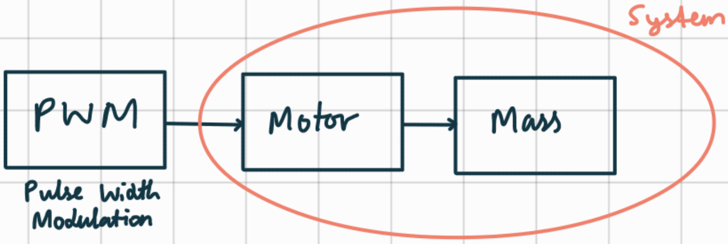
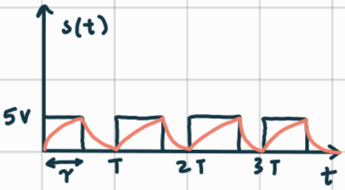


Low Pass Filter

Mass reacts slowly to acceleration



τ: Pulse Duration

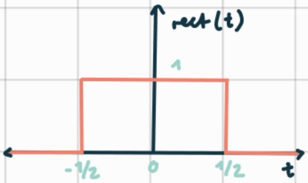
Duty cycle: $DC = \frac{\tau}{T}$

How to describe the PWM signal:

Elementary Signals:

Rectangular Function

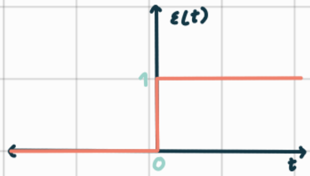
$$\text{rect}(t) = \begin{cases} 1, & \text{for } |t| \leq 0.5 \\ 0, & \text{else} \end{cases}$$



Models Switching Process

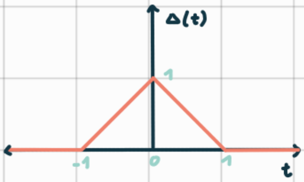
Step Function

$$\varepsilon(t) = \begin{cases} 1, & \text{for } t \geq 0 \\ 0, & \text{else} \end{cases}$$



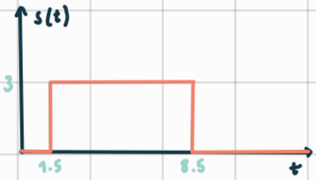
Triangular Function

$$\Delta(t) = \begin{cases} 1 - |t|, & \text{for } |t| \leq 1 \\ 0, & \text{else} \end{cases}$$



Amplitude, Shift, Expansion

$$s(t) = 3 \cdot \text{rect}\left(\frac{t-5}{7}\right)$$



LB: $\frac{t-5}{7} = -\frac{1}{2} \Rightarrow t = 1.5$, UB: $\frac{t-5}{7} = \frac{1}{2} \Rightarrow t = 8.5$

video

Convolution

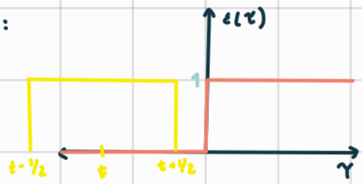
$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

$x(t) = \varepsilon(t)$ switch on at $t=0$ $\quad h(t) = \text{rect}(t)$ short time integrator / low pass

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} \varepsilon(\tau) \cdot \text{rect}(t-\tau)\, d\tau$$

Case 1:



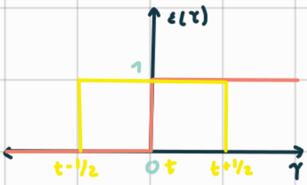
Upper Boundary
$t - \tau = -1/2$
$-\tau = -1/2 - t$
$\tau = t + 1/2$

Lower Boundary
$t - \tau = 1/2$
$\tau = t - 1/2$

$t + 1/2 < 0 \Rightarrow t < -1/2$: $y(t) = 0$ because atleast 1 signal = 0
or: Both signals don't overlap

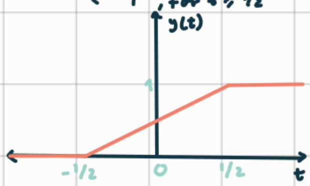
$$y(t) = \begin{cases} 0, & \text{for } t \leq -1/2 \\ t + 1/2, & \text{else} \\ 1, & \text{for } t \geq 1/2 \end{cases}$$

Case 2:

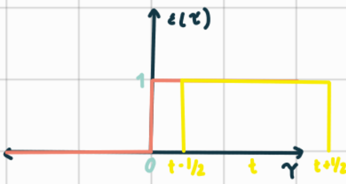


$\left.\begin{array}{l} t + 1/2 \geq 0 \iff t \geq -1/2 \\ t - 1/2 < 0 \iff t < 1/2 \end{array}\right\} -1/2 \leq t < 1/2$

$$y(t) = \int_{-\infty}^{\infty} \varepsilon(\tau) \cdot \text{rect}(t-\tau)\, d\tau$$
$$= \int_{0}^{t+1/2} 1 \cdot 1\, d\tau = [\tau]_{0}^{t+1/2} = t + 1/2$$



Case 3:



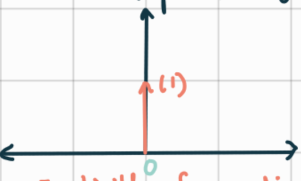
$t - 1/2 \geq 0 \Rightarrow t \geq 1/2$

$$y(t) = \int_{-\infty}^{\infty} \varepsilon(\tau) \cdot \text{rect}(t-\tau)\, d\tau$$
$$= \int_{t-1/2}^{t+1/2} 1 \cdot 1\, d\tau = [\tau]_{t-1/2}^{t+1/2} = t + 1/2 - t + 1/2 = 1$$

