

$$\boxed{y(t) = \int_{-\infty}^{\infty} u(\tau) \cdot g(t-\tau) d\tau = u(t) * g(t) = g(t) * u(t)}$$

Brief derivation of the convolution integral:

An LTI system with the known impulse response g (t) responds to a linear combination of impulses

$$u(t) = c_1 \cdot \delta(t-t_1) + c_2 \cdot \delta(t-t_2)$$

with the same linear combination of impulse responses (superposition principle).

$$y(t) = c_1 \cdot g(t-t_1) + c_2 \cdot g(t-t_2)$$

This relationship can be generalized to any input signals. Because of the hiding property of the Impulse function, any input signal u (t) can be represented as:

$$u(t) = u(t) \cdot \int_{-\infty}^{+\infty} \delta(t-\tau) d\tau = \int_{-\infty}^{+\infty} u(t) \cdot \delta(t-\tau) d\tau = \int_{-\infty}^{+\infty} u(\tau) \cdot \delta(t-\tau) d\tau$$

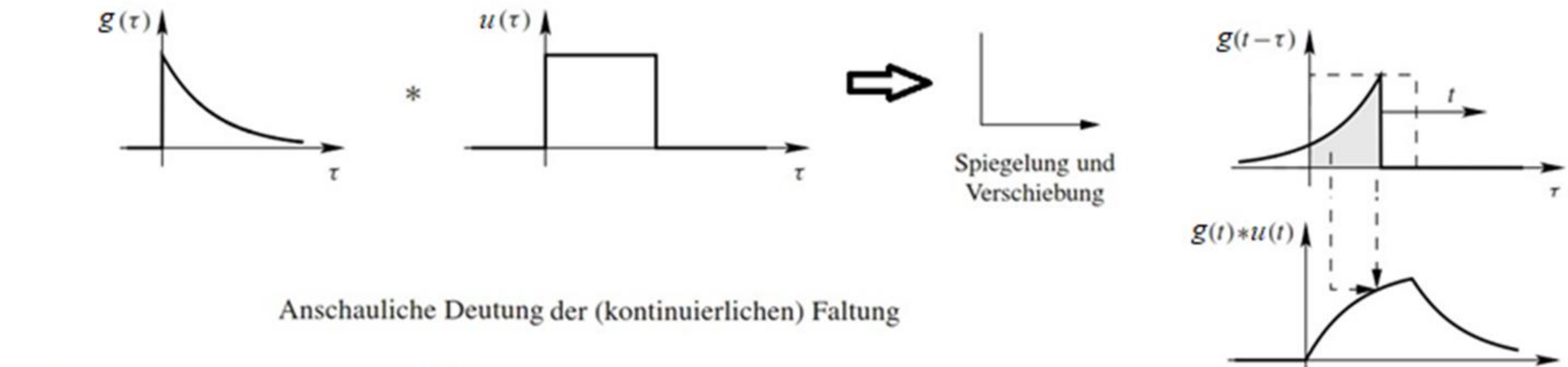
The above equation can clearly be interpreted as a superposition of an infinite number of impulses δ (t - τ) with the weight u (t) together represent the signal u (t). Every single impulse δ (t - τ) now has the system response g (t - τ).
The output signal y (t) results from the superposition of an infinite number of system responses g (t - τ) with the weight u (t), what corresponds to the above convolution product.

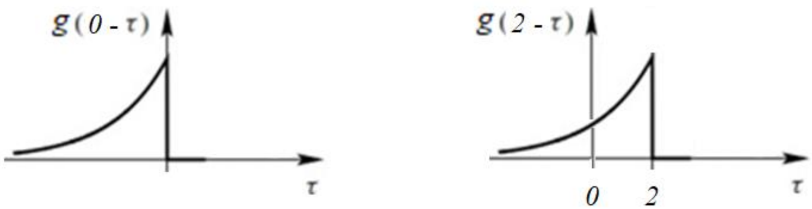
Graphical determination of the convolution integral:

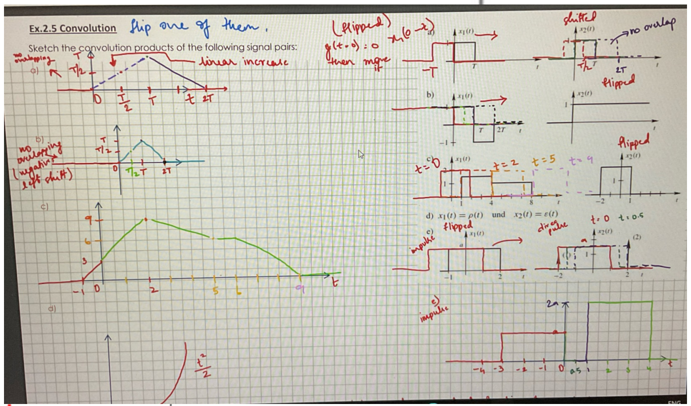
$$y(t) = u(t) * g(t) = \int_{-\infty}^{\infty} u(\tau) \cdot g(t-\tau) d\tau$$

- Sketch the function u(τ)
- Sketch the function g(t - τ) by mirroring the function g (τ) and shift by t**o the right!**
- Compute the product of the two functions u(τ) · g(t - τ) for all τ.
- Determination of the total area under the curve u(τ) · g(t - τ).

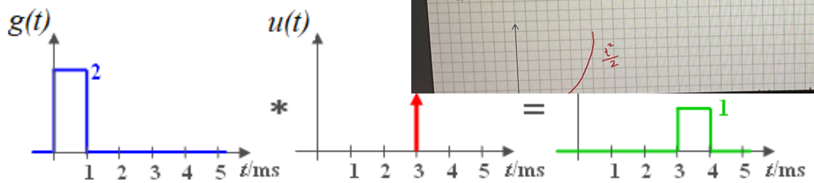
Note: Since the convolution is commutative, it does not matter which function is mirrored and shifted, ie expediently select the simpler function.







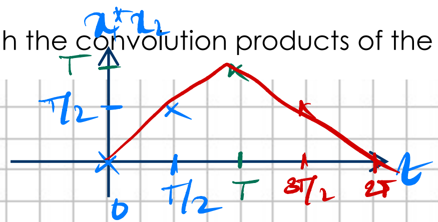
- Convolution with the Dirac pulse:

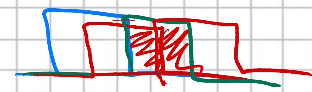


Ex.2.5 Convolution

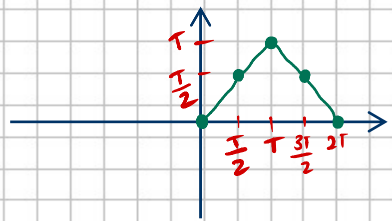
Sketch the convolution products of the following signal pairs:

a)

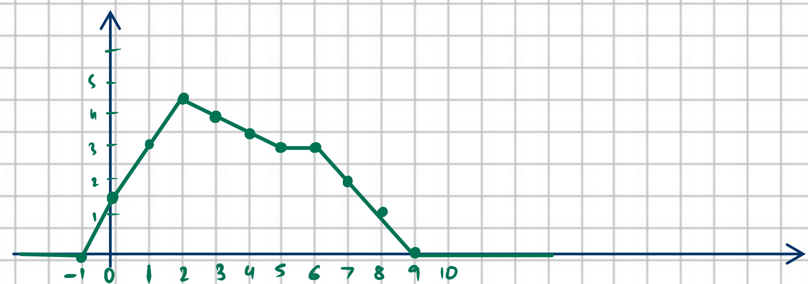




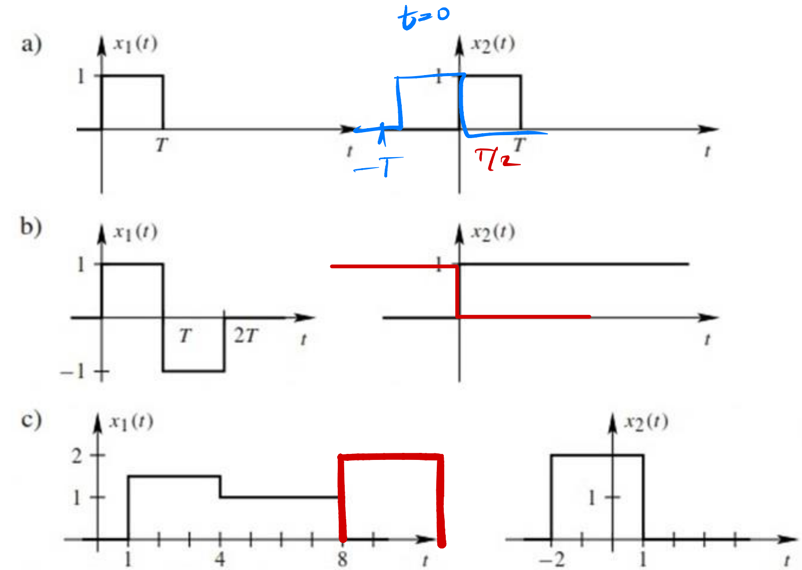
b)



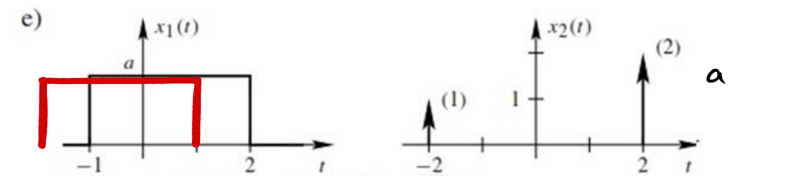
c)

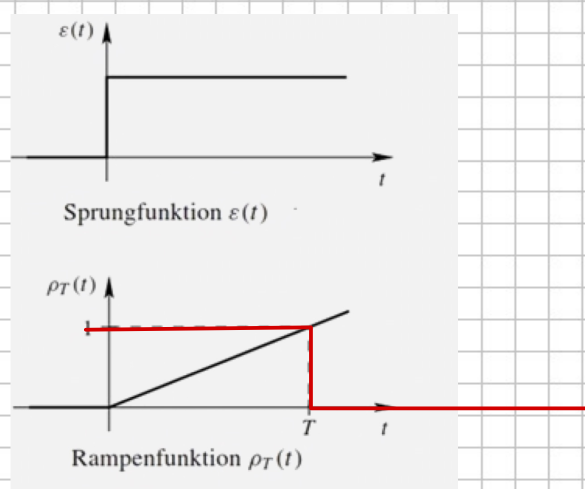


d)

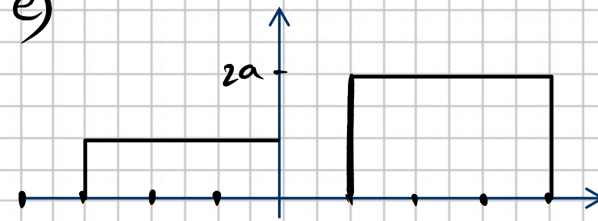


$x_1(t) = \rho(t)$ und $x_2(t) = \varepsilon(t)$





e)



Ex.2.6 Discrete 2D-Convolution

The convolution operation can also be extended to two (location) dimensions. The 2D convolution is the central operation of image processing. A so-called filterKernel k (x, y), e.g. folded into a 3x3 matrix.

$$y(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(i,j)k(x-i,y-j)\,di\,dj \qquad y(x,y) = \sum_{-\infty}^{+\infty}\sum_{-\infty}^{+\infty} u(i,j)k(x-i,y-j)$$

a) Calculate in the two framed areas the convolution of the picture u with the kernels k_i.

u (x, y)

0	**0**	**0**	**1**	**1**	**1**	**1**
0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	0	1	1	1
0	0	0	0	1	1	1
0	0	0	0	1	1	1

*

$k_1(x, y)$

1	**1**	**1**
1	1	1
1	1	1

=

$y_1(x, y)$

0	~~0~~	~~2~~	~~4~~	~~6~~	~~6~~	~~4~~
0	0	3	6	9	9	
0	0	3	6	9	9	
0	0	2	5	8	9	
0	0	1	4	7	9	
0	0	0	3	7	9	
0						

→ 0

* should we complete this

u (x, y)

0	**0**	**0**	**1**	**1**	**1**	**1**
0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	1	1	1	1
0	0	0	0	1	1	1
0	0	0	0	1	1	1
0	0	0	0	1	1	1

*

$k_2(x, y)$

0	**1**	**0**
1	-4	1
0	1	0

=

$y_2(x, y)$

Auto-Correlation function ACF (for energy signals):

- Shows within random signals the temporal bindings

$$\boxed{\varphi_{xx}(\tau) = \int_{-\infty}^{\infty} x(t) \cdot x(t+\tau) dt}$$

The AKF is symmetrical and can be calculated like a convolution operation,

- $\varphi(0)$ is the signal energy E_x.

$$\boxed{\varphi_{xx}(\tau) = \varphi_{xx}(-\tau)} \quad \boxed{\varphi_{xx}(\tau) = x(\tau) * x(-\tau)}$$

- **Use:** Finding hidden repetitions in signals:
  - The AKF of a periodic signal is also periodic with the same period duration
  - The AKF shows a maximum at the time an echo occurs (radar ...)

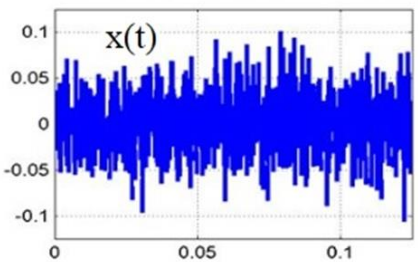
Cross-Correlation function CCF:

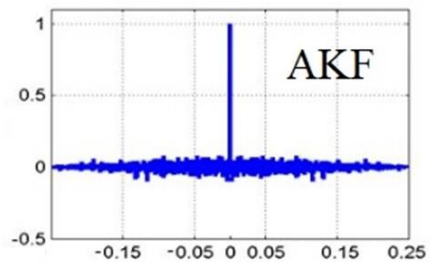
$$\boxed{\varphi_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) \cdot y(t+\tau) dt}$$

$$\boxed{\varphi_{xy}(\tau) = \varphi_{yx}(-\tau)}$$

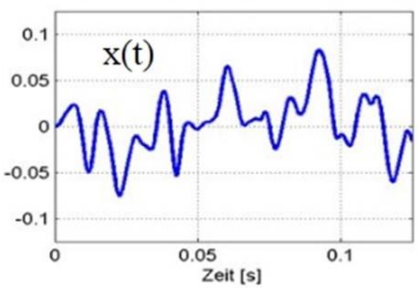
- Examples to the ACF:

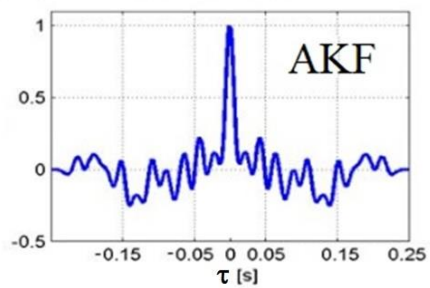
Benachbarte Rausch-Sample sind sich *nicht* ähnlich

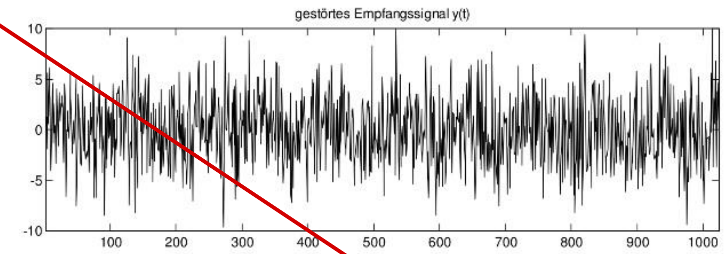


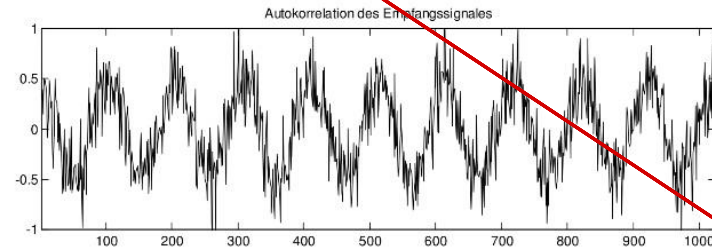


Benachbarte Rausch-Sample sind sich *ziemlich* ähnlich





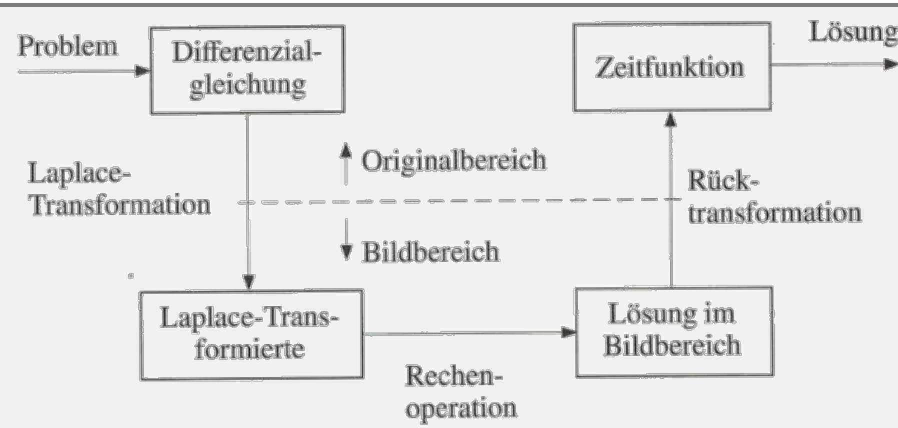




Definition of Laplace Transform

The Laplace transform converts a DE in the **time domain (original domain)** in a simpler algebraic equation in the **s-domain (image domain)** and is **the** key to the treatment of LTI systems!

time domain → s domain



Elementary properties of the ordinary (single-sided) Laplace transform:

- **Definition:** $$\boxed{X(s) = L\{x(t)\} = \int_0^\infty x(t)\cdot e^{-st}dt, \quad \text{mit } x(t) = 0 \quad \text{für } t < 0}$$
  $$x(t) = L^{-1}\{X(s)\} = \frac{1}{2\pi j}\int_{\sigma-\infty}^{\sigma+\infty} X(s)e^{st}ds$$
- **Dumbbell icon:** $x(t) \quad \circ\!-\!\bullet \quad X(s)$
  $X(s) \quad \bullet\!-\!\circ \quad x(t)$
- **Differentiation rule*:** $\dot{x}(t) \quad \circ\!-\!\bullet \quad s\cdot X(s)$
  $\ddot{x}(t) \quad \circ\!-\!\bullet \quad s^2\cdot X(s)$
- **Linearity:** $a_1x_1(t)+a_2x_2(t) \quad \circ\!-\!\bullet \quad a_1X_1(s)+a_2X_2(s)$
- s is "only" a complex number, called the Laplace variable s = σ + jω.

* The above formula only applies if the **left-hand limit** of the function x (t) and all its derivatives (at t = 0_-) are zero! In the case of continuous functions, x (0_-) = x (0) = x (0_+). For more details, see the appendix.

$$\dot{x}(t) \quad \circ\!-\!\bullet \quad s\cdot X(s) - x(0_-)$$
$$\text{mit} \quad x(0_-) = \lim_{h\to 0} x(0-h)$$

3.2 Laplace Transform

Calculate the Laplace transform of the following signals.

a) $x(t) = \varepsilon(t)$

b) $x(t) = e^{-at} \cdot \varepsilon(t)$

d) $x(t) = e^{-4t} \cdot \cos(t) \cdot \varepsilon(t)$

e) $x(t) = 3t^3 \cdot \varepsilon(t)$

f) $x(t) = \int_0^t 3e^{-\tau} d\tau$

g) $x(t) = \begin{cases} 2 & \text{für} \quad 0 \le t < T \\ 0 & \text{sonst} \end{cases}$

h) $x(t) = (t-2) \cdot \varepsilon(t)$

i) $x(t) = (t-2) \cdot \varepsilon(t-2)$

a) $x(t) = \varepsilon(t) \quad X(s) = \mathcal{L}\{\varepsilon(t)\} = \int_0^\infty \varepsilon(t) e^{-st} dt = \int_0^\infty e^{-st} dt = \frac{1}{s}$ (from table)

b) $x(t) = e^{-at} \cdot \varepsilon(t) \circ\!\!\xrightarrow{\mathcal{L}}\!\!\bullet \; X(s) = \frac{1}{s+a}$

d) $x'(t) = \cos(t) \cdot \varepsilon(t) \circ\!\!\xrightarrow{\mathcal{L}}\!\!\bullet \; X'(s) = \frac{s}{s^2+1} \Rightarrow x(t) = e^{-4t} x'(t) = X(s+4) = \frac{s+4}{(s+4)^2+1}$

e) $x'(t) = \frac{t^2}{2} \varepsilon(t) \circ\!\!\xrightarrow{\mathcal{L}}\!\!\bullet \; X'(s) = \frac{1}{s^3}$

$x(t) = 6t \, x'(t) \circ\!\!\xrightarrow{\mathcal{L}}\!\!\bullet \; 6\left(-\frac{d}{ds} X'(s)\right) = -6 \frac{d s^{-3}}{ds} = \frac{18}{s^4}$

i) $x(t) = (t-2)\varepsilon(t-2)$

$t\varepsilon(t-2) - 2\varepsilon(t-2)$

$\Big\downarrow \mathcal{L}$

$X(s) = \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s}$

f) $x(t) = \int_0^t \underbrace{3e^{-\tau}}_{x(\tau)} d\tau \circ\!\!\xrightarrow{\mathcal{L}}\!\!\bullet \; X(s) = \frac{1}{s} X(s) = \frac{3}{s} \cdot \frac{1}{s+1} = \frac{3}{s(s+1)}$

g) $x(t) = \begin{cases} 2 & \text{for } 0 \le t < T \\ 0 & \text{sonst} \end{cases} \quad x(t) = 2[\varepsilon(t) - \varepsilon(t-T)] \circ\!\!\xrightarrow{\mathcal{L}}\!\!\bullet \; 2\left[\frac{1}{s} - \frac{e^{-sT}}{s}\right]$

h) $x(t) = (t-2) \cdot \varepsilon(t) = t\varepsilon(t) - 2\varepsilon(t) \circ\!\!\xrightarrow{\mathcal{L}}\!\!\bullet \; X(s) = \frac{1}{s^2} - \frac{2}{s}$

make graph & check

* why is the other using formula?

Ex.2.8 Laplace transform for initial value problems

Solve the following two differential equations with initial conditions below using the Laplace transform and sketch the respective solution x(t).

a) $\dot{x}(t)+x(t)=0 \qquad x(0\_)=1$

b) $\dot{x}(t)+x(t)=\varepsilon(t) \qquad x(0\_)=0$

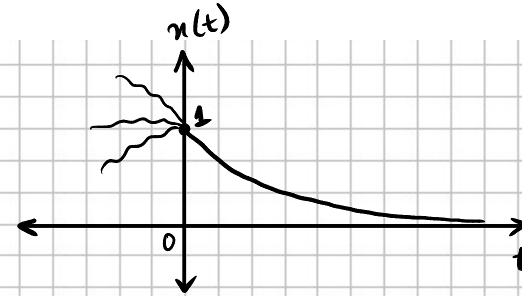
c) $\dot{x}(t)+x(t)=\delta(t) \qquad x(0\_)=0$

algebraic equation

a) DE: $\dot{x}(t)+x(t)=0 \;\circ\!\xrightarrow{\mathcal{L}}\!\bullet$ AE: $sX(s)-X(0_-)+X(s)=0$

$\Rightarrow sX(s)-1+Xs=0$

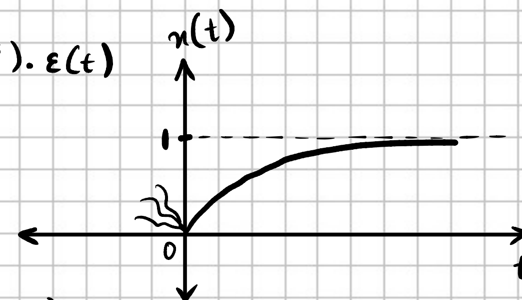
$\Rightarrow X(s)=\dfrac{1}{s+1} \;\bullet\!\xrightarrow{\mathcal{L}^{-1}}\!\circ\; x(t)=e^{-t}\cdot\varepsilon(t)$



b) $\dot{x}(t)+x(t)=\varepsilon(t) \;\circ\!\xrightarrow{\mathcal{L}}\!\bullet$ AE: $s\cdot X(s)-x(0_-)+X(s)=\dfrac{1}{s}$

$\Rightarrow X(s)(s+1)=\dfrac{1}{s}$

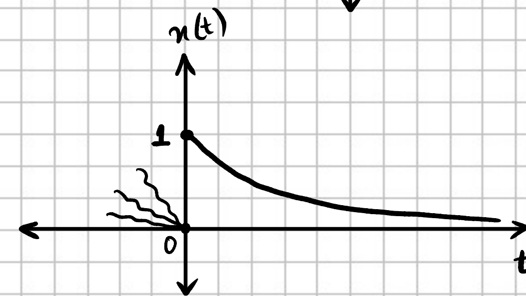
$\Rightarrow X(s)=\dfrac{1}{s(s+1)} \;\bullet\!\xrightarrow{\mathcal{L}^{-1}}\!\circ\; x(t)=1(1-e^{-t})\cdot\varepsilon(t)$



c) $\dot{x}(t)+x(t)=\delta(t) \;\circ\!\xrightarrow{\mathcal{L}}\!\bullet\; sX(s)-x(0_-)+X(s)=1$

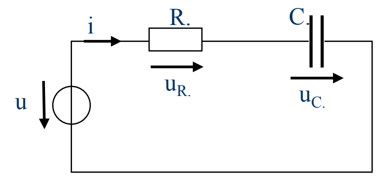
$\Rightarrow X(s)(s+1)=1$

$\Rightarrow X(s)=\dfrac{1}{s+1} \;\bullet\!\xrightarrow{\mathcal{L}^{-1}}\!\circ\; x(t)=e^{-t}\varepsilon(t)$



Ex.2.9 Laplace transform for initial value problems

Before t = 0, the capacitor is charged to $u_C(0_-)$. At t = 0 the circuit is operated with a sinusoidal voltage u(t) of frequency $\omega = 1$ (R = 1 Ohm, C = 1 Farad),
Calculate the exact course of the voltage $u_c(t)$ using the techniques of the Laplace transform.



DE: $\underbrace{RC}_{1} \cdot \dot{U}_c(t) + U_c(t) = u(t)$

$U_R + U_c = u(t)$

$\dot{U}_c(t) + U_c(t) = \sin(\omega t)\,\varepsilon(t)$

$\Rightarrow s V_c(s) - U_c(0-) + U_c(s) = \frac{1}{s^2+1}$

$\Rightarrow V_c(s)(s+1) = \frac{1}{s^2+1} + U_c(0-)$

$\Rightarrow V_c(s) = \frac{1}{(s^2+1)(s+1)} + \frac{U_c(0_-)}{(s+1)}$ PFD

$\Rightarrow V_c(s) = \frac{0.5}{s+1} + \frac{0.5-0.5s}{s^2+1} + \frac{U_c(0_-)}{s+1}$

$\mathcal{L}^{-1}$ $\quad \frac{-0.5(s-1)}{s(s+1)}$ $\quad a=-1$, $b=1$

$U_c(t) = \left[0.5e^{-t} - 0.5\left(\frac{-1}{1} + \frac{1-(-1)}{1}e^{-t}\right) + U_c(0_-)e^{-t}\right]\varepsilon(t)$

* ans doesn't match

how do we get sin, cos terms

* how to do PFD

$\frac{1}{(s^2+1)(s+1)} = \frac{A}{s^2+1} + \frac{B}{s+1}$

$1 = As + A + Bs^2 + B$

$1 = (A+B)(1) + s(A+Bs)$

$A + B = 1$

$A + Bs = 0$

- For LTI systems without prehistory the transfer function is defined as:

$$\boxed{\begin{aligned} y^i(0\_) &= 0 \\ u^i(0\_) &= 0 \end{aligned}}$$ (= zero state)

Laplace ↓

$$\ldots + a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = \ldots + b_2\ddot{u}(t) + b_1\dot{u}(t) + b_0 u(t)$$

$$\Downarrow L$$

$$\ldots + a_2 s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = \ldots + b_2 s^2 U(s) + b_1 s U(s) + b_0 U(s)$$

$$Y(s)\cdot[\ldots + a_2 s^2 + a_1 s + a_0] = U(s)\cdot[\ldots + b_2 s^2 + b_1 s + b_0]$$

Transfer Function:

- It is (in the image domain) the **output in relation to the inout**:

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{\ldots + b_2 s^2 + b_1 s + b_0}{\ldots + a_2 s^2 + a_1 s + a_0}}$$

System Equation:

$$\boxed{Y(s) = G(s)\cdot U(s)}$$

Impulse Response:

(or weight function)

$$\boxed{g(t) \circ\!-\!\bullet\, G(s)}$$

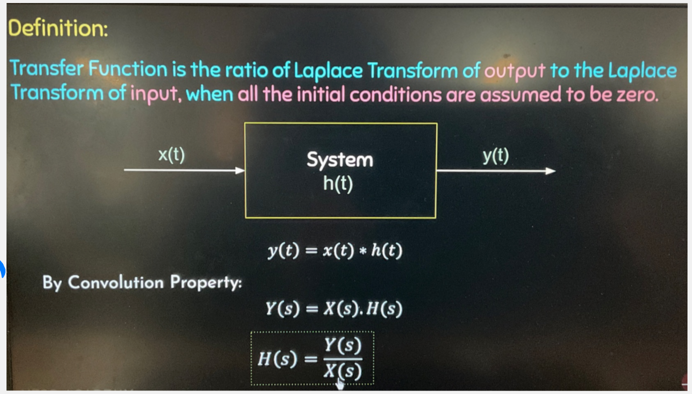
laplace transform of impulse response gives transfer function

Step response:

$$\boxed{h(t) \circ\!-\!\bullet\, G(s)\cdot\tfrac{1}{s}}$$

for step response

$$\Rightarrow \quad h(t) = \int_{\cancel{-\infty}\;0}^{t} g(\tau)\,d\tau$$



Example Transfer function of the RC element, R = 2Ω, C = 1F:

DE: $RC\,\dot{u}_c(t) + u_c(t) = u(t)$

$\circ\!-\!\bullet\; \mathcal{L}$

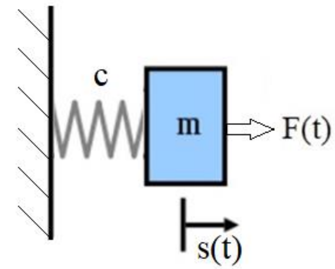
AE: $RCsU_c(s) - \underbrace{RC\,U_c(0_-)}_{=\,0 \text{ at zero state?}} + U_c(s) = U(s)$

$\Rightarrow U_c(s)\,[2s+1] = U(s)$

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{U_c(s)}{U(s)} = \frac{1}{2s+1}}$$

Ex.2.10 Transfer function

a) Give the transfer function G(s) of the system.



DE : $m\ddot{s}(t) + c\,s(t) = F(t)$

$\circ\!\!-\!\!\bullet\ \mathcal{L}$

AE: $ms^2 Y(s) - \overbrace{s \cdot Y(0_-) - \dot{Y}(0_-)}^{0\ *\,?} + cY(s) = U(s)$

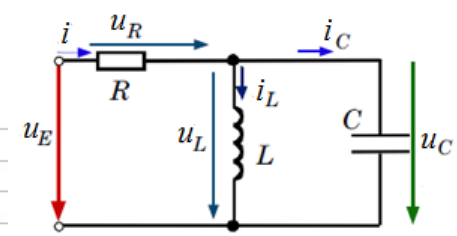
Only for zero state → nothing has happened yet.

$Y(s)\,(ms^2 + c) = U(s)$

$\therefore \boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + c}}$

Transfer function (PT2 element)

a) Give the transfer function G(s) of the system.



$LC\ddot{U}_c(t) + \frac{L}{R}\dot{U}_c(t) + U_c(t) = \frac{L}{R}\dot{U}_E(t)$

zero state

$\circ\!\!-\!\!\bullet\ \mathcal{L}$

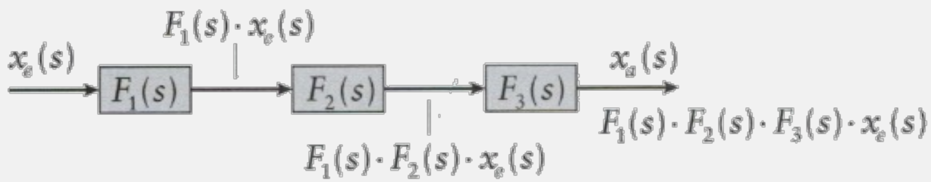
$\Rightarrow LCs^2 Y(s) + \frac{L}{R} s Y(s) + Y(s) = \frac{L}{R} s U(s)$

$\Rightarrow Y(s)\left[LCs^2 + \frac{Ls}{R} + 1\right] = \frac{Ls}{R} U(s)$

$\Rightarrow G(s) = \frac{Y(s)}{U(s)} = \frac{Ls/R}{LCs^2 + Ls/R + 1}$

Transfer functions of a

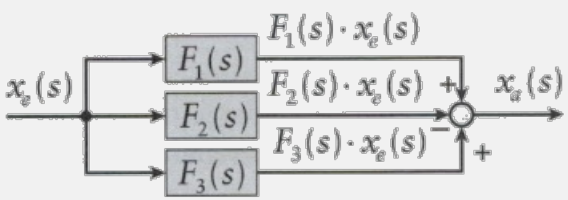
A) Chain structure:



$$x_a(s) = F_1(s)\cdot F_2(s)\cdot F_3(s)\cdot x_e(s) = F(s)\cdot x_e(s)$$

$$F(s) = F_1(s)\cdot F_2(s)\cdot F_3(s)$$

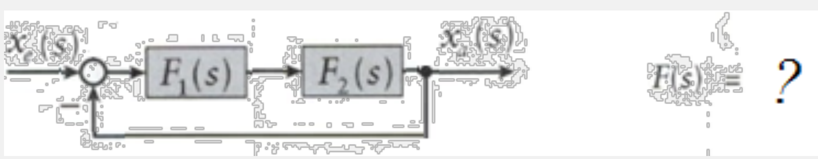
B) Parallel structure:



$$x_a(s) = (F_1(s) - F_2(s) + F_3(s))\cdot x_e(s) = F(s)\cdot x_e(s)$$

$$F(s) = F_1(s) - F_2(s) + F_3(s)$$

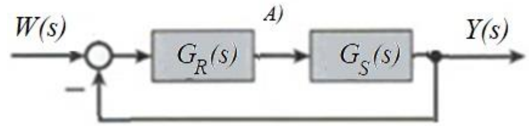
C) Circle structure: control loop



$$x_a(s) = F_2(s)\,F_1(s)\,(x_e(s) - x_a(s))$$

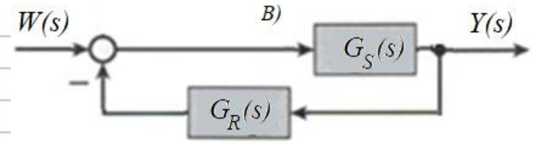
$$F(s) = F_2(s)\,F_1(s)$$

Ex.2.12 Transfer function of complex systems



a) Calculate for the two standard control loops ($G_S(s)$ is the proces transfer function, $G_R(s)$ the controller transfer function) the total transfere function $G_{WY}(s)$, ie

$$G(s) = \frac{Y(s)}{W(s)}$$



A) $Y = G_S G_R (W - Y)$

$Y = G_S G_R W - G_S G_R Y$

$Y(1 + G_S G_R) = G_S G_R W$

$$G(s) = \frac{Y(s)}{W(s)} = \frac{G_S(s)\, G_R(s)}{1 + G_S(s)\, G_R(s)}$$

$(W - G_R Y)\, G_S = Y$

B) $Y = G_S (W - G_R Y)$

$\Rightarrow Y = G_S W - G_S G_R Y$

$\Rightarrow Y(1 + G_S G_R) = G_S W$

$$\Rightarrow G(s) = \frac{Y(s)}{W(s)} = \frac{G_S(s)}{1 + G_S(s)\, G_R(s)}$$

*?

b) For the first standards control loop for a step-like input w(t), calculate the response y(t) at t = ∞ using the limit theorem of the Laplace transform. The shown partial transfer functions are given:

$$G_R(s) = K \qquad G_S(s) = \frac{s+1}{s^2+2s+2}$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\cdot Y(s) = \lim_{s\to 0} \cancel{s}\cdot G(s)\cdot \underbrace{W(s)}_{=\frac{1}{\cancel{s}}} = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{G_S G_R}{1 + G_S G_R}$$

$W(s) = \frac{1}{s}$ $\quad W(s) = \frac{1}{s}$

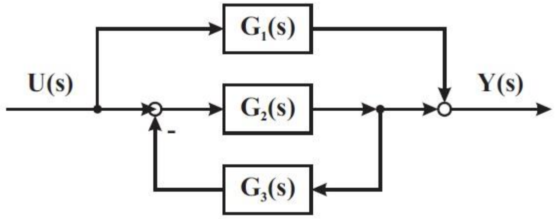
$w(t) = \varepsilon(t)$

*??

$$= \lim_{s\to 0} \frac{K \frac{s+1}{s^2+2s+2}}{1 + \frac{K(s+1)}{s^2+2s+2}} = \frac{\frac{K}{2}}{1 + \frac{K}{2}} = \boxed{\frac{K}{2+K}}$$

2.9 Transfer Function of complex systems

a) Find the overall transfer function $G_{total}(s)$ of the system shown.



