1) Motivation

In class "control systems": Description of plant and controller by transfer functions in frequency domain (= "classical" control theory)

Advantages:

- simple & straightforward handling of connected and cascaded subsystems (e.g. controller and plant)
- Quick access to the basic properties (dynamics, stability)
- Many methods and tools for analysis and design of control loops available
- Many engineers have (basic) experience with classical control theory

Disadvantages:

- Restricted to Linear Time Invariant (LTI) systems
- Usually restricted to Single Input Single Output (SISO) systems (Generalization to Multiple Input Multiple Output (MIMO) systems is possible, but not straight forward)
- only Input/Output behaviour is considered (no internal dynamic properties, information about internal structure is usually lost)
- Controllability and Observability not known (Fukushima)

State Space Methods:

- Direct description of plant (& controller) using algebraic & differential equations which result from modelling (description in "time domain")
- Standarized form of these time-based equations

2) State-Space Description of LTI Systems

2.1) Physical Modelling & Derivation of State Equations

Example: Consider the Electrical Network

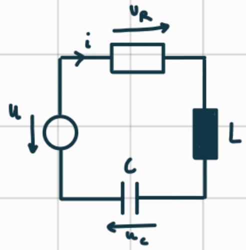
$u: u$, $y: i$

$W_c = \frac{1}{2} C u_c^2$

$W_L = \frac{1}{2} L i^2 = \frac{1}{2} L \left(\frac{u_R}{R}\right)^2$

Can use u_R instead of i as a state variable

Order of System = No. of independent storage elements



· Physical Modelling:

$u = Ri + L\dot{i} + u_c$

$i = C \cdot \dot{u}_c$

· Derive State Equations

To obtain standardized system description in state space form

· Introduce a set of auxilliary variables (the so-called state variables or just states) such that the total energy and/or information stored in the system is completely determined if and only if all state variables are known

we know:

Total Energy stored in the network is determined by

· Capacitor voltage u_c &

· Inductor Current i

We introduce u_c & i as states x_1 & x_2, $x_1 = u_c$, $x_2 = i$

$u = Rx_2 + L\dot{x}_2 + x_1$, $x_2 = C \cdot \dot{x}_1$

Solving for the time derivatives of the states:

$\dot{x}_1 = \frac{x_2}{C}$, $\dot{x}_2 = \frac{u - Rx_2 - x_1}{L}$, $y = x_2$ Summarizing, we get the following state space description of the network above:

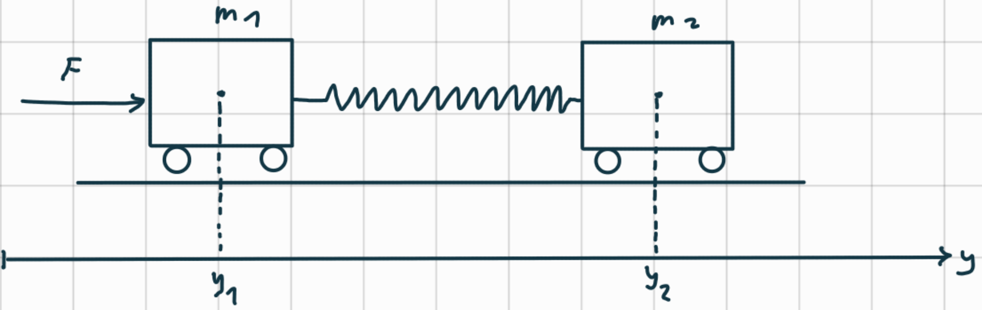
These first-order differential equations in that form (explicitly solved for $\dot{x}_i$, only states and inputs on the right hand side) are called State Differential Eq

To complete system description → output must be expressed in terms of states and inputs

⇒ here: as $y = i$ and $i = x_2$, we get $y = x_2$

Only cover SISO with some exceptions

Another example: 2-mass system



0 P.E.

Assumptions:

· Friction is neglected

· Non-linear spring characteristic

$F_{spring} = K(y_2 - y_1 - l)^3$

$K > 0$

Spring is in forceless state if carts are at a distance l to each other, i.e. if $y_2 - y_1 - l = 0$

$u = F$, $y = y_2$

Physical Modelling:

- We apply Newton's law $F = ma$

Total force on cart 1, $F_1 = F + K(y_2 - y_1 - l)^3$

Total force on cart 2, $F_2 = F - K(y_2 - y_1 - l)^3$

$F + K(y_2 - y_1 - l)^3 = m_1 \cdot \ddot{y}_1 \ldots$ ①

$-K(y_2 - y_1 - l)^3 = m_2 \cdot \ddot{y}_2 \ldots$ ②

State Differential Equations:

$$\left.\begin{aligned} \dot{x}_1 &= \dot{y}_1 = x_2 \,,\quad \dot{x}_3 = \dot{y}_2 = x_4 \\ \dot{x}_3 &= \ddot{y}_1 = \frac{1}{m_1} \cdot (F + K(y_2 - y_1 - l)^3) = \frac{1}{m_1} \cdot (u + K(x_3 - x_1 - l)^3) \\ \dot{x}_4 &= \ddot{y}_2 = \frac{-1}{m_2} \cdot K \cdot (y_2 - y_1 - l)^3 = \frac{-K}{m_2}(x_3 - x_1 - l)^3 \end{aligned}\right\}$$

Introduce State Variables:

"Rule" for introducing state variables (effective in many cases after physical modelling is completed and elimination of all dependent variables): Find all ⋯. All variables and ˙, ¨ till ⋯-1 are state variables.