Dirac Impulse δ

weight of one (1) aka not amplitude

shift, weighten



Explicitly for continuous

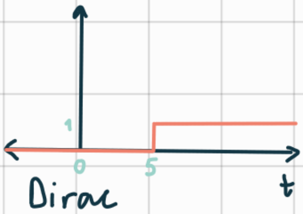
Neutral element of convolution

$$y(t) = x(t) * \delta(t) = x(t)$$

Convolution with shifted Dirac

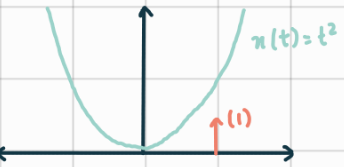
$$y(t) = x(t) * \delta(t-5) = x(t-5)$$

Ex: $\varepsilon(t) * \delta(t-5)$



Multiplication with Dirac

$x(t) = t^2$, $y(t) = x(t) \cdot \delta(t-5)$



$$= x(5) \cdot \delta(t-5)$$

$$= 25 \cdot \delta(t-5)$$

Used for AD Conversion

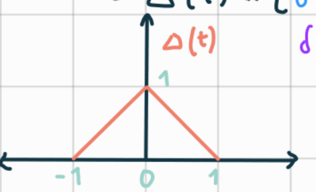
$$y(t) = h(t) * \varepsilon(t)$$

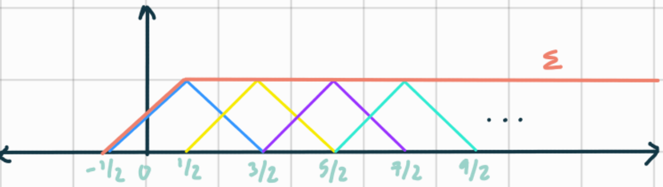
$$= \underline{\text{rect}(t) * \text{rect}(t)} * \sum_{n=0}^{\infty} \delta\left(t - \frac{2n+1}{2}\right)$$

$\text{rect}(t) * \text{rect}(t) = T \cdot \Delta\left(\frac{t}{T}\right)$, where $T = 1$

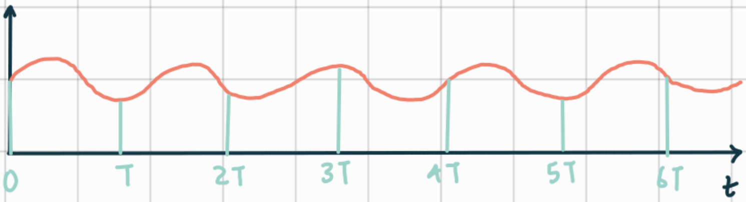
$$y(t) = \Delta(t) * \sum_{n=0}^{\infty} \delta\left(t - \frac{2n+1}{2}\right)$$

$$= \Delta(t) * \left[\delta(t - 1/2) + \delta(t - 3/2) + \delta(t - 5/2) + \delta(t - 7/2) \ldots\right]$$

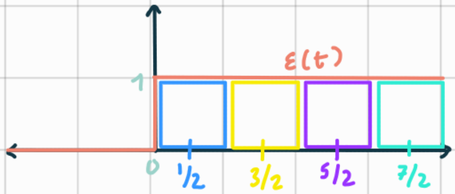




Sampling



Old Example



$$\varepsilon(t) = \text{rect}(t - 1/2) + \text{rect}(t - 3/2) + \text{rect}(t - 5/2) + \text{rect}(t - 7/2) \ldots$$

$$= \text{rect}(t) * \delta(t - 1/2) + \text{rect}(t) * \delta(t - 3/2) + \text{rect}(t) * \delta(5/2) + \text{rect}(t) * \delta(t - 7/2) \ldots$$

Distributive law of convolution

$$\varepsilon(t) = \text{rect}(t) * \left[\delta(t - 1/2) + \delta(t - 3/2) + \delta(t - 5/2) \ldots\right]$$

$$= \text{rect}(t) * \sum_{n=0}^{\infty} \delta\left(t - \frac{2n+1}{2}\right)$$

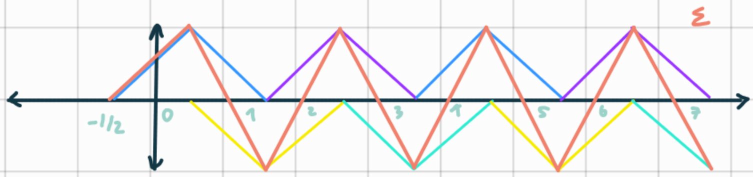
$$y(t) = h(t) * x(t), \quad h(t) = \text{rect}(t)$$

$$x(t) = \text{rect}(t) * \delta(t - 1/2) - \text{rect}(t) * \delta(t - 3/2) \ldots$$

$$= \text{rect}(t) * \sum_{n=0}^{\infty} \delta\left(t - \frac{2n+1}{2}\right)(-1)^n$$

$$y(t) = \text{rect}(t) * \text{rect}(t) * \sum_{n=0}^{\infty} \delta\left(t - \frac{2n+1}{2}\right)(-1)^n$$

$$= \Delta(t) * \sum_{n=0}^{\infty} \left(t - \frac{2n+1}{2}\right)(-1)^n$$



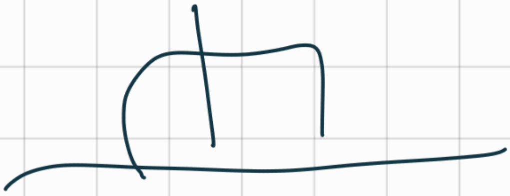
$n=0$: $\Delta(t)\ \delta(t - 1/2)(-1)^0$, $n=1$: $\Delta(t)\ \delta(t - 3/2)(-1)^1$