$Y = Y_1 + Y_2 \qquad Y_1 = G_1 U$

$Y_2 = G_2(U - G_3 Y_2) = U G_2 - G_2 G_3 Y_2$

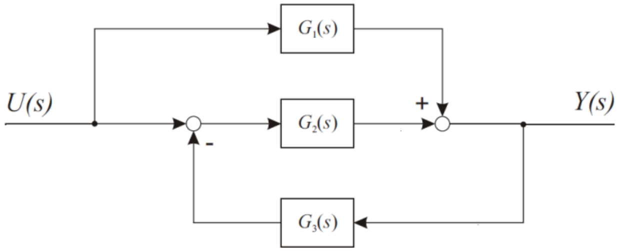
$Y_2(1 + G_2 G_3) = U G_2 \Rightarrow Y_2 = \dfrac{UG_2}{1+G_2G_3}$

$Y = Y_1 + Y_2 = UG_1 + \dfrac{UG_2}{1+G_2G_3}$

$(U - G_3 Y_2)\, G_2$

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = G_1 + \frac{G_2}{1+G_2G_3}}$$

b) Find the overall transfer function $G_{total}(s)$ of the system shown.



$Y_1 = UG_1$

$Y = UG_1 + G_2(U - G_3 Y) \Rightarrow Y = UG_1 + G_2 U - G_3 Y$

$\Rightarrow Y(1 + G_3) = U(G_1 + G_2)$

$\therefore G(s) = \dfrac{Y(s)}{U(s)} = \dfrac{G_1 + G_2}{1 + G_3}$

$= \dfrac{G_1(s) + G_2(s)}{1 + G_3(s)}$

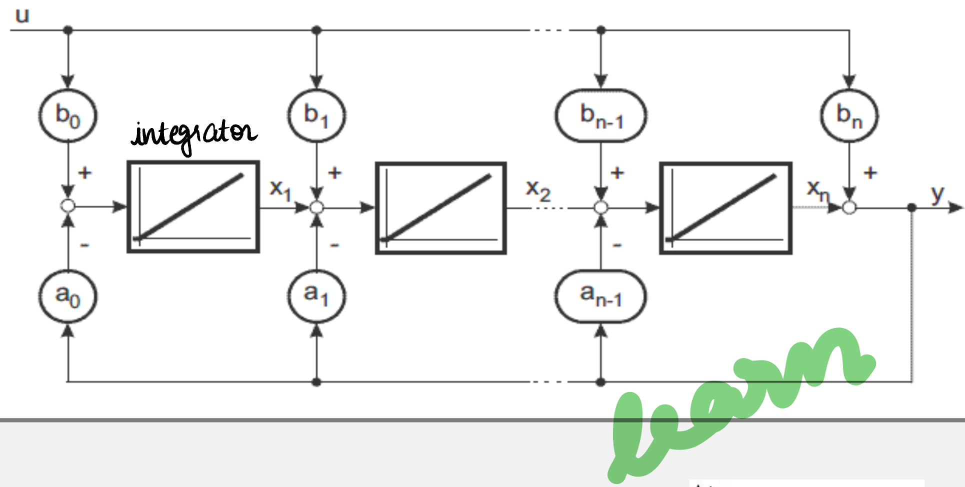
SW Implementation of a Transfer Functions

First canonical structure:

Derivation e.g. in in Otto Foellinger; Control technology.

software

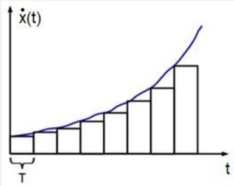
$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + \ldots + b_2 s^2 + b_1 s + b_0}{s^n + \ldots + a_2 s^2 + a_1 s + a_0}$$



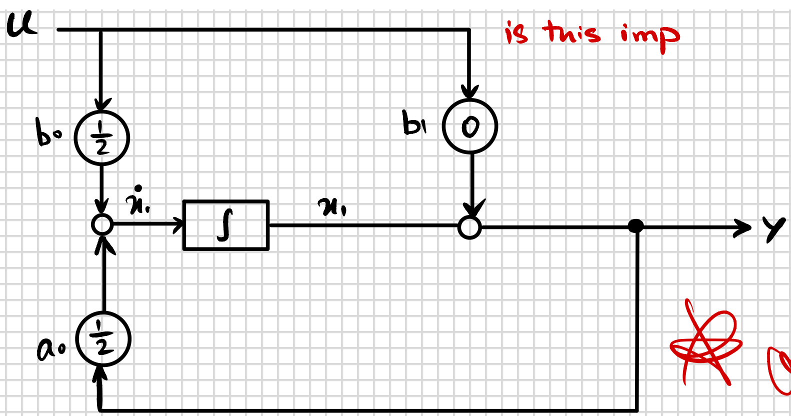
Integration in SW loop:

For example, if you want a filter G (s) in a **SW loop (digital)** realize, the above canonical structures are also used directly as a prototype, if you are doing the integrators for the states x_i replaced by a numerical approximation:
(T is the so-called sampling time, or the time for one loop run.

$$x_i = x_{i\_old} + T \cdot \dot{x}_i$$



- Example: RC element as a digital filter: $G(s) = \frac{Y(s)}{U(s)} = \frac{1}{1+2s} = \frac{\frac{1}{2}}{s+\frac{1}{2}} = \frac{b_0}{s+a_0}$ $\tau = RC = 2\sec$



??

software code

```
x1=0   b1= 0.5   b0=0   a0= 0.5   T=1
while (1) {
y= x1+ b0u ;  [x1+ 0.u]
x1p = b1u - a0y ;
x1 = x1 + T. x1p ;
}
```

5.11 Inverse Laplace transform by Partial fraction decomposition

a) Calculate, for example, the inverse Laplace transform of functions a) to c) using partial fraction decomposition and a correspondence table.

a) $G(s)=\dfrac{5s+11}{(s+5)\cdot(s-2)}$

b) $G(s)=\dfrac{s^2+4s}{(s+1)^2\cdot(s+2)}$

c) $G(s)=\dfrac{2s-3}{s-3}$

ⓐ

$$G(s)=\frac{5s+11}{(s+5)(s-2)}=\frac{A}{s+5}+\frac{B}{s-2}=\frac{As-2A+Bs+5B}{(s+5)(s-2)}$$

By coeff. comparison $A+B=5$ and $5B-2A=11$

$\Rightarrow A=2$, $B=3$

$$\therefore G(s)=\frac{2}{s+5}+\frac{3}{s-2}\ \circ\!\!\overset{\mathcal{L}^{-1}}{\longrightarrow}\!\!\bullet\ g(t)=\left[2e^{-5t}+3e^{2t}\right]\varepsilon(t)$$

ⓑ

$$G(s)=\frac{s^2+4s}{(s+1)^2\cdot(s+2)}=\frac{A}{s+2}+\frac{B}{s+1}+\frac{C}{(s+1)^2}$$

$A=-4$
$B=5$
$C=-3$

$$G(s)=\frac{-4}{s+2}+\frac{5}{s+1}-\frac{3}{(s+1)^2}\ \circ\!\!\overset{\mathcal{L}^{-1}}{\longrightarrow}\!\!\bullet\ g(t)=\left[-4e^{-2t}+5e^{-t}-3te^{-t}\right]\varepsilon(t)$$

ⓒ

$$G(s)=\frac{2s-3}{s-3}=$$

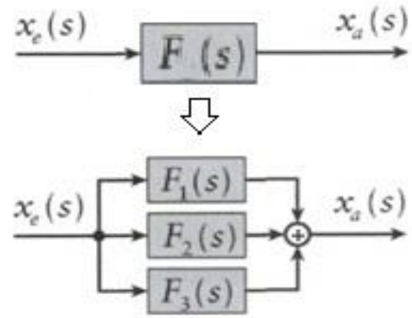
$$\begin{array}{r|l} & 2 \\ \hline s-3 & 2s-3 \\ & 2s-6 \\ \hline & 3 \end{array}$$

$$G(s)=2+\frac{3}{s-3}$$

$$\circ\!\!\overset{\mathcal{L}^{-1}}{\longrightarrow}\!\!\bullet$$

$$g(t)=2\delta(t)+3e^{3t}$$

$$F(s)=\frac{\ldots+b_2s^2+b_1s+b_0}{\ldots+a_2s^2+a_1s+a_0}=c\cdot\frac{\ldots(s-\beta_2)(s-\beta_1)}{\ldots(s-\alpha_2)(s-\alpha_1)}\xrightarrow{\text{PBZ}}F(s)=F_1(s)+F_2(s)+F_3(s)+\ldots$$



pole – den.
zero – num.

Possible partial fractions:

A) Constant: $\boxed{F_1(s)=K}$

B) Simple real pole: $\boxed{F_2(s)=\frac{A}{s-\alpha}}$

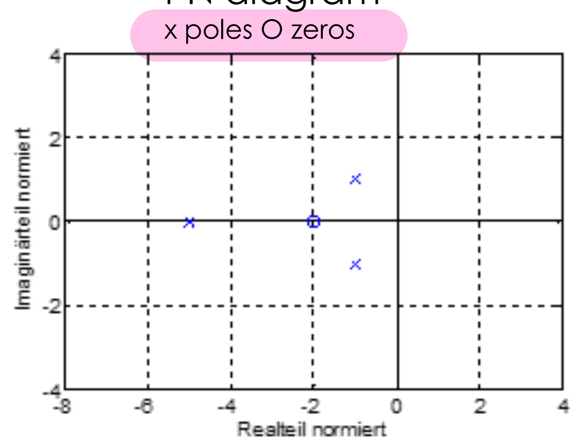
C) Multiple real pole: $\boxed{F_3(s)=\frac{A_n}{(s-\alpha)^n}+\cdots\frac{A_2}{(s-\alpha)^2}+\frac{A_1}{s-\alpha}}$

D) Complex Pole pair: $\boxed{F_4(s)=\frac{As+B}{s^2+\alpha s+\beta}}$ ⇒ DT_2 derived PT_2 [not PT_2]

- E.g: s^2+2s+2

  $s_{\infty 1,2}=\frac{-2\pm\sqrt{4-8}}{2}=-1\pm j$

PN diagram



Impulse responses:

A) Constant: $K \quad \bullet\!-\!\circ \quad K\cdot\delta(t)$

B) Simple real pole: $\frac{1}{s-\alpha} \quad \bullet\!-\!\circ \quad e^{\alpha t}\cdot\varepsilon(t)$

C) Multiple real pole: $\frac{1}{(s-\alpha)^2} \quad \bullet\!-\!\circ \quad t\cdot e^{\alpha t}\cdot\varepsilon(t)$

D) Complex Pole pair: $\frac{1}{s-(\alpha+j\omega)}+\frac{1}{s-(\alpha-j\omega)} \quad \bullet\!-\!\circ \quad e^{\alpha t}(e^{j\omega t}+e^{-j\omega t})\cdot\varepsilon(t)=2e^{\alpha t}\cos(\omega t)\cdot\varepsilon(t)$

$$\frac{1}{s-(\alpha+j\omega)} \quad \bullet\!-\!\circ \quad e^{(\alpha+j\omega)t}\cdot\varepsilon(t)=e^{\alpha t}e^{j\omega t}\cdot\varepsilon(t)$$

Poles determine stability and dynamics of a system, the zeros only its intensity!

- Proportional systems are systems in which there is a directly proportional relationship between input and output.

Transfer Function:

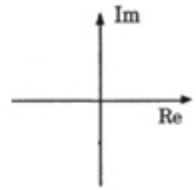
$$\boxed{G(s) = \frac{x_a(s)}{x_e(s)} = K_S}$$

- **DE:**

Due to the proportionality between x_a and x_e The (trivial) DE applies here:

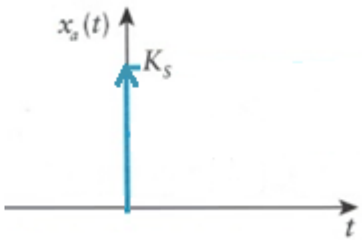
$$x_a(t) = K_S \cdot x_e(t)$$

PN diagram:



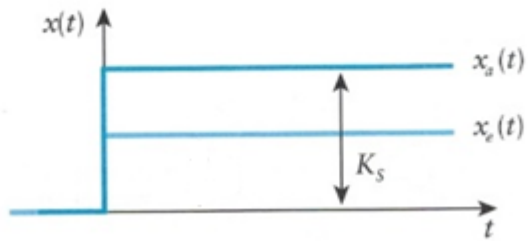
Impulse response:

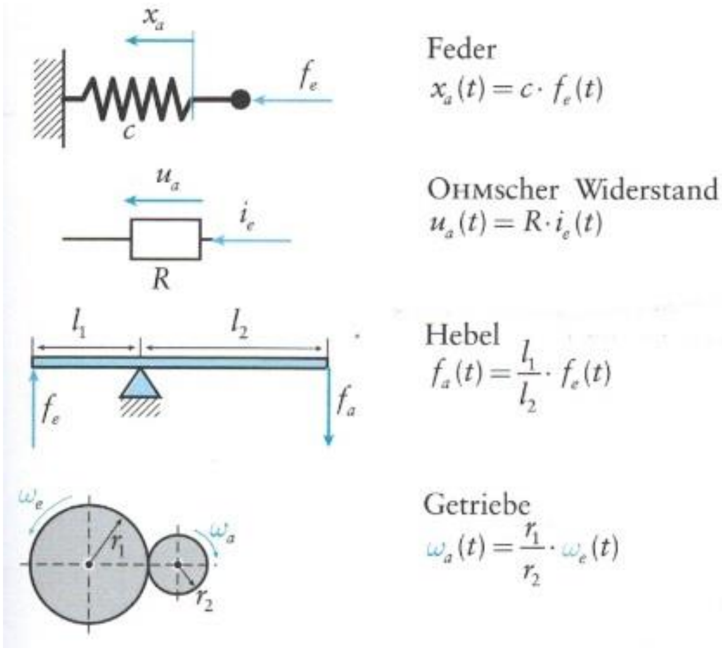
$$\boxed{g(t) = K_S \cdot \delta(t)}$$



Step response:

$$\boxed{h(t) = K_S \cdot \hat{x}_e \cdot \varepsilon(t)}$$





Beispiele für P-Strecken (P-Übertragungsglieder)

The PT$_1$-Element

- In the case of 1st order proportional elements, the system does not react immediately but with a **delay**.

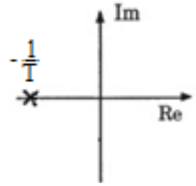
Transfer function:

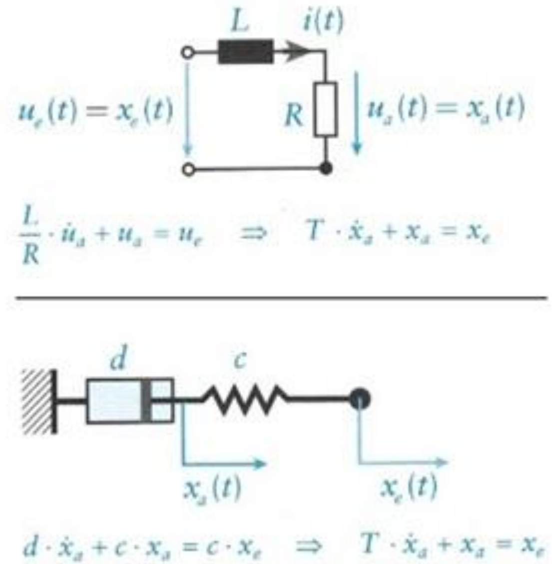
$$\boxed{G(s)=\frac{Y(s)}{U(s)}=\frac{K}{T\cdot s+1}} = \frac{\frac{K}{T}}{s+\frac{1}{T}}$$

- **DE:**

$$T\cdot\dot{y}(t)+y(t)=K\cdot u(t)$$

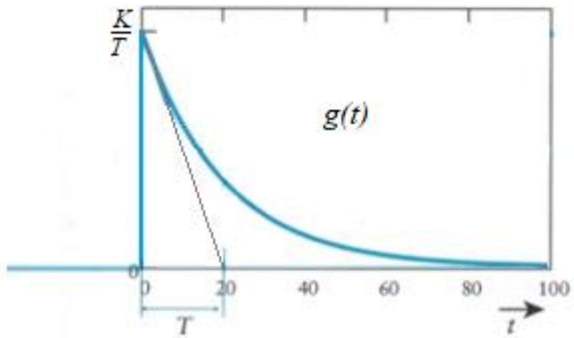
PN diagram:





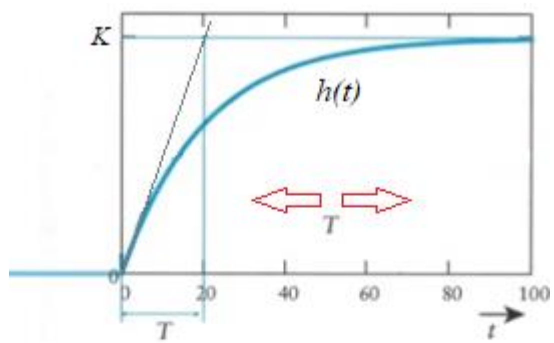
Impulse response:

$$\boxed{g(t)=\frac{K}{T}\cdot e^{-\frac{t}{T}}\cdot\varepsilon(t)}$$



Step response:

$$\boxed{h(t)=K\cdot(1-e^{-\frac{t}{T}})\cdot\varepsilon(t)}$$



- The <u>more negative</u> a pole is, the <u>faster</u> is the reaction <u>positive</u> poles lead to <u>unstable systems</u> (positive exponent!).

The PT_2-Element

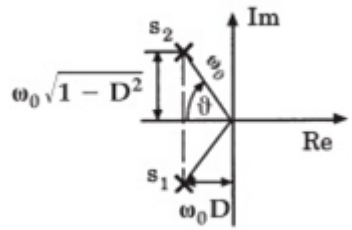
Transfer function:

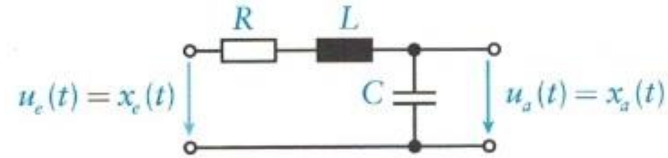
$$G(s)=\frac{K}{a_2\cdot s^2+a_1\cdot s+a_0}$$

- **DE:**

$$a_2\cdot\ddot{y}(t)+a_1\cdot\dot{y}(t)+a_0 y(t)=K\cdot u(t)$$

PN diagram:





$$LC\frac{d^2u_a(t)}{dt^2}+RC\frac{du_a(t)}{dt}+u_a(t)=u_e(t)$$

Normalized transfer function:

- To simplify the analysis, it is better to use a standardized representation the new sizes **Degree of damping D** and (undamped) **Natural frequency ω_0**, you get:

$$G_0(s)=\frac{K\cdot\omega_0^2}{s^2+2D\omega_0\cdot s+\omega_0^2}$$

Impulse response (-1 <D <1):

$$g(t)=K\cdot\frac{\omega_0}{\sqrt{1-D^2}}\cdot e^{-D\omega_0 t}\cdot\sin(\sqrt{1-D^2}\,\omega_0 t)\cdot\varepsilon(t)$$

Step response (-1 <D <1):

$$h(t)=K\cdot[1-\frac{1}{\sqrt{1-D^2}}\cdot e^{-D\omega_0 t}\cdot\sin(\sqrt{1-D^2}\,\omega_0 t+\vartheta)]\cdot\varepsilon(t)\qquad \text{mit}\quad \vartheta=\arccos(D)$$

For other values of D there are real poles and therefore no sine functions ...

1. **PT1 System (First-Order System):**
   - A first-order system is characterized by a single energy storage element (e.g., capacitor or inductor) and one dominant pole in its transfer function. The transfer function of a first-order system is often represented in the form:
     $G(s)=\frac{K}{1+Ts}$
     - K is the system gain.
     - T is the time constant.
   - The response of a first-order system to a step input is characterized by a single exponential curve, and it does not oscillate.
2. **PT2 System (Second-Order System):**
   - A second-order system has two dominant poles in its transfer function, indicating the presence of two energy storage elements. The transfer function of a second-order system is often represented as:
     $G(s)=\frac{K}{1+2\zeta\omega_n s+\omega_n^2 s^2}$
     - K is the system gain.
     - ζ is the damping ratio.
     - ω_n is the natural frequency.
   - The response of a second-order system to a step input can exhibit oscillations. The damping ratio ζ determines the degree of damping and influences the type of response: overdamped (no oscillations), critically damped, or underdamped (oscillatory).

Step Responses of the PT_2-Element Stability for PT_2

Step responses:

$$\boxed{Pole: \quad s_{\infty 1,2} = -D\omega_0 \pm \omega_0 \sqrt{D^2 - 1}}$$

- **D> 1 (crawling behavior):**

  PT_2 = Series connection two different PT_1-Links (T_1 <> T_2):

  $$x_a(t) = [1 - \tfrac{1}{T_1 - T_2} \cdot (T_1 e^{-\frac{t}{T_1}} - T_2 e^{-\frac{t}{T_2}}] \cdot \varepsilon(t)$$

- **D = 1 (aperiodic limit case):**

  PT_2 = Series connection two of the same PT_1-Links ($T_2 = 2T_1$):

  $$x_a(t) = [1 - (1 + \tfrac{t}{T_1}) e^{-\frac{t}{T_1}}] \cdot \varepsilon(t)$$

- **D <1 (stable vibration behavior):**

  At D = 0.707, the shortest possible rise is just achieved without overshooting!

  The frequency to be observed ω and the period T_P are:

  $$\omega = \omega_0 \cdot \sqrt{1 - D^2} \quad ; \quad T_P = \frac{2\pi}{\omega_r}$$

- **D = 0 (limit-stable vibration behavior):**

  Without damping, the step response oscillates permanently with a constant amplitude

  $$x_a(t) = [1 - \cos(\omega_0 t)] \cdot \varepsilon(t)$$

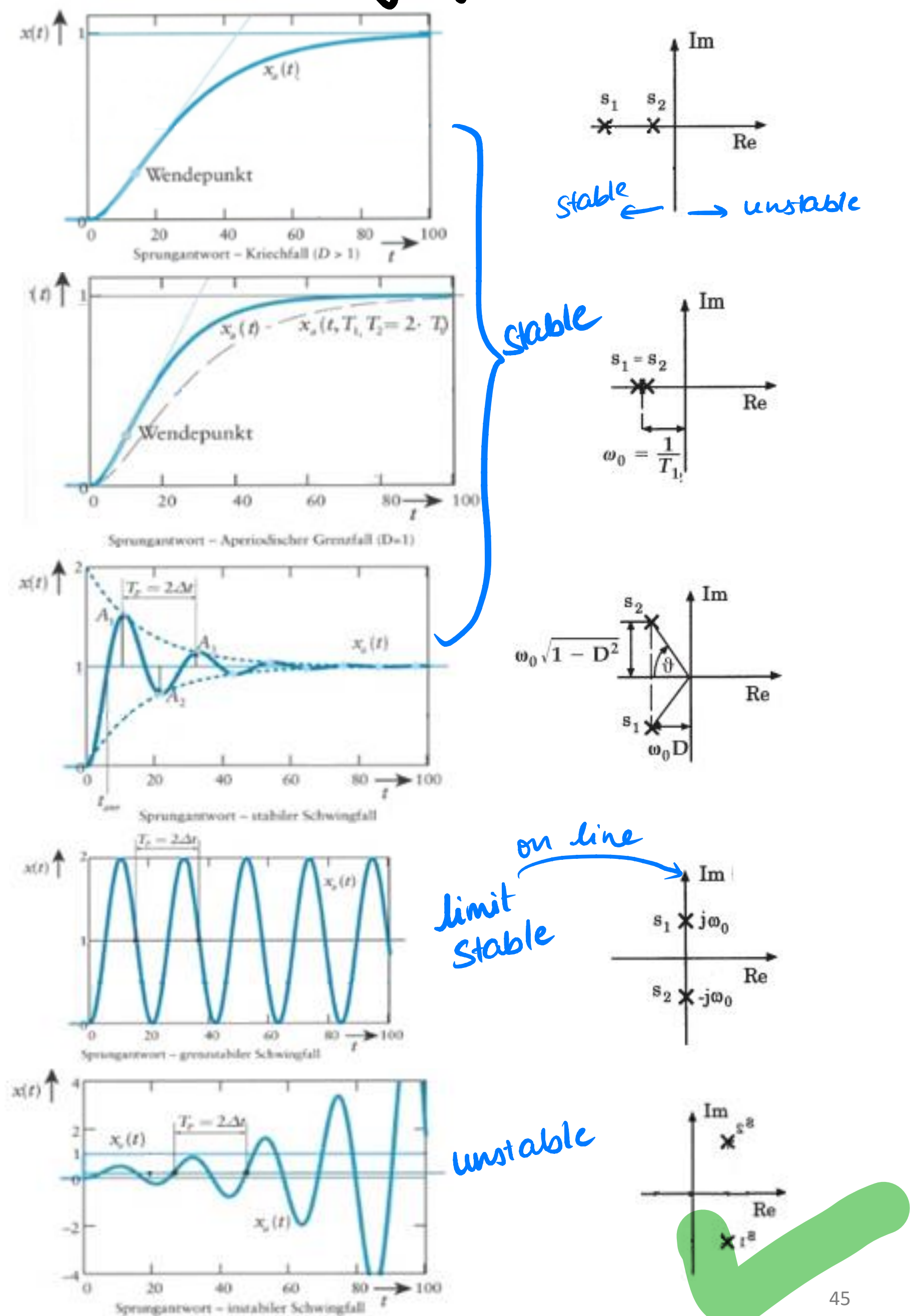
learn!

- **D <0 (unstable vibration behavior):**

  With negative damping, the step response oscillates with increasing amplitude.

  $$x_a(t) = [1 - \frac{1}{\sqrt{1 - D^2}} \cdot e^{-D\omega_0 t} \cdot \sin(\sqrt{1 - D^2}\,\omega_0 t + \vartheta)] \cdot \varepsilon(t)$$

  $$\vartheta = \arccos(D) \quad für \quad -1 < D < 1$$



- **Conclusion:** Complex conjugate poles oscillate, real ones do not, the imaginary part corresponds to the frequency, the real part of the decay

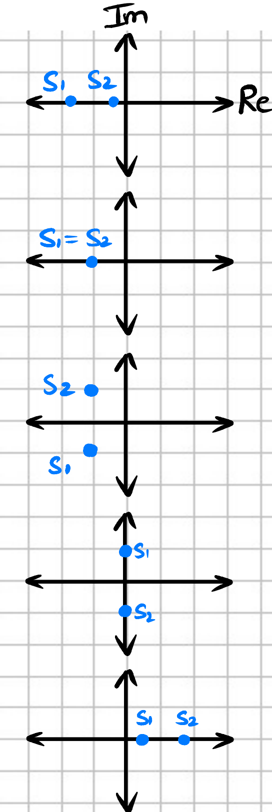
Ex.3.3 Step response of the PT2 system

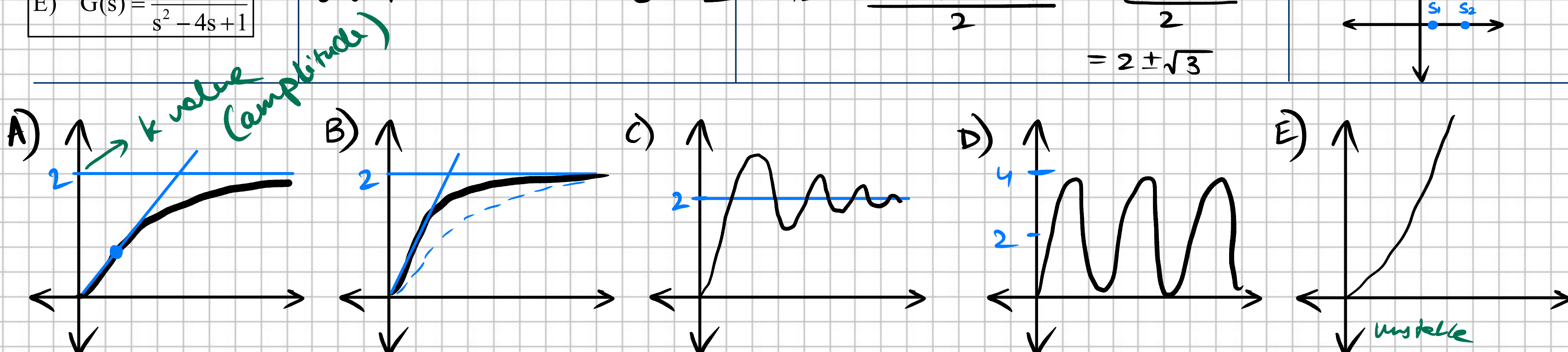
a) Sketch the step responses of the systems:

$$G_0(s) = \frac{K\omega_0^2}{s^2 + 2D\omega_0 \cdot s + \omega_0^2}$$

$$s = \frac{-2D\omega_0 \pm \sqrt{4D^2\omega_0^2 - 4\omega_0^2}}{2} = -D\omega_0 \pm \omega_0\sqrt{D^2-1}$$

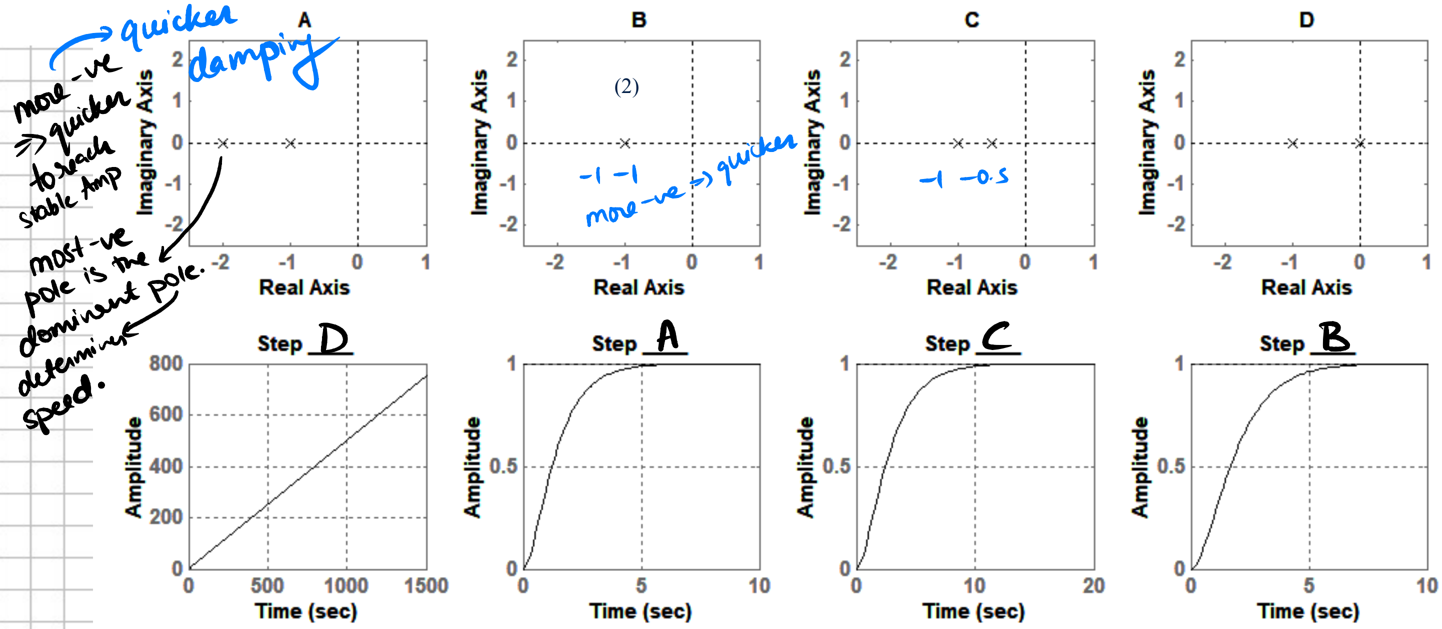
A) $G(s) = \frac{2}{s^2+4s+1}$	$\omega_0 = 1$	$K=2$	$D=2$	$s_{\infty 1,2} = \frac{-4 \pm \sqrt{16-4\cdot 1}}{2} = -2 \pm \sqrt{3}$	
B) $G(s) = \frac{2}{s^2+2s+1}$	$\omega_0 = 1$	$K=2$	$D=1$	$s_{\infty 1,2} = \frac{-2 \pm \sqrt{4-4}}{2} = -1 \pm 0$	
C) $G(s) = \frac{2}{s^2+s+1}$	$\omega_0 = 1$	$K=2$	$D=0.5$	$s_{\infty 1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$	
D) $G(s) = \frac{2}{s^2+1}$	$\omega_0 = 1$	$K=2$	$D=0$	$s_{\infty 1,2} = \frac{-0 \pm \sqrt{0-4}}{2} = \pm i$	
E) $G(s) = \frac{2}{s^2-4s+1}$	$\omega_0 = 1$	$K=2$	$D=-2$	$s_{\infty 1,2} = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$	





Ex.3.4 Step responses of PT2-systems

a) Look at the PN diagrams of 2nd order systems and assign the correct step responses at the top and bottom of each.



quicker slower

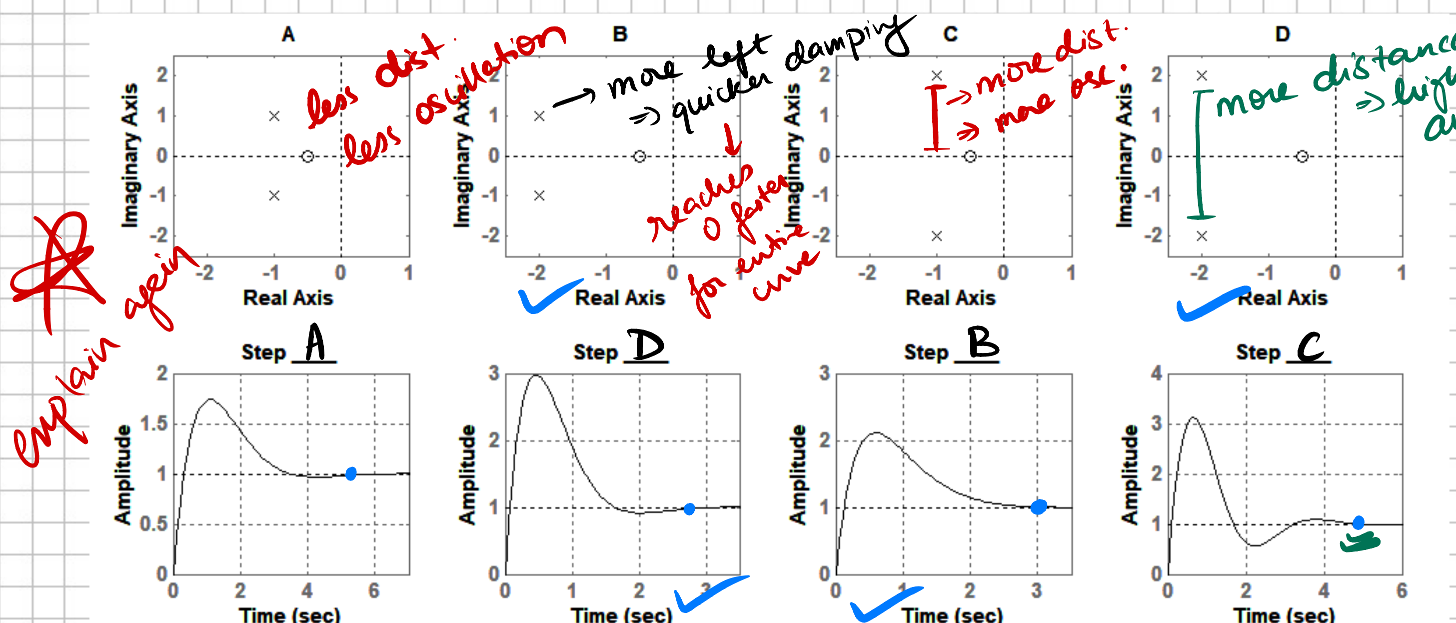
A) $G = \frac{A}{s+2} + \frac{B}{s+1}$

D) $G(s) = \frac{A}{s+1} + \frac{B}{s}$

not limited output

zero goes up,

pole pulls down



C, D = 2 1/s

A, B = 1 1/s

?

Realizability of Transfer Functions

$$m=n \quad G(s)=\frac{4s+3}{2s+1} = \underbrace{2}_{P} + \underbrace{\frac{1}{2s+1}}_{PT_1}$$

$$m>n \quad G(s)=\frac{4s^2+3s+2}{s+1} = \underbrace{4s}_{D} \underbrace{-1}_{P} + \underbrace{\frac{3}{s+1}}_{PT_1}$$

derivative system – causes a problem (not causal)

Transfer functions with numerator degree M = denominator degree N:

- If **M = N**, there will be a jump in the step response, the system is called **jumpable!**
- Conversely, systems with **M <N are not jumpable!**

If M = N, after the PBZ there is a P-element and a ÜF with M <N. the split-off P-element leads to a Jump in the step response.