$x_1 = y_1 \,,\; x_2 = \dot{y}_1 \,,\; x_3 = y_2 \,,\; x_4 = \dot{y}_2$

Output Equations:

$$\underbrace{y = y_2 = x_3}_{\text{Summary}}$$

The state space description above is non-linear because of the $()^3$ term on the RHS of the state differential equations.

However, the state equations of the RLC network are linear

Both systems are time invariant, since there is no explicit dependence on t on the RHS

2.2 Linearization around an operating point

When is a state space description linear and time invariant?
State equations are linear and time invariant (LTI) if the RHS of both the state differential equations and the output equation(s) only contain summands of the form

$a_j x_i$ and/or $b_r \cdot v_l$ where a_j and b_r are constant real coefficients

Not allowed (in LTI state equations) are for example:

- Multiplications of states and/or inputs
- Constant summands
- Non-linear functions like sin, $\sqrt{\ }$, etc. depending on the states and or inputs

General form of LTI State Equations for SISO System(Sheet 6)

Using vectors and matrices, the linear state equations can be written in a very compact form

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⋮

3) Analysis of LTI Systems

3.1) Solution of State Equations

Consider:

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}u \quad (*)$$
$$y = \underline{c}^T\underline{x} + du \quad (**)$$

We are looking for the evolution of $\underline{x}(t)$ for $t \geq 0$

for given initial state $\underline{x}(0) = \underline{x}_0$ and known input signal $u(t)$ for $t \geq 0$

As $(*)$ represents a linear (excited) system of differential equations with constant coefficients

→ The stated initial value problem has a unique solution (which always exists.) (Picard - Lindelöf Theorem)

Solution candidate:

$$\underline{x}(t) = \underline{e}^{\underline{A}t} \cdot \underline{x}_0 + \int_0^t \underline{e}^{\underline{A}(t-\tau)} \cdot \underline{b} \cdot u(\tau)\,d\tau \quad (\square)$$

In order to show that $(\square)$ solves the problem, we have to prove that

- it satisfies the differential equation $(*)$
- it satisfies the initial condition $\underline{x}(0) = \underline{x}_0$

Differential Equation:

$$(\square) \rightarrow \dot{\underline{x}}(t) = \frac{d}{dt}\left(\underline{e}^{\underline{A}t}\underline{x}_0\right) + \frac{d}{dt}\int_0^t \underline{e}^{\underline{A}(t-\tau)}\,\underline{b}\,u(\tau)\,d\tau$$

Using the following 2 properties of the matrix exponential function

- $\frac{d}{dt}\left(\underline{e}^{\underline{A}t}\right) = \underline{A} \cdot \underline{e}^{\underline{A}t}$
- $\underline{e}^{\underline{A}(t-\tau)} = \underline{e}^{\underline{A}t - \underline{A}\tau} = e^{\underline{A}t} \cdot e^{-\underline{A}\tau}$

we obtain

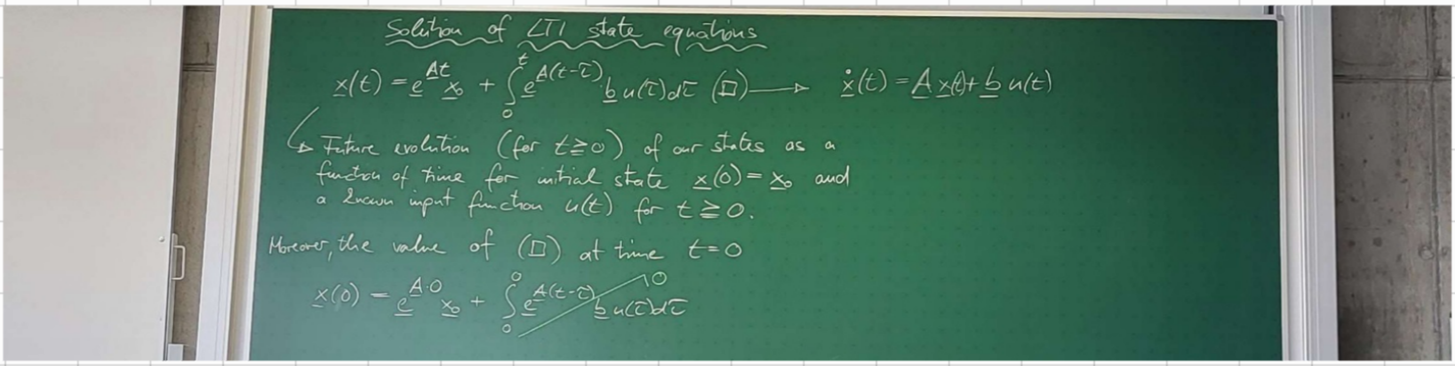
$$\dot{\underline{x}}(t) = \underline{A}\,\underline{e}^{\underline{A}t} \cdot \underline{x}_0 + \frac{d}{dt}\left[\underline{e}^{\underline{A}t} \cdot \int_0^t \underline{e}^{-\underline{A}\tau} \cdot \underline{b} \cdot u(\tau)\, d\tau\right.$$