$n=2$: $\Delta(t)\ \delta(t - 5/2)(-1)^2$, $n=3$: $\Delta(t)\ \delta(t - 7/2)(-1)^3$

Similarity Theorem of the Fourier Transform

Ex: $x(t) = 2 \cdot si(3\pi t)$ | $si(\pi t) = sinc(t)$

$si(\pi t) = \frac{\sin(\pi t)}{\pi t}$, $si(0) = 1$



Sketch the spectrum by magnitude and phase :

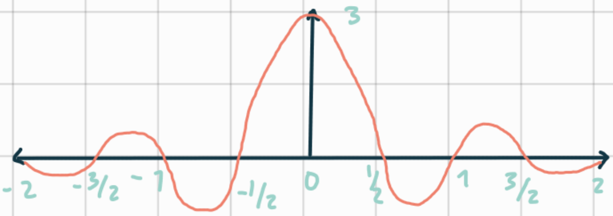
Magnitude: $|X(f)| = \sqrt{Re(X(f))^2 + Im(X(f))^2} = \frac{2}{3}\,rect$?

Phase: $\varphi(f) = \arctan \frac{Im(X(f))}{Re(X(f))} = 0$

Parseval Theorem

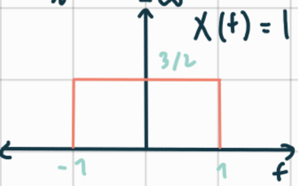
$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$

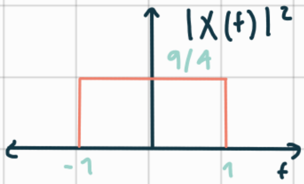
The energy of a signal can be calculated in time and image domain



Ex: $x(t) = 3\,si(2\pi t) \circ\!\!-\!\!\bullet \frac{3}{2}\,rect\left(\frac{f}{2}\right) = X(f)$

$E_x = \int_{-\infty}^{\infty} |3\,si(2\pi t)|^2 dt$ Integration too complicated



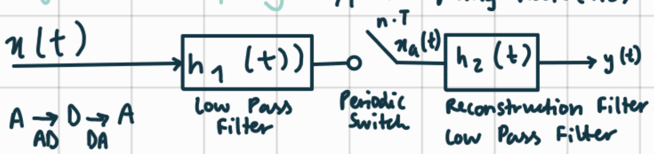


$E_x = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} \frac{9}{4}\,rect\left(\frac{f}{2}\right) df = \int_{-1}^{1} \frac{9}{4} df = \frac{9}{2}$

PDF : Power Density Function

Digital Sampling

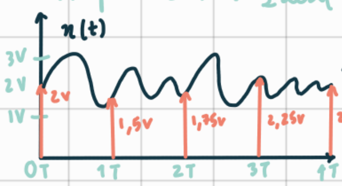
T: Sampling Time (s)
$1/T$: Sampling Rate (Hz)



Simple Case : Ideal Sample

Time Domain : $x_a(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT)$

Frequency Domain: $X_a(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$



$x_a(t) = 2V \cdot \delta(t) + 1{,}5V \cdot \delta(t - T) + 1{,}75V \cdot \delta(t - 2T) \ldots$

Digital signal : $x_a[n] = [2; 1{,}5; 1{,}75 \ldots]$

Alternative Representation in Frequency Domain

$X_a(f) = X(f) * \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$

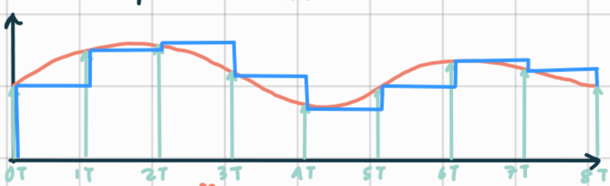
$\circ\!\!-\!\!\bullet\; a \cdot e^{-2\pi i f t_0}$

$X_a(f) = 2 \cdot e^{-2\pi i f \cdot 0} + 1{,}5 \cdot e^{-2\pi i f \cdot T} + 1{,}75 \cdot e^{-2\pi i f \cdot 2T} + \ldots$

$\Rightarrow X_a(f) = \sum_{n=-\infty}^{\infty} x_a(nT) \cdot e^{-2\pi i f \cdot nT}$

Non-ideal Sample : Sample and Hold

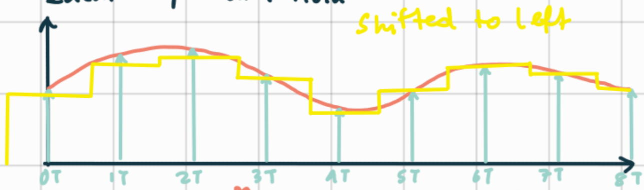
Real Sample and Hold:



$x_a(t) = \underbrace{\left[x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT)\right]}_{\text{Ideal Sample}} * rect\left(\frac{t}{T}\right) * \delta\left(t - \frac{T}{2}\right)$

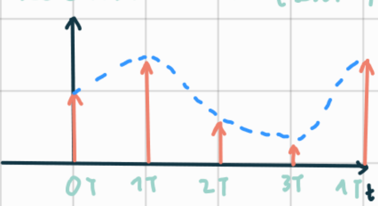
Ideal Sample and Hold:

Shifted to left



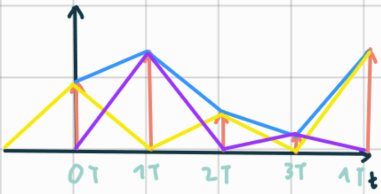
$x_a(t) = \underbrace{\left[x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT)\right]}_{\text{Ideal Sample}} * rect\left(\frac{t}{T}\right) * \delta\left(t - \frac{T}{2}\right)$

Reconstruction (Interpolation)



- We get a continuous signal via the envelope function
- Envelope functions are smooth
- Low Pass Filters form envelope functions

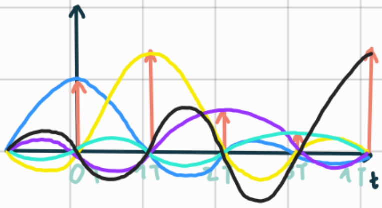
Linear Interpolation



Linear Interpolated Envelope

$$y(t) = x_a(t) * \Delta\left(\frac{t}{T}\right)$$

Ideal Interpolation



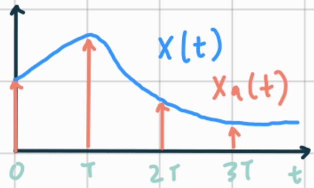
$$y(t) = x_a(t) * si\left(\pi \frac{t}{T}\right)$$

Sum of all si functions leads to the envelope

Frequency Domain

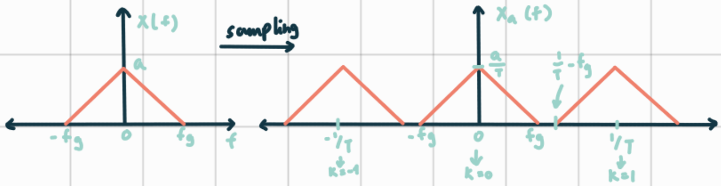
Miss

Domain: $X(t) \xrightarrow{\text{sampling}} X_a(t) = X(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT)$



$$X_a(f) = X(f) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$$

Example



So that the spectra don't overlap

$$\frac{1}{T} - f_g \geq f_g \Rightarrow 2f_g \leq \frac{1}{T}$$

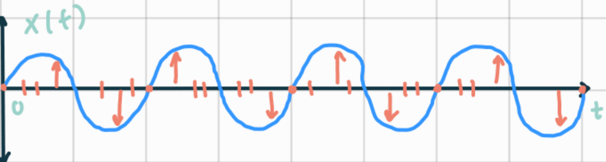
Sampling Theorem (Nyquist Theorem)

$f_g \leq \frac{1}{2T}$ with $f_s = \frac{1}{T}$ ← sample rate

$f_g \leq \frac{1}{2} f_s \mid 2 f_g \leq f_s$

Violation of Sampling Theorem

Sampling Theorem in Time Domain Representation



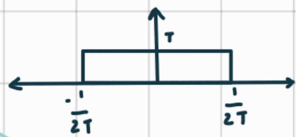
Atleast one sampling per half wave

y(t) Envelope function of analog signal

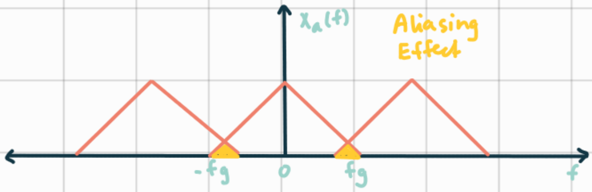
New Example: Reconstruction via si-Function

Sum of all shifted si-functions → signal envelope

We know: $si\left(\pi \frac{t}{T}\right) \circ\!\!-\!\!\bullet\ T \cdot rect(T \cdot f)$

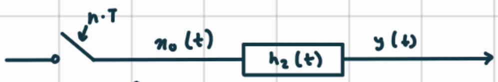


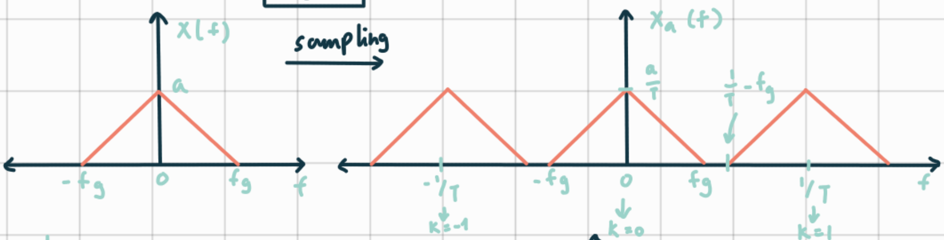
Anti-Example $f_g > \frac{1}{2T}$:



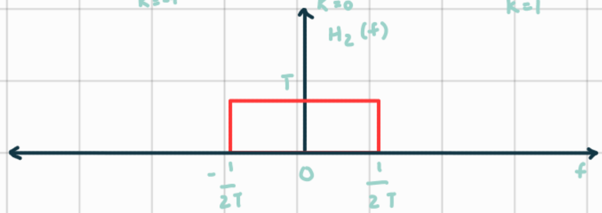
$h_2(t) = si\left(\pi \frac{t}{T}\right)$

Reconstruction in frequency-domain via ideal interpolation



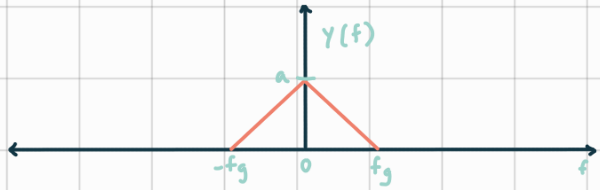


Multiplication



Sampling Theorem

$f_g \leq \frac{1}{2T}$



Discrete Convolution

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(m) \cdot h(n-m)$$

Model $x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$

LTI

≈ LSI → Linear Shift Invariant

Example: $x(n) = \delta(n) + \frac{1}{2}\delta(n-1)$

$h(n) = \sum_{k=0}^{\infty} (\frac{1}{2})^k \cdot \delta(n-k) = (\frac{1}{2})^n \cdot \varepsilon(n)$ ← Time discrete unit step

a) Draw $h(n)$ with specification of all characteristic values

b) Is $h(n)$ stable according to BIBO? Bounded Input Bounded Output

To prove: $\sum_{n=-\infty}^{n=\infty} |h(n)| < \infty$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} (\tfrac{1}{2})^n \cdot \varepsilon(n) = \sum_{n=0}^{\infty} (\tfrac{1}{2})^n = \frac{1}{1-1/2} = 2$$

$2 < \infty$, Hence Proved, $h(n)$ is stable

c) Is $h(n)$ causal?

Yes, because $h(n) = 0$, $n < 0$

Anti-Causal: $h(n) = 0$, $n > 0$

Non-Causal: $h(n) \neq 0$, $n > 0$ and $n < 0$

d) Determine $y(n) = x(n) * h(n) = h(n) * x(n)$

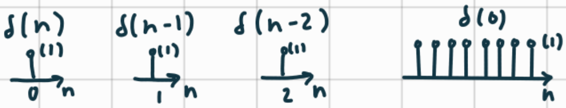
$$y(n) = \left[\sum_{k=0}^{\infty} (\tfrac{1}{2})^k \cdot \delta(n-k)\right] * \left[\delta(n) + \tfrac{1}{2}\delta(n-1)\right]$$

$= h(n) + \frac{1}{2} h(n-1)$

=

$= \delta(n) + (\frac{1}{2})^{n-1} \cdot \varepsilon(n-1) = \delta(n) + h(n-1)$

$= \varepsilon(n) - (\frac{1}{2})^{n-2} \varepsilon(n-2)$



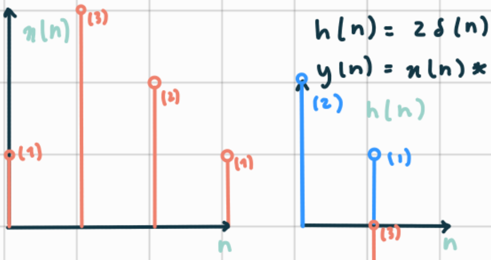
Example:

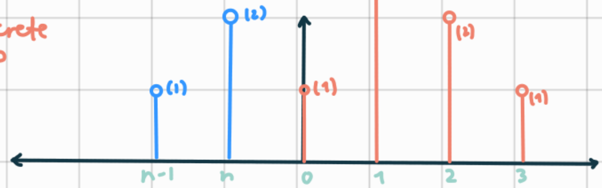
$x(n) = \delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)$

$x = [\ 1 \quad 3 \quad 2 \quad 1\ ]$

$h(n) = 2\delta(n) + \delta(n-1)$

$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m) \cdot h(n-m)$ ← shift + flip





$n < 0$, $y(n) = 0$, $n > 4$, $y(n) = 0$

Move to position n during convolution

$n = 0$, $y(0) = 1 \cdot 2 = 2$

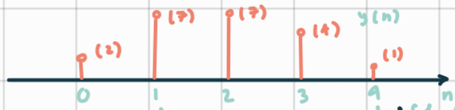
$n = 1$, $y(1) = 1 \cdot 1 + 2 \cdot 3 = 7$

$n = 2$, $y(2) = 1 \cdot 3 + 2 \cdot 2 = 7$

$n = 3$, $y(3) = 1 \cdot 2 + 2 \cdot 1 = 4$

$n = 4$, $y(4) = 1 \cdot 1 = 1$

$y(n) = 2 \cdot \delta(n) + 7\delta(n-1) + 7\delta(n-2) + 4\delta(n-3) + \delta(n-4)$