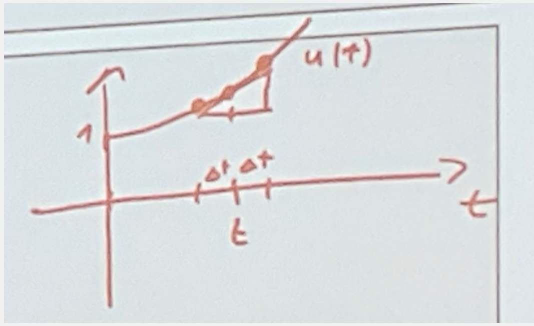
Transfer functions with numerator degree M > denominator degree N:

- If **M> N,** the system is **not causal** and thus **not realizable!**

If M> N, a **Polynomial division**be performed. It results in a sum of**Powers in s, a P-term** and one ÜF with M <N. The split-off powers in s correspond to derivatives of the Input signal in the time domain. The derivative of a signal u (t) can be math. Are described as:

$$y(t)=\frac{du(t)}{dt} \approx \frac{u(t+\Delta t)-u(t-\Delta t)}{2\cdot\Delta t}$$

This means that input signals that lie in the future are also used to calculate ideal derivatives. This is **not possible!**



Perfect D System $G(s)=s$ "not causal"

Real D system $G(s)=\frac{s}{Ts+1}$ $u \rightarrow \boxed{D} \rightarrow \boxed{PT_1} \rightarrow y$ M=N

Stability criterions in the time domain:

→ bounded input bounded output

BIBO-Stability:

- A continuous LTI system is exactly BIBO-stable when its impulse response g(t) can be absolutely integrated, i.e. the step response remains finite in any case.

area under curve not limited

$\Downarrow$

not stable

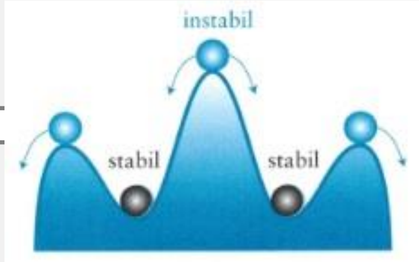
$$\int_0^{\infty} |g(t)|\,dt < \infty$$

$$|Y(t)| = \left| \int_{-\infty}^{+\infty} g(\tau)\cdot u(t-\tau)\,d\tau \right| \le \int_{-\infty}^{+\infty} |g(\tau)|\cdot|u(t-\tau)|\,d\tau \le \int_{-\infty}^{\infty} |g(\tau)|\cdot u_{max}\,d\tau$$

$$= u_{max} \int_{-\infty}^{\infty} |g(\tau)|\,d\tau < \infty$$

Asymptotically-Stability:

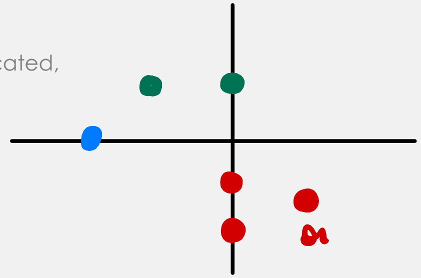
$$\lim_{t\to\infty} g(t) = 0$$



Stability criterion in the s-domain:

A continuous LTI system is

- **stable**,
  if all poles of the system function in the **open left s-half plane** lie, ie there are no poles on the imaginary axis or in the right s-half plane are located,
- **borderline or quasi-stable**, / limit stable
  if all poles lie in the left half-plane and only simple poles appear on the imaginary axis,
- **unstable**,
  as soon as a pole lies in the open right half-plane or a multiple pole lies on the imaginary axis.



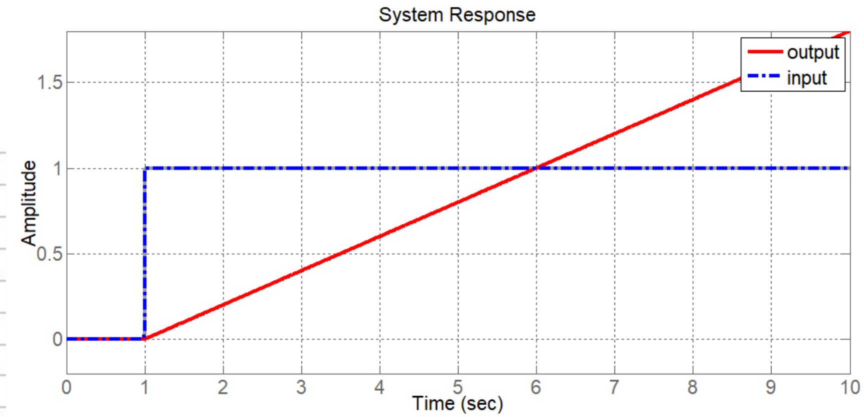
Ex.2.16 System identification:

If you are measuring a pure I-element, you will receive the system answer shown here.

$$G(s) = \frac{1}{T_N s}$$

to read out

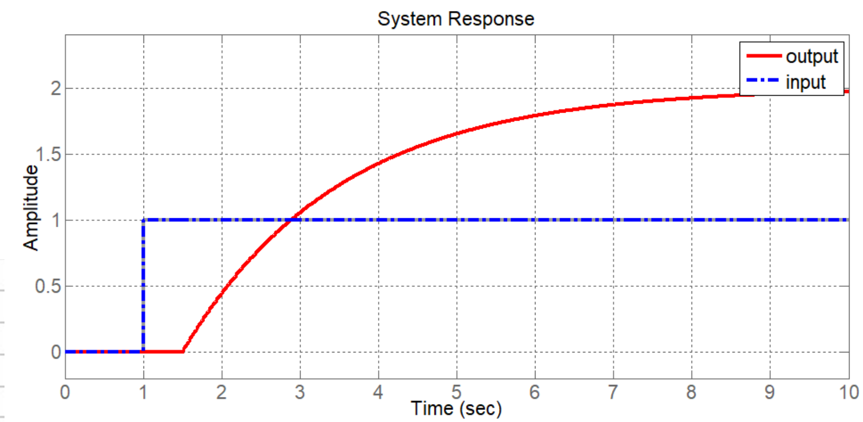
??



a) Determine the integration time constant T from this measurement$_N$.

If you are measuring an unknown system, you will receive the following system response.

$$G(s) = ?$$



a) Give a transfer function G (s) that describes this system.
(Note: this time it does not have to consist only of the quotient of two polynomials in s).

Existence of the Laplace Transform

- A Laplace transform X(s) exists only if the **Laplace integral converges** (i.e. is finite for certain values of s).

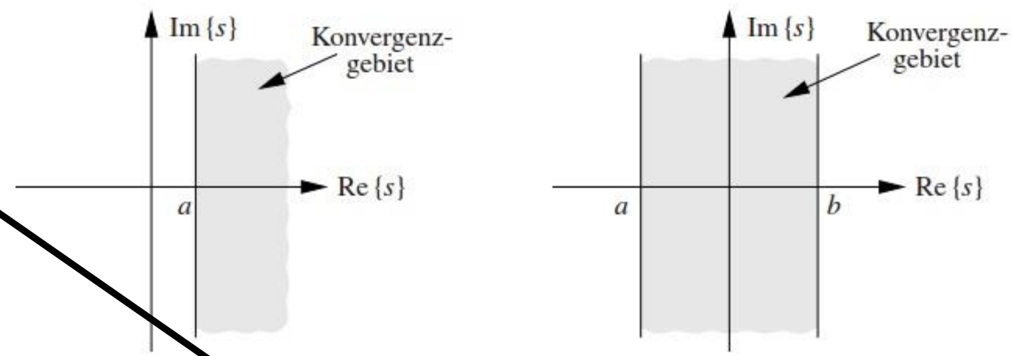
$$X(s) = L\{x(t)\} = \int_0^\infty x(t) \cdot e^{-st} dt, \qquad mit\ x(t) = 0 \quad für\ t < 0$$

- This condition is only fulfilled if | x(t) | for t→∞ does **not grwo faster than an exponential function**!
  In this case the function x (t) can be estimated with | **x (t)** | **<k · e** ^{at} .
- Only if the **Re (s)> a**, the corresponding Laplace integral converges.

- Explanation:
  With $x(t) = e^{\lambda t} \cdot \varepsilon(t) = e^{(a+j\omega)t} \cdot \varepsilon(t)$
  surrendered:

$$X(s) = \int_0^\infty e^{\lambda t} \cdot \varepsilon(t) \cdot e^{-st} dt = \int_0^\infty e^{(\lambda - s)t} dt = \frac{1}{\lambda - s} \cdot e^{-(s-\lambda)t} \Big|_0^\infty$$

$$= \lim_{t \to \infty} \frac{1}{\lambda - s} \cdot e^{-(s-\lambda)t} - \frac{1}{\lambda - s} \cdot e^{-(s-\lambda)0}$$



a: einseitige Transformation

b: zweiseitige Transformation

Konvergenzgebiete der Laplace-Transformation

- For **Re(s - λ)> 0:**
  strives for the exponential function for t →∞ to zero, the integral is therefore convergent, the Laplace transform **exists** for this area of the s-level.
- For **Re(s - λ) <0:**
  if the exponential function tends towards infinity, the integral is accordingly not convergent that Laplace transform **exists** for this area of the s-level **not.**
- The Laplace integral is convergent for the entire area of the s-plane with a gray background.
  In the cases that are interesting from the point of view of system theory, the convergence of the Laplace integral can be assumed at least in part of the s-plane. Of the**Convergence area** the Laplace transformation is therefore useful for the computation of technically interesting cases of subordinate Meaning.
- Only with the Fourier transformation does the convergence range of the Laplace transformation become important again ...

Ex.2.17 Existence of the Laplace Transform

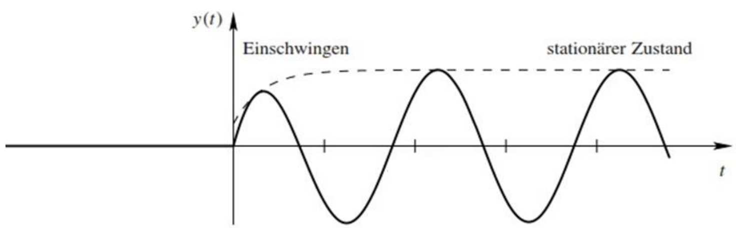
$$\mathcal{L}_{II}\{x(t)\} = \int_{-\infty}^{+\infty} x(t) \cdot e^{-st} dt$$

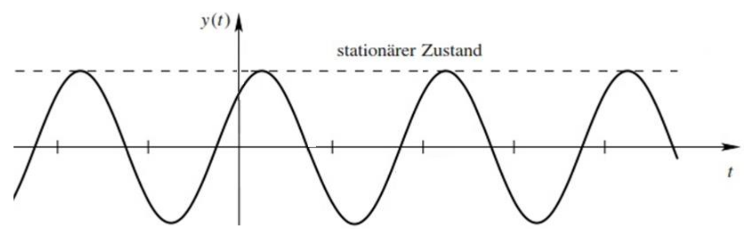
2.3 System Description via the Fourier Transform

Why the Fourier Transform now?

Motivation 1:

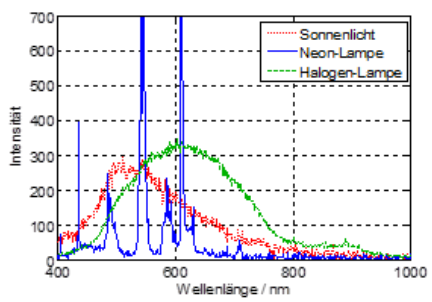
- If only **the stationary behavior of a system** (i.e. without transient processes) is of interest, the Fourier transform is much more suitable (**= frequency response**)

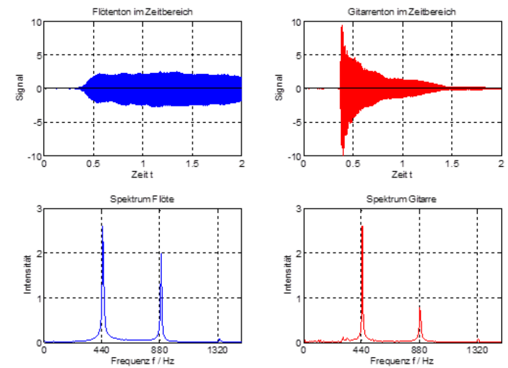




Motivation 2:

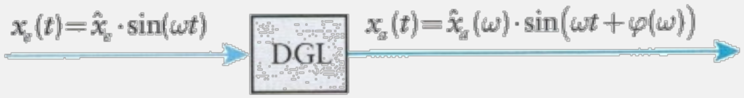
- The Fourier transformation is the generalization of the Fourier series and thus provides important information about the **spectral composition of a signal** (**= spectrum**).



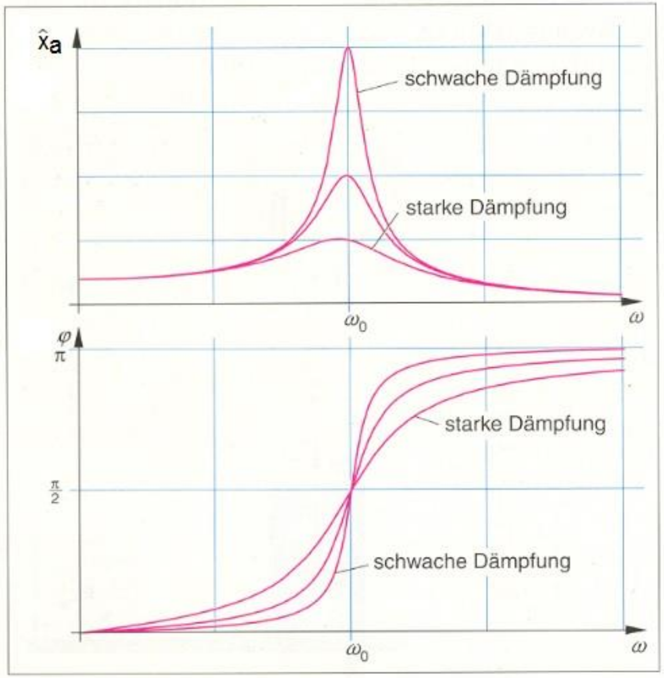


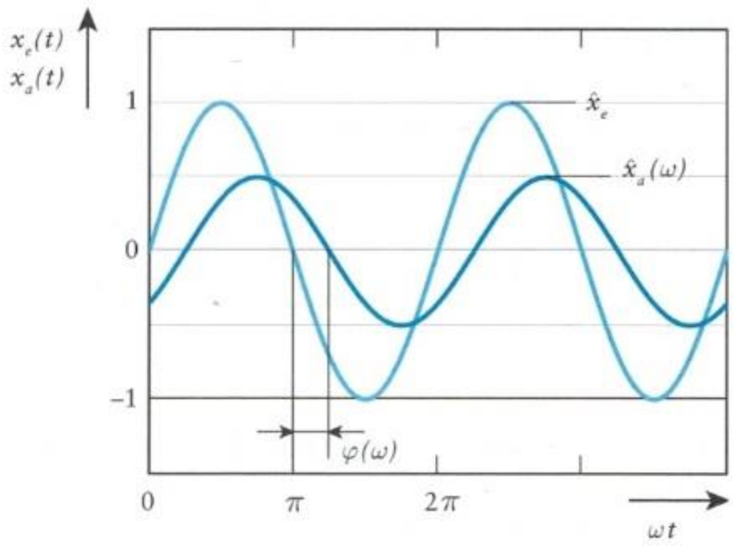
System responses with sinusoidal excitation:

- If a linear system is excited with a constant sinusoidal signal, the output signal oscillates with the **same frequency**, but generally with a **different amplitude and phase!**



Frequency response: Harmonic excitation in the time domain





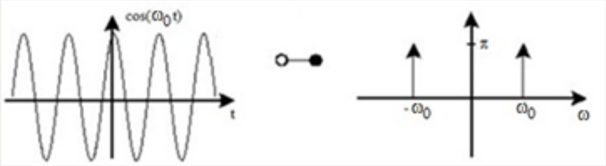
Definition:

$$X(j\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

$$x(t) = F^{-1}\{X(j\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

(t) → (jω)

- **Differentiation rule:**
  $$\dot{x}(t) \quad \circ\!-\!\bullet \quad j\omega \cdot X(j\omega)$$
  $$\ddot{x}(t) \quad \circ\!-\!\bullet \quad (j\omega)^2 \cdot X(j\omega)$$
- **Shift:**
  $$x(t+t_0) \quad \circ\!-\!\bullet \quad X(j\omega) \cdot e^{j\omega t_0}$$
- Harmonic signal:
  $$\cos(\omega_0 t) \quad \circ\!-\!\bullet \quad \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$



Frequency response:

$$G(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = \frac{\ldots + b_2 (j\omega)^2 + b_1 j\omega + b_0}{\ldots + a_2 (j\omega)^2 + a_1 j\omega + a_0}$$

~G(s) in laplace (Transfer function)

System equation:

$$Y(j\omega) = G(j\omega) \cdot U(j\omega) \qquad Y(j\omega) = |G(j\omega)| \cdot e^{j\angle G(j\omega)} \cdot U(j\omega)$$

- Harmonic excitation:
  $$Y(j\omega) = |G(j\omega)| \cdot e^{j\angle G(j\omega)} \cdot \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$
  $$= |G(j\omega_0)| \cdot \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] \cdot e^{j\angle G(j\omega_0)}$$
  $$\bullet\!-\!\circ \quad y(t) = |G(j\omega_0)| \cdot \cos\left[\omega_0 t + \angle G(j\omega_0)\right]$$

Amplitude response:

$$|G(j\omega)| = \frac{|Y(j\omega)|}{|U(j\omega)|}$$

Phase response:

$$\varphi(j\omega) = \angle G(j\omega)$$

Comparison of Laplace and Fourier Transform

If you want to hide the transient processes, you would actually have to apply the two-sided Laplace transform (with all its convergence problems):

Fourier Transform:

$$X(j\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}dt$$

$t \to j\omega$

$Y(j\omega)$

- **System equation:**

$$Y(j\omega) = G(j\omega)\cdot U(j\omega)$$

Laplace transformation (double-sided):

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}dt$$

$t \to s$

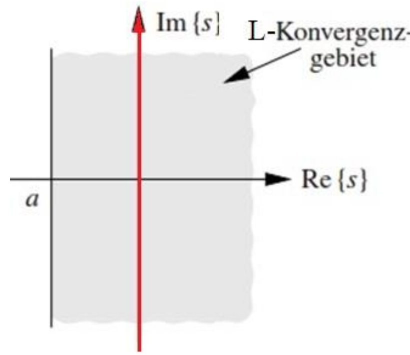
$Y(s)$

$$Y(s) = G(s)\cdot U(s)$$

Limited to $s = j\omega$, both transformations are absolutely identical.

For the important signal class, whose two-sided Laplace convergence region includes includes the imaginary axis, and which includes **all causal and decaying signals**, both transforms exist **with the substitution:**

$$X_{Fourier}(j\omega) = X_{Laplace}(s)\big|_{s=j\omega}$$



Frequency Response:

$$\dot{x}(t) \circ\!-\!\bullet\ j\omega\cdot X(j\omega)$$

For the important system class whose Laplace convergence region includes the imaginary axis, and to which **all stable systems** belong, the so-called frequency response is obtained directly from the transfer function **with the substitution:**

$$\boxed{G(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = G(s)\big|_{s=j\omega}} = \frac{-b_2\omega^2 + b_1 j\omega + b_0}{-a_2\omega^2 + a_1 j\omega + a_0}$$

Sine-Signals (non-causal):

For the important signal class of (non-causal) constant sine signals actually exist the Fourier transform (with the help of **distributions**) !

$$\boxed{X_{Fourier}(j\omega) = \int_{-\infty}^{\infty} \cos(\omega_0 t)e^{-j\omega t}dt} = \tfrac{1}{2}\int_{-\infty}^{\infty}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)e^{-j\omega t}dt = \tfrac{1}{2}\int_{-\infty}^{\infty}\left(e^{-j(\omega-\omega_0)t} + e^{-j(\omega+\omega_0)t}\right)dt = \boxed{\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]}$$

Fourier transform of the non-causal constant one:

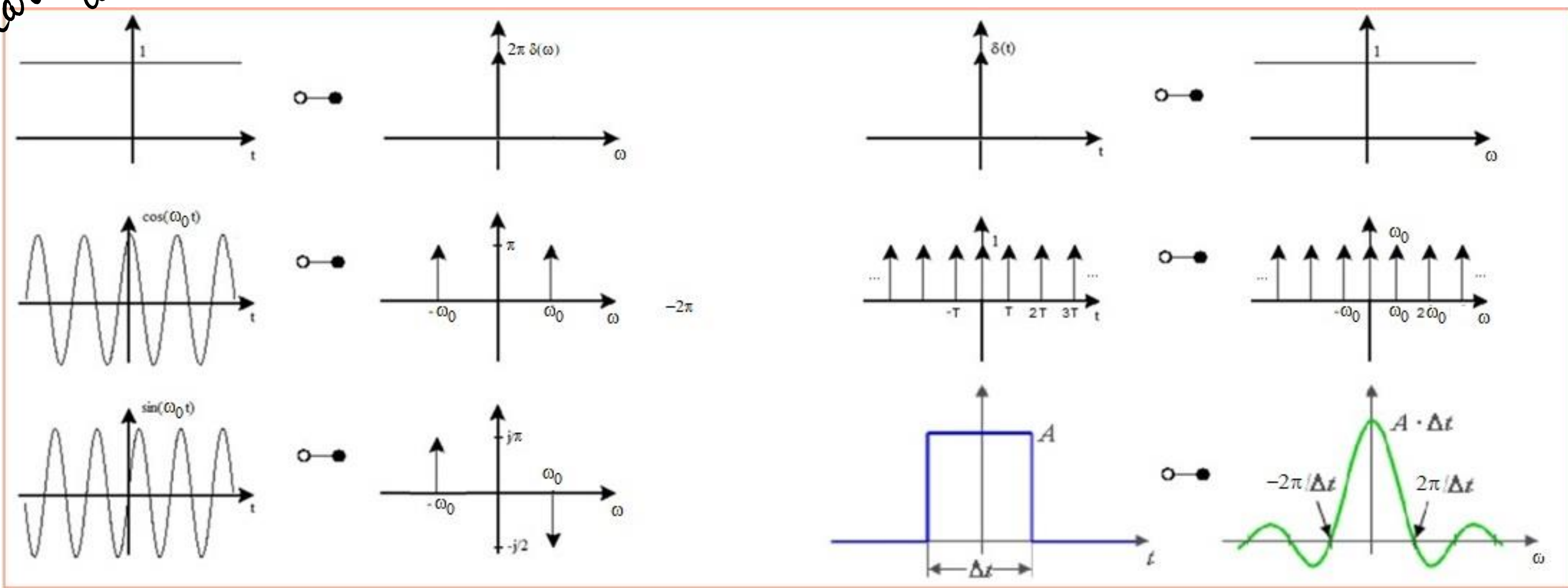
$$F\{1\} = \int_{-\infty}^{\infty} 1 \cdot e^{-j\omega t}\,dt = \lim_{t_0 \to \infty} \int_{-t_0}^{t_0} e^{-j\omega t}\,dt = \lim_{t_0 \to \infty} \frac{1}{-j\omega} e^{-j\omega t}\Big|_{-t_0}^{t_0} = \lim_{t_0 \to \infty} \frac{1}{-j\omega}\left(e^{-j\omega t_0} - e^{j\omega t_0}\right) = \lim_{t_0 \to \infty} 2t_0 \frac{\sin(\omega t_0)}{\omega t_0} = \quad 2\pi\delta(\omega)$$

$$L\{1\} = \int_{-\infty}^{\infty} 1 \cdot e^{-st}\,dt = \lim_{t_0 \to \infty} \int_{-t_0}^{t_0} e^{-st}\,dt = \lim_{t_0 \to \infty} \frac{-1}{s} e^{-st}\Big|_{-t_0}^{t_0} = \lim_{t_0 \to \infty} \frac{1}{s}\left(e^{-st_0} - e^{st_0}\right) \Rightarrow \infty \quad ϟ$$

- **<u>Conclusion:</u>** The two-sided Laplace transform for x(t) = 1 <u>does not exist</u>, the Fourier transform, (introducing distributions) but does!!!

Physically of course there are no negative frequencies, but the Euler equation leads to clearer calculations and therefore also introduces the negative frequency!

learn all



Frequency Response:

- For stable systems, the transfer function is used to obtain the **so-called frequency response after the substitution:**

$$\boxed{G(j\omega)=\frac{Y(j\omega)}{U(j\omega)}=G(s)\Big|_{s=j\omega}} \qquad =\frac{-b_2\omega^2+b_1 j\omega+b_0}{-a_2\omega^2+a_1 j\omega+a_0}$$

The frequency response is therefore a complex function depending on the frequency variable ω or $j\omega$. $G(j\omega)=|G(j\omega)|\cdot e^{j\varphi(j\omega)}$

Amplitude Response:

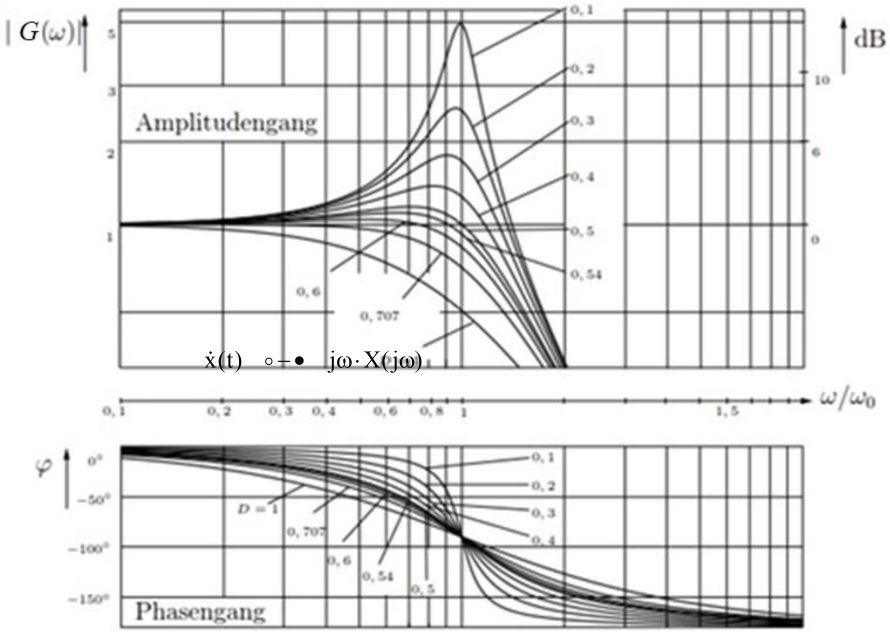
- Describes for <u>**each**</u> **sinusoidal** signal with the <u>frequency ω</u> the ratio of the output to the input amplitude!

$$\boxed{|G(j\omega)|=\frac{|Y(j\omega)|}{|U(j\omega)|}}$$

Phase Response:

- describes the time shift between the output and input oscillation!

$$\boxed{\varphi(j\omega)=\angle G(j\omega)}$$



The (pseudo) unit Decibel (dB):

$$a(j\omega)=20\lg|G(j\omega)|=10\lg|G(j\omega)|^2$$

It is only used to rrescale extremely large value ranges.

Pegel	0 dB	3 dB	4.8 dB	7 dB	10 dB	20 dB
Betrag	1	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{10}$	10
Leistung	1	2	3	5	10	100

Elementare Umrechnungen von Pegel in Betrags- und Leistungsgrößen

Complex Calculation

Cartesian representation: Polar representation:

$$z = x + iy$$

$$z = r(\cos\varphi + i\sin\varphi)$$

Addition:

- Rule: real and Imaginary parts add

$$z_1 + z_2 = (x + iy) + (u + iv) \; = (x + u) + i(y + v)$$

Subtraction:

- Rule: real and Imaginary parts subtract

$$z_1 - z_2 = (x + iy) - (u + iv) \; = (x - u) + i(y - v)$$

Multiplication:

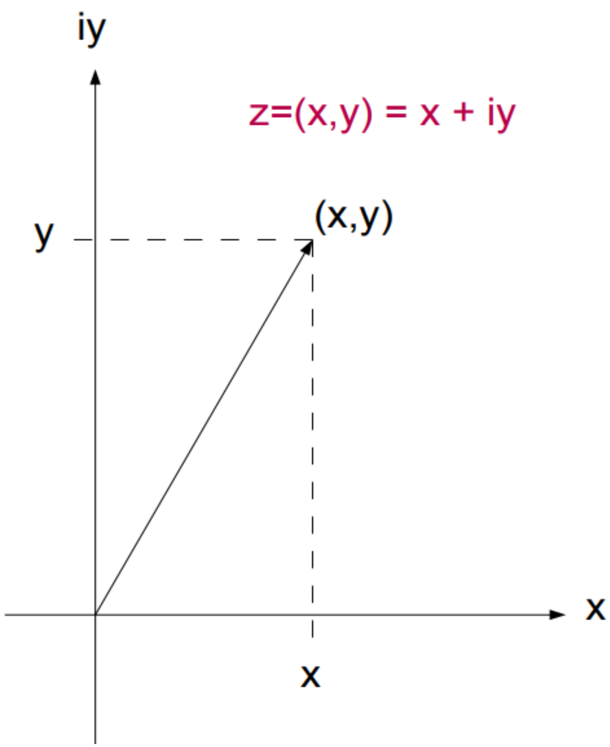
- Rule: multiply amounts and add angles

$$z_1 \cdot z_2 = r_1 r_2 (\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$$

Division:

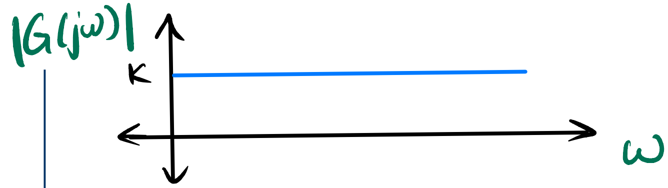
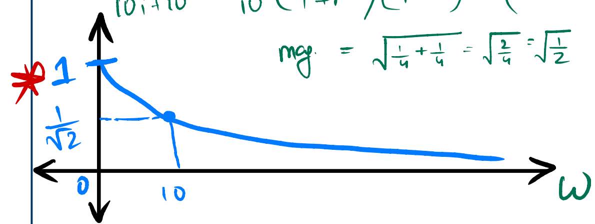
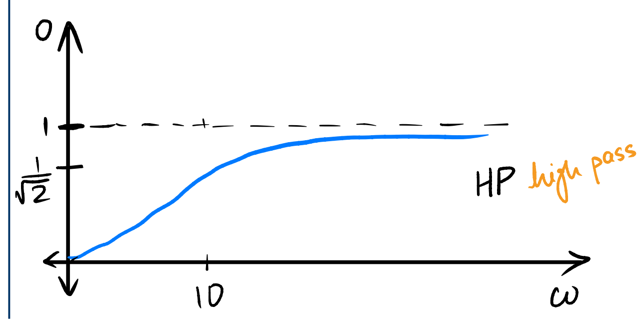
- Rule: divide amounts and subtract angles

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2))$$



$$|z| := \sqrt{x^2 + y^2}$$

Frequency Responses of the Basic Elements

Transfere Funct.	Frequency response	Amplitude response $\lvert G(j\omega)\rvert$	Phase response $\varphi(j\omega)$
$P: \quad G(s) = K$	$G(j\omega) = K$		
$PT_1: \quad G(s) = \frac{10}{s+10}$	$G(j\omega) = \frac{10}{j\omega + 10}$	$\frac{10}{10i+10} = \frac{10}{10}\left(\frac{1}{i+1}\right)\left(\frac{i-1}{i-1}\right) = \left(\frac{i-1}{-1-1}\right) = \frac{1-i}{2}$	
		mag. $= \sqrt{\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}}$	
			
	$\omega = 0 \Rightarrow \lvert G(j\omega)\rvert = 1$		
	$\omega = \infty \Rightarrow \lvert G(j\omega)\rvert = 0$		
$PT_2: \quad G(s) = \frac{100}{s^2 + 20Ds + 100}$	$G(j\omega) = \frac{100}{-\omega^2 + 20Dj\omega + 100}$		
	$D = 0 \quad D = 0.5 \quad D = 1$		
$DT_1: \quad G(s) = \frac{s}{s+10}$	$G(j\omega) = \frac{j\omega}{j\omega + 10}$		

Low pass filter:
low freq. ratio ⇒ amplitude = 1
high freq. ratio ⇒ amplitude = 0

$j = -1$

The Bode-Plot

Bode-Plot (-Diagram):

- Is a special representation of the **frequency response G(jω) in a <u>logarithmic</u> scale**.