$$= \underline{A}\,\underline{e}^{\underline{A}t} \cdot \underline{x}_0 + \underline{A}\,\underline{e}^{\underline{A}t} \cdot \int_0^t \underline{e}^{-\underline{A}\tau}\, \underline{b}\, u(\tau)\, d\tau + \underline{e}^{\underline{A}t} \cdot \left(\underline{e}^{-\underline{A}\tau} \cdot \underline{b} \cdot u(\tau)\right)$$

$$= \underline{A}\,\underline{e}^{\underline{A}t} \cdot \underline{x}_0 + \underline{A} \int_0^t \underline{e}^{-\underline{A}(t-\tau)}\, \underline{b}\, u(\tau)\, d\tau + \underbrace{\underline{e}^{\underline{A}\cdot 0}}_{\underline{I}} \cdot \underline{b} \cdot u(t)$$

$$= \underline{A}\left(\underline{e}^{\underline{A}t} \cdot \underline{x}_0 + \int_0^t \underline{e}^{-\underline{A}(t-\tau)}\, \underline{b}\, u(\tau)\, d\tau\right) + \underline{b}\, u(t)$$

$= \underline{A}\; \underline{x}(t) + \underline{b}\, u(t)$ $\Rightarrow$ solution candidate satisfies state differential equation



$\rightsquigarrow \underline{x}(0) = \underline{x}_0$

shows that $(\square)$ also meets the initial condition.
$\Rightarrow$ $(\square)$ solves the initial value problem
Since according to the Picard - Lindelöf theorem only
1 solution exists
$\Rightarrow$ $(\square)$ is the unique solution

3.2 Interpretation of the Solution to the LTI State Equations

Obviously the matrix exponential function $\underline{e}^{\underline{A}t}$ is the most significant form in $(\square)$. For the sake of simplicity, we restrict the following analysis to the case where $\underline{A}$ is diagonalizable (all fundamental results remain valid for the general case).

$\tilde{\underline{M}} = \underline{T}^{-1}\, \underline{M}\, \underline{T}$ with $\det(\underline{T}) \neq 0$

For $\underline{T} = \underline{V} = [\,\underline{v}_1 \;\; \underline{v}_2 \; \dots \; \underline{v}_n\,]$

$$\rightsquigarrow \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \ddots\ \lambda_n \end{bmatrix} = V^{-1} M V$$

↝ Using the matrix of eigenvectors of $\underline{A}$

$\underline{V} = [\underline{v_1}\ \underline{v_2} \dots v_n]$ we can transform

$\underline{A}$ to diagonal form :

$$\underline{V}^{-1} \cdot \underline{A} \cdot \underline{V} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \ddots\ \lambda_n \end{bmatrix} \rightsquigarrow \underline{A} = \underline{V} \cdot \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \ddots\ \lambda_n \end{bmatrix} \cdot \underline{V}^{-1}$$

↝ The matrix exponential function $\underline{e}^{\underline{A}t}$ can be written as

$$\underline{e}^{\underline{A}t} = \underline{e}^{\underline{V} \cdot \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \ddots\ \lambda_n \end{bmatrix} \cdot \underline{V}^{-1} t} = e^{\underline{V} \cdot \begin{bmatrix} \lambda_1 t & 0 & 0 \\ 0 & \lambda_2 t & 0 \\ 0 & 0 & \ddots\ \lambda_n t \end{bmatrix} \cdot \underline{V}^{-1}} \qquad \left[\underline{e}^{\underline{T} \cdot \underline{M} \cdot \underline{T}^{-1}} = \underline{T} \cdot \underline{e}^{\underline{M}} \cdot \underline{T}^{-1} \right]$$

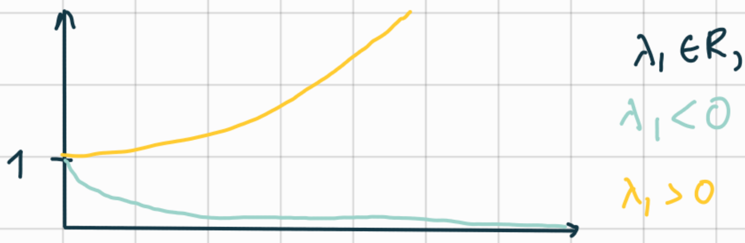
↝

$$\underline{e}^{\underline{A}t} = \underline{V} \cdot \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & \ddots\ e^{\lambda_n t} \end{bmatrix} \cdot \underline{V}^{-1}$$

⇒ obviously, the scalar functions of the form $e^{\lambda_i t}$ (where λ_i : eigenvalue of $\underline{A}$) play a central role in the solution of the state equations. In particular, the system response to a non-zero initial state $\underline{x}_0 = \underline{x}(0) \neq \underline{0}$ and zero excitation $u(t) = 0$ for all $t \geq 0$ (= the solution of the homogenous state differential equation $\dot{\underline{x}} = \underline{A}\,\underline{x}$) is

$$\underline{x}_h(t) = \underline{e}^{\underline{A}t} \cdot \underline{x_0} = \underline{V} \cdot \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & \ddots\ e^{\lambda_n t} \end{bmatrix} \cdot \underline{V}^{-1} \cdot \underline{x_0}$$

$$x_{h,1}(t) = c_{11} \cdot e^{\lambda_1 t} + c_{12} \cdot e^{\lambda_2 t} + \dots + c_{1n} e^{\lambda_n t}$$
$$\vdots$$
$$x_{h,m}(t) = c_{m1} \cdot e^{\lambda_1 t} + c_{m2} \cdot e^{\lambda_2 t} + \dots c_{mn} e^{\lambda_n t}$$



