Task 18

We consider the LTI SISO system

$$\underline{\dot{x}} = \begin{bmatrix} -1 & 5 \\ 7 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x}$$

a) The system is completely controllable, since the controllability matrix

$$\underline{Q}_c = \begin{bmatrix} \underline{b} & \underline{A}\underline{b} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 7 \end{bmatrix}$$

is regular, because $\det\left(\underline{Q}_c\right) = 7 \neq 0$.

b) The system is completely observable, since the observability matrix

$$\underline{Q}_o = \begin{bmatrix} \underline{c}^T \\ \underline{c}^T \underline{A} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 7 & -3 \end{bmatrix}$$

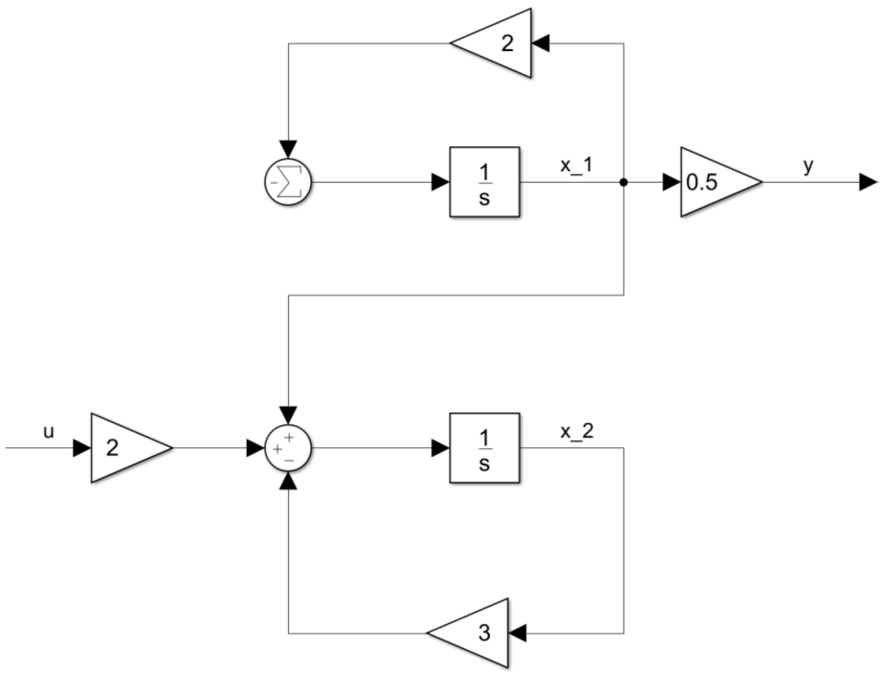
is regular, because $\det\left(\underline{Q}_o\right) = -7 \neq 0$.

Task 19

We consider the LTI SISO system

$$\underline{\dot{x}} = \begin{bmatrix} -2 & 0 \\ 1 & -3 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0.5 & 0 \end{bmatrix} \underline{x}$$

a) Block diagram:



It is obvious from the block diagram that the system is neither completely controllable nor completely observable:

- The input u has no influence on the state x_1 → not controllable.
- The state x_2 has no impact on the output y → not observable.

b)

$\underline{Q}_c = [\underline{b} \quad \underline{A}\underline{b}] = \begin{bmatrix} 0 & 0 \\ 2 & -6 \end{bmatrix}$ → not completely controllable, since $\det\left(\underline{Q}_c\right) = 0$

$\underline{Q}_o = \begin{bmatrix} \underline{c}^T \\ \underline{c}^T \underline{A} \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ -1 & 0 \end{bmatrix}$ → not completely observable, since $\det\left(\underline{Q}_o\right) = 0$

c)

$$\underline{\dot{\tilde{x}}} = \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix} \underline{\tilde{x}} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix} \underline{\tilde{x}}$$

$\underline{\tilde{Q}}_c = \begin{bmatrix} 2 & -6 \\ 2 & -6 \end{bmatrix}$ → not completely controllable, since $\det\left(\underline{\tilde{Q}}_c\right) = 0$

$\underline{\tilde{Q}}_o = \begin{bmatrix} 0.5 & -0.5 \\ -1 & 1 \end{bmatrix}$ → not completely observable, since $\det\left(\underline{\tilde{Q}}_o\right) = 0$

d)

Applying the regular state transformation

$$\underline{\tilde{x}} = \underline{T} \cdot \underline{x}$$

with $\det(\underline{T}) \neq 0$ to the original system

$$\underline{\dot{x}} = \underline{A}\underline{x} + \underline{b}u$$
$$y = \underline{c}^T \underline{x} + du$$

yields the transformed system

$$\underline{\dot{\tilde{x}}} = \underline{T}\underline{A}\underline{T}^{-1}\underline{\tilde{x}} + \underline{T}\underline{b}u$$
$$y = \underline{c}^T \underline{T}^{-1}\underline{\tilde{x}} + du$$

1. Controllability:

Controllability matrix of the original system:

$$\underline{Q}_c = \begin{bmatrix} \underline{b} & \underline{A}\underline{b} & \cdots & \underline{A}^{n-1}\underline{b} \end{bmatrix}$$

Controllability matrix of the transformed system:

$$\begin{aligned}
\underline{\tilde{Q}}_c &= \begin{bmatrix} \underline{\tilde{b}} & \underline{\tilde{A}}\underline{\tilde{b}} & \cdots & \underline{\tilde{A}}^{n-1}\underline{\tilde{b}} \end{bmatrix} \\
&= \begin{bmatrix} \underline{T}\underline{b} & \underline{T}\underline{A}\underline{T}^{-1}\underline{T}\underline{b} & \cdots & \underline{T}\underline{A}\underline{T}^{-1} \cdot \underline{T}\underline{A}\underline{T}^{-1} \cdot \ldots \cdot \underline{T}\underline{A}\underline{T}^{-1}\underline{T}\underline{b} \end{bmatrix} \\
&= \underline{T}\begin{bmatrix} \underline{b} & \underline{A}\underline{b} & \cdots & \underline{A}^{n-1}\underline{b} \end{bmatrix} \\
&= \underline{T} \cdot \underline{Q}_c
\end{aligned}$$

Since for square matrices with the same dimensions $\underline{M}_1$ and $\underline{M}_2$ the relationship

$$\det(\underline{M}_1\underline{M}_2) = \det\left(\underline{M}_1\right) \cdot \det(\underline{M}_2)$$

holds, we get

$$\det\left(\underline{\tilde{Q}}_c\right) = \det(\underline{T}) \cdot \det(\underline{Q}_c)$$

As $\det(\underline{T}) \neq 0$ for any regular state transformation, this means that $\det\left(\underline{\tilde{Q}}_c\right) \neq 0$ if and only if $\det(\underline{Q}_c) \neq 0$. Consequently, the system resulting after the state transformation is completely controllable if and only if the original system is completely controllable.

2. Observability:

The proof for observability can be shown analogously to controllability by using the observability matrices of the original and the transformed system.