Convolution via shift & summation

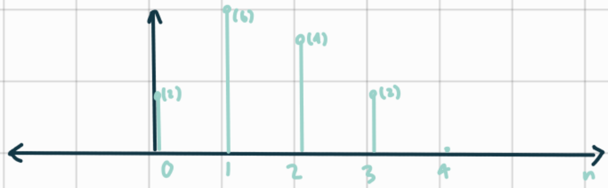
$y(n) = x(n) * h(n)$

$= x(n) * [2\delta(n) + \delta(n-1)]$

Distributive law

$= x(n) * 2\delta(n) + x(n) * \delta(n-1)$

$= 2x(n) + x(n-1)$ Convolution with Delta Impulse equals shift in time



This method is much faster if $x(n)$ or $h(n)$ have few values $\neq 0$

Non-casual Filter

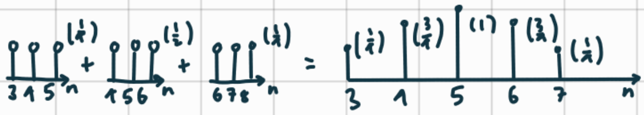
$x(n) = \delta(n-3) + \delta(n-4) + \delta(n-5)$

Low pass filter, casual:

$h(n) = \frac{1}{4}\delta(n) + \frac{1}{2}\delta(n-1) + \frac{1}{4}\delta(n-2)$

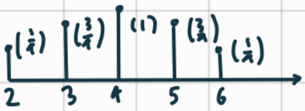
$y(n) = x(n) * h(n)$

$= x(n) * \left[\frac{1}{4}\delta(n) + \frac{1}{2}\delta(n-1) + \frac{1}{4}\delta(n-2)\right]$



Center at 5 not 4, which is why:

Non-casual $h(n) = \frac{1}{4}\delta(n+1) + \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n-1)$



Fourier Transformation for Time Discrete Signals

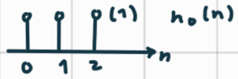
Exercise: $X(k) = \sum_{k=0}^{\infty} x(n) \cdot \underbrace{\cos\left(\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right)}_{\text{Basis function } h_k(n)}$ Discrete Cosine Transformation DCT

a) Derive 3 Filters for $0 \le k < N$, $N = 3$, $0 \le n \le N$

$k=0$, $n=0$: $\cos\left(\frac{\pi}{3}\left(0+\frac{1}{2}\right)\cdot 0\right) = 1$

$n=1$: $\cos\left(\frac{\pi}{3}\left(1+\frac{1}{2}\right)\cdot 0\right) = 1$

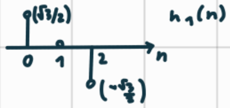
$n=2$: $\cos\left(\frac{\pi}{3}\left(2+\frac{1}{2}\right)\cdot 0\right) = 1$



$k=1$, $n=0$: $\cos\left(\frac{\pi}{3}\left(0+\frac{1}{2}\right)\cdot 1\right) = \sqrt{3}/2$

$n=1$: $\cos\left(\frac{\pi}{3}\left(1+\frac{1}{2}\right)\cdot 1\right) = 0$

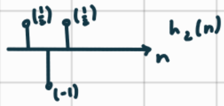
$n=2$: $\cos\left(\frac{\pi}{3}\left(2+\frac{1}{2}\right)\cdot 1\right) = -\frac{\sqrt{3}}{2}$



$k=2$, $n=0$: $\cos\left(\frac{\pi}{3}\left(0+\frac{1}{2}\right)\cdot 2\right) = 1/2$

$n=1$: $\cos\left(\frac{\pi}{3}\left(1+\frac{1}{2}\right)\cdot 2\right) = -1$

$n=2$: $\cos\left(\frac{\pi}{3}\left(2+\frac{1}{2}\right)\cdot 2\right) = 1/2$



Cut off Frequency

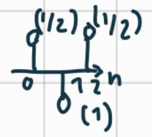
$\sqrt{3}\,\sin(2\pi f T) = \frac{\sqrt{3}}{2}$

$f_{c1} = \frac{1}{2\pi T}\overbrace{\arcsin\left(\frac{1}{\sqrt{2}}\right)}^{\pi/4}$

$f_{c1} = \frac{\pi}{2\pi \cdot 4T} = \frac{1}{8T}$

$f_{c2} = f_1 - f_{c1}$

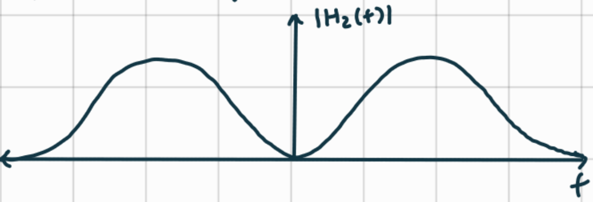
$h_2(n) = \frac{1}{2}\delta(n) - \delta(n-1) + \frac{1}{2}\delta(n-2)$



$H_2(n) = \frac{1}{2}e^{-j2\pi f 0 T} - e^{-j2\pi f T} + \frac{1}{2}e^{-j2\pi f \cdot 2T}$

$= e^{-j2\pi f T}\left(\frac{1}{2}e^{j2\pi f T} - 1 + \frac{1}{2}e^{-j2\pi f T}\right)$