$\lambda_1, \lambda_2 = \lambda_1^*$

$\leadsto \lambda_1 = a + jb, \lambda_2 = a - jb$

$\leadsto x_{1,h}(t) = \ldots (c+jd) \cdot e^{(a+jb)t} + (c-jd) \cdot e^{(a-jb)t}$

$= \ldots e^{at} ((c+jd) e^{jbt} + (c-jd) e^{-jbt})$

$= \ldots e^{at} (c(e^{jbt} + e^{-jbt}) + j\cancel{d}(e^{jbt} - e^{-jbt}))$

$= \ldots e^{at} (2c \cos(bt) - 2d \sin(bt))$

$\lambda = a \pm jb$

State space controller is more powerful than PI and PID, allows changing of all eigenvalues instead of only limited parameters.

Clearly, if $Re\{\lambda_i\} < 0$ for <u>all</u> $i = 1, 2, \ldots, n$ it follows that $\lim_{t \to \infty} e^{\lambda_i t} = 0$ for all $i = 1, 2, \ldots, n$.

$\Rightarrow \lim_{t \to \infty} \underline{x}_h(t) = \underline{V} \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & & 0 \end{bmatrix} \underline{V}^{-1} \underline{x}_0 = \underline{0}$ for any arbitrary initial state x_0

However, if any one eigenvalue has a positive real part, i.e. if $Re\{\lambda_i\} > 0$ for atleast one i then

$\lim_{t \to \infty} e^{\lambda_i t} \to \infty$ for this i

In that case, there surely exists initial conditions x_0 such that

$\lim_{t \to \infty} |\underline{x}_{j,h}(t)| \longrightarrow \infty$

for atleast one j, i.e. atleast one state exceeds all bounds with rising time t

i and j different because eigen movement might not correspond to the same state

Moreover, it can be shown that the second term in the general solution formula ($\square$)

$\int_0^t \underline{e}^{\underline{A}(t-\tau)} \underline{b}\, u(\tau)\, d\tau$ remains bounded for all bounded inputs $u(t)$

if $Re\{\lambda_i\} < 0 \ \forall \ i \in N$

$$\left.\begin{array}{l} \text{i.e. } Re\{\lambda_i\} < 0 \ \forall i \in N \\ \& \ |u(t)| < \infty \ \forall t \geq 0 \end{array}\right\} \Rightarrow \left| \int_0^t \underline{e}^{\underline{A}(t-\tau)} \underline{b}\, u(\tau)\, d\tau \right| < \infty \ \forall \ t \geq 0$$

If A is not diagonalizable then convert matrix to Jordan Form
e^{λ} matrix complicated, will also have $te^{\lambda_i t}$, $t^2 e^{\lambda_i t}$, etc.

As all these main results remain valid if $\underline{A}$ is not diagonalizable, they give rise to the following stability definition (In Aux Sheet 12)

3.3 Asymptotic Stability of LTI Systems (Aux. Sheet 12)

Example: Asymptotic stability & solution of the state equations

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}}_{\underline{A}} \underline{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\underline{b}} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{c^T} \underline{x}$$

- Asymptotic stability:
  - As $\underline{A}$ is a triangular matrix $\Rightarrow \lambda_1 = -1,\ \lambda_2 = -2$
  - Verification:

$$\det(\lambda \underline{I} - A) \overset{!}{=} 0 \rightarrow \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda+1 & 1 \\ 0 & \lambda+2 \end{bmatrix}\right) = (\lambda+1)(\lambda+2) \overset{!}{=} 0, \ \lambda_1 = -1, \lambda_2 = -2$$

Check if eigenvalues can also be calculated by $\det(A - I\lambda) \overset{!}{=} 0$

$\Rightarrow$ As both eigenvalues have negative real parts
$\rightsquigarrow$ The system is asymptotically stable

- We want to find the future evolution of the state vector $\underline{x}(t)$ for $t \geq 0$ for the initial condition $\underline{x}(0) = \underline{x_0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and for the step input $u(t) = \varepsilon(t - 1\,\text{sec})$

$$\rightsquigarrow \underline{x}(t) = \underbrace{\underline{e}^{\underline{A}t} x_0}_{x_h(t)} + \underbrace{\int_0^t \underline{e}^{\underline{A}(t-\tau)} \underline{b}\, u(\tau)\, d\tau}_{x_{ih}(t)}$$

A matrix is diagonalizable if all its eigenvalues are unique

(1) Homogenous solution $\underline{x}_h(t)$

$$\text{Using } \underline{A} = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \underset{\uparrow}{=} \underbrace{\begin{array}{c} v_1 \ v_2 \\ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}}_{\underline{V}} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_{\underline{V}^{-1}}$$

