Task 13

$$\dot{x} = k_1 e^{\frac{k_2}{x}} - k_3 u \quad \rightarrow \quad f(x,u) = k_1 e^{\frac{k_2}{x}} - k_3 u$$
$$y = \arctan(x) \quad \rightarrow \quad h(x,u) = \arctan(x)$$

a) Equilibrium points:

$$f(x_{OP}, u_{OP}) = k_1 e^{\frac{k_2}{x_{OP}}} - k_3 u_{OP} = 0$$

$$\overset{k_1 \neq 0}{\Longleftrightarrow} \quad e^{\frac{k_2}{x_{OP}}} = \frac{k_3}{k_1} u_{OP}$$

For $k_2 \neq 0$ the exponential function can take any positive value apart from 1 if x_{OP} varies. Thus, we can solve this expression for x_{OP} if the right hand side is strictly positive and unequal to 1, which certainly is the case for $k_1 > 0, k_3 > 0$, $u_{OP} > 0$ and $u_{OP} \neq \frac{k_1}{k_3}$:

$$x_{OP} = \frac{k_2}{\ln\left(\frac{k_3}{k_1} u_{OP}\right)}$$

Consequently, for each constant $u_{OP} > 0$ apart from $u_{OP} = \frac{k_1}{k_3}$ the system has the unique equilibrium point $\left(x_{OP} = \frac{k_2}{\ln\left(\frac{k_3}{k_1} u_{OP}\right)}, u_{OP}\right)$.

b) We can linearize the system around an equilibrium point, if the right hand sides of the state equations are twice continuously differentiable functions in x and u in a neighborhood of the equilibrium point.

Since

- $\frac{\partial^2 f}{\partial x^2}(x,u) = \frac{k_1 k_2}{x^3} e^{\frac{k_2}{x}} (2 + \frac{k_2}{x})$
- $\frac{\partial^2 f}{\partial u^2}(x,u) = 0$
- $\frac{\partial^2 h}{\partial x^2}(x,u) = \frac{-2x}{(1+x^2)^2}$
- $\frac{\partial^2 h}{\partial u^2}(x,u) = 0$

exist and are continuous for all $x \neq 0$ (where the system has no equilibrium point), we can linearize the particular system at hand around all equilibrium points.

c) Determine linearized system description:

- $\frac{\partial f}{\partial x}(x_{OP}, u_{OP}) = -\frac{k_1 k_2}{x_{OP}^2} e^{\frac{k_2}{x_{OP}}} = -\frac{k_3}{k_2} u_{OP} \left[\ln\left(\frac{k_3}{k_1} u_{OP}\right)\right]^2$
- $\frac{\partial f}{\partial u}(x_{OP}, u_{OP}) = -k_3$
- $\frac{\partial h}{\partial x}(x_{OP}, u_{OP}) = \frac{1}{1+x_{OP}^2} = \frac{\left[\ln\left(\frac{k_3}{k_1} u_{OP}\right)\right]^2}{k_2^2 + \left[\ln\left(\frac{k_3}{k_1} u_{OP}\right)\right]^2}$
- $\frac{\partial h}{\partial u}(x_{OP}, u_{OP}) = 0$

→
$$\Delta\dot{x} = -\frac{k_3}{k_2} u_{OP} \left[\ln\left(\frac{k_3}{k_1} u_{OP}\right)\right]^2 \Delta x - k_3 \Delta u$$
$$\Delta y = \frac{\left[\ln\left(\frac{k_3}{k_1} u_{OP}\right)\right]^2}{k_2^2 + \left[\ln\left(\frac{k_3}{k_1} u_{OP}\right)\right]^2} \Delta x$$

where:

$$\Delta x = x - \frac{k_2}{\ln\left(\frac{k_3}{k_1} u_{OP}\right)}$$
$$\Delta u = u - u_{OP}$$
$$\Delta y = y - \arctan\left(\frac{k_2}{ln\left(\frac{k_3}{k_1} u_{OP}\right)}\right)$$

Task 14

$$\underline{\dot{x}} = \begin{bmatrix} 2x_1^2 - 7x_2 + x_1x_2x_3 + ux_1 \\ 2x_1 + x_2x_3 \\ x_1x_2 + 4x_3 + ux_2 \end{bmatrix} \quad \rightarrow \quad \underline{f}(\underline{x}, u) = \begin{bmatrix} 2x_1^2 - 7x_2 + x_1x_2x_3 + ux_1 \\ 2x_1 + x_2x_3 \\ x_1x_2 + 4x_3 + ux_2 \end{bmatrix}$$

$$y = x_1 + 3x_3 \quad \rightarrow \quad h(\underline{x}, u) = x_1 + 3x_3$$

a) Equilibrium point:

$$\underline{f}(\underline{x}_{OP}, u_{OP}) = \begin{bmatrix} 2x_{1,OP}^2 - 7x_{2,OP} + x_{1,OP}x_{2,OP}x_{3,OP} + u_{OP}x_{1,OP} \\ 2x_{1,OP} + x_{2,OP}x_{3,OP} \\ x_{1,OP}x_{2,OP} + 4x_{3,OP} + u_{OP}x_{2,OP} \end{bmatrix} = \underline{0}$$

Substituting $x_{1,OP} = x_{2,OP} = 1$ we obtain

$$\begin{bmatrix} -5 + x_{3,OP} + u_{OP} \\ 2 + x_{3,OP} \\ 1 + 4x_{3,OP} + u_{OP} \end{bmatrix} = \underline{0} \rightarrow x_{3,OP} = -2, u_{OP} = 7$$

→ The requested equilibrium point is at $\left(\underline{x}_{OP} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, u_{OP} = 7 \right)$.

b) Determine linearized state equations in matrix form:

- $$\frac{\partial \underline{f}}{\partial \underline{x}}(\underline{x}_{OP}, u_{OP}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}_{\substack{\underline{x}=\underline{x}_{OP} \\ u=u_{OP}}}$$

$$= \begin{bmatrix} 4x_{1,OP} + x_{2,OP}\, x_{3,OP} + u_{OP} & -7 + x_{1,OP}\, x_{3,OP} & x_{1,OP}\, x_{2,OP} \\ 2 & x_{3,OP} & x_{2,OP} \\ x_{2,OP} & x_{1,OP} + u_{OP} & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -9 & 1 \\ 2 & -2 & 1 \\ 1 & 8 & 4 \end{bmatrix}$$

- $$\frac{\partial \underline{f}}{\partial u}(\underline{x}_{OP}, u_{OP}) = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{bmatrix}_{\substack{\underline{x}=\underline{x}_{OP} \\ u=u_{OP}}} = \begin{bmatrix} x_{1,OP} \\ 0 \\ x_{2,OP} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- $$\frac{\partial h}{\partial \underline{x}}(\underline{x}_{OP}, u_{OP}) = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix}_{\substack{\underline{x}=\underline{x}_{OP} \\ u=u_{OP}}} = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$$

- $$\frac{\partial h}{\partial u}(\underline{x}_{OP}, u_{OP}) = 0$$

→
$$\Delta\underline{\dot{x}} = \begin{bmatrix} 9 & -9 & 1 \\ 2 & -2 & 1 \\ 1 & 8 & 4 \end{bmatrix} \Delta\underline{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Delta u$$
$$\Delta y = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \Delta x$$

where:

$$\Delta x = x - \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$
$$\Delta u = u - 7$$
$$\Delta y = y + 5$$