The representation of the logarithmic frequency characteristic has the **great advantage**that the resulting amplitude and phase response **systems connected in series** through (graphic) **addition** the individual amplitude or phase responses can be formed!

$$G_{ges}(j\omega) = G_1(j\omega) \cdot G_2(j\omega) \qquad \rightarrow \qquad G_{ges}(j\omega) = |G_1(j\omega)| \cdot |G_2(j\omega)| \cdot e^{j(\varphi_1(j\omega)+\varphi_2(j\omega))}$$

Amplitude response of a series connection:

$\lg(a \cdot b) = \lg a + \lg b$

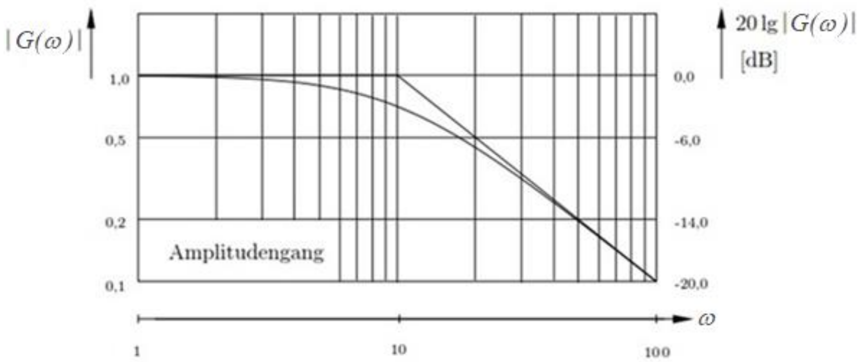
$$20\lg|G(j\omega)| = 20\lg|G_1(j\omega)| + 20\lg|G_2(j\omega)|$$

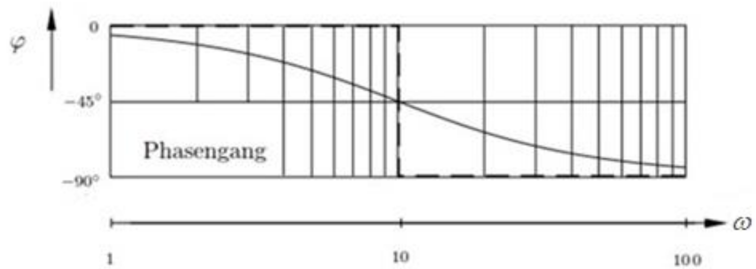
Phase response of a series connection:

$$\varphi(j\omega) = \varphi_1(j\omega) + \varphi_2(j\omega)$$

<u>ExampleE</u>: Bode diagram and its straight line approximation of an RC element (PT_1-System):

$$G(s) = \frac{10}{s+10} \qquad \rightarrow \qquad G(j\omega) = \frac{10}{j\omega+10}$$



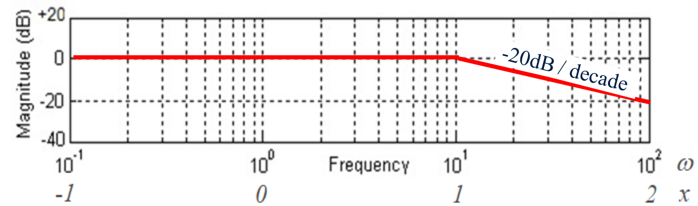
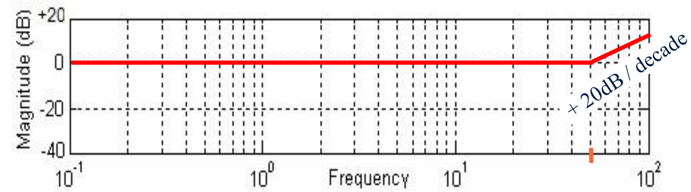
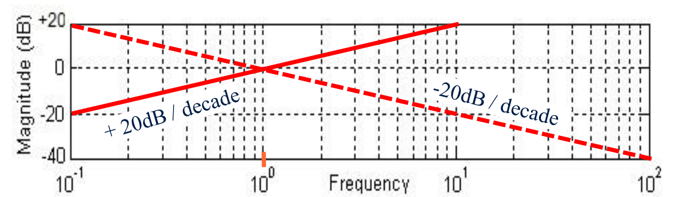
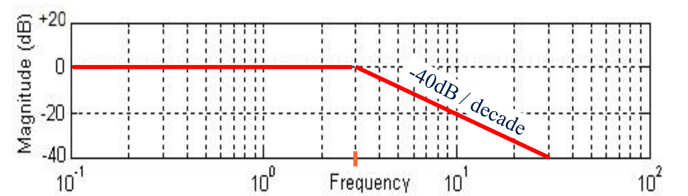
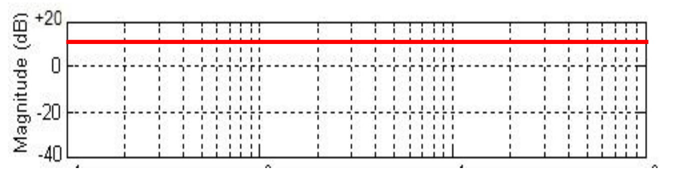


Approximation of the Amplitude Response

$$G_{PT_2}(s) = \frac{1}{\frac{s^2}{\omega_0^2} + \frac{2D\,s}{\omega_0} + 1}$$

$$\lg\left(\tfrac{a\cdot b}{c}\right) = \lg a + \lg b - \lg c$$

$$\boxed{G(s) = \frac{6\cdot s\cdot(s+50)}{(s+10)(s^2+6s+9)}} = \frac{6\cdot 50\cdot s\cdot(\frac{s}{50}+1)}{10\cdot 9\cdot(\frac{s}{10}+1)\cdot(\frac{s^2}{3^2}+\frac{2s}{3}+1)} \Rightarrow \left|G(j\omega)\right| = \frac{6\cdot 50\cdot\left|j\omega\right|\cdot\left|(\frac{j\omega}{50}+1)\right|}{10\cdot 9\cdot\left|(\frac{j\omega}{10}+1)\right|\cdot\left|(\frac{-\omega^2}{3^2}+\frac{2j\omega}{3}+1)\right|}$$

$G(s)$	$\lvert G(j\omega)\rvert$	$\lvert G(j\omega)\rvert_{dB} = 20\lg\lvert G(j\omega)\rvert$	Amplitude response
$PT_1 : \frac{1}{(\frac{s}{10}+1)}$	$\frac{1}{\lvert\frac{j\omega}{10}+1\rvert} = \begin{cases} 1 & \omega \ll 10 \\ \frac{1}{\sqrt{2}} & \omega = 10 \\ \frac{10}{\omega} & \omega \gg 10 \end{cases}$ $(10 = \omega_a = \lvert s_\infty\rvert)$	$= \begin{cases} 0dB & \omega \ll 10 \\ -3dB & \omega = 10 \\ 20\lg 10 - 20\lg\omega & \omega \gg 10 \end{cases}$ x	
$DT_1 : (\frac{s}{50}+1)$	$\lvert\frac{j\omega}{50}+1\rvert = \begin{cases} 1 & \omega \ll 50 \\ \sqrt{2} & \omega = 50 \\ \frac{\omega}{50} & \omega \gg 50 \end{cases}$ $(50 = \omega_b = \lvert s_0\rvert)$	$= \begin{cases} 0dB & \omega \ll 50 \\ +3dB & \omega = 50 \\ 20\lg\omega - 20\lg 50 & \omega \gg 50 \end{cases}$	
$D: \quad s$ $\left(I: \quad \frac{1}{s}\right)$	$= \lvert j\omega\rvert = \omega$ $\left(= \lvert\frac{1}{j\omega}\rvert = \frac{1}{\omega}\right)$	$= 20\lg\omega$ $(= -20\lg\omega)$	
$PT_2 : \frac{1}{(\frac{s^2}{3^2}+\frac{2s}{3}+1)}$	$\frac{1}{\lvert\frac{-\omega^2}{3^2}+\frac{2j\omega}{3}+1\rvert} = \begin{cases} 1 & \omega \ll 3 \\ \approx \pm 1 & \omega = 3 \\ \left(\frac{3}{\omega}\right)^2 & \omega \gg 3 \end{cases}$ $(3 = \omega_0 = \lvert s_{\infty 1,2}\rvert)$	$= \begin{cases} 0dB & \omega \ll 3 \\ \approx \pm 0dB & \omega = 3 \\ 40\lg 3 - 40\lg\omega & \omega \gg 3 \end{cases}$	
$P: \quad \frac{6\cdot 50}{10\cdot 9}$	$= +3{,}33$	$= 20\lg 3{,}33 = +10{,}5dB$	

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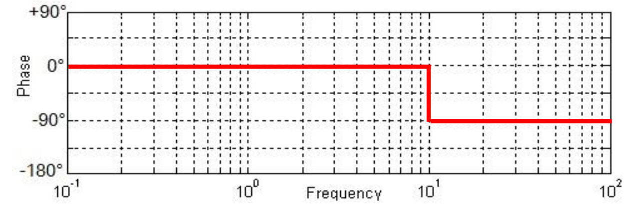
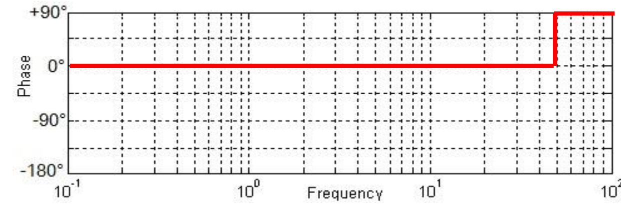
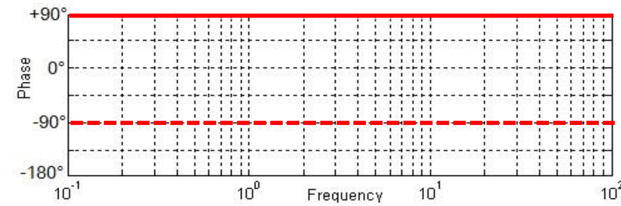
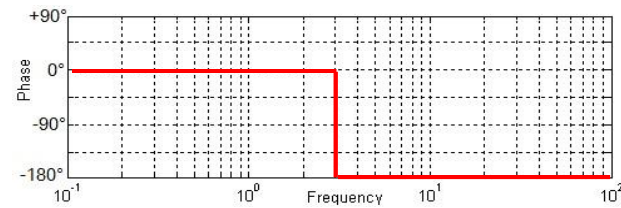
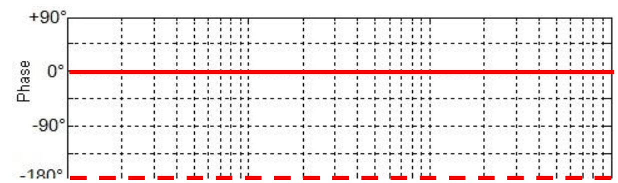
Approximation of the Phase Response

$$G_{PT_2}(s) = \frac{1}{\frac{s^2}{\omega_0^2} + \frac{2Ds}{\omega_0} + 1}$$

$$\boxed{G(s) = \frac{6 \cdot s \cdot (s+50)}{(s+10)(s^2+6s+9)}} = \frac{6 \cdot 50 \cdot s \cdot (\frac{s}{50}+1)}{10 \cdot 9 \cdot (\frac{s}{10}+1) \cdot (\frac{s^2}{3^2} + \frac{2s}{3} + 1)} \Rightarrow \varphi(j\omega) = \angle\left(\tfrac{6 \cdot 50}{10 \cdot 9}\right) + \angle(j\omega) + \angle(\tfrac{j\omega}{50}+1) - \angle(\tfrac{j\omega}{10}+1) - \angle(\tfrac{-\omega^2}{3^2} + \tfrac{2j\omega}{3} + 1)$$

$G(s)$	$\phi(j\omega) = \angle G(j\omega) = \arctan\left(\frac{\mathrm{Im}\{G(j\omega)\}}{\mathrm{Re}\{G(j\omega)\}}\right)$	Phase response
$PT_1 : \frac{1}{(\frac{s}{10}+1)}$	$\varphi(j\omega) = \angle \frac{1}{(\frac{j\omega}{10}+1)} = \begin{cases} 0 & \omega << 10 \\ -45° & \omega = 10 \\ -90° & \omega >> 10 \end{cases}$	
$DT_1 : (\frac{s}{50}+1)$	$\varphi(j\omega) = \angle\left(\frac{j\omega}{50}+1\right) = \begin{cases} 0 & \omega << 50 \\ +45° & \omega = 50 \\ +90° & \omega >> 50 \end{cases}$	
$D: \quad s$ $\left(I: \quad \frac{1}{s}\right)$	$\varphi(j\omega) = \angle(j\omega) = +90° \quad (immer)$ $\varphi(j\omega) = \angle\left(\frac{1}{j\omega}\right) = -90° \quad (immer)$	
$PT_2 : \frac{1}{(\frac{s^2}{3^2} + \frac{2s}{3} + 1)}$	$\varphi(j\omega) = \angle \frac{1}{(\frac{-\omega^2}{3^2} + \frac{2j\omega}{3} + 1)} = \begin{cases} 0 & \omega << 3 \\ -\frac{\pi}{2} & \omega = 3 \\ -\pi & \omega >> 3 \end{cases}$	
$P: \quad \frac{6 \cdot 50}{10 \cdot 9}$	$\varphi(j\omega) = \angle(3{,}33) = 0 \quad (immer)$	

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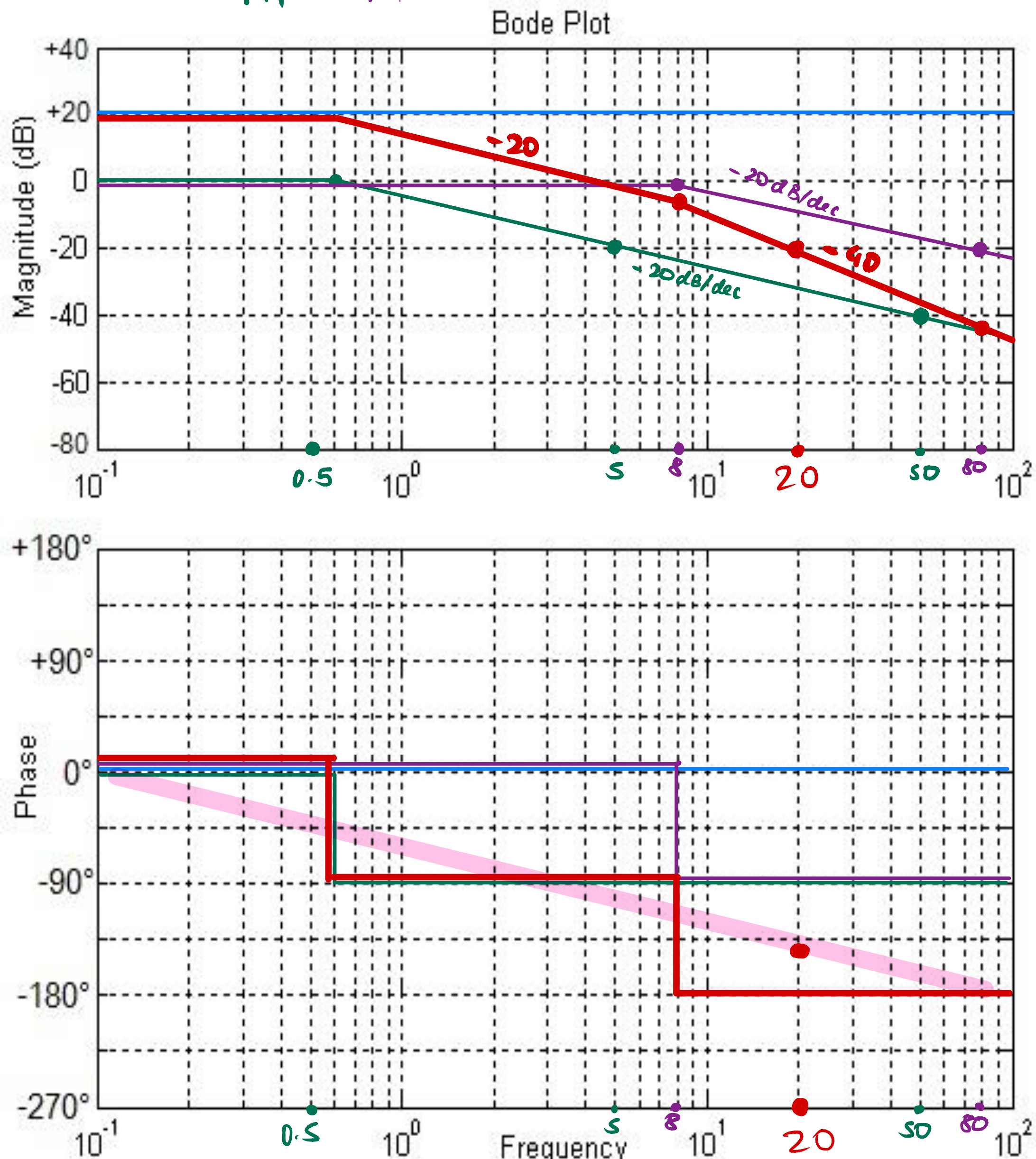
2.18 Bode Diagram I.

a) Sketch approximately the Bode diagram for the following system:

b) The system is excited with the shown signal u(t):
Which steady-state response y(t) is to be expected?

$$G(s) = \underbrace{10}_{P} \cdot \underbrace{\frac{1}{\left(\frac{s}{0.5}+1\right)}}_{PT_1} \cdot \underbrace{\frac{1}{\left(\frac{s}{8}+1\right)}}_{PT_1}$$

$$u(t) = 12V \cdot \sin(20\tfrac{1}{s} \cdot t)$$



b) $u(t) = 12V\sin\left(20\frac{1}{s} \cdot t\right)$

$y(t) = A \sin\left(20\frac{1}{s}t + P\right)$

$|G(j20)| = -20$

$20 \log x = -20$

$\log x = -1$

$x = 10^{-1} = 0.1$

$\Rightarrow A = 0.1 \times 12$
$= 1.2$

$\angle G(j20) = 150° = P$

Where is the formula for $|G(j20)| = 20\log X$

Ex.2.19 Bode Diagram II

Sketch the Bode diagram of the following system:

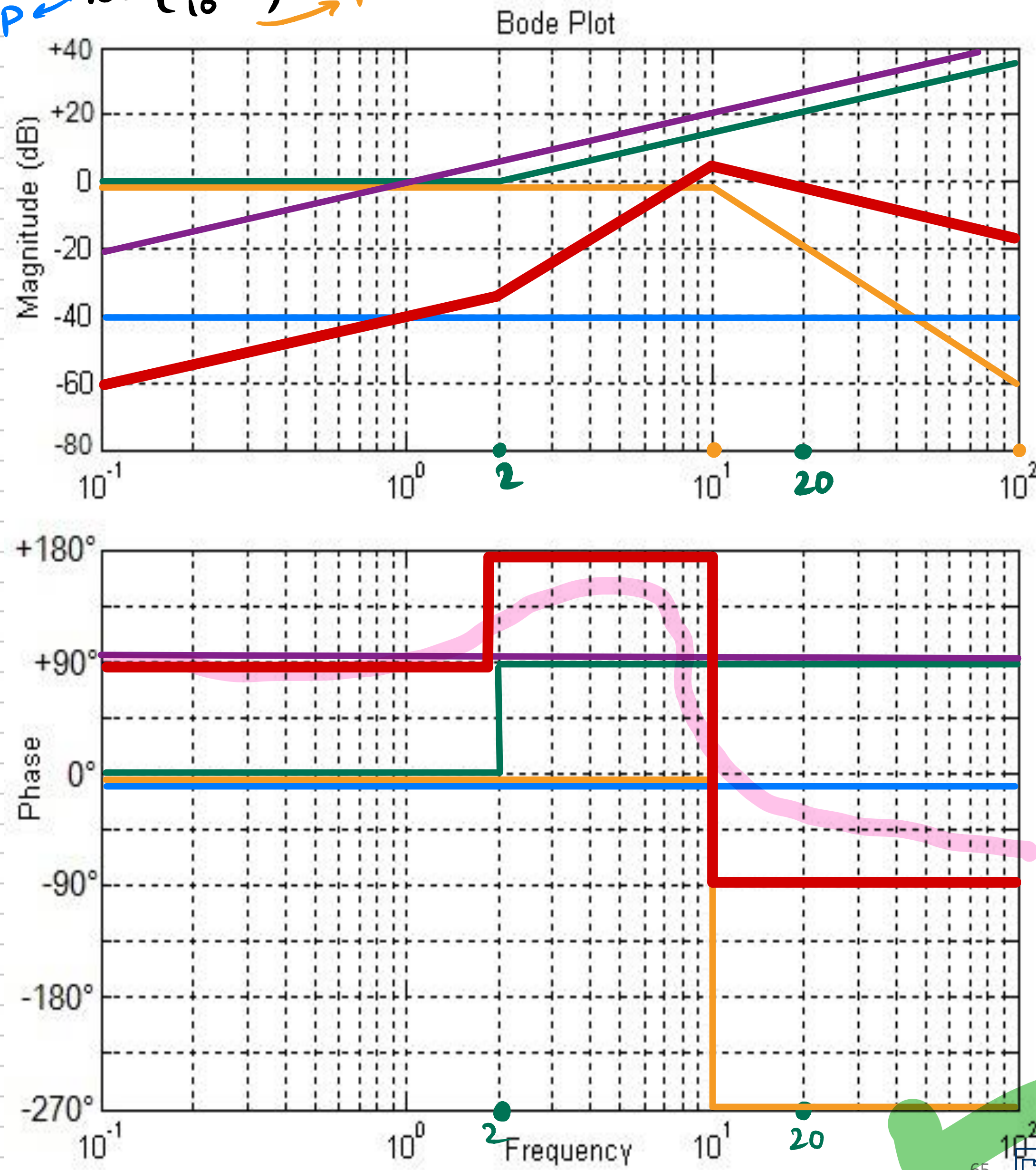
$$G(s) = \frac{5 \cdot s \cdot (s+2)}{(s+10)^3} = \frac{s \cdot s \cdot 2\left(\frac{s}{2}+1\right)}{10^3\left(\frac{s}{10}+1\right)^3} = \frac{s\left(\frac{s}{2}+1\right)}{100\left(\frac{s}{10}+1\right)^3}$$

D, DT_1, P, PT_1

$20\log\left(\frac{1}{100}\right)$

$20 \cdot -2 = -40$

Practice!



Overview PT_2-Element

Übertragungsfunktion

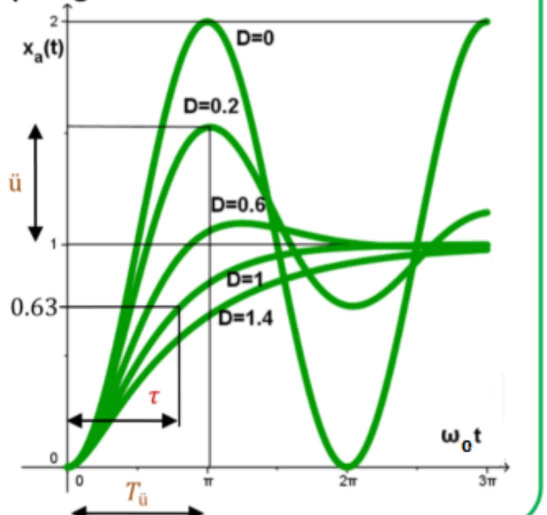
$$\omega_0^2\,\ddot{x}_a + 2D\,\omega_0\,\dot{x}_a + x_a = \omega_0^2 * x_e \qquad G(s) = \frac{1}{1 + \frac{2D}{\omega_0}s + \frac{1}{\omega_0^2}s^2}$$

D: Dämpfungsgrad
ω_0: natürliche Kreisfrequenz

oszillatorischer Ausgleichsvorgang $0 < D < 1$	aperiodischer Grenzfall $D = 1$	aperiodischer Ausgleichsvorgang $D > 1$
schwingfähiges System kompl. konj. Lösungen (imaginär)	nicht schwingfähiges System reelle Lösungen (2 PT1-Glieder) $\frac{1}{(s+a)(s+b)}$	

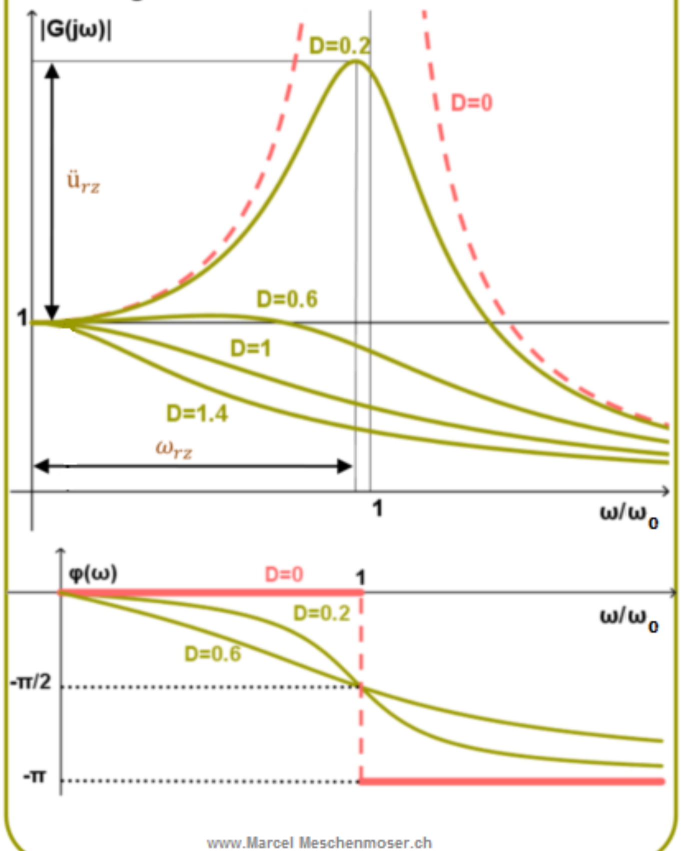
Resonanzfrequenz	$\omega_{rz} = \omega_0\sqrt{1-2D^2}$
Resonanzüberhöhung	$ü_{rz} = \frac{1}{2D\sqrt{1-D^2}}$
also bei	$\lvert D\rvert < \frac{1}{\sqrt{2}}$

Sprungantwort

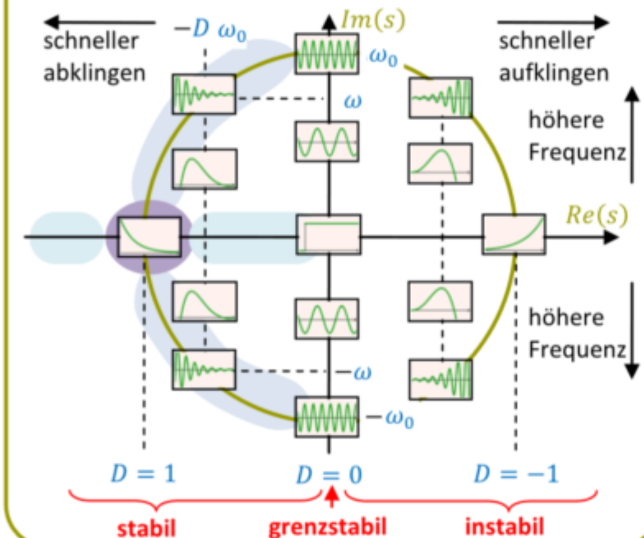


Absolute Dämpfung	$\sigma = D * \omega_0$
Abklingzeitkonstante (Zeit bis auf 37%)	$\tau = \frac{1}{\sigma}$
Kreisfrequenz der gedämpften Schwingung	$\omega = \omega_0\sqrt{1-D^2} = \frac{\pi}{T_ü}$
Periodendauer der gedämpften Schwingung	$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0\sqrt{1-D^2}}$
Überschwingzeit $T_ü$	halbe Periodendauer
Überschwingweite	$ü = e^{-\frac{\pi D}{\sqrt{1-D^2}}}$

Bodediagramm



Gauss'sche Zahlenebene

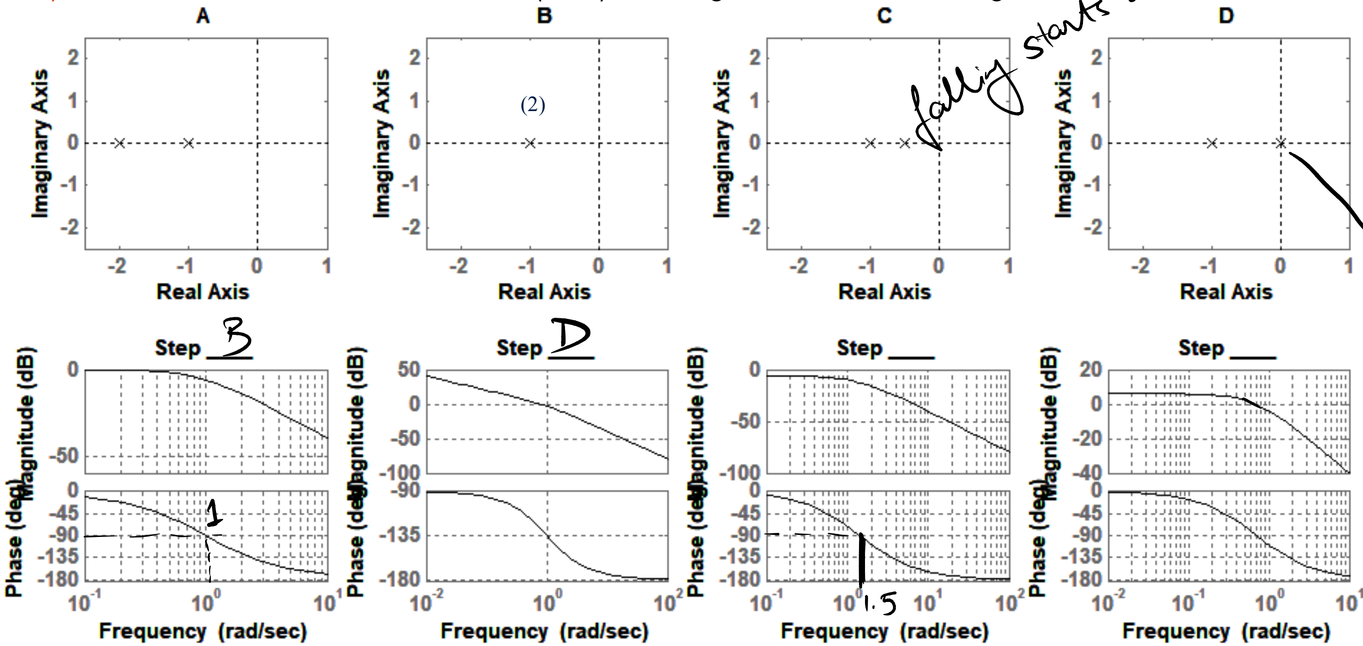


Polstellen	$s_{1,2} = \omega_0\left(-D \pm \sqrt{D^2-1}\right)$
Betrag	$\lvert G(j\omega)\rvert = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{2D\omega}{\omega_0}\right)^2}}$
Winkel	$\arg(G(j\omega)) = -\operatorname{atan}\left(\frac{\frac{2D\omega}{\omega_0}}{1+\frac{\omega^2}{\omega_0^2}}\right)$

Ex.2.20 Bode Diagram III

a) In each case state the transfer function (K = 1) and assign the correct Bode diagrams.

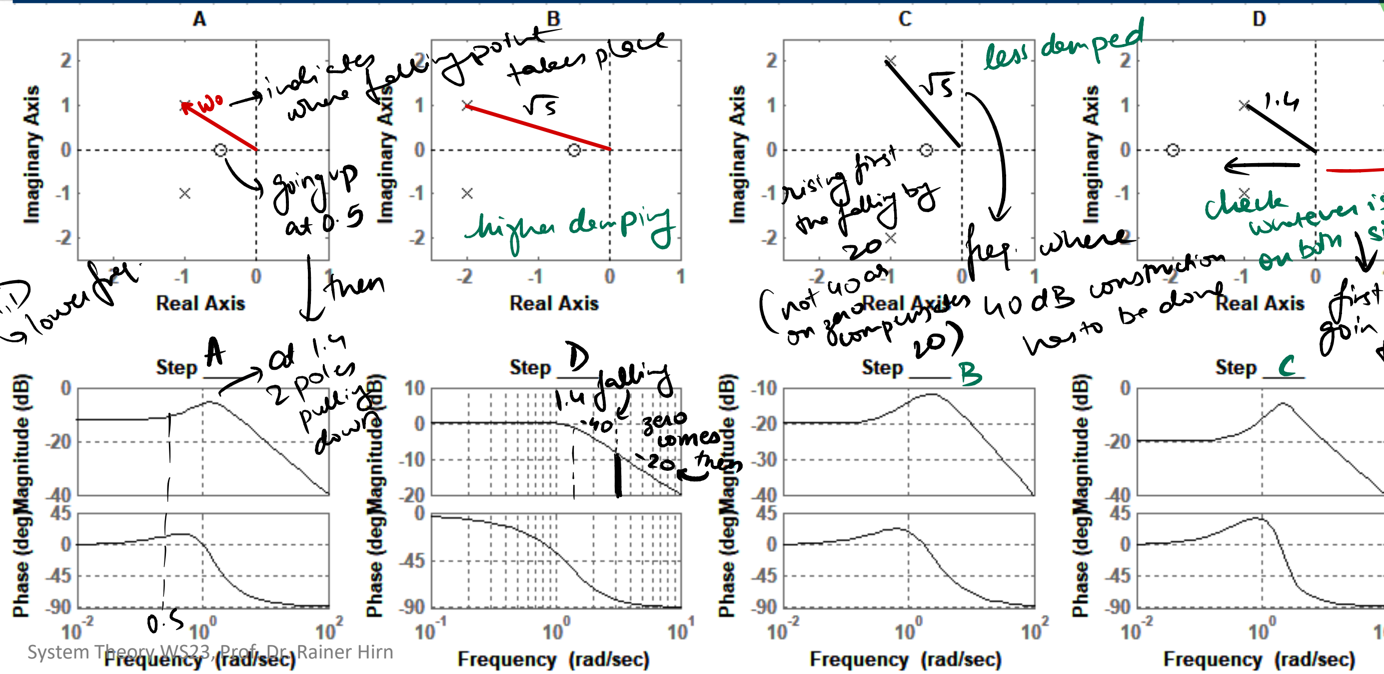
falling starts at 0.5



every pole pulling down by 20 dB

at 0: permanently inclining

practise



A, D slower freq.

then

(not 40 as zero compensates 20)

freq. where 40 dB construction has to be done

first going down then going up

1) $\omega_0 = \sqrt{1^2+1^2} = \sqrt{2}$

2) $\omega_0 = \sqrt{2^2+1^2} = \sqrt{5}$

3) $\omega_0 = \sqrt{1^2+2^2} = \sqrt{5}$

4) $\omega_0 = \sqrt{1^2+1^2} = \sqrt{2}$

1 pole $\searrow -20$

2 pole $\searrow -40$

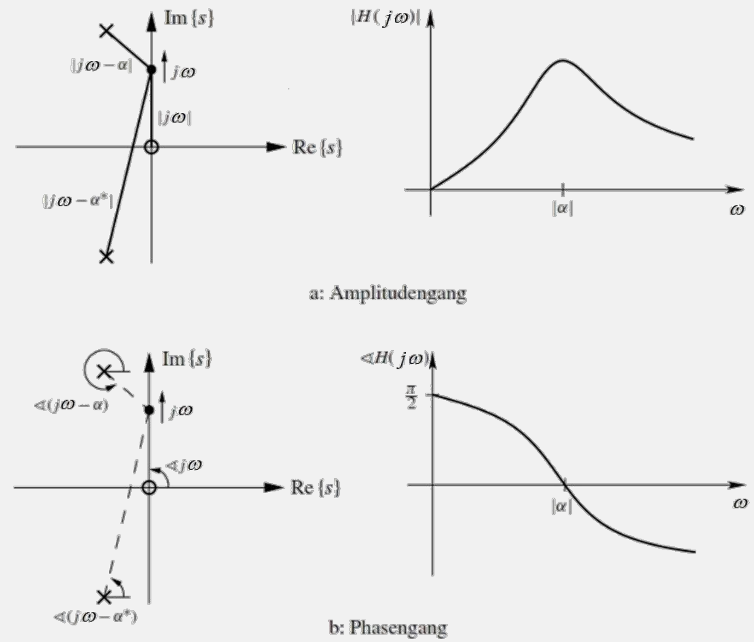
1 zero $\nearrow +20$

Frequency Response Design

A desired one **amplitude response** can be roughly specified using the pole-zero diagram:

$$|G(j\omega)| = c \cdot \frac{|j\omega - \beta_1| \cdot |j\omega - \beta_2| \cdot \ldots}{|j\omega - \alpha_1| \cdot |j\omega - \alpha_2| \cdot \ldots}$$

Because the individual terms can **graphically as a distance** of considered frequency point on the imaginary axis of the Poles and zeros are understood ...



Abschätzung des Frequenzganges über die Pol- und Nullstellenlage

The **Phase response** can be estimated in a similar way:

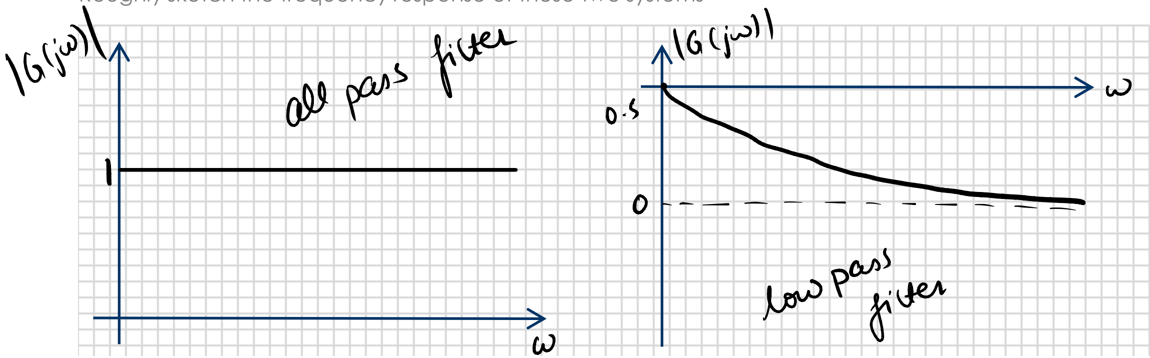
$$\angle G(j\omega) = \angle c + \sum_{i=1}^{M} \angle(j\omega - \beta_i) - \sum_{i=1}^{N} \angle(j\omega - \alpha_i)$$

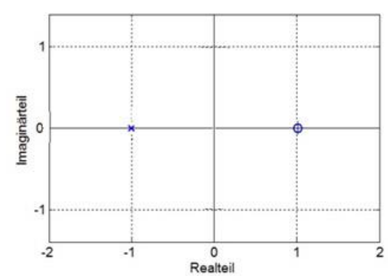
The respective phases of the individual terms add up here.
The contributions from zeros are positive and the contributions from Poland negative one,

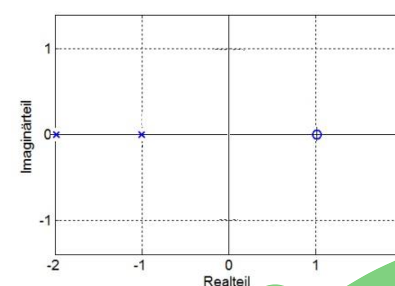
- Example:

Roughly sketch the frequency response of these two systems

$$G(s) = \frac{s-1}{s+1} \qquad G(s) = \frac{s-1}{(s+2)(s+1)}$$







Minimum Phase Systems

- Have **no** zeros in the right s-half plane.
- Minimum-phase systems have the shortest possible **signal propagation times** (signal delays)!

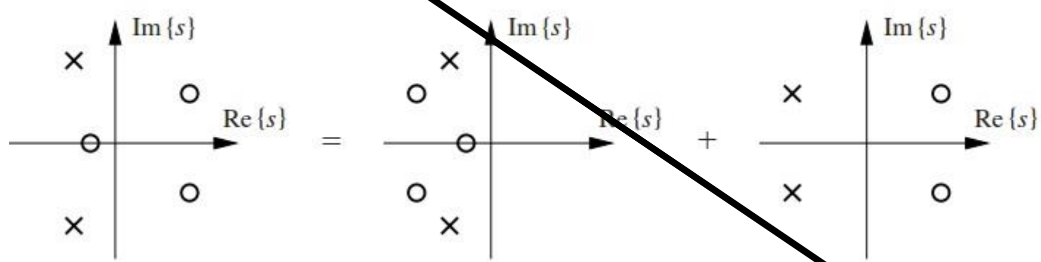
Allpass Systems

- Have a **constant** amplitude response,
- Their poles and zeros are **mirror images** to each other, they only cause a frequency-dependent phase-shifting effect.

- **Each** (all-pass) **system** can be **decomposed** into a minimum-phase and an all-pass component**!**

Example 1:

$$G(s) = G_M(s) \cdot G_A(s)$$



Zerlegung eines allpaßhaltigen Systems

Example 2:

Locus Curve:

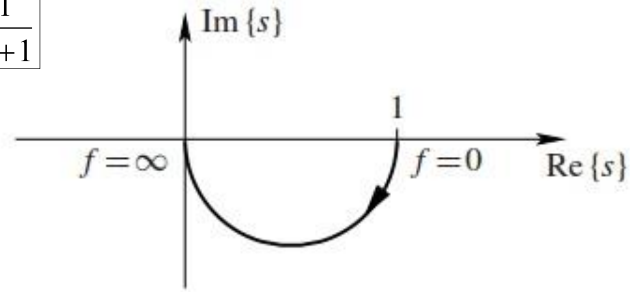
- Here, all values of the frequency response G(jω) are plotted directly in the complex plane, while the parameter ω runs through the range from 0 to ∞.