See linear algebra chapter

We get

$$\underline{A}t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -t & 0 \\ 0 & -2t \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\underline{e}^{\underline{A}t} = \underline{e}^{\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}-t&0\\0&-2t\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}} = \begin{bmatrix}1&1\\0&1\end{bmatrix}\cdot \underline{e}^{\begin{bmatrix}-t&0\\0&-2t\end{bmatrix}}\begin{bmatrix}1&-1\\0&1\end{bmatrix} = \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}e^{-t}&0\\0&e^{-2t}\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}$$

$$= \begin{bmatrix}e^{-t}&e^{-2t}\\0&e^{-2t}\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix} = \begin{bmatrix}e^{-t}&e^{-2t}-e^{-t}\\0&e^{-2t}\end{bmatrix}$$

$$\rightsquigarrow \underline{x}_h(t) = \underline{e}^{\underline{A}t}\, x_0 = \begin{bmatrix}e^{-t}&e^{-2t}-e^{-t}\\0&e^{-2t}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}e^{-2t}-e^{-t}\\e^{-2t}\end{bmatrix}$$

$$\underline{e}^{\underline{T}\underline{M}\underline{T}^{-1}} = \underline{T}\underline{e}^{\underline{M}}\underline{T}^{-1}$$

$$e^{A(t-\tau)} = e^{At-A\tau} = e^{At}\cdot e^{-A\tau}$$

(2) System response to input signal

$$\underline{x_{in}}(t) = \int_0^t \underline{e}^{\underline{A}(t-\tau)}\, \underline{b}\, u(\tau)\, d\tau$$

$$= \underline{e}^{\underline{A}t}\cdot \int_0^t \underline{e}^{-A\tau}\begin{bmatrix}1\\0\end{bmatrix}\varepsilon(\tau-1\,\text{sec})\, d\tau$$

$$\rightsquigarrow \underline{x_{in}}(t) = \underline{e}^{\underline{A}t}\cdot\left[\int_0^1 \underline{e}^{-\underline{A}\tau}\begin{bmatrix}1\\0\end{bmatrix}\cdot 0\, d\tau + \int_1^t \underline{e}^{-\underline{A}\tau}\cdot\begin{bmatrix}1\\0\end{bmatrix}\cdot 1\, d\tau\right] \text{ for } t \geq 1\,\text{sec}$$

$$= \underline{\varepsilon(t-1\,\text{sec})}\, \underline{e}^{\underline{A}t}\cdot\int_1^t \underline{e}^{-\underline{A}\tau}\cdot\begin{bmatrix}1\\0\end{bmatrix} d\tau \quad \underline{\text{for } t \geq 0}$$

$$= \varepsilon(t-1\,\text{sec})\, \underline{e}^{\underline{A}t}\cdot\int_1^t \begin{bmatrix}e^{\tau}\\0\end{bmatrix} d\tau$$

$$= \varepsilon(t-1\,\text{sec})\begin{bmatrix}e^{-t}&e^{-2t}-e^{-t}\\0&e^{-2t}\end{bmatrix}\begin{bmatrix}e^{t}-e\\0\end{bmatrix}$$

$$= \varepsilon(t-1\,\text{sec})\begin{bmatrix}e^{-t}(e^{t}-e)\\0\end{bmatrix}$$

$$= \varepsilon(t-1\,\text{sec})\begin{bmatrix}1-e^{1-t}\\0\end{bmatrix} \quad \text{for } t \geq 0$$

One eigen movement and step input can be seen here

$\Rightarrow$ The total solution is

$$\underline{x}(t) = \underline{x_h}(t) + \underline{x_{in}}(t)$$

$$= \begin{bmatrix}e^{-2t}-e^{-t}\\e^{-2t}\end{bmatrix} + \varepsilon(t-1\,\text{sec})\begin{bmatrix}1-e^{1-t}\\0\end{bmatrix} \quad \text{for } t \geq 0$$

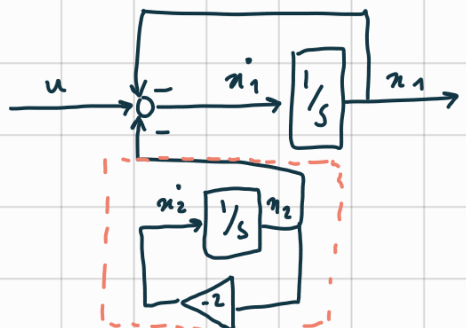
$\rightsquigarrow$ See Aux. Sheet 13 for a plot of the results

3.4 Controllability

Motivation: Consider the example LTI system from the last section

$$\underline{\dot{x}} = \begin{bmatrix}-1&-1\\0&-2\end{bmatrix}\underline{x} + \begin{bmatrix}1\\0\end{bmatrix}u$$

(The output equation is of no interest when discussing controllability)



Obviously, the input u cannot influence the state x_2. This is also visible in the solution of the state equations illustrated on Aux. Sheet 13. We speak about a system, which is <u>not (completely) controllable</u>.

General Definition: Controllability

The LTI System with the state differential equation

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}$$

is called <u>(completely) controllable</u> iff for any arbitrary initial state $\underline{x}_0 = \underline{x}(0)$ there exists an input function $\underline{u}(t)$ which drives the system to the final state $\underline{0}$ in finite time

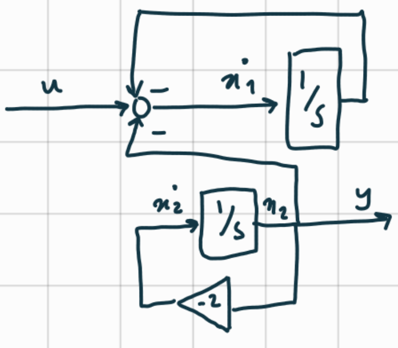
Note:

1) The above definition implies that a completely controllable LTI System can be driven from any arbitrary state to any desired state in any non-zero time.
2) Iff a system is completely controllable, we can find a suitable feedback controller which allows to freely adjust <u>all</u> eigenvalues of the closed-loop system.