$|H_2(n)| = \left|\frac{1}{2}\cdot 2\cdot\cos(2\pi f T) - 1\right|$

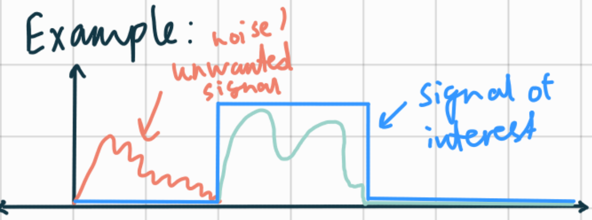


Bandpass

FIR Filter Design

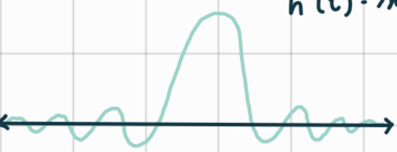
(Finite Impulse Response)

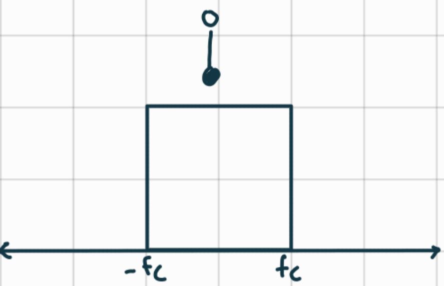
Example:



We know:

$h(t) = si(ft)$





$H(f) = rect\left(\frac{f}{2f_c}\right)$

$si(\pi t)$ o—● $rect(f)$

similarity theorem $x(bt)$ o—● $\frac{1}{|b|}\cdot X\left(\frac{f}{b}\right)$

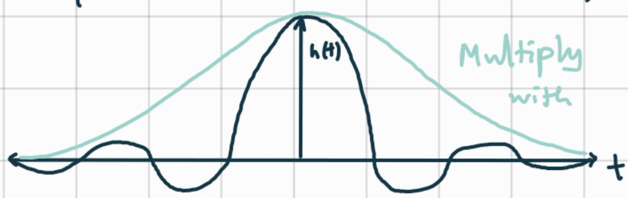
here $b = 2\cdot f_c$

Problems:

- Steep slopes (infinitely high)
  → infinite circuit effort
- Impulse response is infinite
- Not casual

Solutions:

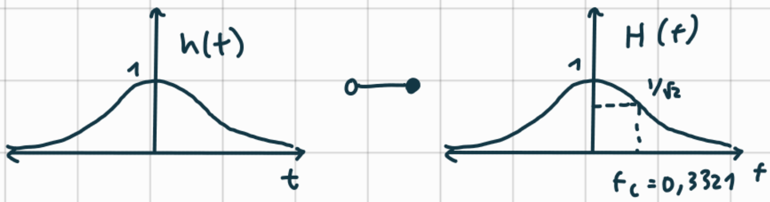
- Limit the response in time typically with a window function (ex: Hann, Hamming, Kaiser etc)



Time limited impulse response has an infinite spectrum

Example Exercise:

$h(t) = e^{-\pi t^2}$ o—● $H(f) = e^{-\pi f^2}$



a) Determine b ($b>0$) so that $f_c = 1\,kHz$

Similarity theorem: $x(bt)$ o—● $\frac{1}{|b|}X\left(\frac{f}{b}\right)$

$x(t) = e^{-\pi t^2} \rightarrow x(bt) = e^{-\pi (bt)^2}$ o—● $\frac{1}{|b|}e^{-\pi\left(\frac{f}{b}\right)^2}$

We know: $h(f_c) = e^{-\pi\cdot 0{,}3321^2} = \frac{1}{\sqrt{2}}$

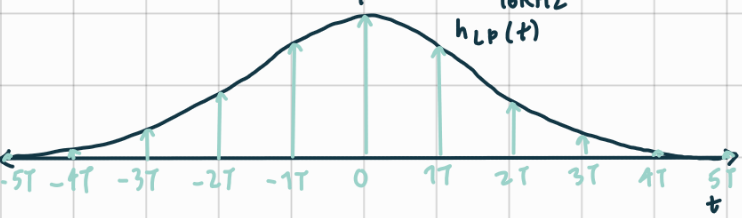
$e^{-\pi\left(\frac{1\,kHz}{b}\right)^2} = \frac{1}{\sqrt{2}} \Rightarrow \frac{1\,kHz}{b} = 0{,}3321 \Rightarrow b = 3010\,Hz$

neglect Hz

$\Rightarrow h_{LP}(t) = 3010\cdot e^{-\pi(3010\,Hz\cdot t)^2}$ LP $\hat{=}$ Low Pass

$H_{LP}(f) = 1\cdot e^{-\pi\left(\frac{f}{3010\,Hz}\right)^2}$

b) Determine $h(n)$ for $T = \frac{1}{16\text{kHz}}$ and $|n| \le 5$



$n=0$: $h(0) = 3010$

$n=1, n=-1$: $h(1) = 3010 \cdot e^{-\pi\left(3010\,\text{Hz} \cdot \frac{1}{16\text{kHz}}\right)^2}$