Very often the consideration of the reciprocal frequency response 1 / H (f) helps to determine the locus:

$\frac{1}{H(f)}$	$H(f)$
Gerade durch Ursprung	Gerade durch Ursprung
Kreis durch Ursprung	Gerade nicht durch Ursprung
Gerade nicht durch Ursprung	Kreis durch Ursprung
Kreis nicht durch Ursprung	Kreis nicht durch Ursprung

Eigenschaften der konformen Abbildung der Inversion

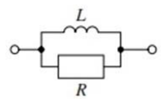
$$G(s) = \frac{1}{s+1}$$



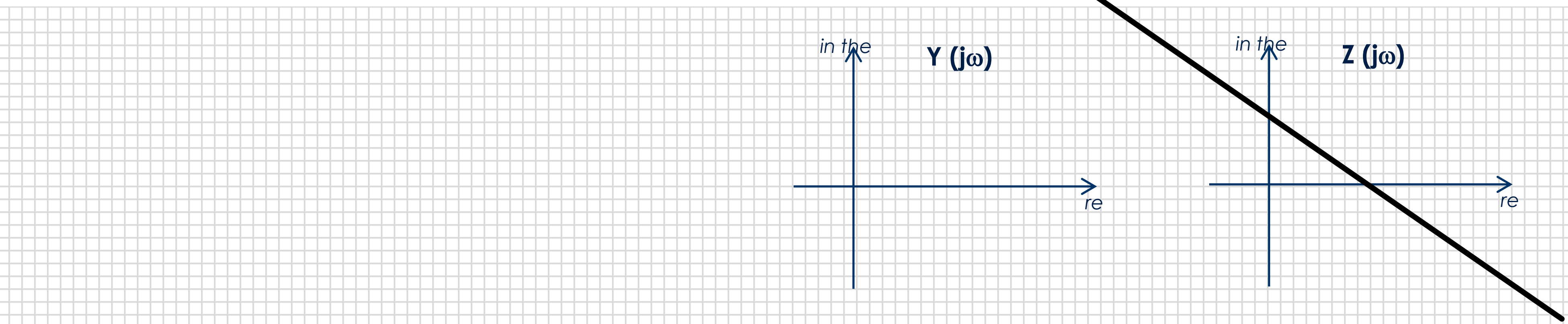
Ortskurve Tiefpaß erster Ordnung

Task:

The admittance this parallel connection reads: $Y(j\omega) = \frac{1}{Z(j\omega)} = \frac{1}{R} + \frac{1}{j\omega L}$

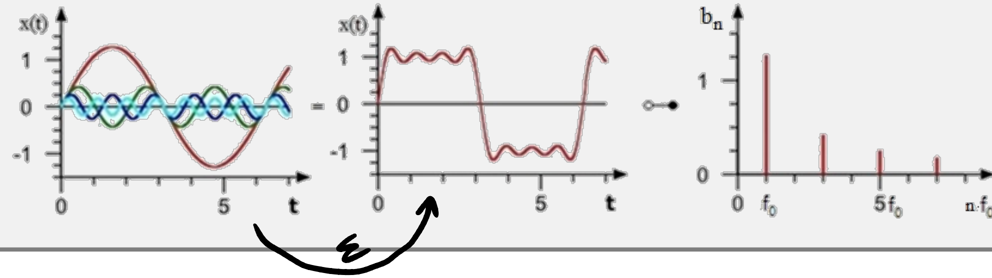


First draw the locus of the admittance Y (jω) and then the locus of the impedance Z (jω).



Fourier Series Expansion:

Only integer multiples of the fundamental frequency are needed!



Three forms of representation:

periodic signal

- **Sine-cosine representation:** $$\boxed{x_p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \quad , \quad b_0 = 0}$$
- **Cosine representation:** $$\boxed{x_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \phi_n) \quad , \quad A_n = \sqrt{a_n^2 + b_n^2} \quad \tan\phi_n = \frac{b_n}{a_n}}$$
- **Complex representation:** $$\boxed{x_p(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{jn\omega_0 t} \quad , \quad c_n = \frac{a_n - jb_n}{2} = \frac{A_n}{2} \cdot e^{-j\phi_n} \qquad c_{-n} = \frac{a_n + jb_n}{2} = \frac{A_n}{2} \cdot e^{j\phi_n}}$$

coefficients

- **Fourier series coefficients:**

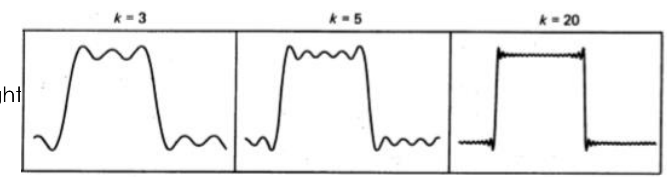
$$c_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cdot e^{-jn\omega_0 t} dt \qquad a_n = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cos(n\omega_0 t) dt \qquad b_n = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \sin(n\omega_0 t) dt$$

The coefficients indicate with what weight and with what displacement the **Fundamental wave** and the individual **Harmonics** occur in the approximation.
The Fourier series can be used as a **discrete spectrum** be interpreted.

Helpful: As you can see, the are complex Fourier coefficients c_n and c_{-n} of a real signal always complex conjugate to each other!

Gibb's Phenomenon:

In the case of jumps, one always remains even with an infinite number of harmonics **Error of approx. 9%** the jump height



Fourier Transform as Spectrum of a Signal

- The Fourier transform can be understood as a **boundary transition of the complex Fourier series.**
  Because a non-periodic function x (t) can be used as a periodic function with an infinitely long period $T_0 \to \infty$ can be understood:

Definition of Fourier Transformation:

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Inverse Fourier transformation:

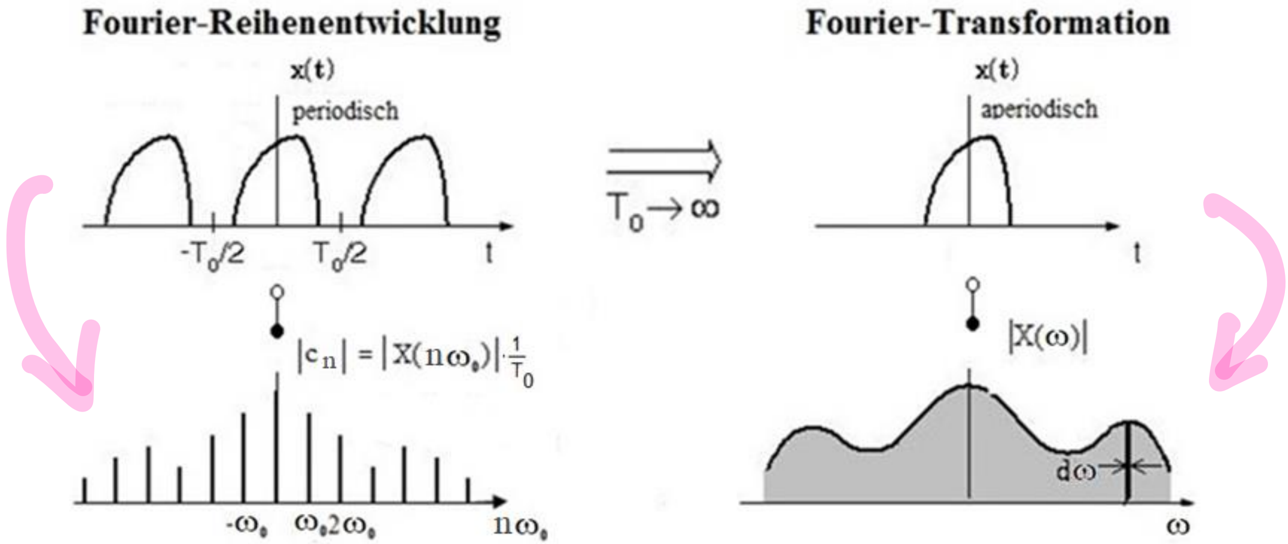
$$x(t) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Spectrum:

- is called the Fourier transform of a signal x(t).
- The spectrum gives the **spectral composition** of the signal, i.e. how strong and with what phase shift each frequency is contained.
- The spectrum of one **non-periodic** signal is **continuous!**
- The spectrum of one **periodic** Signal is **discrete** (it consists only of Dirac impulses)!

Since the spectrum is a complex function, one needs for the representation **Amount and phase**.

Note: Exist in the literature **version with f** as a frequency variable!

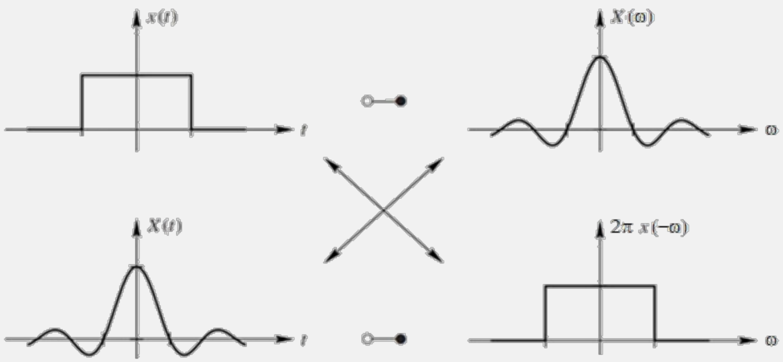


Duality:

$$x(t) \;\circ\!-\!\bullet\; X(\omega)$$

$$X(t) \;\circ\!-\!\bullet\; 2\pi \cdot x(-\omega)$$

This takes place Return transfer. the FT generally in the same way as the outward transformation.



Symmetry:

$$x(t) = x_g(t) + x_u(t) + jx'_g(t) + jx'_u(t)$$

$$X(j\omega) = \underbrace{X_g(j\omega) + X'_u(j\omega)}_{X(j\omega)} + \underbrace{jX'_g(j\omega) + jX_u(j\omega)}_{jX(j\omega)}$$

Band Limitation:

- A signal x (t) is called **band limited**if its spectrum is X (ω) from a certain cut-off frequency ω_G disappears:

  X (ω) = 0 for |ω| $\geq \omega_G$.

- Because of the **duality**, a **time-limited Signal is not band limited** and a **band limited signal is not time-limited!**

Parseval's Theorem (this formula only applies to energy signals!):

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = E_x$$

I.e..the signal**energy** of a signal can be **both in the time domain and in the frequency domain** to calculate.

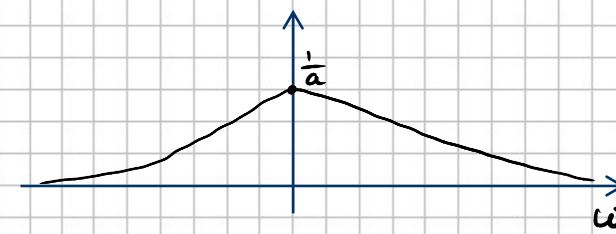
Ex.2.21 Spectrum I.

a) Determine and sketch the spectrum (→ fourier) of the decaying causal e-function x(t) as magnitude and phase.

$x(t) = e^{-at} \cdot \varepsilon(t), \quad a > 0$

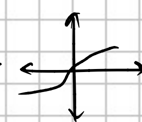
$$x(t) = e^{-at} \cdot \varepsilon(t) \quad \overset{\mathcal{L}}{\circ\!\!-\!\!\bullet} \quad \left.\frac{1}{s+a}\right|_{s=j\omega} = \frac{1}{j\omega + a} = X(j\omega)$$

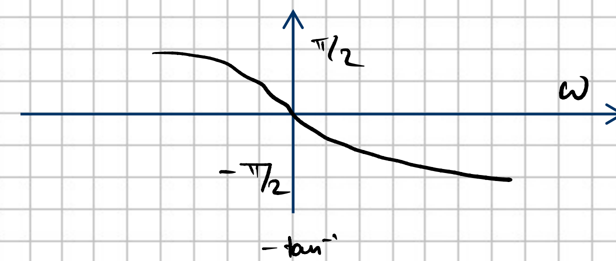
$$|X(j\omega)| = \frac{|1|}{|j\omega + a|} = \frac{1}{\sqrt{\omega^2 + a^2}}$$



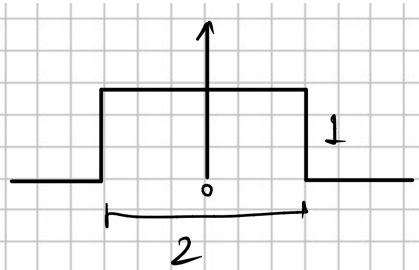
$$\angle X(j\omega) = \angle \text{num} - \angle \text{den} = 0 - \tan^{-1}\frac{\omega}{a}$$

$\tan^{-1}$:





b) Calculate and sketch the spectrum of the rectangular function of width 2 and height 1 as magnitude and phase.

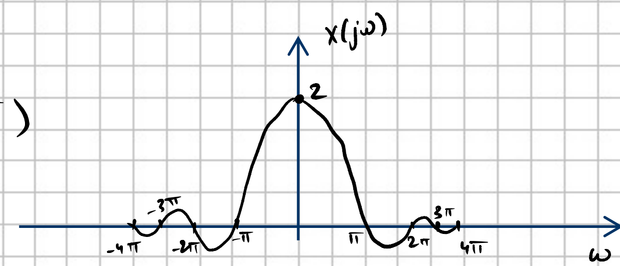


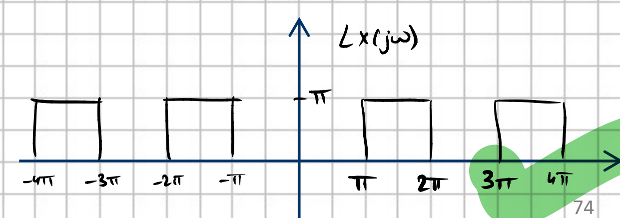
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-1}^{1} 1 \cdot e^{-j\omega t} dt = \left.\frac{e^{-j\omega t}}{-j\omega}\right]_{-1}^{1} = \frac{1}{j\omega}\left(e^{-j\omega t} - e^{j\omega t}\right)$$

$$= \frac{+2j\sin(\omega)}{+j\omega} = \frac{2\sin\omega}{\omega}$$

$$X(j\omega) = |2|\, si\left(\frac{2\omega}{2}\right) = 2\frac{\sin\omega}{\omega}$$



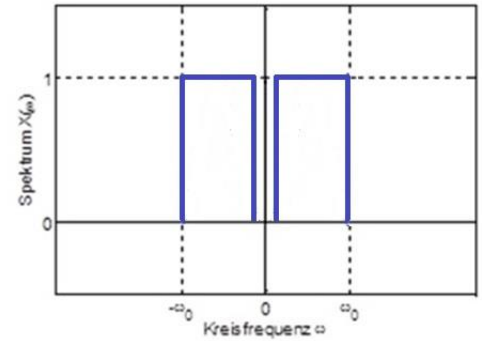


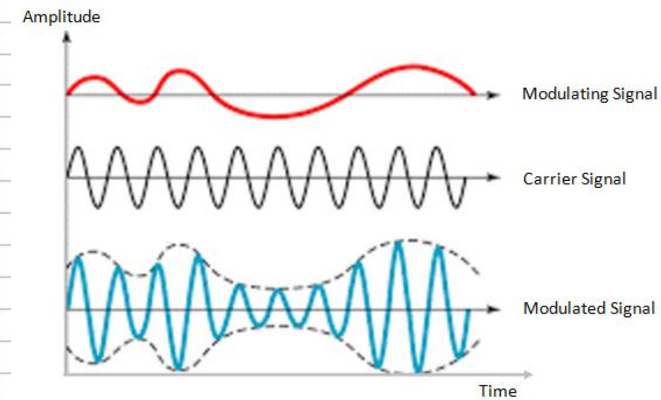
Ex 2.22 Spectrum II (AM broadcasting)

a) In an AM transmitter, in principle, a low-frequency audio signal x(t), whose spectrum X(jω)is shown is multiplied by a high frequency carrier signal (this is called modulation).
I.e. the antenna signal is e.g.:

$$y(t) = x(t) \cdot \cos(5\omega_0 t)$$

Calculate the spectrum Y(jω) of the antenna signal and sketch it.



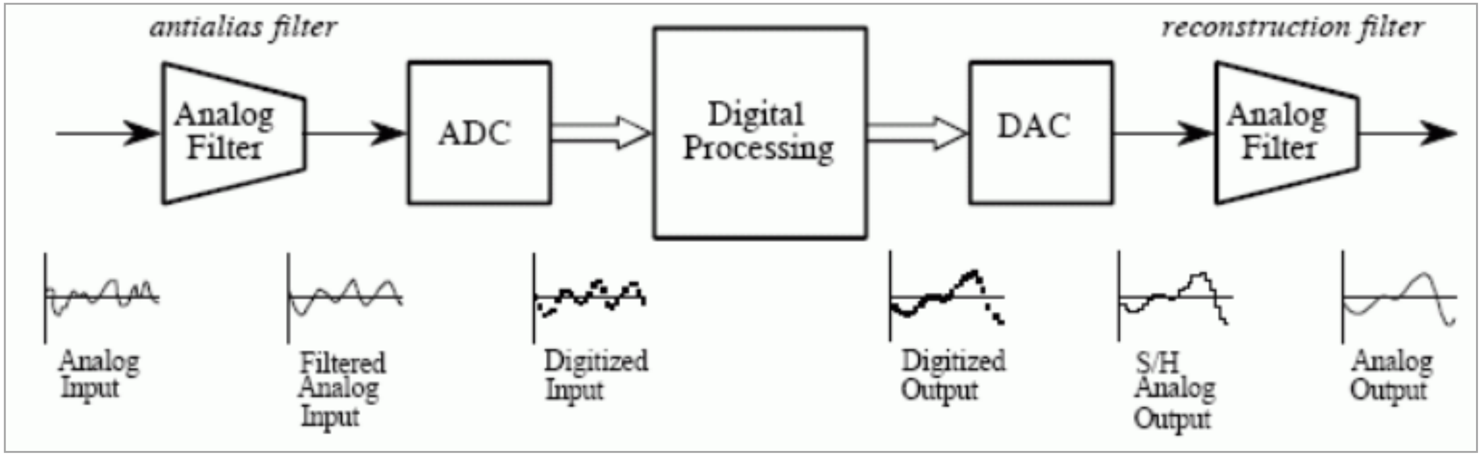


3.1 Basic Properties of Time-Discrete Signals

Digital Signal Processing

- Deals with the generation and processing of digital signals using digital systems.
- A **digital signal** is **discreet in time and values!**

Typical signals and components in a digital system:



- The design of time-discrete algorithms for the easy implementation of systems in software is the subject of the following chapters.

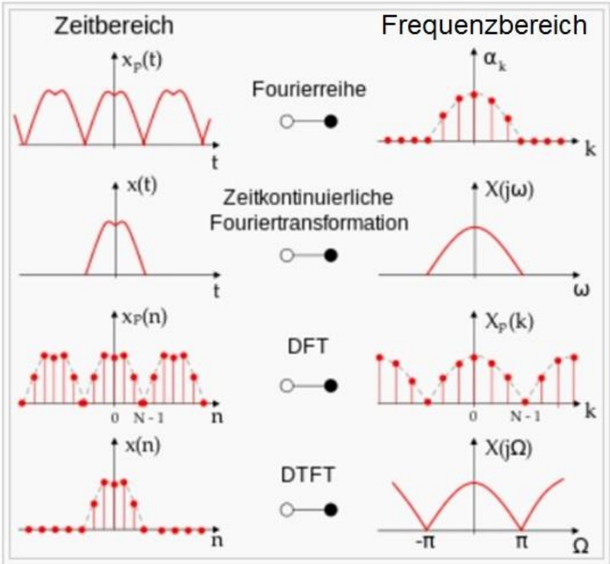
The Transformations of Systems Theory

An overview of the most important transformations („**tools**") **of systems theory:**

- Hint: The **z-transformation** is only a Laplace transformation version adapted to the notation of discrete systems (integrals become sums)!

		Zeitbereich	
		kontinuierliches Signal $x(t)$	diskretes Signal $x[k]$
Bildbereich	komplex	**Laplace-Transformation** $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$ $x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$	***z*-Transformation** $X(z) = \sum_{k=-\infty}^{\infty} x[k]\, z^{-k}$ $x[k] = \frac{1}{2\pi j} \oint_C X(z)\, z^{k-1}\, dz$
Frequenzbereich	kontinuierlich	**Fourier-Transformation** $X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$ $x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$	**Zeitdiskrete Fourier-Transformation** $X(f) = \sum_{k=-\infty}^{\infty} x[k]\, e^{-j2\pi T f k}$ $x[k] = T \int_{-1/2T}^{1/2T} X(f)\, e^{j2\pi T f k}\, df$
	diskret	**Fourier-Reihe** $X_n = \frac{1}{T_P} \int_{-T_P/2}^{T_P/2} x(t)\, e^{-j2\pi \frac{nt}{T_P}}\, dt$ $x(t) = \sum_{n=-\infty}^{\infty} X_n\, e^{j2\pi \frac{nt}{T_P}}$	**Diskrete Fourier-Transformation** $X[n] = \sum_{k=0}^{N-1} x[k]\, e^{-j2\pi \frac{kn}{N}}$ $x[k] = \frac{1}{N} \sum_{n=0}^{N-1} X[n]\, e^{j2\pi \frac{kn}{N}}$

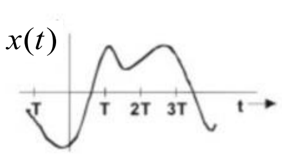
Discrete Signal $\circ\!-\!\bullet$ *Periodically Spektrum*
Periodically Signal $\circ\!-\!\bullet$ *Discrete Spektrum*



Overview Continuous vs. Discrete Systems

Continuous:

- Signal:



- Differential-Eq.: $\boxed{a_2\ddot{y}(t)+a_1\dot{y}(t)+a_0y(t)=b_1\dot{u}(t)+b_0u(t)}$

$$\Updownarrow L$$

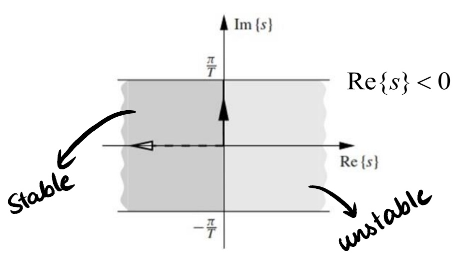
- TF: $\boxed{G(s)=\dfrac{Y(s)}{U(s)}=\dfrac{b_1s+b_0}{a_2s^2+a_1s+a_0}}$

- LaplaceTransf.: $\boxed{X(s)=\int_0^\infty x(t)e^{-st}dt}$

- Convolution: $\boxed{y(t)=g(t)*u(t)=\int_{-\infty}^{\infty}g(\tau)\cdot u(t-\tau)d\tau}$

- Frequency resp.: $\boxed{G(j\omega)=G(s)\big|_{s=j\omega}}$

- Stability:

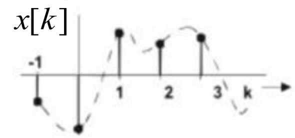


$$z=e^{sT}$$

$$\Longleftrightarrow$$

Discrete:

Signal:



Abkü.: $x[k]=x(kT)$

Differences-Eq.: $\boxed{\tilde{a}_2y[k+2]+\tilde{a}_1y[k+1]+\tilde{a}_0y[k]=\tilde{b}_1u[k+1]+\tilde{b}_0u[k]}$

$$\Updownarrow Z$$

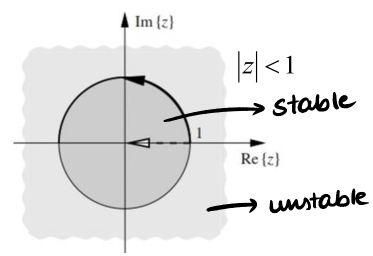
TF: $\boxed{G(z)=\dfrac{Y(z)}{U(z)}=\dfrac{\tilde{b}_1z+\tilde{b}_0}{\tilde{a}_2z^2+\tilde{a}_1z+\tilde{a}_0}}$

z-Transform .: $\boxed{X(z)=\sum_{k=0}^{\infty}x[k]\cdot z^{-k}}$ Abkü.: $z=e^{sT}$

Convolution: $\boxed{y[k]=g[k]*u[k]=\sum_{n=-\infty}^{\infty}g(n)\cdot u(k-n)}$

Frequency resp.: $\boxed{G(e^{j\Omega})=G(z)\big|_{z=e^{j\Omega}}}$ Abkü.: $\Omega=\omega T$

Stability:



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e.g.: (forward) difference quotient: e.g. bilinear transformation: e.g. recursive solution of a difference-q.:

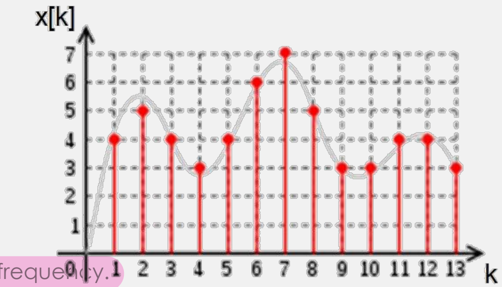
$$\left.\frac{dy(t)}{dt}\right|_{t=kT}\approx\frac{y((k+1)T)-y(kT)}{T}=\frac{y[k+1]-y[k]}{T}$$

$$G(z)=G(s)\big|_{s=\frac{2}{T}\cdot\frac{z-1}{z+1}}$$

$$\boxed{y[k]=\tfrac{1}{\tilde{a}_2}\left(-\tilde{a}_1y[k-1]-a_0y[k-2]+\tilde{b}_1u[k-1]+\tilde{b}_0u[k-2]\right)}$$

The Sampling Process

Sampling – reduction of continuous time signal to a discrete time signal



- is **taking signal samples**'of a continuous signal x(t) at a time interval T, the following signal is generated:

$$x[k] = x(t)\big|_{t=kT} \qquad k \in Z$$

- **x [k]** is called a **time-discrete signal**, **signal sequence** or short **samples**.

  the time interval T between two samples as the sampling interval, the reciprocal value $f_a = 1 / T$ as the sampling rate or sampling frequency.

  T   1/T

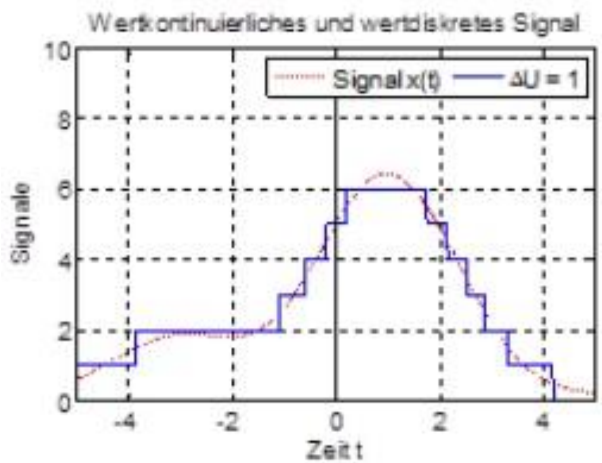
Quantization error

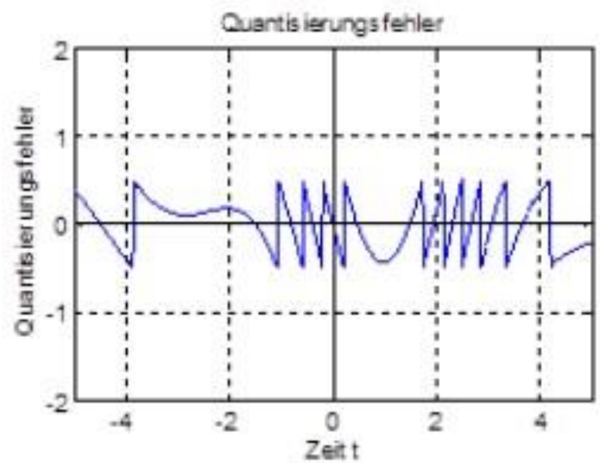
- is the **rounding errors**, which arises when the values are assigned in the AD converter, as this ionly works with a **limited resolution**.

  He will **usually neglected** or only viewed as a weak noise signal.

  For example, a 10-bit AD converter can only handle 2^{10}= Determine 1024 different values. At 5VMeasuring range is the minimum possible resolution, for example:

$$\Delta U = \frac{5V}{2^{10}} = \frac{5V}{1024} \approx 5mV$$





Spectrum of a sampled signal:

- Math. the simplest way to represent the sampling process is a **multiplication** of the time signal x(t) with a **impulse train** (time interval T):

$$x_a(t) = x(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

- This multiplication in the time domain corresponds to a convolution with the spectrum of this pulse comb in the spectral domain.
  This is again an impulse comb (only with the factor ω_a multiplied)! - it follows:

$$X_a(j\omega) = \frac{1}{2\pi} X(j\omega) * \left(\omega_a \cdot \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_a) \right) = \frac{1}{T} \cdot \sum_{n=-\infty}^{\infty} X\big(j(\omega - n\omega_a)\big)$$

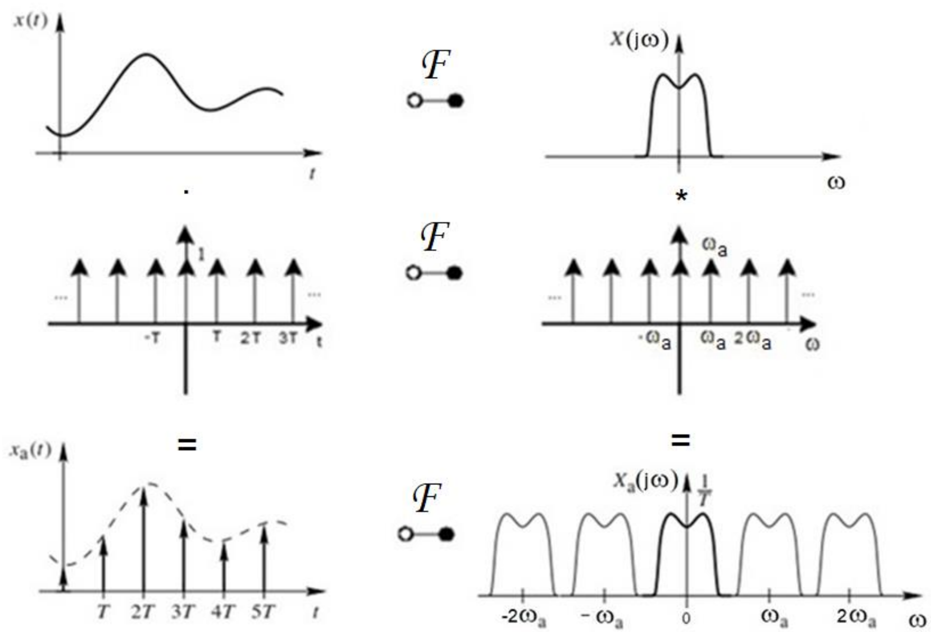
"Sampling repeats a signal spectrum infinitely periodically in ω_a!"

and weighted with 1 / T.

- Description of sampling by multiplication with a Dirac pulse train:

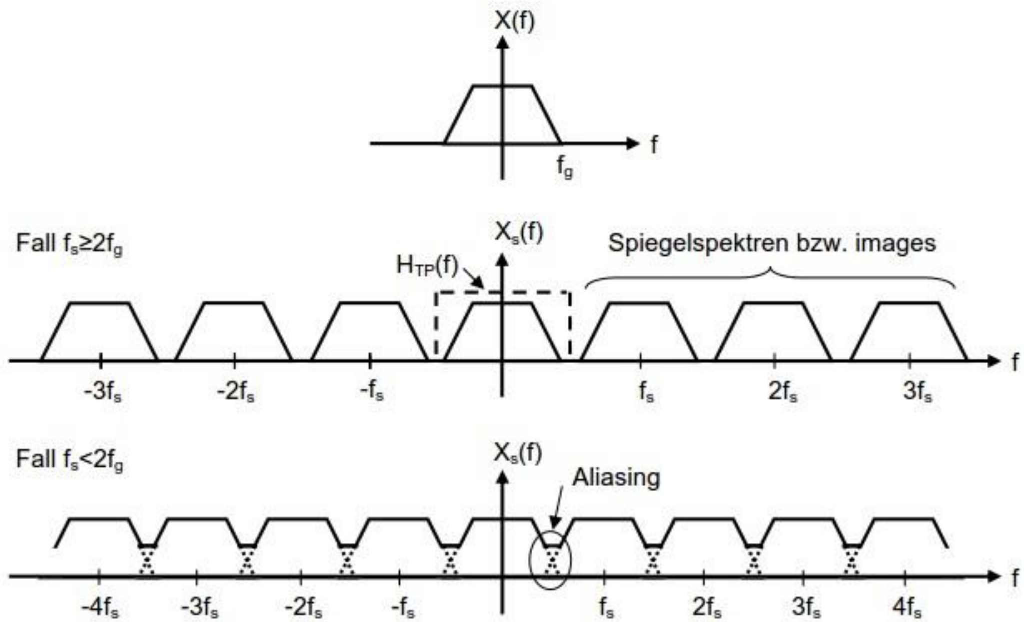
$$\sum_{k=-\infty}^{\infty} \delta(t - kT) \quad \circ\!-\!\bullet \quad \omega_a \cdot \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_a)$$

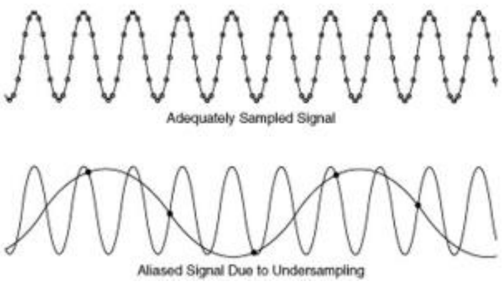
$$\omega_a = 2\pi f_a = \frac{2\pi}{T}$$



Undersampling (Aliasing)

- occurs when a signal is sampled **too slowly.**
  This results in an overlap of the new sidebands, which leads to a loss of information content in the signal - this cannot be **reversed**!





Nyquist-Shannon Sampling Theorem:

- Every band-limited signal x(t) can be clearly represented by means of **samples x[k] = x (kT)**, the **sampling frequency ω_a must be greater than twice the cutoff frequency ω_G (maximum frequency) of the signal**

$$\boxed{\omega_a \geq 2\omega_g}$$

Sampling freq. ≥ 2 cutoff freq.

- Headquarters **condition** for the 'Scannability'of a signal is the **Band limitation**, therefore an analog so-called. **Anti-aliasingLow pass** upstream!

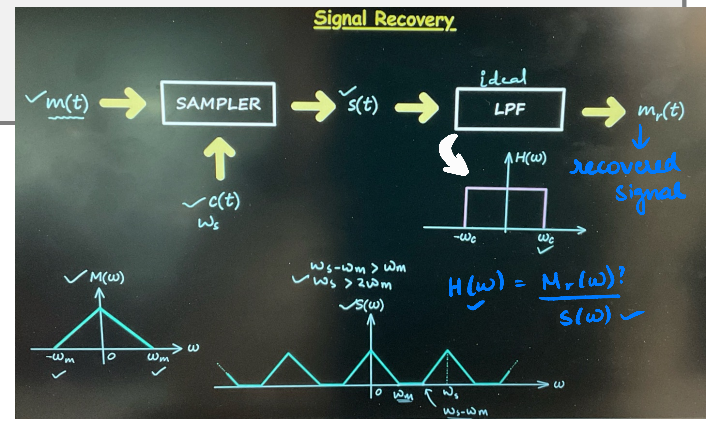
Signal reconstruction

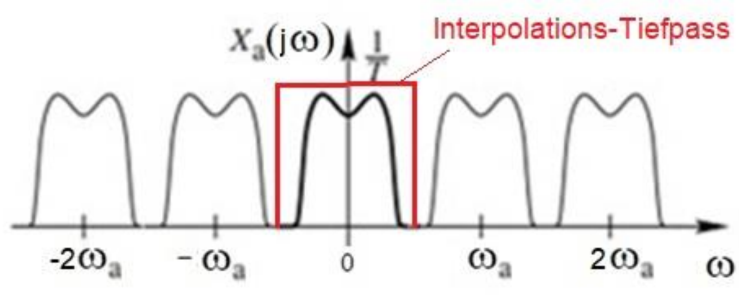
- from the weighted pulse train $x_a(t)$ to the analog signal x(t) is achieved by eliminatiing the **periodic continuations** of the spectrum with the aid of an analog **low pass filter** (Interpolation low pass).

  An ideal would be suitable Low pass with the cutoff frequency equal to half the sampling frequency.
  His **Impulse response would be an si function:**

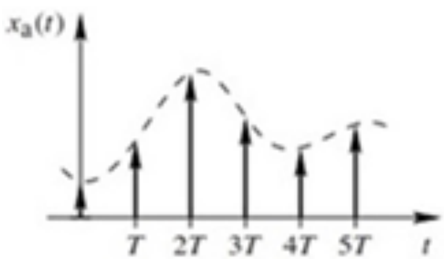
$$G_{TP}(j\omega) = T \cdot rect\left(\tfrac{\omega T}{2\pi}\right) \quad \bullet\!-\!\circ \quad g_{TP}(t) = si\left(\pi \tfrac{t}{T}\right)$$

discrete → analog





- The ideally reconstructed time signal x(t) is therefore simply a superposition of impulse responses of this filter weighted with the sampled values, i.e. si-functions! In reality this is **not possible** because of the lack of causality!