→ Controllability criterion for LTI SISO systems (see Aux-Sheet 14)

3.5 Observability

Motivation:



$$\dot{\underline{x}} = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x}$$

Obviously, the output y does not depend on x_1. Thus, even if we know the input signal $u(t)$ we will never be able to determine the value of $x_1(t)$ if the initial value $x_{1,0} = x_1(0)$ is unknown.

We speak about a system, which is <u>not (completely) observable</u>.

General Definition: Observability

The LTI System

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}$$
$$y = \underline{C}\,\underline{x} + \underline{D}\,\underline{u}$$

is called completely observable, iff any initial state $\underline{x}_0 = \underline{x}(0)$ can be uniquely determined for known input $\underline{u}(t)$ $(t \geq 0)$ by measuring $\underline{y}(t)$ over a finite time period.

Note:

Iff a system is (completely) observable, then we can design an algorithm which is estimating all state variables and where the error dynamics between the estimated and the actual states can be freely adjusted.

Observability criterion for LTI SISO systems

$\rightsquigarrow$ See Aux. Sheet 14

3.6 Relationship Between State Equations and Input-output (I/O) Transfer Function for LTI SISO Systems

Apply Laplace Transform to

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}u$$
$$y = \underline{c}^T\underline{x} + du$$

$$\rightsquigarrow \quad s\underline{X}(s) - \underline{x}(0) = \underline{A}\,\underline{X}(s) + \underline{b}\,U(s) \qquad (*)$$
$$Y(s) = \underline{c}^T\underline{X}(s) + d\,U(s) \qquad (**)$$

We only want to consider the input-output behaviour

$\rightsquigarrow$ $\underline{x}_0 = \underline{x}(0) = \underline{0}$

$(*) \rightsquigarrow s\,\underline{X}(s) = \underline{A}\,\underline{X}(s) + \underline{b}\,U(s)$

$\Leftrightarrow (s\,\underline{I} - \underline{A})\,\underline{X}(s) = \underline{b}\,U(s)$

Solving for $\underline{X}(s)$:

$$\underline{X}(s) = (s\underline{I} - \underline{A})^{-1}\,\underline{b}\,U(s)$$

Substituting into $(**)$:

$$Y(s) = [\underline{c}^T (s\underline{I} - \underline{A})^{-1}\underline{b} + d]\,U(s)$$

$$\rightarrow \boxed{F(s) = \frac{Y(s)}{U(s)} = \underline{c}^T (s\underline{I} - \underline{A})^{-1}\underline{b} + d}$$

Example 1:

completely observable & controllable system (see Task 18 Ex. Sheet 7)

$$\underline{\dot{x}} = \underbrace{\begin{bmatrix} -1 & 5 \\ 7 & -3 \end{bmatrix}}_{\underline{A}} \underline{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\underline{b}} u$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\underline{c}^T} \underline{x}$$

$$\rightarrow F(s) = c^T (s\underline{I} - \underline{A})^{-1}\underline{b}$$

$$= [0 \;\; 1] \begin{bmatrix} s+1 & -5 \\ -7 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

· Dots because they will be multiplied by 0

$$= [0 \;\; 1] \cdot \frac{1}{(s+1)(s+3)-35} \cdot \begin{bmatrix} \cdot & \cdot \\ 7 & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{7}{s^2+4s-32} = \frac{7}{(s-4)(s+8)}$$

We observe:

Both eigenvalues of $\underline{A}$ ($\lambda_1 = 4$, $\lambda_2 = -8$) are also poles of $F(s)$

Example 2:

Input-Output transfer function of the system

$$\underline{\dot{x}} = \underbrace{\begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}}_{\underline{A}} \underline{x} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\underline{b}} u, \quad y = \underbrace{[0 \;\; 1]}_{\underline{c}^T} \underline{x}$$

which is not completely observable (see Section 3.5)

$$\rightsquigarrow F(s) = \underline{c}^T (s\underline{I} - \underline{A})^{-1}\underline{b} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \frac{1}{(s+1)(s+2)} \cdot \begin{bmatrix} \cdot & \cdot \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}$$

We observe:

The eigenvalue $\lambda_1 = -1$ is not a pole of $F(s)$ and the order of $F(s)$ is smaller than the order of the state space system. The reason is that the eigenvalue $\lambda_1 = -1$ (which is not visible at the output y, see block diagram in Section 3.5) is cancelled by a zero in $F(s)$.

