Exercise Image Processing

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Sheet 1

In this exercise we cover the sections *homogeneous coordinates*, *2D straight line* and *2nd order curves*. The questions are small-part and can be seen as examples of potential exam problems. The formulas needed to complete the tasks are as follows:

A 2D straight line in homogeneous coordinates is given by the equation

$$\mathbf{l}^\top \overline{\mathbf{x}} = 0\,.$$

The parameters are given by two different points $\overline{\mathbf{x}}_1$, $\overline{\mathbf{x}}_2$ on the line

$$\mathbf{l} = \overline{\mathbf{x}}_1 \times \overline{\mathbf{x}}_2\,.$$

The intersection point results in:

$$\overline{\mathbf{x}} = \mathbf{l}_1 \times \mathbf{l}_2\,.$$

In homogeneous coordinates, 2nd order curves are described by the following equation:

$$\overline{\mathbf{x}}^\top \mathbf{C} \overline{\mathbf{x}} = 0 \quad , \quad \text{mit} \qquad \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}\,,$$

where $\mathbf{C}$ is a symmetric matrix.

Curves of 2nd order are completely given by the following three characteristic quantities:

$$\Delta = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{vmatrix}\,, \quad \delta = \begin{vmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{vmatrix}\,, \quad s = c_{11} + c_{22}\,.$$

This results in different real curves:

- $\delta > 0$ and $s\Delta < 0$: Ellipse, if $c_{11} = c_{22}$ and $c_{12} = 0$: circle.
- $\delta < 0$ and $\Delta \neq 0$: Hyperbola.
- $\delta = 0$ and $\Delta \neq 0$: Parabola.
- $\delta < 0$ and $\Delta = 0$: pair of straight lines.
- $\delta = 0$ and $\Delta = 0$: parallel lines.

If $\Delta \neq 0$ results in the center $\mathbf{m}$ of the curve to:

$$\mathbf{m} = \begin{bmatrix} \delta_2/\delta \\ \delta_3/\delta \end{bmatrix}\,, \quad \text{wobei} \quad \delta_2 = \begin{vmatrix} c_{12} & c_{13} \\ c_{22} & c_{23} \end{vmatrix}\,, \quad \delta_3 = \begin{vmatrix} c_{13} & c_{11} \\ c_{23} & c_{12} \end{vmatrix}\,. \tan(2\theta) = \frac{2c_{12}}{c_{11} - c_{22}}\,.$$

Every curve of 2nd order can be obtained by a displacement $\mathbf{x}' = \mathbf{x} - \mathbf{m}$ and a rotation $\mathbf{x}'' = \mathbf{R}\mathbf{x}' = [\cos\theta, -\sin\theta; \sin\theta, \cos\theta]\,\mathbf{x}'$ into the normal form $a^* x''^2 + b^* y''^2 + c^* = 0$.

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Task 1.1: 2nd order curves

The following parameterization of a 2nd order curve is given:

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{11} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & -5 \\ -9 & -5 & 102 \end{bmatrix} .$$

a) What kind of curve do the coefficients of the matrix $\mathbf{C}$ describe? Use the characteristic quantities Δ, δ and s to answer the question.

b) Calculate the displacement vector $\mathbf{m}$ that transforms the curve into normal form. What quantities do the parameters c_{13} and c_{23} correspond to?

c) Set up the normal form $\tilde{\mathbf{C}}$ (diagonal matrix!) of $\mathbf{C}$ by converting the coordinates $\mathbf{x}$ into transformed coordinates $\mathbf{x}'$. To do this, first set up the transformation matrix $\mathbf{T}$ for homogeneous coordinates $\overline{\mathbf{x}} = \mathbf{T}\overline{\mathbf{x}}'$ and then calculate from it $\tilde{\mathbf{C}} = \mathbf{T}^\top\mathbf{C}\mathbf{T}$.