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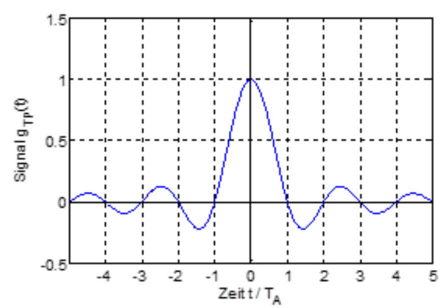

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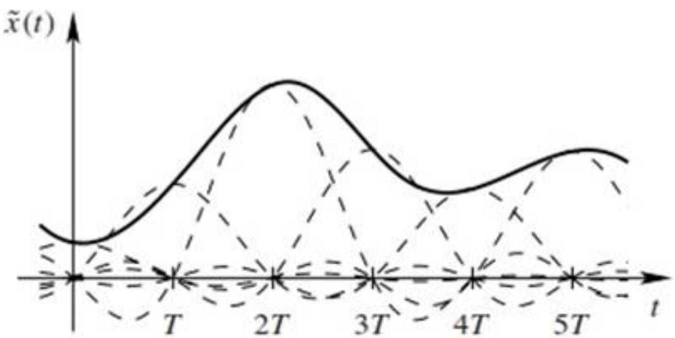


1. A Zero-Hold-Element (ZOH)

- first creates an analog **step signal x_H** in the real DA-converter,

- **Consideration in the time domain:**

  The holding element thus generates a rectangle of width T from an impulse, ie its impulse response =

  $$g(t)=\varepsilon(t)-\varepsilon(t-T)$$

  The signal at the holding element output $x_H(t)$ is thus the **folding** the one with the x (kT) weighted Dirac impulse comb and the above impulse response:

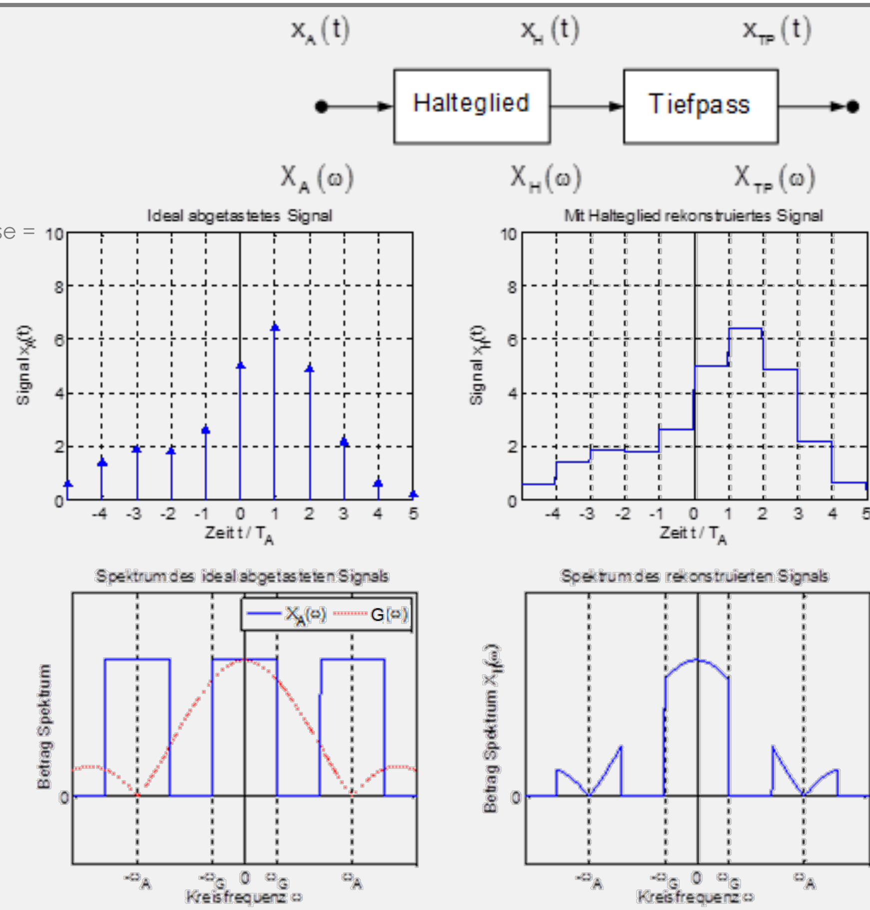
  $$x_H(t)=\left(\sum_{k=-\infty}^{\infty}x(kT)\delta(t-kT)\right)*\left(\varepsilon(t)-\varepsilon(t-T)\right)$$

- **Consideration in the spectral range:**

  The spectrum of the staircase signal $x_H(t)$ is therefore the ideal **periodic signal spectrum $X_a(j\omega)$ multiplied by the si spectrum G (jω)** of the sample and hold element:

  $$X_H(j\omega)=X_a(j\omega)\cdot G(j\omega)\quad \text{mit}\quad G(j\omega)=\tfrac{2}{\pi}\,\mathrm{si}(\tfrac{\omega T}{2})\cdot e^{-j\frac{\omega T}{2}}$$

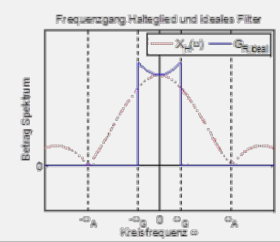
  This slightly attenuates higher frequencies in the baseband,
  the sidebands are already more strongly suppressed, which makes the subsequent low-pass filtering easier!



2. Analog Interpolation Low pass:

- Is required to remove the remaining (already weakened) sidebands of the staircase signal.
- **ZOH compensation:**

  The above-mentioned ZOH can be reduced in the higher frequency components either by additional digital or analog inverse filtering can be undone in the baseband!

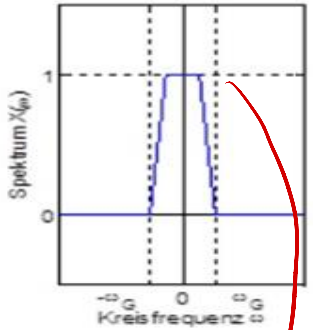


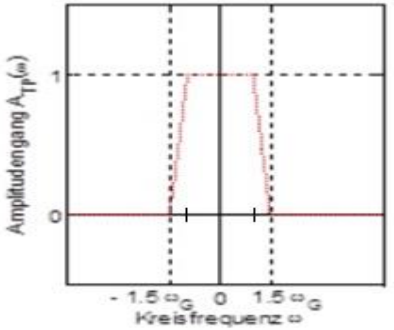
Ex.3.1 Sampling with a filter with finite slopes

Given is the spectrum X (jω) an analog signal x (t) which is to be sampled.
For later reconstrucion, an analogue low pass filter is to be used, which has the amplitdue response response $A_{TP}(j\omega)$ as shown.

a) First, sketch the resulting spectrum when x (t) at a sampling frequency of $3\omega_G$ ab is keyed.

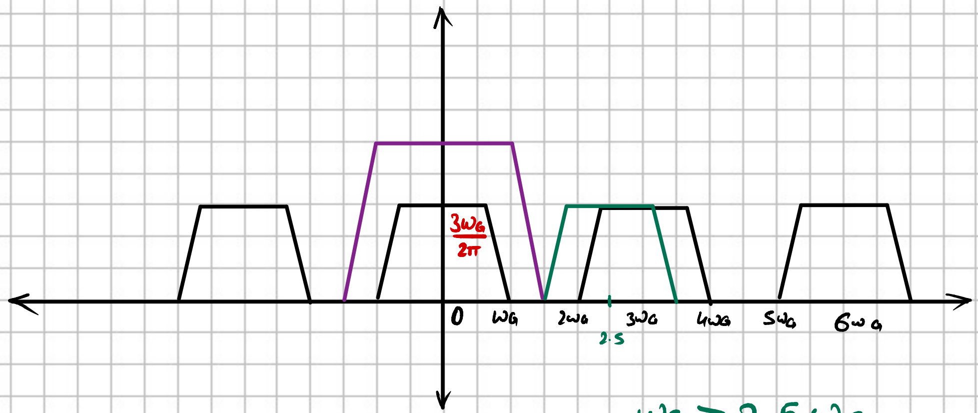
b) What is the minimum sampling frequency that must be selected for this Reconstruction low pass no aliasing occurs at the output?





$1 \rightarrow \frac{3\omega_G}{2\pi}$

2/3 ?



$\omega_S \geq 2.5\,\omega_G$

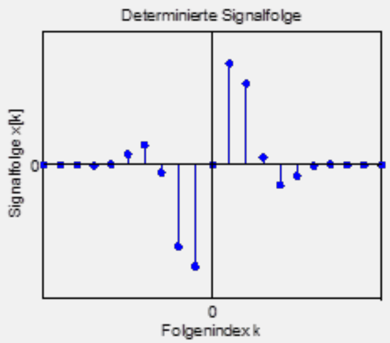
Determined signals

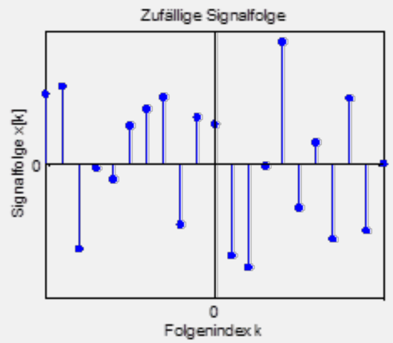
- i.e. math. Either **explicitly** described

$$x[k]=10\cdot r^{k}\cdot\sin(\Omega_0 k)$$

- or just **implicitly** as so-called **difference equation** e.g. with the Initial condition $x[0] = x_0$:

$$x[k]=2\cdot x[k-1]-4\cdot x[k-2]$$





Random signals

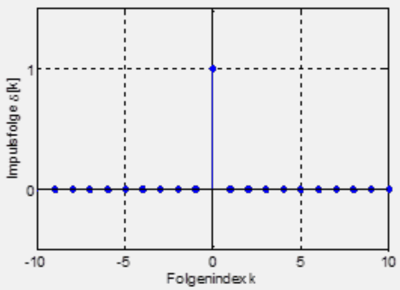
- cannot be specified exactly, only statistical properties are known for them.

Pulse Signal:

- Is not defined as a distribution, only as

$$\delta[k]=\begin{cases}1 & \mathit{für} \quad k=0\\ 0 & \mathit{für} \quad \mathit{ganzahlige}\, k\neq 0\end{cases}$$

It is zero for k <0 and is therefore a causal consequence.

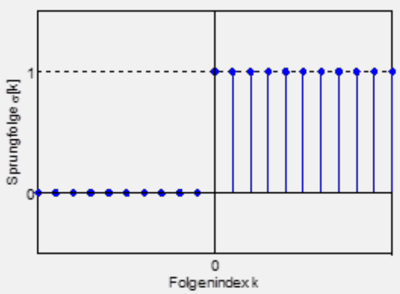


Step Signal:

- Is not defined as a distribution but just as

$$\sigma[k]=\varepsilon[k]=\begin{cases}0 & \mathit{für} \quad k<0\\ 1 & \mathit{für} \quad k\geq 0\end{cases}$$

It is zero for k <0 and is therefore a causal consequence.



Ex.3.2 Discrete-time Signals

k- → right shift

k+ → left shift

-k+ → right shift

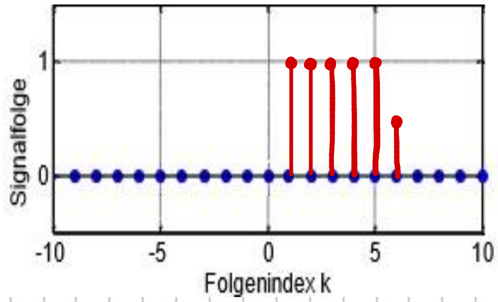
Given is a time-discrete signal x[k].

a) Sketch the two signals

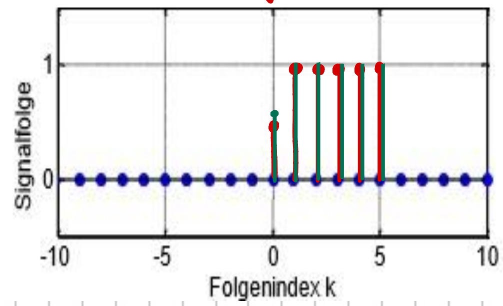
$y_1[k] = x[k-2]$

$y_2[k] = x[4-k]$

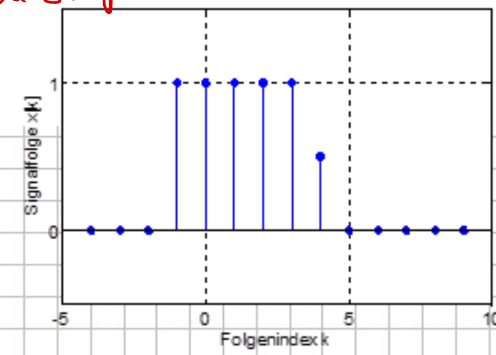
$y_1[k]$



$y_2[k]$



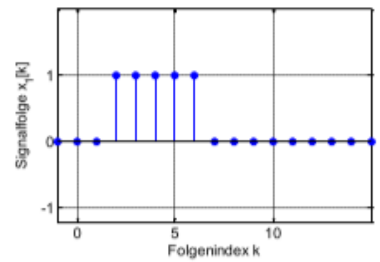
$x[k]$

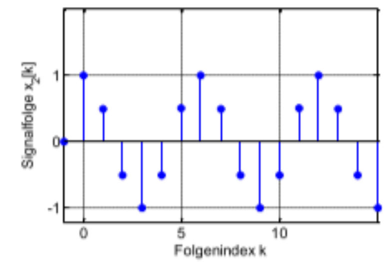


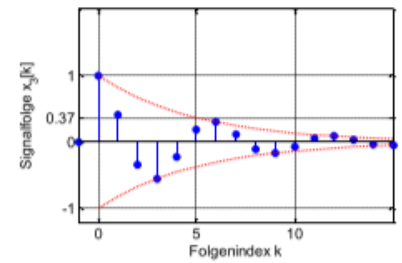
b) Give a math. explicit expression of the following signals.

Are they causal?

Are they energy or power signals?







$$x_1[k] = \delta[k-2] + \delta[k-3] + \delta[k-4] + \delta[k-5] + \delta[k-6]$$

$$= \varepsilon(t-2) - \varepsilon(t-7)$$

$$x_2[k] = \underbrace{1}_{\text{amplitude}} \cdot \cos\left[2\pi/6 \cdot k\right] \, \varepsilon[k]$$

→ to make it causal

Area under power signal = ∞

$$x_3[k] = e^{-\frac{1}{5}k} \cos\left(\frac{2\pi}{6} \cdot k\right) \cdot \varepsilon[k]$$

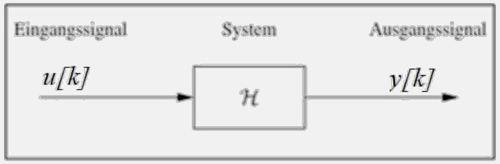
how $e^{-\frac{1}{5}}$

The Difference Equation

Description of time-discrete systems in the time domain:

- in the **continuous time** one describes systems with **differential equations**.
- in the **diskrete time** one describes systems with d**ifference equations.**

  Algorithms for digital signal processing usually work with a clock with **fixed sampling time T.**

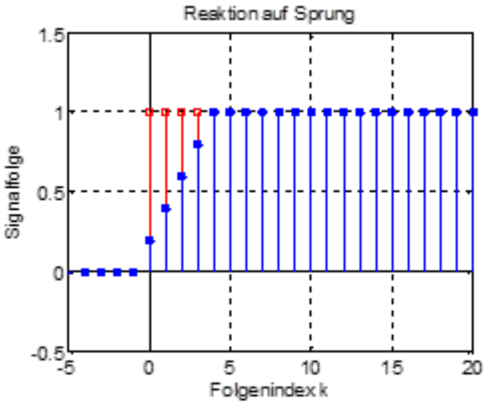


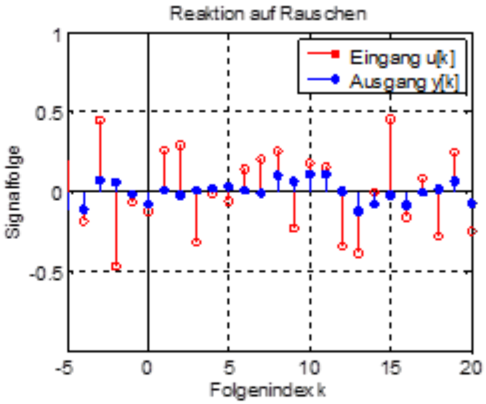
Example Moving average filter for signal smoothing

$$y[k] = \frac{1}{5} \cdot \left(u[k] + u[k-1] + u[k-2] + u[k-3] + u[k-4] \right)$$

The following noise suppressor is more memory-saving
(= digital low pass filter)

$$y[k] = \frac{4}{5} \cdot y[k-1] + \frac{1}{5} \cdot u[k]$$





Impulse and step responses:

- Provide both (filter) algorithms **linear difference equations** which, as in continuous time, must first settle in and are already clearly described by their impulse and step responses!

Discrete-time approximation of a DE:

not relevant

$$a_2\ddot{y}(t)+a_1\dot{y}(t)+a_0y(t)=b_2\ddot{u}(t)+b_1\dot{u}(t)+b_0u(t)$$

- is obtained by replacing the differentials for example by (backward) -**difference quotients** over **a sampling period T**:

$$\left.\frac{dy(t)}{dt}\right|_{t=kT}\approx\frac{y[k]-y[k-1]}{T}$$

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=kT}\approx\frac{\frac{y[k]-y[k-1]}{T}-\frac{y[k-1]-y[k-2]}{T}}{T}=\frac{y[k]-2y[k-1]+y[k-2]}{T^2}$$

The result is a sum of sequence values which are each shifted by one sample value. In general, the current output signal is a function of the past input and output values and the current input signal.

Difference Equation:

- Analogous to the continuous time, the description of linear time-invariant **systems in discrete time** leads to linear **difference equations of Nth order with constant coefficients:**
- $$\boxed{\tilde{a}_2y[k-2]+\tilde{a}_1y[k-1]+\tilde{a}_0y[k]=\tilde{b}_2u[k-2]+\tilde{b}_1u[k-1]+\tilde{b}_0u[k]}$$

  or:

  $$\boxed{y[k]=-\tilde{a}_1y[k-1]-\tilde{a}_2y[k-2]+\tilde{b}_0u[k]+\tilde{b}_1u[k-1]+\tilde{b}_2u[k-2]} \qquad (\tilde{a}_0=1)$$

System Response in the Time Domain

1) Direct recursive solution of the difference equation:

$$y[k]=\sum_{m=0}^{M} d_m \cdot u[k-m]-\sum_{n=1}^{N} c_n \cdot y[k-n]$$

This recursive solution method is completely sufficient for computer implementations. However, it is not a closed solution.

2) Discrete Convolution (aperiodic):

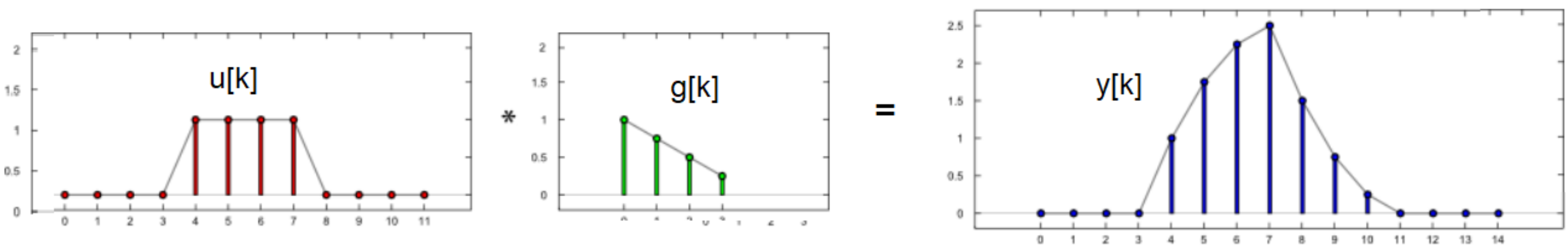
- the **Impulse response g[k]** of a time-discrete system is again the response to an impulse.

  Any input signal u [k] can now be described as a weighted sum of pulses: $u[k]=\sum_{\kappa=-\infty}^{\infty} u[\kappa]\cdot\delta[k-\kappa]$

- The system response y[k] is thus simply the superposition of impulse responses weighted with u[k] **= convolution**

$$\boxed{y[k]=g[k]*u[k]=\sum_{n=-\infty}^{\infty} g[n]\cdot u[k-n]}$$

$$y(t)=\int_{-\infty}^{\infty} g(\tau)\cdot u(t-\tau)d\tau$$



The z-Transform

The **equivalent** to the Laplace **transform** in discrete time is the **z transformation**.

- Derivation:

$$x_A(t) = \sum_{k=0}^{\infty} x(kT)\cdot\delta(t-kT)$$

$$X_A(s) = L\left\{\sum_{k=0}^{\infty} x(kT)\cdot\delta(t-kT)\right\} = \sum_{k=0}^{\infty} x(kT)\cdot L\{\delta(t-kT)\} = \sum_{k=0}^{\infty} x(kT)\cdot e^{-skT} = \sum_{k=0}^{\infty} x[k]\cdot(e^{sT})^{-k} \qquad q.e.d.$$

Definition of the ordinary (one-sided) z-Transform:

$$\boxed{X(z) = Z\{x[k]\} = \sum_{k=0}^{\infty} x[k]\cdot z^{-k} \quad mit\ x[k]=0 \quad für\ k<0} \quad \boxed{z = e^{sT} \quad \in C}$$

$$x[k] = \frac{1}{2\pi j}\oint_C X(z)\cdot z^{k-1}dz$$

- **Dumbbell symbol:** $x[k] \quad \circ\!-\!\bullet \quad X(z)$
- **Correspondence:** $\delta[k]\circ\!-\!\bullet 1 \qquad a^k\cdot\varepsilon[k]\circ\!-\!\bullet \dfrac{z}{z-a}$
- **Shift rule:** $x[k-k_0] \quad \circ\!-\!\bullet \quad z^{-k_0}\cdot X(z)$

- Further correspondence: **See Annex:**

Ex.3.3 z-Transform (calculation rules)

Compute the z-transforms of the following sequences using the properties of the z-transform:

a) $x[k] = k^2 \cdot \varepsilon[k]$
b) $x[k] = k \cdot a^k \cdot \varepsilon[k]$
c) $x[k] = (k-k_0) \cdot a^{k-k_0} \cdot \varepsilon[k-k_0]$

a) $x[k] = k^2 \varepsilon[k]$

$x'[k] = k \cdot \varepsilon[k] \circ\!\!\overset{z}{-\!\!-}\!\!\bullet\; X(z) = \frac{z}{(z-1)^2}$

$k \cdot x'[k] = -z \frac{d}{dz} X(z) = -z \frac{d}{dz} \frac{z}{(z-1)^2}$

$= -z\left(\frac{1(z-1)^2 - z\,2(z-1)}{(z-1)^4}\right)$

$= \frac{-z(z^2 - \cancel{2z} + 1 - 2z^2 + \cancel{2z})}{(z-1)^4} = \frac{-z(-z^2+1)}{(z-1)^4}$

$= \frac{z^3 - z}{(z-1)^4}$

b) $x[k] = k \cdot a^k \cdot \varepsilon[k]$

$x'[k] = a^k \varepsilon[k] \circ\!\!\overset{z}{-\!\!-}\!\!\bullet\; X'(z) = \frac{z}{z-a}$

$x[k] = k \cdot x'[k] \circ\!\!\overset{z}{-\!\!-}\!\!\bullet\; X(z) = -z \frac{d}{dz} X'(z)$

$\therefore X(z) = -z \frac{d}{dz}\left(\frac{z}{z-a}\right)$

$= -z\left(\frac{1(z-a) - z(1)}{(z-a)^2}\right)$

$= \frac{-z(\cancel{z} - a - \cancel{z} + \ )}{(z-a)^2} = \frac{za}{(z-a)^2}$

c) $x[k] = (k-k_0)a^{k-k_0}\varepsilon[k-k_0]$

$\parallel$

$z^{-k_0} X(z)$

$= z^{-k_0} \frac{za}{(z-a)^2}$

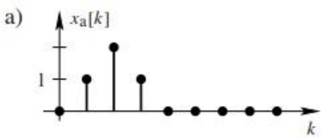
$x[k] = k a^k \varepsilon[k]$

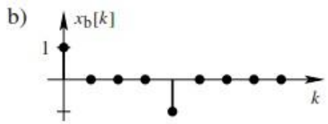
$\updownarrow$

$X(z) = \frac{za}{(z-a)^2}$

z-Transform (Signal values)

Calculate the z-transform of the following signals.





a) $x[k] = \delta[k-1] + 2\delta[k-2] + \delta[k-3]$

$\updownarrow$

$X(z) = z^{-1} + 2z^{-2} + z^{-3}$

b) $x[k] = \delta[k] - \delta[k-4]$

$\updownarrow$

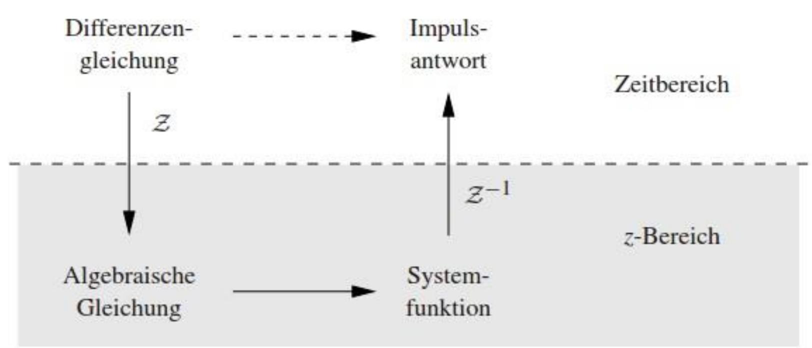
$X(z) = 1 - z^{-4}$

Difference equation:

$$\ldots + c_2 y[k-2] + c_1 y[k-1] + c_0 y[k] = \ldots + d_2 u[k-2] + d_1 u[k-1] + d_0 u[k]$$

$$\Downarrow Z$$

$$\ldots + c_2 z^{-2} Y(z) + c_1 z^{-1} Y(z) + c_0 Y(z) = \ldots + d_2 z^{-2} U(z) + d_1 z^{-1} U(z) + d_0 U(z)$$



Zusammenhang der Systembeschreibungen im Zeit- und Bildbereich

z-Transfer Function:

- is the ratio of output to input in the image area:

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\ldots + d_2 z^{-2} + d_1 z^{-1} + d_0}{\ldots + c_2 z^{-2} + c_1 z^{-1} + c_0} \cdot \frac{z^N}{z^N} = \frac{\ldots + b_2 z^2 + b_1 z + b_0}{\ldots + a_2 z^2 + a_1 z + a_0} \quad \text{mit} \quad a_i = c_{N-i} \quad und \quad b_i = d_{N-i}$$

System equation: $\boxed{Y(z) = G(z) \cdot U(z)}$

Impulse response: $g[k] \circ\!-\!\bullet\ G(z)$

same as laplace $g(t) \circ\!-\!\bullet\ G(s)$

Convolution: $$y[k] = g[k] * u[k] = \sum_{n=-\infty}^{\infty} g[n] \cdot u[k-n]$$

Attention:

- As you can see, the z-Transfer Function exists in **two forms** (once in **z** and **z^{-1}**) with mirrored coefficients!

Stability of Time-Discrete Systems

Between the s-area and z-area the formal relationship holds (is <u>only</u> a substitution):

$$\boxed{z = e^{sT} = e^{T\cdot \mathrm{Re}\{s\}} \cdot e^{jT\cdot \mathrm{Im}\{s\}}}$$

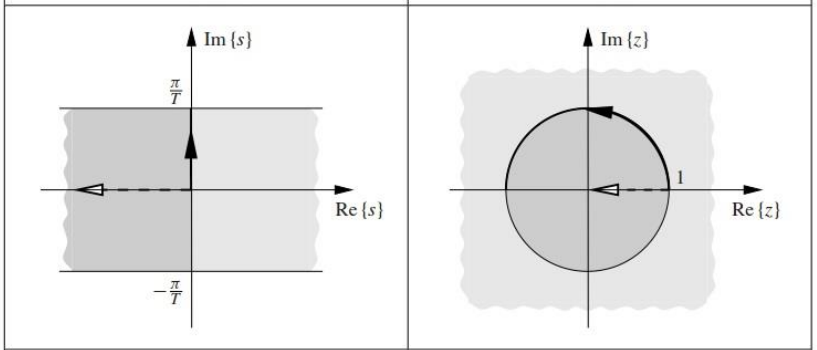
$$0_s \xrightarrow{z=e^{sT}} 1_z \qquad -\infty_s \xrightarrow{z=e^{sT}} 0_z \qquad \pm\frac{j\pi}{T}_s \xrightarrow{z=e^{sT}} -1_z$$

- The above relationship can be interpreted mathematically as mapping of the complex s-plane into the complex z-plane.

Stability

- Of a time-discrete exists when **all poles of the z-transfer function lie inside the unit circle**!

s-Bereich		*z*-Bereich	



linke Halbebene	$\mathrm{Re}\{s\} < 0$	Innere des Einheitskreises	$\|z\| < 1$
imaginäre Achse	$\mathrm{Re}\{s\} = 0$	Einheitskreis	$\|z\| = 1$
rechte Halbebene	$\mathrm{Re}\{s\} > 0$	Äußere des Einheitskreises	$\|z\| > 1$
Spiegelung an reeller Achse	s^*	Spiegelung an reeller Achse	z^*
Spiegelung an imaginärer Achse	$-s^*$	Spiegelung am Einheitskreis	$\frac{1}{z^*}$

Zusammenhang zwischen *s*- und *z*-Bereich

Convergence area of the one-sided z-Transform:

- is the area **outside a circle** enclosing the largest pole in magnitude (completely analogous to the Laplace transformation)

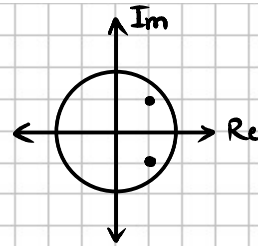
Ex.3.4 Stability analysis of time-discrete systems

Check the following systems for stability:

a) $G(z) = \frac{z^2}{z^2 + z + 0.5}$ b) $y[k] = y[k-1] + y[k-2] + \overset{u}{\cancel{x}}[k]$ c) $g[k] = \varepsilon[k] - \varepsilon[k-3]$

a) $z_{\infty 1,2} = \frac{-1 \pm \sqrt{1-2}}{2} = -\frac{1}{2} \pm \frac{i}{2} \Rightarrow$ Stable ✓

$|z_{\infty 1,2}| < 1 \Rightarrow$ Stable



b) $y[k] = y[k-1] + y[k-2] + u[k]$

$Y(z) = z^{-1} Y(z) + z^{-2} Y(z) + U[z]$

$U(z) = Y(z) - z^{-1} Y(z) - z^{-2} Y(z)$

$\therefore G(z) = \frac{Y(z)}{U(z)} = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{1}{1 - \frac{1}{z} - \frac{1}{z^2}} = \frac{z^2}{z^2 - z - 1}$

$z = \frac{+1 \pm \sqrt{1-4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} > 1 \Rightarrow$ Unstable

c) $g[k] = \varepsilon[k] - \varepsilon[k-3]$

$G(z) = \frac{z}{z-1} - z^{-3}\frac{z}{z-1} = \frac{z^4 - z}{z^3(z-1)}$ $z = 0 / 1$

$\Rightarrow$ stable

Alternate

in time domain BIBO stability: $\int_0^\infty |g(t)|\,dt < \infty \rightarrow$ stable

in discrete world BIBO stability: area under curve $< \infty$ $\rightarrow$ $\sum_{k=0}^{\infty} |g[k]| < \infty$

here Σ $1+1+1 = 3 < \infty \rightarrow$ BIBO stable

Ex 3.5 Inverse z-Transformthrough partial fraction decomposition

a) Determine the time signal x [k] using the partial fraction decomposition:

$$X(z)=\frac{3z^2-4z}{(z-1)(z-2)} = \frac{3z^2-4z}{z^2-2z-z+2} = \frac{3z^2-4z}{z^2-3z+2}$$

$$X(z) = \frac{3z^2-4z}{(z-1)(z-2)}$$

$$\Rightarrow \frac{X(z)}{z} = \frac{3z-4}{(z-1)(z-2)} = \frac{1}{z-1} + \frac{2}{z-2}$$

$$\Rightarrow X(z) = \frac{z}{z-1} + \frac{2z}{z-2}$$

$\bullet\!-\!\circ\ z^{-1}$

$$\Rightarrow x[k] = \varepsilon[k] + 2\cdot 2^k\cdot\varepsilon[k]$$

$$\begin{array}{r|l} & 3 \\ \hline z^2-3z+2 & 3z^2-4z \\ & -(3z^2-9z+6) \\ \hline & 0+5z-6 \end{array}$$

This logic does not work

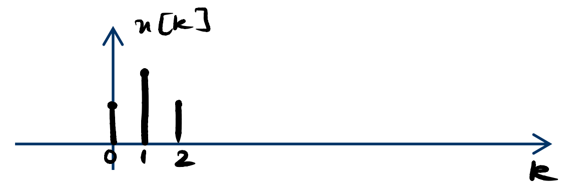
Inverse z-transformation by Polynomial division

a) Calculate and plot the time signal x [k] using the Polynomial division:

$$X(z)=\frac{z^2+2z+1}{z^2} = 1 + \frac{2}{z} + \frac{1}{z^2}$$

$\bullet\!-\!\circ\ z^{-1}$

$$x[k] = \delta[k] + 2\delta[k-1] + \delta[k-2]$$



Discrete-time systems

a) What is the difference equation of this system?

b) How must M be chosen so that the system is causal?

$$G(z)=\frac{z^M+1}{z^2+2z-1}$$

$$G(z) = \frac{z^M+1}{z^2+2z-1} = \frac{Y(z)}{U(z)}$$

$$Y(z)z^2 + Y(z)(2z) - Y(z) = U(z)z^M + U(z) \quad \therefore \div z^2$$

$\bullet\!-\!\circ\ z^{-1}$

$$\delta[k+2]y[k] + 2\delta[k+1]y[k] + y[k] = \delta[k+M]u[k] + u[k] \;✗$$

✗ need a causal system

IMP

$n > d$ ⇒ non causal

$m \leq 2$ → causal

$$\Rightarrow Y(z) + Y(z)\left(\frac{2}{z}\right) - \frac{Y(z)}{z^2} = U(z)\cdot z^{M-2} + \frac{U(z)}{z^2}$$

$\bullet\!-\!\circ\ z^{-1}$

$$y[k] + 2y[k-1] - y[k-2] = u[k-M+2] + u[k-2]$$

Exact (Matched)-Transform

$$z = e^{sT} \qquad bzw. \qquad s = \frac{1}{T}\cdot \ln z$$

$\ln z = \ln e^{sT}$
$\Rightarrow \ln z = sT$
$\Rightarrow s = \frac{1}{T} \ln z$

Since the transformation between s and z areas is an e-function, polynomials in s and z are unfortunately only mapped into so-called transcendent functions.