The general behaviour between the eigenvalues of $\underline{A}$ and the poles of $F(s)$ depending on controllability / observability of a system is summarized on $\rightsquigarrow$ Aux. sheet 15

4) State-space Controller Design

Fundamental objectives which are usually addressed by control engineering schemes:

- Compensation of initial errors (including compensation of shortly acting disturbances & stabilization)
- Enforce a desired output value or output signal
- Compensation of permanently acting disturbances

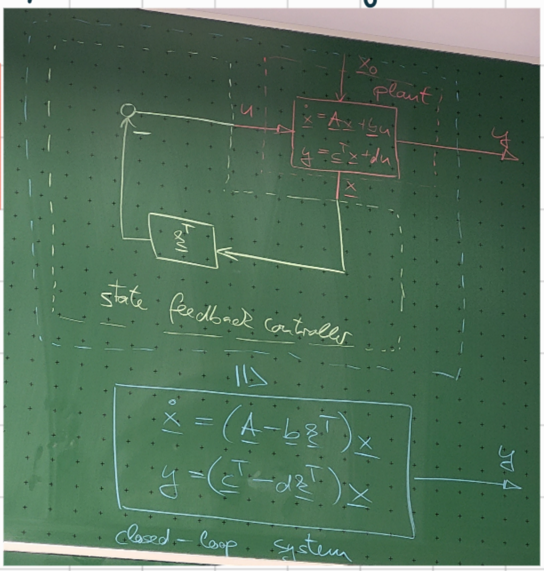
4.1) (Complete) Linear State Feedback

4.1.1) Structure of Control Law

First, we focus on compensating arbitrary initial errors $\underline{x}_0 = \underline{x}(0) \neq \underline{0}$ of the system

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}u \qquad (*)$$
$$y = \underline{c}^T\underline{x} + du \qquad (**)$$

- Compensation of arbitrary unknown errors
  $\Rightarrow$ feedback is necessary
- All information about the current condition of the system is contained in state vector $\underline{x}$
  $\Rightarrow$ state feedback $u = g(\underline{x})$, where $g(\cdot)$ is some nonlinear function
- In order to stay in the LTI system domain
  $\Rightarrow$ approach: <u>linear</u> state feedback
  $$u = -\underline{k}^T\underline{x} = -k_1x_1 - k_2x_2 - \dots - k_nx_n \quad (\square)$$
  with constant gain vector $\underline{k}^T = [k_1\ k_2 \dots k_n]$
- Substituting the control law $(\square)$ into the state equations $(*)$ - $(**)$ yields the closed-loop system
  $$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}\,(-\underline{k}^T\underline{x}) \rightarrow \boxed{\dot{\underline{x}} = (\underline{A} - \underline{b}\,\underline{k}^T)\underline{x}}$$
  $$y = \underline{c}^T\underline{x} + d\,(-\underline{k}^T\underline{x}) \qquad \boxed{y = (\underline{c}^T - d\,\underline{k}^T)\underline{x}}$$



$\rightarrow$ Applying $u = -k^T x$ yields a linear homogenous closed-loop system with the dynamic matrix
$$\underline{\tilde{A}} = \underline{A} - \underline{b}\,\underline{k}^T$$
$\Rightarrow$ Compensation of initial errors is achieved if closed-loop system is asymptotically stable, since then
$$\lim_{t \to \infty} \underline{x}(t) = \underline{0}$$

If and how fast initial errors will decay is determined by the eigenvalues of the <u>closed-loop system matrix</u>:
$$\underline{\tilde{A}} = \underline{A} - \underline{b}\cdot\underline{k}^T$$

4.1.2 Pole Placement Design (Should be called Eigenvalue Placement Design)

We want to find $\underline{k}^T$ such that applying
$u = -\underline{k}^T \cdot \underline{x}$
to (*) - (**) results in a closed-loop system with desired eigenvalues $\lambda_{c,1}, \lambda_{c,2} \dots \lambda_{c,n}$.

Design equation:
$$\det(\lambda \underline{I} - \tilde{\underline{A}}) \overset{!}{=} \prod_{c=1}^{n} (\lambda - \lambda_{c,i})$$
$$\rightarrow \boxed{\det(\lambda \underline{I} - \underline{A} + \underline{b}\,\underline{k}^T) \overset{!}{=} (\lambda - \lambda_{c,1})(\lambda - \lambda_{c,2}) \dots (\lambda - \lambda_{c,n})}$$

Solving the design equations & summary of design procedure
$\rightarrow$ See Aux. Sheet 16

Example: Design of state feedback controller for
$$\dot{\underline{x}} = \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}}_{\underline{A}} \underline{x} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\underline{b}} u$$
such that the closed-loop eigenvalues $\lambda_{c,1} = -3 = \lambda_{c,2}$ results.

- Check for controllability:
$$\underline{Q}_c = [\underline{b} \quad \underline{A}\cdot\underline{b}] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \rightarrow \det(\underline{Q}_c) = 1 \neq 0 \rightarrow \text{completely controllable}$$

- Design equation
$$\det(\lambda \underline{I} - \underline{A} + \underline{b}\,\underline{k}^T) \overset{!}{=} (\lambda+3)^2 \text{ with } \underline{k}^T = [k_1 \quad k_2]$$
$$\underline{b}\,\underline{k}^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}[k_1 \quad k_2] = \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix}$$
$$\rightarrow \det\left(\begin{bmatrix} \lambda+1 & 0 \\ -1 & \lambda-1 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} \lambda+1+k_1 & k_2 \\ -1 & \lambda-1 \end{bmatrix}\right)$$
$$= (\lambda+1+k_1)(\lambda-1) + k_2 = \lambda^2 + k_1\lambda - 1 - k_1 + k_2 \overset{!}{=} \lambda^2 + 6\lambda + 9$$
$$\Rightarrow k_1\lambda = 6\lambda \Rightarrow k_1 = 6, \quad -k_1 + k_2 - 1 = 9 \Rightarrow k_2 = 9 + 1 + k_1 = 16$$