Task 1.2: Calculate straight lines, intersections and tangents

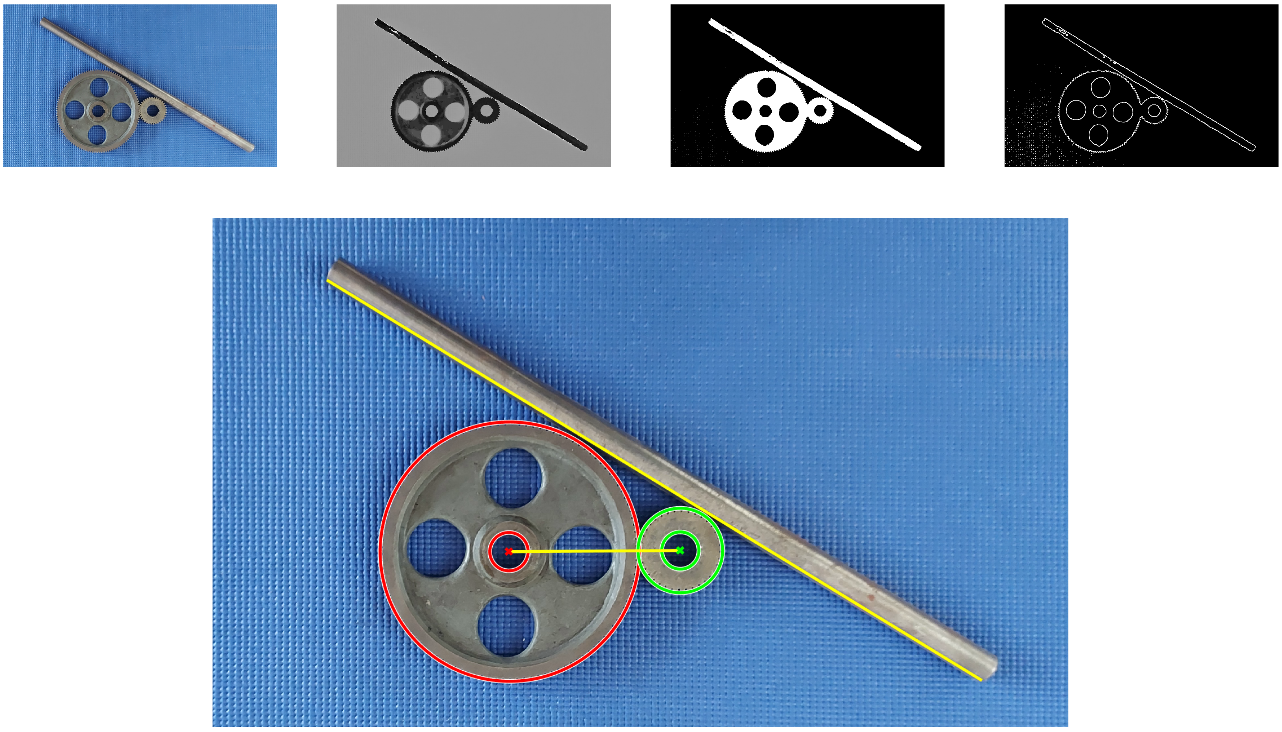

Figure 1: Segmentation via thresholding on the hue channel of the image and subsequent contour extraction via morphological operators results in the red, green and yellow curves in the image.

A local coordinate system with origin in the center $\mathbf{m}_r = [0, 0]^\top$ of the red circles is assumed (x-axis points to the right, y-axis up). The center of the green circles has the coordinates $\mathbf{m}_g = [5.5, 0]^\top$. The large red outer circle of the gear has the following parameters:

$$\mathbf{C}_r = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{11} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -16 \end{bmatrix} .$$

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a) Determine the straight line $\mathbf{l}_{rg}$ through the two centers of the circles for homogeneous coordinates.

b) Normalize the parameters of $\mathbf{l}_{rg}$ to Hessian normal form: $\mathbf{l}^\top \overline{\mathbf{x}} = \mathbf{n}^\top \mathbf{x} - d = 0$.

c) The intersection of the two yellow straight lines results in $\mathbf{x}_s = [14, 0]^\top$. Calculate the straight line $\mathbf{l}_s$, which results from the two points of contact $\mathbf{x}_1$ and $\mathbf{x}_2$ of the tangents $\mathbf{l}_1 = \mathbf{C}_r \mathbf{x}_1$ and $\mathbf{l}_2 = \mathbf{C}_r \mathbf{x}_2$ to the big red circle by the intersection $\mathbf{x}_s$. First make a sketch with all the variables and then think of the formula for the straight line $\mathbf{l}_s$. Note: It must be valid: $\mathbf{l}_s^\top \overline{\mathbf{x}}_1 = \mathbf{l}_s^\top \overline{\mathbf{x}}_2 = \mathbf{l}_1^\top \overline{\mathbf{x}}_s = \mathbf{l}_2^\top \overline{\mathbf{x}}_s$.

d) Now calculate the points of contact of the tangents to the circle.

e) Calculate the parameters of the tangents $\mathbf{l}_{1/2}$ and check if the intersection point of these tangents is the same as the given intersection point $\mathbf{x}_s = [14, 0]^\top$. (This must be the case if you have calculated correctly).