- However, you can do the exact transformation, for example, by a **Taylor series** development with cut-off after the linear term:

$$s = \frac{1}{T}\cdot \ln z \approx \frac{2}{T}\cdot\left\{\frac{z-1}{z+1} + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \ldots\right\}$$

Bilinear (Tustin) -Transform / Trapezoidal Rule:

$$\boxed{s = \frac{2}{T}\cdot\frac{z-1}{z+1} \qquad bzw. \qquad z = \frac{1+s\frac{T}{2}}{1-s\frac{T}{2}}}$$

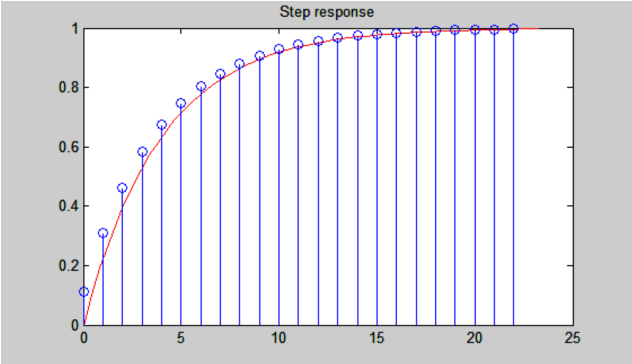
Because of its simplicity, the bilinear transformation is the **most widely used method** to obtain a digital algorithm (Euler methods are less precise)!
With the filter design, however, there is a slight frequency distortion (so-called. wraping). For more details see literature ...

example (T = 1):

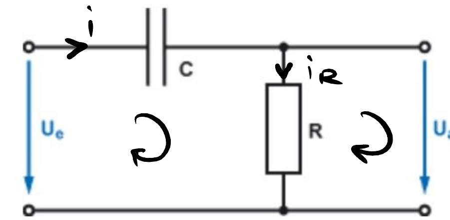
$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{1}{4s+1}} \qquad \Rightarrow \quad G(z) = \left.\frac{1}{4s+1}\right|_{s=\frac{2}{T}\cdot\frac{z-1}{z+1}} = \frac{1}{4\frac{2}{T}\cdot\frac{z-1}{z+1}+1} = \frac{z+1}{8z-8+z+1} = \frac{z+1}{9z-7}\cdot\frac{z^{-1}}{z^{-1}} = \frac{1+z^{-1}}{9-7z^{-1}}$$

$$Y(z)\cdot(9-7z^{-1}) = U(z)\cdot(1+z^{-1}) \quad \bullet\!-\!\circ \quad 9y[k]-7y[k-1] = u[k]+u[k-1]$$

$$\boxed{y[k] = \tfrac{7}{9}y[k-1] + \tfrac{1}{9}u[k] + \tfrac{1}{9}u[k-1])}$$



Ex.3.6 Filter Realization by using Bilinear Transform



A digital filter is to be developed:

a) First determine the DE and the transfer function G (s) of this analog high-pass filter. (Assume: C = 2µF and R = 1MOhm, ie τ = RC = 2 sec.)

b) Convert this analog filter into a digital filter using the bilinear transformation (for T = 0.4 sec).

c) Finally, give the difference equation to be coded.

a) $u = v_e(t)$ $\quad y = u_a(t)$ $\quad x = v_c(t)$

$i = C\frac{dv_c}{dt}$ $\quad U_c + U_R = U_e$ $\quad R = \frac{U_R}{i_R} = \frac{U_R}{i}$ $\quad U_R = U_a$

$\dot{x} = \frac{dv_c}{dt} = \frac{i}{C} = \frac{U_R}{RC} = \frac{u_e - u_c}{RC}$

∴ DE ⇒ $\dot{u}_c = \frac{u_a}{RC}$ ⇒ $u_a = \dot{v}_c(RC)$

⇒ $u_a = (\dot{u}_e - \dot{u}_a)(RC)$

⇒ $RC\dot{y} + y = RC\dot{u}$ ⇒ $2\dot{y} + y = 2\dot{u} \overset{\mathcal{L}}{\circ\!\!-\!\!\bullet} 2sY(s) + Y(s) = 2sU(s)$

⇒ $G(s) = \frac{Y(s)}{U(s)} = \frac{2s}{2s+1}$ ⇒ DT_1 ⇒ H.P filter

DT_1: $\omega\uparrow$ ⇒ Mag $\uparrow$ ⇒ HP

both ↓ bandpass

PT_1: $\omega\uparrow$ ⇒ Mag $\downarrow$ ⇒ LP

all DT_1s HP?

b) $G(s) = \frac{2s}{2s+1}$ ⇒ $G(z) = \left.\frac{2s}{2s+1}\right|_{s=\frac{2}{T}\frac{z-1}{z+1}}$

$$= \frac{2\times\frac{2}{0.4}\frac{(z-1)}{z+1}}{2\times\frac{2}{0.4}\left(\frac{z-1}{z+1}\right)+1} = \frac{(z-1)}{(z-1)+\frac{(z+1)}{10}} = \frac{z-1}{z+0.1z-1+0.1} = \frac{z-1}{1.1z-0.9}$$

c) $G(z) = \frac{z-1}{1.1z-0.9}$ ⇒ $Y(z)(1.1z-0.9) = (z-1)U(z)$ ⇒ $Y(z)\left(1.1 - \frac{0.9}{z}\right) = \left(1-\frac{1}{z}\right)U(z)$

$\bullet\!\!-\!\!\circ\ z^{-1}$

$1.1\,y[k] - 0.9\,y[k-1] = u[k] - u[k-1]$

SW code ⇒ $y[k] = \frac{0.9}{1.1}y[k-1] + \frac{u[k]}{1.1} - \frac{u[k-1]}{1.1}$

Pulse Invariant Transform:

If the impulse response is to be simulated exactly, one must sample the impulse response of the continuous-time system and subject this sequence to the z-transformation, ie

$$\boxed{G(z) = \cdot Z\left\{ L^{-1}\{G(s)\}\Big|_{t=kT} \right\}}$$

Step Invariant Transform:

If the step response is to be reproduced exactly, the step response of the time-continuous system must be sampled, this derivation and this sequence must be subjected to the z-transformation. The derivation can expediently be carried out in the z range (a multiplication by (z-1) / z), ie one obtains

$$\boxed{G_{HS}(z) = \frac{z-1}{z} \cdot Z\left\{ L^{-1}\left\{\frac{G(s)}{s}\right\}\Bigg|_{t=kT} \right\}}$$

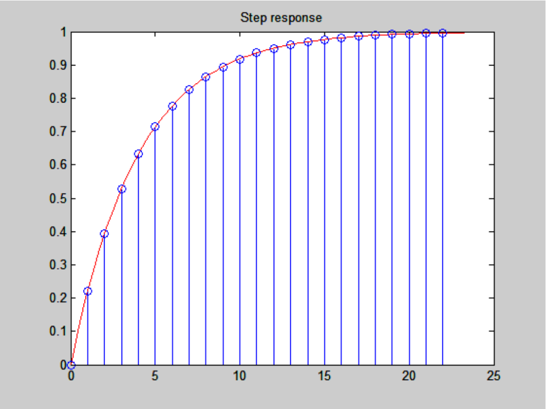
Example: Step invariant transform

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{1}{4s+1}} \quad \Rightarrow \quad H(s) = \frac{G(s)}{s} = \frac{1}{s(4s+1)} \quad \bullet\!-\!\circ \quad h(t) = (1-e^{-\frac{t}{4}}) \cdot \varepsilon(t)$$

$$\Rightarrow \quad h(t)\big|_{t=kT} = (1-e^{-\frac{kT}{4}}) \cdot \varepsilon(kT) = \left(1-(e^{-\frac{T}{4}})^k\right) \cdot \varepsilon(kT) \qquad c = e^{-\frac{T}{4}}$$

$$\Rightarrow \quad h[k] = (1-c^k) \cdot \varepsilon[k] \quad \circ\!-\!\bullet \quad H(z) = \frac{z}{z-1} - \frac{z}{z-c} \quad \Rightarrow \quad \boxed{G(z) = \tfrac{z-1}{z} \cdot H(z) = \frac{1-c}{z-c}}$$

$$\Rightarrow \quad \boxed{y[k] = (1-c) \cdot u[k-1]) + c \cdot y[k-1]}$$



Ex.3.7 Step-Invariant Transform

A controller development led to the following time-continuous I-controller: $G(s) = \frac{2}{s}$

a) Perform the discretizationby using the step invariant transform and again give the difference equation of the time-discrete controller. This time the sampling time is T = 500ms.

ⓐ ⓑ

a) $G(s) = \frac{2}{s}$

$$h(s) = \frac{2}{s^2} \xrightarrow{\mathcal{L}^{-1}} h(t) = 2 \cdot t \cdot \varepsilon(t)$$

$$h(t)\Big|_{t=kT} = 2 \cdot kT \cdot \varepsilon(kT) \quad \text{where } T = 500\,ms \Rightarrow h(t) = 1000\,k\ \varepsilon[k]$$

$$x[k] = x(kT)$$

$$Z\left(h(t)\Big|_{t=kT}\right) \Rightarrow 1000\,k\ \varepsilon[k] \xrightarrow{Z} H(z) = 1000 \frac{z}{(z-1)^2}$$

$$G(z) = \frac{z-1}{z} \cdot 1000 \cdot \frac{z}{(z-1)^2} = \boxed{\frac{1000}{z-1}} = \frac{Y(z)}{U(z)}$$

ⓑ $z\,Y(z) - Y(z) = 1000\,U(z)$

$$\Rightarrow Y(z) - \frac{Y(z)}{z} = \frac{1000}{z} U(z)$$

z^{-1}

$$y[k] - y[k-1] = 1000\,u[k-1]$$

$$\Rightarrow y[k] = y[k-1] + 1000\,u[k-1] \Rightarrow \text{Difference equation}$$

Direct Form I:

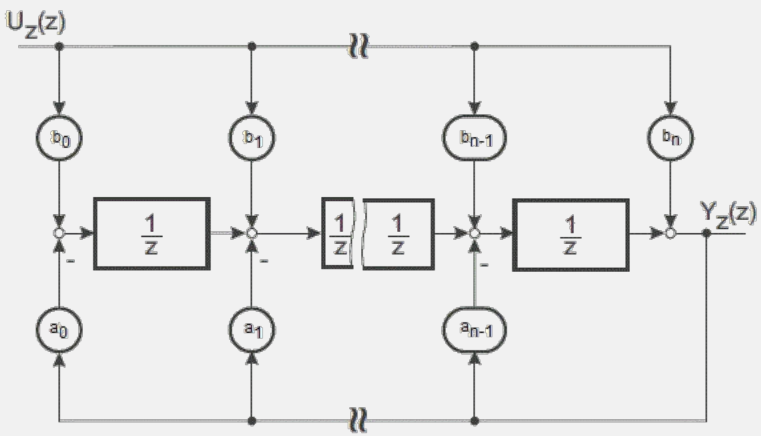
$$G(z)=\frac{Y(z)}{U(z)}=\frac{\ldots+b_2z^2+b_1z+b_0}{\ldots+a_2z^2+a_1z+a_0}\cdot\frac{z^{-N}}{z^{-N}}=\frac{\ldots+d_2z^{-2}+d_1z^{-1}+d_0}{\ldots+c_2z^{-2}+c_1z^{-1}+c_0}$$

$$Y(z)\cdot(\ldots+c_2z^{-2}+c_1z^{-1}+c_0)=U(z)\cdot(\ldots+d_2z^{-2}+d_1z^{-1}+d_0)\quad \bullet\!-\!\circ \quad c_0y[k]=d_0u[k]+d_1u[k-1]+d_2u[k-2]\ldots-c_1y[k-1]-c_2y[k-2]\ldots$$

Analogous to continuous time, there are canonical ones **Block diagrams for simulation or as implementationPrototypes:**(Derivation see literature)

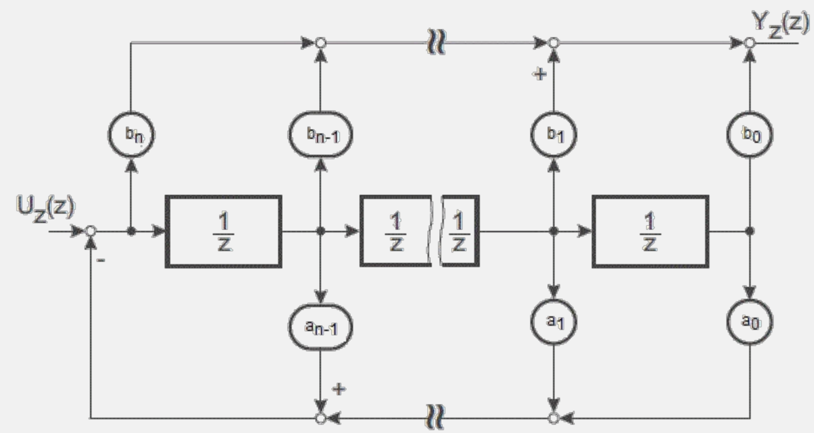
Canonical Transposed Direct Form II:

- (Observation normal form)



Direct Form II:

(Controll normal form):



- The **block z^{-1}** describes a **delay (Memory) by one clock pluse.**

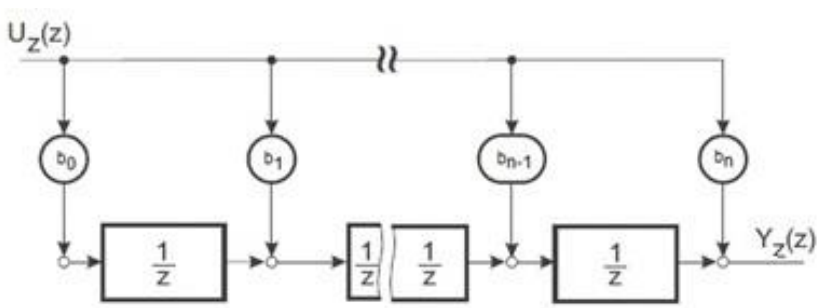
The direct forms II come with the **Minimum number** to save, but are not the first choice, especially for fixed point arithmetic (better direct form I or other cascade structures ...)

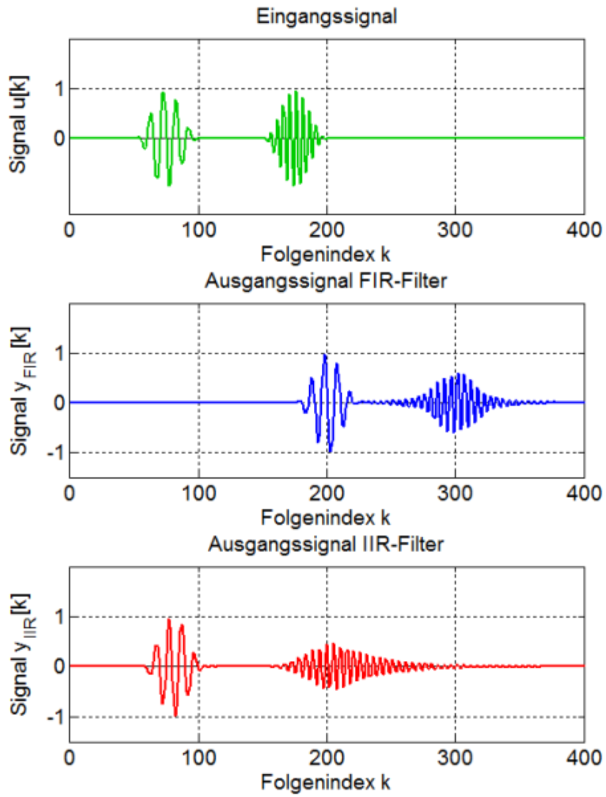
FIR Filter

- have <u>no</u> feedbacks and therefore have a **<u>finite</u>** impulse response **(finite impulse response).**
- they are **<u>always stable</u>** (because they have <u>only poles at the origin or no feedback</u>) and have a **<u>linear phase</u>**.

$$\boxed{G_{FIR}(z) = \frac{\ldots + b_2 z^2 + b_1 z + b_0}{z^N}}$$

Because of their linear phase, they always have one <u>constant group delay</u> and are therefore very popular, especially when designing digital filters ...





IIR Filter

- have feedback and therefore have a **<u>infinite</u>** long impulse response **(Infinite Impulse Response).**
- They **can be unstable** and **do not** have a **linear phase**.

$$\boxed{G_{IIR}(z) = \frac{\ldots + b_2 z^2 + b_1 z + b_0}{\ldots + a_2 z^2 + a_1 z + a_0}}$$

With the same filter order, however, they have a higher slope than FIR filters ...

The Discrete Time Fourier Transformation (DTFT)

The for discrete signals adapted **Fourier Transform** is the **Discrete Time Fourier Transform**

- Derivation:

$$x_a(t) = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta(t - kT)$$

$$X_a(j\omega) = F\{x_a(t)\} = \int_{-\infty}^{+\infty} \left(\sum_{k=-\infty}^{\infty} x[k] \cdot \delta(t-kT) \right) \cdot e^{-i\omega t} dt = \sum_{k=-\infty}^{\infty} x[k] \cdot \int_{-\infty}^{+\infty} \delta(t-kT) \cdot e^{-i\omega t} dt = \sum_{k=-\infty}^{\infty} x[k] \cdot e^{-i\omega kT} \qquad q.e.d.$$

Definition of the discrete-time Fourier transform:

$$\boxed{X(j\Omega) = \sum_{k=-\infty}^{\infty} x[k] \cdot e^{-i\Omega k}} \qquad \boxed{\Omega = \omega T = \omega \tfrac{2\pi}{\omega_A}}$$

Inverse:

$$x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(j\Omega) \cdot e^{j\Omega k} d\Omega$$

- **Ω = normalized angular frequency** (to become independent of T!)
- I.e. that the **spectrum** of a time-discrete signal is **continuous and <u>periodically</u>** in 2π.

If x[k] is the time-discrete signal sampled by a signal x(t), then the spectrum of x[k] (except for the factor 1 / T) is the superposition of **infinite number of periodic continuations** of the Fourier spectrum $X_F(j\omega)$ from x(t)!

This also applies if the sampling theorem is disregarded, but then again leads to spectral overlap of the sidebands ...

$$X(j\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_F(j(\omega - n \cdot \omega_a)) \Big|_{\omega = \frac{\Omega}{T}}$$

Normierte Kreisfrequenz Ω	Kreisfrequenz ω
- π	- $\omega_A/2$
- π/2	- $\omega_A/4$
0	0
π/2	$\omega_A/4$
π	$\omega_A/2$

Discrete time Impulse function:

$$x[k]=\delta[k] \quad \circ\!-\!\bullet \quad X(j\Omega)=1$$

discrete → DTFT

Discrete-time rectangular pulse:

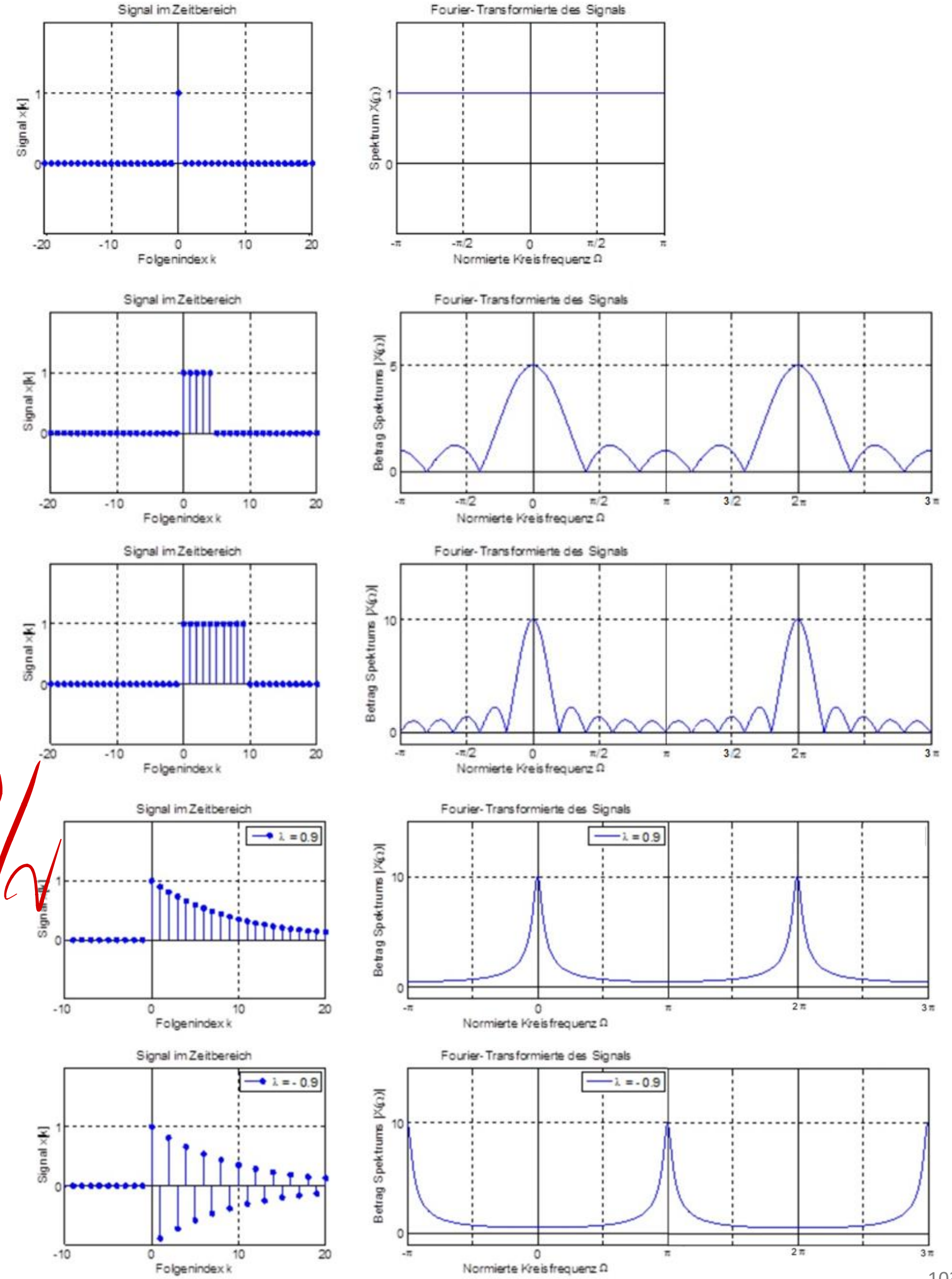
$$x[k]=\varepsilon[k]-\varepsilon[k-\kappa] \quad \circ\!-\!\bullet \quad X(j\Omega)=e^{-j\frac{\Omega(\kappa-1)}{2}}\cdot\frac{\sin(\frac{\kappa\cdot\Omega}{2})}{\sin(\frac{\Omega}{2})}$$

Discrete-time exponential sequence:

$$x[k]=\lambda^{\kappa}\cdot\varepsilon[k] \quad \circ\!-\!\bullet \quad X(j\Omega)=\frac{1}{1-\lambda\cdot e^{-j\Omega}}$$

$e^{i\theta}=\cos\theta-i\sin\theta$

$\cos n=\frac{e^{-n}+e^{n}}{2}$



Ex.3.9 Discrete-Time Fourier Transformation

A time-discrete signal x[k] has the following spectrum. $X(j\Omega) = \dfrac{1}{1-e^{-j\Omega}}$

a) Determine and sketch the time signal x[k].

b) Determine the spectra of the following signals using the theorems of the discrete-time Fourier transform:

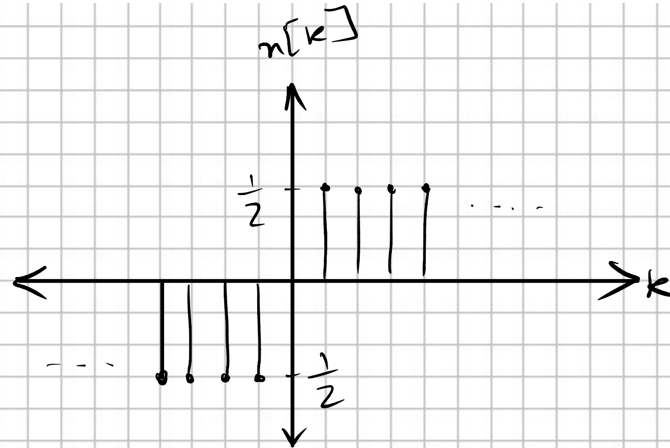
→ Fourier transform of signal = spectrum of signal

$x_1[k] = x[k-5]$ $x_2[k] = x[-k]$ $x_3[k] = k\cdot x[k]$ $x_4[k] = x[k] * x[k]$ $x_5[k] = x[k] - x[k-1]$

(a) $X(j\Omega) = \dfrac{1}{1-e^{-j\Omega}}$

$X'(j\Omega) = \pi\delta(\Omega) + \dfrac{1}{1-e^{-j\Omega}} \xrightarrow{DTFT^{-1}} x[k] = \varepsilon[k]$

$X(j\Omega) = X'(j\Omega) - \dfrac{2}{2}\pi\delta(\Omega) \xrightarrow{DTFT^{-1}} x[k] = \varepsilon[k] - \dfrac{1}{2}$



(b) $\rightarrow x_1[k] = x[k-5]$

$X_1(j\Omega) = X(j\Omega)\cdot e^{-j\Omega 5}$

$= \dfrac{1}{1-e^{-j\Omega}} e^{-j\Omega 5}$

$\rightarrow x_2[k] = x[-k]$

$X_2(j\Omega) = X(-j\Omega) = \dfrac{1}{1-e^{j\Omega}}$

$\rightarrow x_3[k] = k\cdot x[k]$

$X_3(j\omega) = j\,\dfrac{dX(j\Omega)}{d\Omega} = j\,\dfrac{d}{d\Omega}\left(\dfrac{1}{1-e^{-j\Omega}}\right)$

$= \ldots$

$\rightarrow x_4[k] = x[k] * x[k]$

$X_4(j\omega) = X(j\Omega)\cdot X(j\Omega)$

$= \dfrac{1}{(1-e^{-j\Omega})^2}$

$\rightarrow x_5[k] = x[k] - x[k-1]$

$X_5(j\omega) = X(j\Omega) - X(j\Omega)\cdot e^{-j\Omega 1}$

$= \dfrac{1}{1-e^{-j\Omega}}\left(1-e^{-j\Omega}\right)$

The Frequency Response of Discrete Systems

Frequency response:

$$G(j\Omega) = \frac{Y(j\Omega)}{U(j\Omega)}$$

- It is a complex function of the frequency and the discrete time **Fourier transform of the impulse response g [k]** of a system.

Calculation from the z-transfer function:

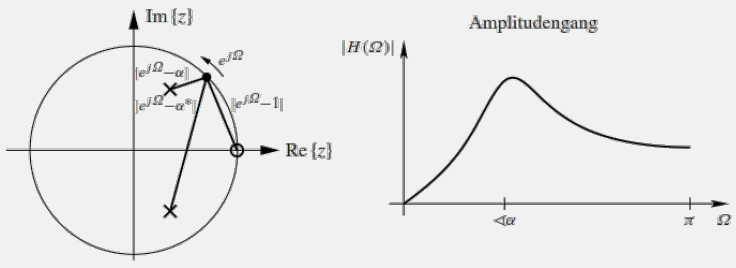
- For the important signal and system class whose z-convergence area includes the unit circle, and which uncludes **all causal and decaying signals** and **all stable systems** both transforms exist **with the substitution:**

$$\boxed{G(j\Omega) = G(z)\big|_{z=e^{j\Omega}}}$$

Amplitude response:

- describes the ratio of the output to the input amplitudefor every sinusoidal signal with the normalized frequency Ω :

$$|G(j\Omega)| = \frac{|Y(j\Omega)|}{|U(j\Omega)|}$$



Phase response:

- describes the temporal phase shift between the output and the input oscillation.

$$\varphi(j\Omega) = \angle G(j\Omega)$$

In the s-plane the frequency response corresponded to the course of the s-TF along the imaginary axis,
In the z-plane the frequency response corresponds to the **course of the z-TF along the unit circle**!

Ex.3.10 Frequency response of a digital filter

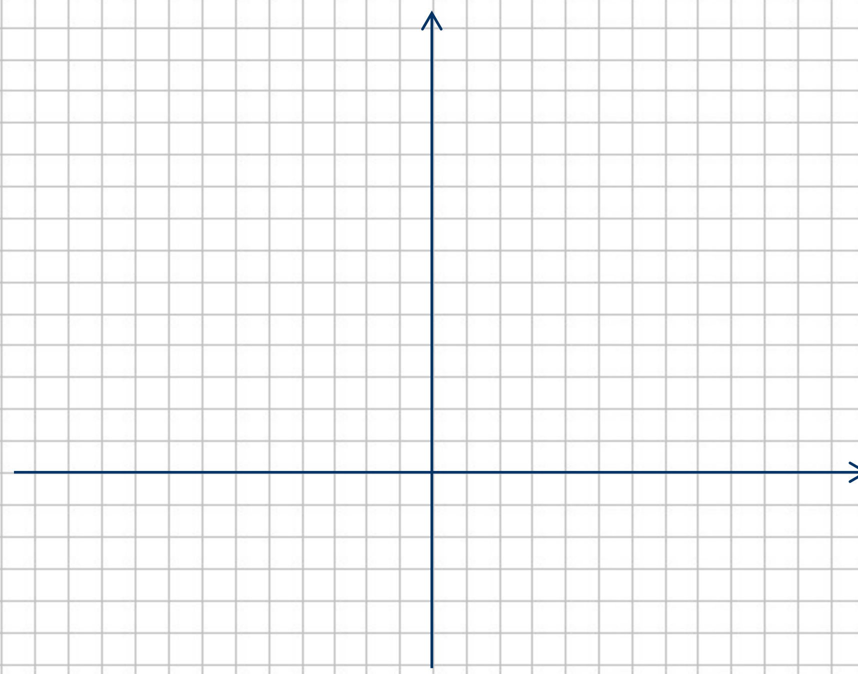
The difference equation of a digital filter for forming the mean value is given:

$$y[k] = \frac{1}{3}\left(u[k] + u[k-1] + u[k-2]\right)$$

a) Give its transfer function.
b) Sketch the PZ diagram of the filter.
c) Give its frequency response.
d) Calculate and sketch the magnitude of the frequency response.

a)

to be completed



- It directly calculates (except for a prefactor) the **Fourier coefficients of the discrete-time Fourier series:** $c_n = \frac{1}{N \cdot T} \sum_{k=0}^{N-1} x[k] \cdot e^{-j\left(2\pi \frac{n}{N} k\right)}$

Discrete Fourier Transform (DFT):

$$\boxed{X[n] = DFT\{x[k]\} = \sum_{k=0}^{N-1} x[k] \cdot e^{-j\left(2\pi \frac{n}{N} k\right)}} \quad \textit{für} \quad k = 0, \ldots, N-1$$

Inverse DFT:

$$\boxed{x[k] = DFT^{-1}\{X[n]\} = \frac{1}{N} \sum_{n=0}^{N-1} X[n] \cdot e^{j\left(2\pi \frac{n}{N} k\right) 2\pi \frac{kn}{N}}}$$

- The DFT computes „**from N Samples** x[k], **N complex Fourier coefficients** X[n] "

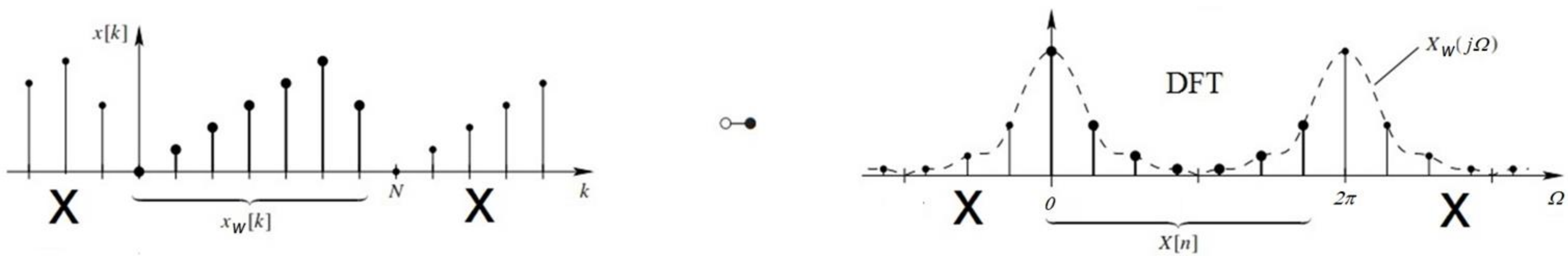
Or:

"It provides N values (of a period) **of the spectrum of a periodically continued time signal of length N. "**

- The following applies: $X[n] = X_W(j\Omega)\big|_{\Omega = \frac{2\pi}{N} \cdot n}$

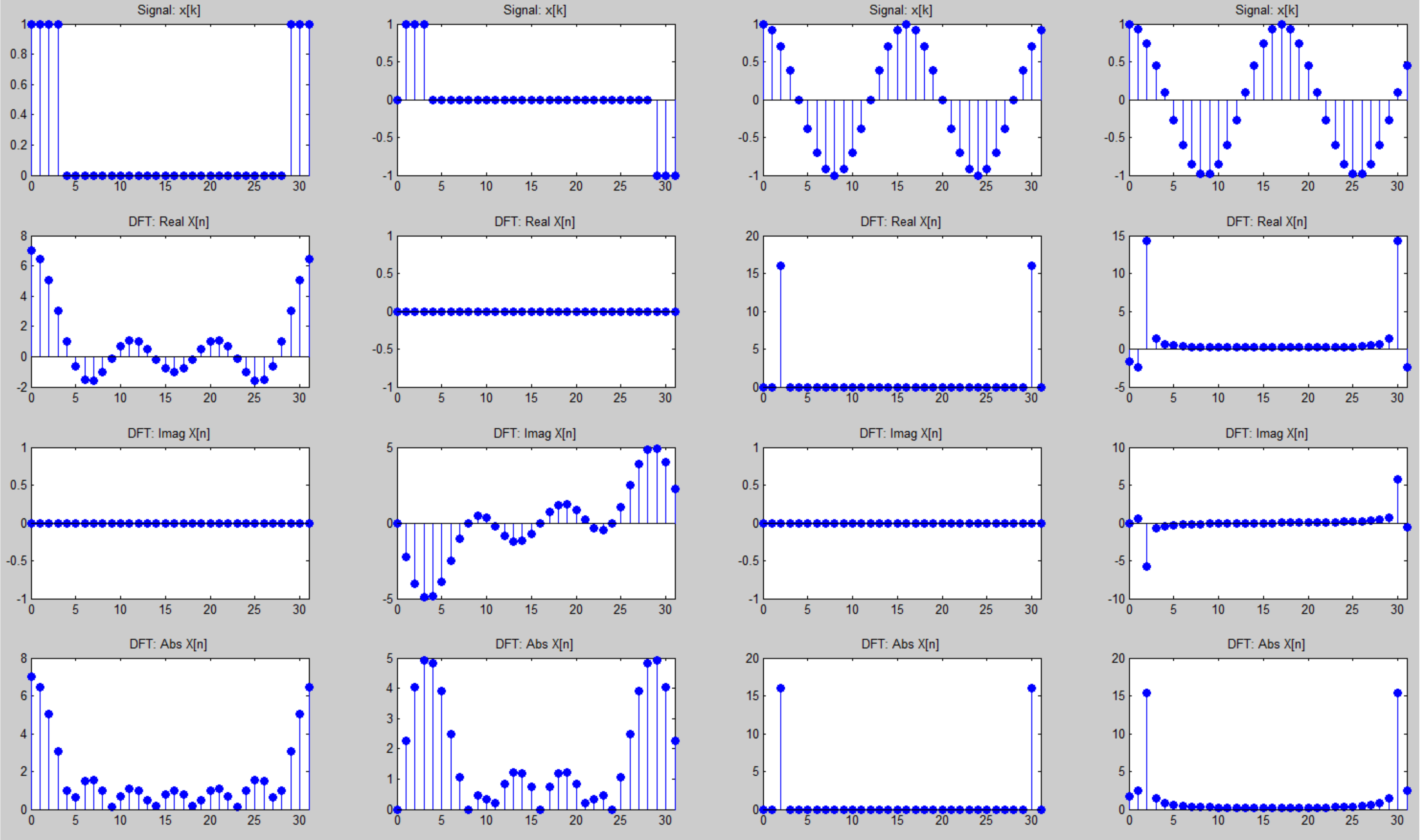
Hints:

- The DFT "only" works with a signal section x[k], but the results correspond to those of the **infinitely periodically continued time signal!**
- Tip: For real For signals x[k], due to the symmetry properties of the Fourier series, the calculation of N / 2 DFT values is alrready sufficient!



Diskretes periodisches Zeitsignal mit zugehörigem diskreten Spektrum

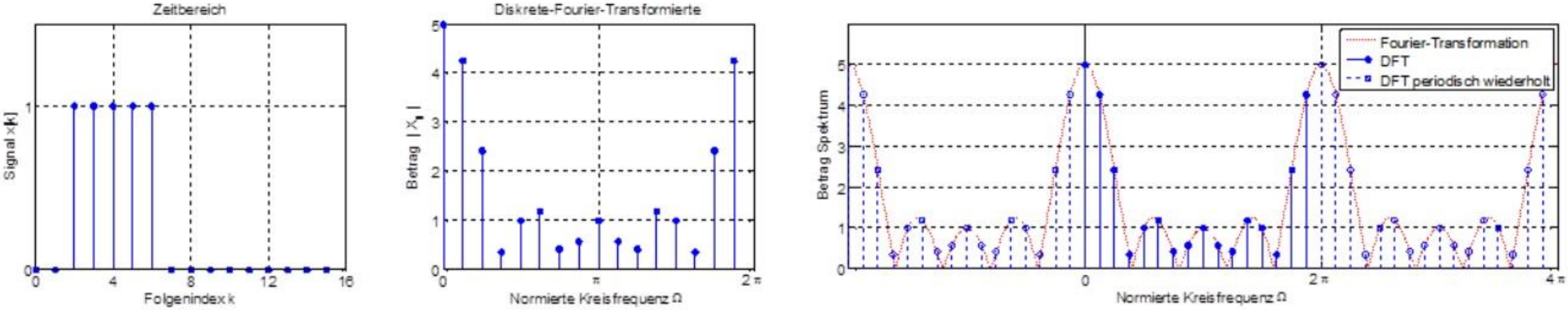
Some Examples of DFT / FFT (N = 32)



The DFT as Spectral Analysis Tool

Example: Rectangular sequence x [k] with N = 16 signal values.

$$x[k] = \varepsilon[k-2] - \varepsilon[k-7] \underset{DFT}{\circ\!-\!\bullet} X[n] = \sum_{k=2}^{6} 1 \cdot e^{-j\left(2\pi \frac{n}{N} k\right)} \qquad x[k] \underset{ZDFT}{\circ\!-\!\bullet} X(j\Omega) = e^{-j4\Omega} \cdot \frac{\sin\left(\frac{5\Omega}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}$$



Limits of DFT:

- 1) The signal x[k] is just a sample of x(t). Signal frequencies that are too high lead to **Aliasing**.
- 2) The signal section $x_w[k] = x[k] \cdot w[k]$, which is used for the DFT, arises in the time domain by multiplication with a window function w(t). This leads to the so-called.**Leakage-Effect.**
- 3) Only N values X [n] of the spectrum are available, but $X_{W.}(j\omega)$ is a continuous function, i.e. the **spectral resolution is limited**.

