Exercise Image Processing
Sample Solution

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Sheet 4

In this exercise, we cover the chapters *image decomposition*, *Fourier transformation*, and *singular value decomposition*. The questions are small-part and can be seen as examples of potential exam problems. Also use the formulary for the exam to work through the problems.

Task 4.1: Image Decomposition

4.1a)

Each gray value of an image can be described as a projection of the entire image onto a base image. What does such a base image look like for a gray value at a certain image location (m, n)?

Answer: It has the value 1 at the point (m, n) and the value 0 at all other grid points.

4.1b)

The singular value decomposition decomposes an image into a weighted sum of base images. What is the property of these base images? How many base images are obtained when decomposing an image of size 3×4?

Answer: The base images are orthogonal to each other and separable. You get 3 base images.

Task 4.2: Fourier transform

4.2a)

What does a wave number vector describe? What information is completely lost in the wavenumber space?

Answer: The wavenumber vector describes the direction and the wavelength of a cosine wave. The location information is completely lost.

4.2b)

Calculate the 2D DFT $\hat{G}_{0,0}$ for $[u,v]^\top = [0,0]^\top$. To which statistical value does this Fourier coefficient correspond to?

Answer: From the defining equation of the 2D DFT

$$G_{m,n} \circ\!\!-\!\!\bullet \hat{G}_{u,v} = \frac{1}{MN} \sum_{m=0}^{M-1} \left(\sum_{n=0}^{N-1} G_{m,n} w_N^{-nv} \right) w_M^{-mu} ,$$

it is quickly seen that the result of $\hat{G}_{0,0}$ corresponds to the mean value of the image:

$$\hat{G}_{0,0} = \frac{1}{MN} \sum_{m=0}^{M-1} \left(\sum_{n=0}^{N-1} G_{m,n} \right).$$

4.2c)

What is the effect of the limited size of an image in Fourier space? How can the occurring discontinuities at the borders of the image be influenced, so that there are less interference effects along the horizontal and vertical and vertical axes?

Answer: The spatial limit of an image of size $M \times N$ limits the discrete Fourier space to $M \times N$ discrete wavenumber vectors. The border effects due to the location limitation can be reduced by a cosine window (or Hamming window). These windows realize a weighting decreasing towards the borders of the image.

4.2d)

Each image $\mathbf{G}$ can be reconstructed over the base images $\mathbf{B}^{m,n}$ without loss using the following formula:

$$\mathbf{G} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} G_{m,n} \mathbf{B}^{m,n} .$$

The back transformation of the discrete Fourier transform for each pixel is given by the following prescription:

$$G_{m,n} = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \hat{G}_{u,v} w_N^{nv} w_M^{mu} .$$

It specifies how each individual pixel in image space $G_{m,n}$ is reconstructed without loss by a weighted sum over all complex-valued elements $w_N^{nv} w_M^{mu}$, weighted by the complex-valued Fourier coefficients $\hat{G}_{u,v}$. Combining both reconstruction rules, the reconstruction of an image $\mathbf{G}$ over a weighted sum of complex-valued basis images $\hat{\mathbf{B}}^{u,v}$ takes the following form:

$$\mathbf{G} = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \hat{G}_{u,v} \hat{\mathbf{B}}^{u,v} .$$

Derive the formula for computing the basis images $\hat{\mathbf{B}}^{u,v}$ as a function of the basis images $\mathbf{B}^{m,n}$ and the complex-valued elements $w_N^{nv} w_M^{mu}$.

Answer: • Substitute the equation of the back transformation into the equation of the image over base images:

$$\mathbf{G} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} G_{m,n} \mathbf{B}^{m,n} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left(\sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \hat{G}_{u,v} w_N^{nv} w_M^{mu} \right) \mathbf{B}^{m,n} .$$

- Convert this formula into the following form:

$$\mathbf{G} = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \hat{G}_{u,v} \left(\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} w_N^{nv} w_M^{mu} \mathbf{B}^{m,n} \right) .$$

- Reading the formula of the complex-valued base images:

$$\hat{\mathbf{B}}^{u,v} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} w_N^{nv} w_M^{mu} \mathbf{B}^{m,n} .$$

Task 4.3: Singular Value Decomposition

Whether a filter mask is separable can be checked with a singular value decomposition of the filter mask.

4.3a)

What condition must a matrix satisfy if it is separable?

Answer: $\mathbf{M} = \mathbf{a}\mathbf{b}^\top$

4.3b)

What condition must the singular value decomposition of a separable matrix satisfy?

Answer: Except for the largest singular value, all other singular values must be zero.

4.3c)

Check which of the two filter masks $\mathbf{M}_1$ or $\mathbf{M}_2$ is separable:

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{M}_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Note: If the matrix is square, then you can use the eigenvalue decomposition.

Answer: The eigenvalues of the matrix $\mathbf{M}_1$ must be determined:

$$det\left(\lambda \mathbf{I} - \mathbf{M}_1\right) = 0 \rightsquigarrow \lambda_{1,2} = 1.$$

The eigenvalue $\lambda_2 = 1 \neq 0$ is nonzero. Thus, the filter mask $\mathbf{M}_1$ is not separable.
The eigenvalues of the matrix $\mathbf{M}_2$ result in $\lambda_1 = 2$ and $\lambda_2 = 0$. Consequently, the filter mask $\mathbf{M}_2$ is separable:

$$\mathbf{M}_2 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} .$$