⇒ Control law:

$$u = -\underline{k}^T \cdot \underline{x} = -[6 \quad 16]\,\underline{x} = -6x_1 - 16x_2$$

4.2 Luenberger Observer = Kalman Filter ???

Luenberger observer with special parameterization of $\underline{l}$ to minimize noise

Issue: Usually we can't measure all state variables

→ state feedback can't be implemented

Remedy: State observer that estimates the states based on u and y

We consider LTI SISO systems with equations (no feedthrough)

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{b}\,u$$
$$y = \underline{c}^T \underline{x}$$

Note: If the original plant has a non-zero feedthrough d

⇒ Construct a measurement signal without feedthrough by using

$$\tilde{y} = y - d \cdot u$$

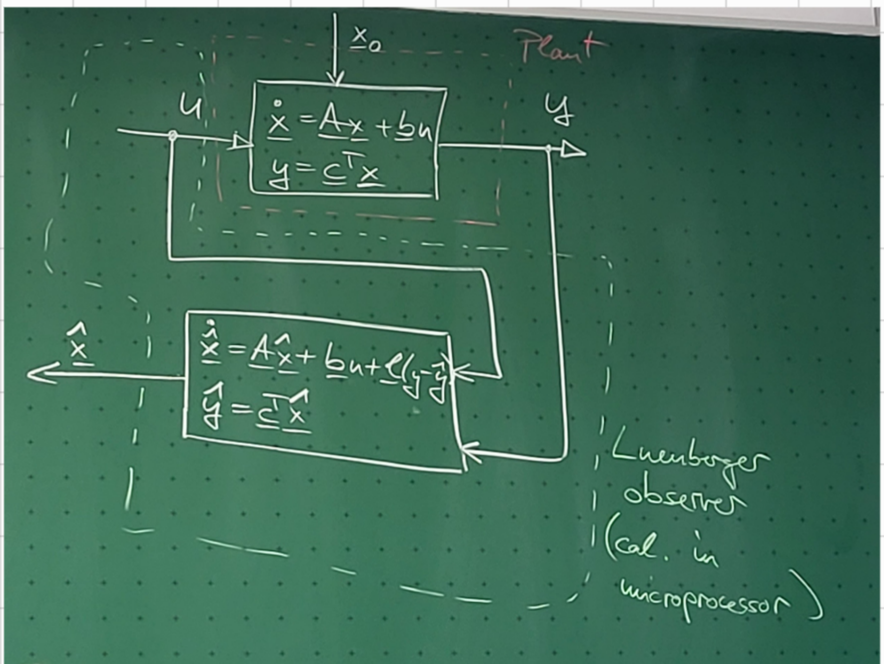
instead of y

⇒ Observer design can be accomplished with respect to $\tilde{y}$, since

$$\tilde{y} = y - d \cdot u = \underline{c}^T x + du - du = \underline{c}^T x$$

Observer algorithm:

$$\underbrace{\dot{\hat{\underline{x}}} = \underline{A}\,\hat{\underline{x}} + \underline{b}u}_{\text{(Simulated) Plant Model}} \underbrace{+ \underset{(n \times 1)}{\underline{l}}\,(y - \hat{y})}_{\text{Correction of Plant Model}}$$
$$\hat{y} = \underline{c}^T \hat{\underline{x}}$$



The correction term $\underline{l}(y-\hat{y})$ must ensure that errors between the true states $\underline{x}$ and the estimated states $\hat{\underline{x}}$ decay with time. In order to see how errors evolve over time, we consider the differential equation for the so-called estimation error

$$\underline{e} = \underline{x} - \hat{\underline{x}}:$$

$$\begin{aligned}\dot{\underline{e}} = \dot{\underline{x}} - \dot{\hat{\underline{x}}} &= \underline{A}\underline{x} + \underline{b}u - \underline{A}\hat{\underline{x}} - \underline{b}u - \underline{l}(y-\hat{y}) \\ &= \underline{A}(\underline{x}-\hat{\underline{x}}) - \underline{l}(\underline{c}^T\underline{x} - \underline{c}^T\hat{\underline{x}}) \\ &= (\underline{A} - \underline{l}\,\underline{c}^T)\underbrace{(\underline{x}-\hat{\underline{x}})}_{\underline{e}}\end{aligned}$$

$$\Rightarrow \quad \dot{\underline{e}} = (\underline{A} - \underline{l}\,\underline{c}^T)\underline{e}$$

$\Rightarrow$ Linear homogenous equation for the estimation error $\underline{e} = \underline{x} - \hat{\underline{x}}$ with the system matrix

$$\underline{F} = \underline{A} - \underline{l}\,\underline{c}^T$$

Obvious idea: Choose parameter vector $\underline{l}$ such that $\underline{F}$ has desired eigenvalues $\lambda_{0,1}, \lambda_{0,2} \dots \lambda_{0,n}$

The design procedure for calculating $\underline{l}$ is similar to determining $\underline{k}$ for a state feedback controller

$\rightarrow$ See Aux. Sheet 17