Fast Fourier Transform (FFT):

- Is an algorithm for **<u>extremely</u> efficient calculation of** the Discrete Fourier Transform. N must always be a **power of two**!
- The computational effort of the **FFT algorithm, for example with N = 1024, is only 1% of the effort of the direct DFT!**

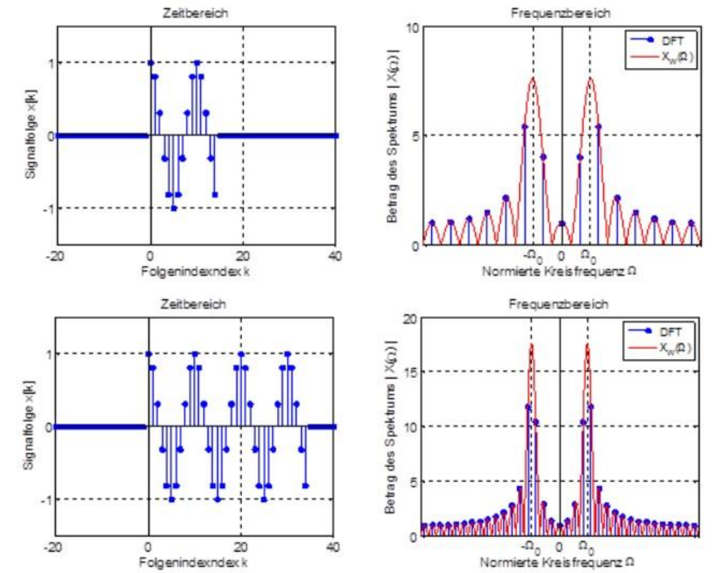
  This is achieved through consistent multiple use of recurring intermediate results … (for further details see literature)

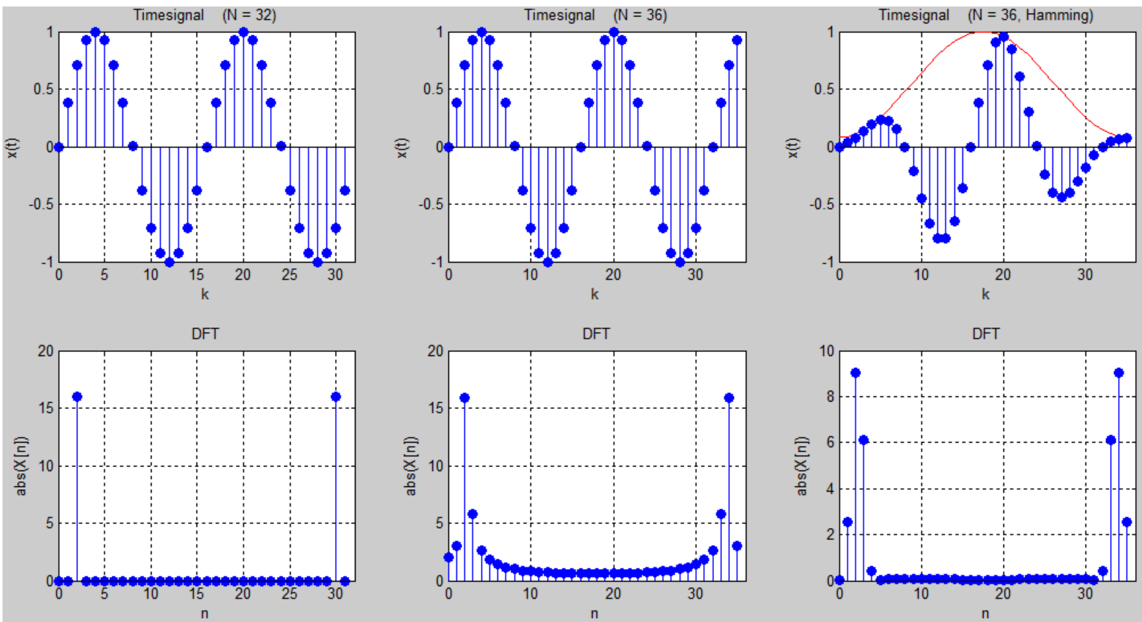
Notes on the Practical Application of DFT / FFT

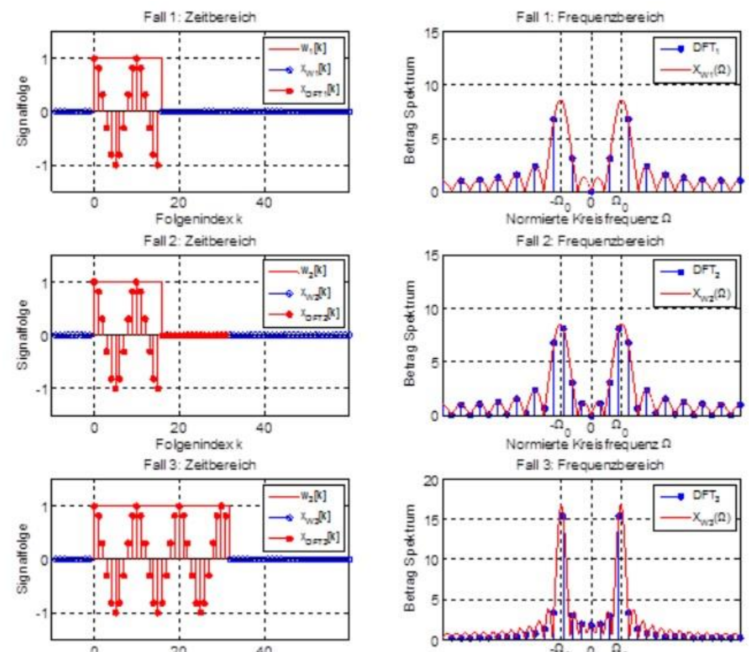
- **1) Consistent use of anti-aliasing filters!**
- **2) Observation period N as long as possible**
  - Basically this increases the spectral resolution.
  - Of the Leakage-Effect at the edges is reduced, because only if the signal x [k] is inherent is shorter than the observation window, or the window lies exactly in such a way that it is periodic by itself the continued signal is created Leakage-Effect less strong (almost impossible in practice).
- **3) Use of edge-smoothing window functions**

  Multiplication with bell-shaped window functions leads to a reduction in the LeakageEffect (e.g. Hamming-Window, HannWindows, Blackman windows ...)
- **4) Zero Padding (Zero-Filling)**

  To increase the frequency resolution, additional zeros can be appended to x [k]
  The spectrum can thus be drawn more finely (= interpolation in the spectral range), the real one
  However, this increases the information content **not**. Of theLeakage-Effect can thereby **not** get better!





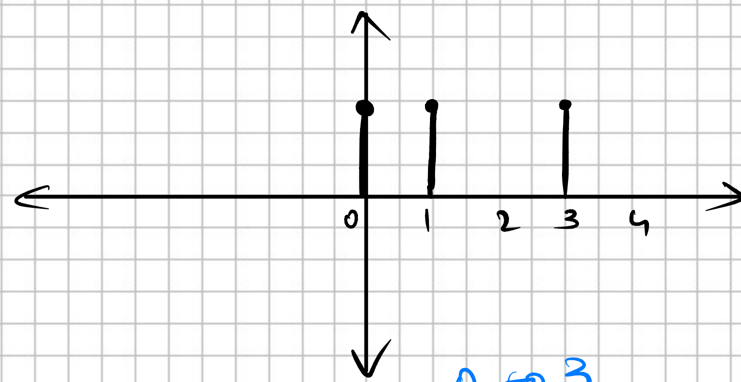


Ü 3.11 Discrete Fourier Transformation (DFT or FFT)

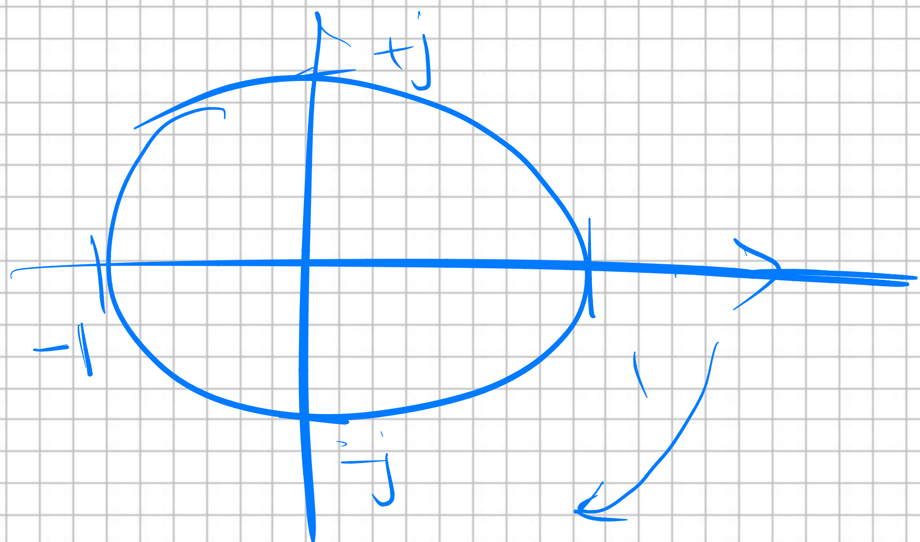
As an exercise, calculate the DFT of length N = 4 of the following signal: $x[k] = \delta[k] + \delta[k-1] + \delta[k-3]$

$$X[n] = DFT\{x[k]\} = \sum_{k=0}^{N-1} x[k] \cdot e^{-j\left(2\pi \frac{n}{N} k\right)}$$

$k=0 \quad (\delta[0] + \delta[-1] + \delta[-3])\, e^{0}$



$k=1 \quad \delta[1] + \delta[0] + \delta[-2]\, e^{-j\ldots}$



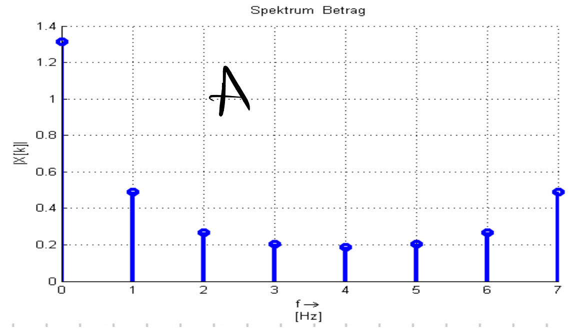
$e^{-2\pi \frac{k}{4}}$ 0 to 3

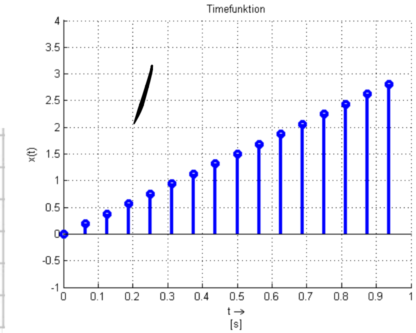
$e^{-\pi \frac{k}{2}}$

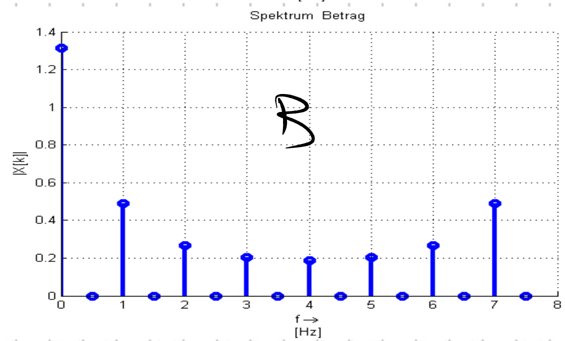
0	e^{-0}	$= 1$
1	$e^{-\pi/2}$	$= -j$
2	$e^{-\pi}$	$= -1$
3	$e^{-\pi \frac{3}{2}}$	$2j$

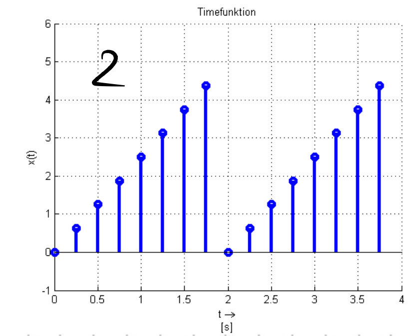
Ex 3.12 Discrete Fourier Transformation (DFT or FFT)

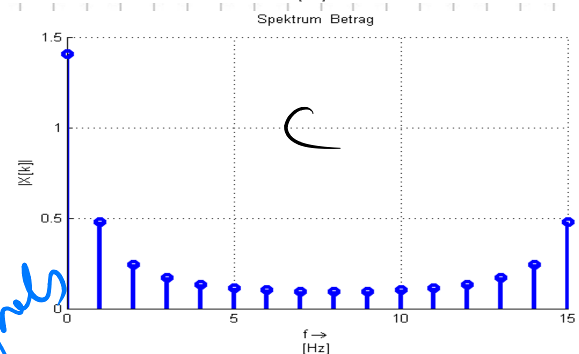
a) The spectra $X_a[k]$ to $X_e[k]$ were calculated using a DFT. Assign to each spectrum the time function and complete the table below.

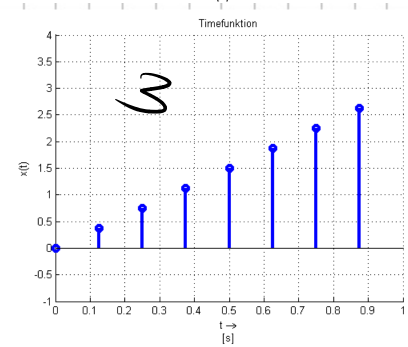


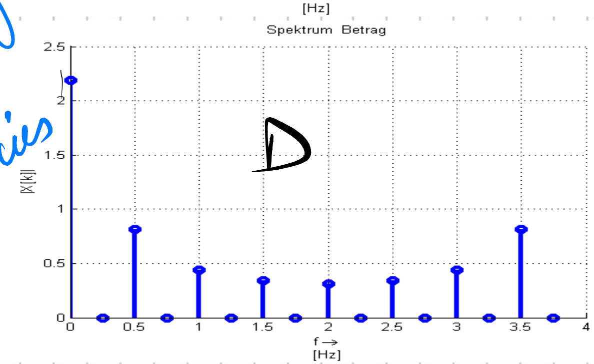


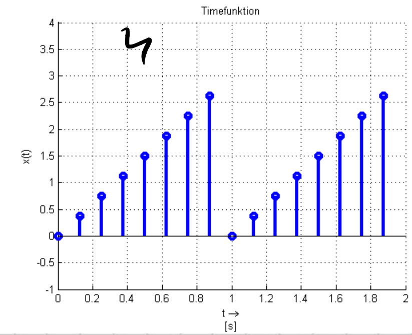












Spektrum	Zeit-funktion	Warum passen diese Funktionen zusammen?	N	T_s [s]	F_s [Hz]
C	1		16	$\frac{1}{16}$	16
D	2		16	$\frac{4}{16}$	4
A	3		8	$\frac{1}{8}$	8
	4		16	$\frac{2}{16}$	8

① match no of signals

② match frequencies

Appendix:

Have fun and good luck!

Construction and the essential contents of this lecture were taken from the standard works:

- [1] T. Frey, M. Bossert, “Signal and System Theory”, Vieweg + publishing house Teubner, 2nd edition, 2008
- [2] B. Girod, R. Rabenstein, A. Stenger, "Introduction to Systems Theory", publisher Teubner,, 2007
- [3] M. Strohmann et al .: "System Theory Online", I-Net of the Karlsruhe University of Applied Sciences - Technology and Economics, 2016

Other literatur used and recommended:

- [3] S. Wyrsch, "Signals and Systems I & II", Teaching materials, HSR, 2011.
- [4] M. Meyer, "Signalverarbeitung", Vieweg, 2000.
- [5] M. Rupf, “Signals and Systems SiSy", Script ZHAW, 2011.
- [6] H. Schmidt-Walter, "Taschenbuch der Elektrotechnik", Verlag Harri Deutsch, 9th edition. 2
- [7] I. Rennert, B. Bundschuh, "Signals and Systems", Verlag Hanser, 2013
- [8] BP Lathi, "Linear Systems other Signals ”, 2nd edition, Oxford University Press, 2005.
- [9] H.-W. Schüssler, "Networks, Signals and Systems", Springer Verlag, 1990
- ...

Important MATLAB commands for Systems Theory

Mathematische Darstellung	Matlab − Darstellung
$a(x) = a_N x^N + \ldots + a_1 x + a_0$	`a = [aN aN−1 … a1 a0]`
$x[k],\ k = k_1, k_1+1, \ldots k_2$	`x = [ x[k1] x[k1+1] … ]` `k = [ k1 : k2]`
$x(t),\ t_1 \le t \le t_2$	`x = [ x(t1) x(t1+ΔT) … ]` `t = [ t1 : ΔT : t2]`

Grundlegende Berechnungen

Polynommultiplikation, Faltung	$c(x) = a(x) \cdot b(x)$	`c = conv(a, b)`
Polynomdivision, Entfaltung	$\frac{c(x)}{a(x)} = q(x) + \frac{r(x)}{a(x)}$	`[q, r] = deconv(c, a)`
Nullstellenbestimmung von Polynomen	$a(x) = \prod_i (x - r_i)$	`r = roots(a)` `a = poly(r)`
Umrechnung von Polynom- und Produktdarstellung	$\frac{b(x)}{a(x)} = k \frac{\prod_i (x-z_i)}{\prod_i (x-p_i)}$	`[b, a] = zp2tf(z, p, k)`^{SP} `[z, p, k] = tf2zp(b, a)`^{SP}
Partialbruchzerlegung	$\frac{b(x)}{a(x)} = \sum_i \frac{r_i}{x-p_i} + \sum_j k_j x^j$	`[r, p, k] = residue(b, a)` `[b, a] = residue(r, p, k)`
Polynomauswertung an gegebener Stelle	$y_0 = a(x_0)$	`y0 = polyval(a, x0)`

Berechnungen diskreter Systeme

Systemdarstellung	$H(z) = \frac{b(z)}{a(z)}$	`b, a`
Berechnung Systemantwort (Differenzengleichung)	$y[k] \circ\!\!-\!\!\bullet \frac{b(z)}{a(z)} \cdot X(z)$	`y = filter(b, a, x)`
Bestimmung Impulsantwort (numerische Rücktransf.)	$h[k] \circ\!\!-\!\!\bullet \frac{b(z)}{a(z)}$	`h = impz(b, a)`^{SP}
Partialbruchzerlegung für z-Transformierte	$\frac{b(z)}{a(z)} = \sum_i \frac{r_i \cdot z}{z-p_i} + \sum_j k_j z^{-j}$	`[r, p, k] = residuez(b, a)`^{SP} `[b, a] = residuez(r, p, k)`^{SP}
Korrelation	$z[k] = \sum_i x^*[i] \cdot y[i+k]$	`z = xcorr(x, y)`^{SP}

Berechnungen kontinuierlicher Systeme

Systemdarstellung	$H(s) = \frac{b(s)}{a(s)}$	`H = tf(b, a)`^{CS,1}
numerische Berechnung Systemantwort	$y(t) \circ\!\!-\!\!\bullet H(s) \cdot X(s)$	`[y, ty] = lsim(H, x, tx)`^{CS}
Bestimmung Impulsantwort	$h(t) \circ\!\!-\!\!\bullet H(s)$	`[h, t] = impulse(H)`^{CS}
Bestimmung Sprungantwort	$h_\varepsilon(t) \circ\!\!-\!\!\bullet \frac{1}{s} H(s)$	`[he, t] = step(H)`^{CS}

Zusammenschaltung kontinuierlicher Systeme

		Matlab	Octave
Reihenschaltung	$H_1(s) \cdot H_2(s)$	`series(H1, H2)`^{CS}	`sysmult(H1, H2)`
Parallelschaltung	$H_1(s) + H_2(s)$	`parallel(H1, H2)`^{CS}	`sysadd(H1, H2)`
Rückkopplung	$\frac{H_1(s)}{1+H_1(s) \cdot H_2(s)}$	`feedback(H1, H2)`^{CS}	`sysfb(H1, H2)`^{2}

Graphische Darstellung

	diskret	kontinuierlich
reelles Signal	`stem(x)`	`plot(t, x)`
Ortskurve (komplexes Signal)	–	`plot(x)`

Berechnung und Darstellung

Pol-Nullstellen-Diagramm	`zplane(b, a)`^{SP}	`pzmap(H)`^{CS}
Frequenzgang	`freqz(b, a)`^{SP}	`freqs(b, a)`^{SP}
Gruppenlaufzeit	`grpdelay(b, a)`^{SP}	–
Bodediagramm	–	`bode(H)`^{CS}
Impulsantwort	`impz(b, a)`^{SP}	`impulse(H)`^{CS}
Sprungantwort	–	`step(H)`^{CS}

Weitere Befehle:

Tiefpaß-Bandpaß-Transformation	`lp2bp`^{SP}
Bilineare Transformation	`bilinear`^{SP}

^{SP} bezeichnet Funktionen, die bei Matlab zur ‘Signal Processing Toolbox’ gehören
^{CS} bezeichnet Funktionen, die bei Matlab zur ‘Control System Toolbox’ gehören

Definition

$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\Omega k}$	$x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) \cdot e^{j\Omega k}\, d\Omega$

Eigenschaften der zeitdiskreten Fourier-Transformation

	Zeitbereich	Frequenzbereich
Linearität	$c_1\, x_1[k] + c_2\, x_2[k]$	$c_1\, X_1(\Omega) + c_2\, X_2(\Omega)$
Faltung	$x[k] * y[k]$	$X(\Omega) \cdot Y(\Omega)$
Multiplikation	$x[k] \cdot y[k]$	$\frac{1}{2\pi}\, X(\Omega) \circledast Y(\Omega)$
Verschiebung	$x[k-k_0]$	$X(\Omega) \cdot e^{-j\Omega k_0}$
Modulation	$e^{j\Omega_0 k} \cdot x[k]$	$X(\Omega - \Omega_0)$
lineare Gewichtung	$k \cdot x[k]$	$j \frac{d}{d\Omega} X(\Omega)$
Zeitinversion	$x[-k]$	$X(-\Omega)$
konj. komplex	$x^*[k]$	$X^*(-\Omega)$
Realteil	$x_R[k]$	$X_{g^*}(j\omega)$
Imaginärteil	$j\ x_I[k]$	$X_{u^*}(j\omega)$
Parsevalsches Theorem	$\sum_{k=-\infty}^{\infty} x[k] \cdot y^*[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) \cdot Y^*(\Omega)\, d\Omega$	

$\boxed{\textit{Frequency Response}: \quad G(j\Omega) = G(z)\big|_{z=e^{j\Omega}}}$ $\quad \Omega = \omega T = \omega \cdot \frac{2\pi}{\omega_A}$

(only if G(z) is stable!)

$\boxed{E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt}$ $\boxed{P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\, dt}$ $\boxed{\begin{aligned} x_g(t) &= \tfrac{1}{2}[x(t) + x(-t)] \\ x_u(t) &= \tfrac{1}{2}[x(t) - x(-t)] \end{aligned}}$ $\boxed{\text{DFT}: \quad X[n] = \sum_{k=0}^{N-1} x[k] \cdot e^{-j\left(2\pi \frac{n}{N} k\right)}}$

Korrespondenzen der zeitdiskreten Fourier-Transformation

Nr.	$x[k]$	$X(\Omega)$
1	$\delta[k]$	1
2	1	$2\pi\ \delta(\Omega)$
3	$\text{III}_N[k]$	$\frac{2\pi}{N}\ \text{III}_{\frac{2\pi}{N}}(\Omega)$
4	$\varepsilon[k]$	$\pi\ \delta(\Omega) + \frac{1}{1 - e^{-j\Omega}}$
5	$\text{sgn}[k]$	$\frac{1}{j \cdot \tan\left(\frac{\Omega}{2}\right)}$
6	$\text{rect}_N[k]$	$e^{-j\frac{(N-1)\Omega}{2}} \cdot \frac{\sin\left(\frac{N\Omega}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}$
7	$e^{j2\pi T f_0 k} = e^{j\Omega_0 k}$	$2\pi\ \delta(\Omega - \Omega_0)$
8	$\cos(2\pi T f_0 k) = \cos(\Omega_0 k)$	$\pi\left[\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)\right]$
9	$\sin(2\pi T f_0 k) = \sin(\Omega_0 k)$	$\pi\ j\left[\delta(\Omega + \Omega_0) - \delta(\Omega - \Omega_0)\right]$

$\text{III}_N[k] = \sum_{n=-\infty}^{+\infty} \delta[k - nN]$

X(Ω) is periodically in 2π!
Table shows only the interval -π< Ω< π

All tables in this appendix are practically 1: 1 taken from:
T. Frey, M. Bossert, "**Signal and systems theory**", Publisher Vieweg + Teubner, 2nd edition, 2008

Z-Transform (single-sided)

Definition

$$X(z) = \sum_{k=0}^{\infty} x[k]\, z^{-k}$$

$$x[k] = \frac{1}{2\pi j} \oint_{\mathcal{C}} X(z)\, z^{k-1}\, dz \qquad \mathcal{C}\text{: pos. orientierte Kurve in } \mathcal{K}$$

Eigenschaften und Rechenregeln

	Zeitbereich	Bildbereich
Linearität	$c_1\, x_1[k] + c_2\, x_2[k]$	$c_1\, X_1(z) + c_2\, X_2(z)$
Faltung	$x[k] * y[k]$	$X(z) \cdot Y(z)$
Dämpfung	$a^k \cdot x[k]$	$X\left(\frac{z}{a}\right)$
lineare Gewichtung	$k \cdot x[k]$	$-z \cdot \frac{d}{dz} X(z)$
konj. komplexes Signal	$x^*[k]$	$X^*(z^*)$
Zeitinversion	$x[-k]$	$X\left(\frac{1}{z}\right)$
diskrete Ableitung	$x[k] - x[k-1]$	$X(z) \cdot \frac{z-1}{z}$
diskrete Integration	$\sum_{i=0}^{k} x[i]$	$X(z) \cdot \frac{z}{z-1}$
periodische Fortsetzung	$\sum_{i=0}^{\infty} x[k - i N_p]$	$X(z) \cdot \frac{1}{1-z^{-N_p}}$
Upsampling	$x\left[\frac{k}{N}\right]$	$X(z^N)$
Verschiebung links	$x[k+k_0],\ k_0 > 0$	$z^{k_0}\, X(z) - \sum_{i=0}^{k_0-1} x[i]\, z^{k_0-i}$
Verschiebung rechts	$x[k-k_0],\ k_0 > 0$	$z^{-k_0}\, X(z)$
Anfangswertsatz	$x[0] = \lim_{z\to\infty} X(z)$, falls Grenzwert existiert	
Endwertsatz	$\lim_{k\to\infty} x[k] = \lim_{z\to 1} (z-1)\, X(z)$, falls $X(z)$ nur Pole mit $\lvert z\rvert < 1$ oder bei $z = 1$	

Korrespondenzen der z-Transformation

Nr.	$x[k]$	$X(z)$
1	$\delta[k]$	1
2	$\delta[k-k_0]$	z^{-k_0}
3	$\varepsilon[k]$	$\frac{z}{z-1}$
4	$k \cdot \varepsilon[k]$	$\frac{z}{(z-1)^2}$
5	$a^k \cdot \varepsilon[k]$	$\frac{z}{z-a}$
6	$\binom{k}{m}\, a^{k-m} \cdot \varepsilon[k]$	$\frac{z}{(z-a)^{m+1}}$
7	$\sin(\Omega_0 k) \cdot \varepsilon[k]$	$\frac{z \cdot \sin(\Omega_0)}{z^2 - 2z\cdot\cos(\Omega_0) + 1}$
8	$\cos(\Omega_0 k) \cdot \varepsilon[k]$	$\frac{z\cdot[z - \cos(\Omega_0)]}{z^2 - 2z\cdot\cos(\Omega_0) + 1}$
9	$a^k \cdot \varepsilon[-k-1]$	$-\frac{z}{z-a}$
10	$a^{\lvert k\rvert},\quad \lvert a\rvert < 1$	$\frac{z\cdot\left(a - \frac{1}{a}\right)}{(z-a)\left(z - \frac{1}{a}\right)}$
11	$\frac{1}{k!} \cdot \varepsilon[k]$	$e^{\frac{1}{z}}$

$z = e^{sT}$

$$\text{Bilinear Transf.:}\quad s = \frac{2}{T} \cdot \frac{z-1}{z+1} \quad or \quad z = \frac{1 + s\frac{T}{2}}{1 - s\frac{T}{2}}$$

$$\text{Step-Inv.Transf.:}\quad G(z) = \frac{z-1}{z} \cdot Z\left\{ L^{-1}\left\{ \frac{G(s)}{s} \right\}\Big|_{t=kT} \right\}$$

Definition

$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t}\, d\omega$

Eigenschaften der Fourier-Transformation

	Zeitbereich	Frequenzbereich
Linearität	$c_1\, x_1(t) + c_2\, x_2(t)$	$c_1\, X_1(j\omega) + c_2\, X_2(j\omega)$
Faltung	$x(t) * y(t)$	$X(j\omega) \cdot Y(j\omega)$
Multiplikation	$x(t) \cdot y(t)$	$\frac{1}{2\pi} X(j\omega) * Y(j\omega)$
Verschiebung	$x(t - t_0)$	$X(j\omega) \cdot e^{-j\omega t_0}$
Modulation	$e^{j\omega_0 t} \cdot x(t)$	$X(j\,[\omega - \omega_0])$
lineare Gewichtung	$t \cdot x(t)$	$-\frac{d}{d(j\omega)} X(j\omega)$
Differentiation	$\frac{d}{dt}\, x(t)$	$j\omega \cdot X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{1}{j\omega}\, X(j\omega) +$ $\pi\, X(0)\, \delta(\omega)$
Skalierung	$x(at)$	$\frac{1}{\lvert a\rvert} \cdot X\left(\frac{j\omega}{a}\right)$
Zeitinversion	$x(-t)$	$X(-j\omega)$
konj. komplex	$x^*(t)$	$X^*(-j\omega)$
Realteil	$x_R(t)$	$X_{g^*}(j\omega)$
Imaginärteil	$j\, x_I(t)$	$X_{u^*}(j\omega)$
Dualität	$X(t)\ \ [X(jt)]$	$2\pi\, x(-\omega)$
Parsevalsches Theorem	$\int_{-\infty}^{\infty} x(t) \cdot y^*(t)\, dt = \frac{1}{2\pi}$	$\int_{-\infty}^{\infty} X(j\omega) \cdot Y^*(j\omega)\, d\omega$

Korrespondenzen der Fourier-Transformation

Nr.	$x(t)$	$X(j\omega)$
1	$\delta(t)$	1
2	1	$2\pi\ \delta(\omega)$
3	$\text{III}_T(t)$	$\frac{2\pi}{\lvert T\rvert}\ \text{III}_{\frac{2\pi}{T}}(\omega)$
4	$\varepsilon(t)$	$\pi\ \delta(\omega) + \frac{1}{j\omega}$
5	$\text{sgn}(t)$	$\frac{2}{j\omega}$
6	$\frac{1}{\pi t}$	$-j\ \text{sgn}(\omega)$
7	$\text{rect}\left(\frac{t}{T}\right)$	$\lvert T\rvert \cdot \text{si}\left(\frac{T}{2}\,\omega\right)$
8	$\text{si}\left(\pi\,\frac{t}{T}\right)$	$\lvert T\rvert \cdot \text{rect}\left(\frac{T}{2\pi}\,\omega\right)$
9	$\Lambda\left(\frac{t}{T}\right)$	$\lvert T\rvert \cdot \text{si}^2\left(\frac{T}{2}\,\omega\right)$
10	$\text{si}^2\left(\pi\,\frac{t}{T}\right)$	$\lvert T\rvert \cdot \Lambda\left(\frac{T}{2\pi}\,\omega\right)$
11	$e^{j\omega_0 t}$	$2\pi\ \delta(\omega - \omega_0)$
12	$\cos(\omega_0 t)$	$\pi\left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right]$
13	$\sin(\omega_0 t)$	$\pi j\left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right]$
14	$e^{-a^2 t^2}$	$\frac{\sqrt{\pi}}{a}\ e^{-\frac{\omega^2}{4a^2}}$
15	$e^{-\frac{\lvert t\rvert}{T}}$	$\frac{2\,T}{1+(T\omega)^2}$

Frequency Response: $G(j\omega) = G(s)\big|_{s=j\omega}$

(only if G(s) is stable!)

$\text{si}(t) = \frac{\sin(t)}{t}$ $\quad \text{III}_T(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$

Laplace Transform (single-sided)

Definition

$$X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) \cdot e^{st}\, ds\,, \quad \sigma \in \mathcal{K}$$

Eigenschaften und Rechenregeln $x(t) = 0,\ t < 0$

	Zeitbereich	Bildbereich
Linearität	$c_1\, x_1(t) + c_2\, x_2(t)$	$c_1\, X_1(s) + c_2\, X_2(s)$
Faltung	$x(t) * y(t)$	$X(s) \cdot Y(s)$
Multiplikation	$x(t) \cdot y(t)$	$X(s) * Y(s)$
Verschiebung rechts	$x(t - t_0)\,,\ t_0 > 0$	$e^{-st_0} \cdot X(s)$
Verschiebung links	$x(t + t_0)\,,\ t_0 > 0$	$e^{st_0} \cdot X(s) - \int_0^{t_0} x(t) \cdot e^{-s(t+t_0)} dt$
Dämpfung	$e^{at} \cdot x(t)$	$X(s - a)$
Gewichtung	$t \cdot x(t)$ $t^2 \cdot x(t)$	$-\frac{d}{ds} X(s)$ $\frac{d^2}{ds^2} X(s)$
Differentiation	$\dot{x}(t)$ $\ddot{x}(t)$	$s \cdot X(s) - x(0_-)$ $s^2 \cdot X(s) - s \cdot x(0_-) - \dot{x}(0_-)$
Integration	$\int_0^t x(\tau)\, d\tau$	$\frac{1}{s} \cdot X(s)$
Skalierung	$x(at)$	$\frac{1}{\lvert a \rvert} \cdot X\left(\frac{s}{a}\right)$
konj. komplexes Signal	$x^*(t)$	$X^*(s^*)$
Anfangswertsatz	$x(0_-) = \lim_{s \to \infty} s \cdot X(s)\,,$ falls $x(0_-)$ existiert	
Endwertsatz	$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s \cdot X(s)\,,$ falls $\lim_{t \to \infty} x(t)$ existiert	

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot y(t - \tau)\, d\tau$$

$$G_{PT_2}(s) = \frac{K \cdot \omega_0^2}{s^2 + 2D\omega_0 s + \omega_0^2} = \frac{K}{\frac{s^2}{\omega_0^2} + \frac{2D \cdot s}{\omega_0} + 1}$$

Korrespondenzen der Laplace-Transformation

Nr.	$X(s)$	$x(t)$
1	1	$\delta(t)$
2	$\frac{1}{s}$	$\varepsilon(t)$
3	$\frac{1}{s^2}$	$t \cdot \varepsilon(t)$
4	$\frac{1}{s^3}$	$\frac{t^2}{2} \cdot \varepsilon(t)$
5	$\frac{1}{s + a}$	$e^{-at} \cdot \varepsilon(t)$
6	$\frac{1}{(s + a)^2}$	$t\, e^{-at} \cdot \varepsilon(t)$
7	$\frac{1}{s\,(s + a)}$	$\frac{1}{a}\left(1 - e^{-at}\right) \cdot \varepsilon(t)$
8	$\frac{1}{(s + a)\,(s + b)}$	$\frac{1}{a - b}\left(e^{-bt} - e^{-at}\right) \cdot \varepsilon(t)$
9	$\frac{1}{s\,(s + a)^2}$	$\frac{1}{a^2}\left[1 - (1 + at)\, e^{-at}\right] \cdot \varepsilon(t)$
10	$\frac{1}{s\,(s + a)\,(s + b)}$	$\frac{1}{ab}\left[1 + \frac{1}{a - b}\left(b\, e^{-at} - a\, e^{-bt}\right)\right] \cdot \varepsilon(t)$
11	$\frac{s}{(s + a)^2}$	$(1 - at)\, e^{-at} \cdot \varepsilon(t)$
12	$\frac{s}{(s + a)\,(s + b)}$	$\frac{1}{a - b}\left(a\, e^{-at} - b\, e^{-bt}\right) \cdot \varepsilon(t)$
13	$\frac{s + a}{s\,(s + b)}$	$\left(\frac{a}{b} + \frac{b - a}{b}\, e^{-bt}\right) \cdot \varepsilon(t)$
14	$\frac{1}{(s + a)\,(s + b)^2}$	$\frac{1}{(a - b)^2}\left(e^{-at} - e^{-bt} + (a - b)\, t\, e^{-bt}\right) \cdot \varepsilon(t)$
15	$\frac{\omega}{s^2 + \omega^2}$	$\sin \omega t \cdot \varepsilon(t)$
16	$\frac{s}{s^2 + \omega^2}$	$\cos \omega t \cdot \varepsilon(t)$