$= 2693 = h(-1)$

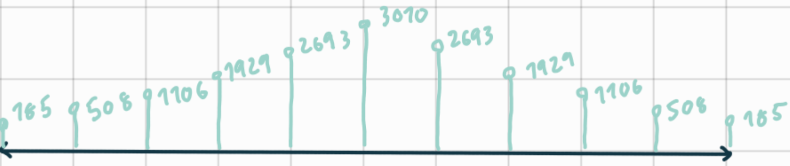
$n=2, n=-2$: $h(2) = h(-2) = 1929$

$n=3, n=-3$: $h(3) = h(-3) = 1106$

$n=4, n=-4$: $h(4) = h(-4) = 508$

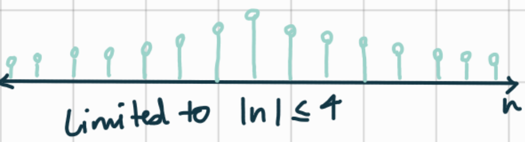
$n=5, n=-5$: $h(5) = h(-5) = 185$

c) Sketch of $h(n)$ for $|n| \le 5$

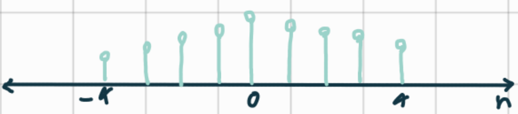


d) Determine the maximum deviation / error between the infinite impulse and the response limit to $|n| \le 4$

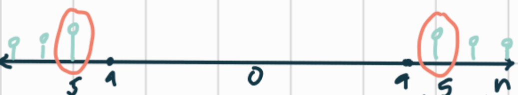
Infinite



Limited to $|n| \le 4$



Error



Maximum error: 185 at $|n| = 5$

e) Sketch the block diagramm for a casual realization of $h(n)$

with multiplication, delay, addition

$\rightarrow (a) \rightarrow$ $\rightarrow \boxed{z^{-1}} \rightarrow$ $\rightarrow (+) \rightarrow$

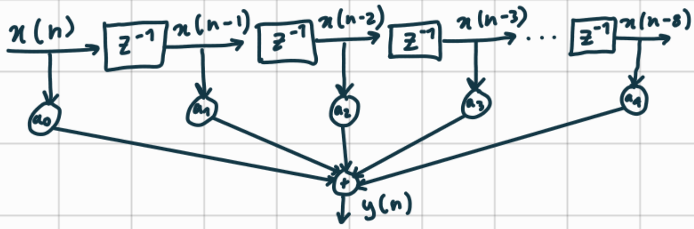
$h(n) = [a_0 \; a_1 \; a_2 \; \dots \; a_8]$

$a_0 = a_8 = 508$, $a_1 = a_7 = 1106$

$a_2 = a_6 = 1929$, $a_3 = a_5 = 2693$

$a_4 = 3010$

$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$



Time discrete convolution

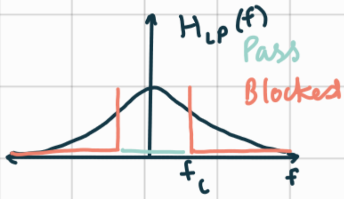
$y(n) = h(n) * x(n) = \sum_{m=-\infty}^{\infty} h(m)\, x(n-m)$

here m: $a_0 \dots a_8$

$\Rightarrow y(n) = \sum_{m=0}^{8} h(m) \cdot x(n-m)$

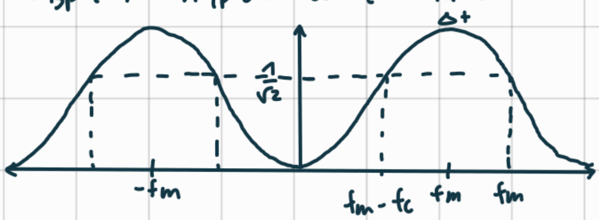
$= h(0) \cdot x(n-0) + h(1) \cdot x(n-1) + \dots h(8)\, x(n-8)$

f) Bandpass filter with center frequency $f_m = 4\,\text{KHz}$. Derive $h_{BP}(t)$, $H_{BP}(f)$ and the bandwidth Δf



$H_{BP}(f) = H_{TP}(f) * \frac{1}{2}\left[\delta(f-f_m) + \delta(f+f_m)\right]$

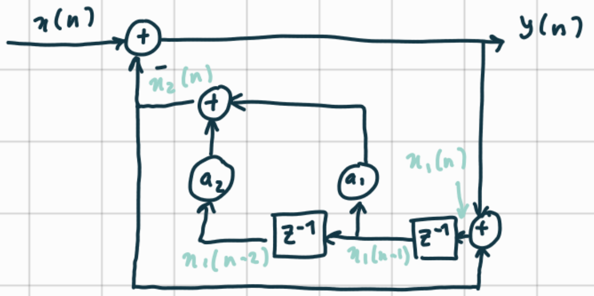
$h_{BP}(t) = h_{TP}(t) \cdot \cos(2\pi f_m T)$



$\Delta f = 2 f_c = 2\,\text{KHz}$

Z - Transform

Predictor of 2^{nd} order in "closed loop" config.



a) Determine the z-transform of the transfer function

- Determine intermediate signals after all adder

$x_1(n) = y(n) + x_2(n) \;\circ\!\!-\!\!\bullet\; Y(z) + X_2(z) \; \ldots (1)$

$x_2(n) = a_2 \cdot x_1(n-2) + a_1 \cdot x_1(n-1) \;\circ\!\!-\!\!\bullet\; X_2(z) = a_2 \cdot z^{-2} \cdot X(z) + a_1 \cdot z^{-1} \cdot X(z) \; \ldots (2)$

$y(n) = x(n) - x_2(n) \;\circ\!\!-\!\!\bullet\; X(z) - X_2(z) \; \ldots (3)$

Z-Transform: $a \cdot x(n - n_0) \;\circ\!\!-\!\!\bullet\; a \cdot z^{-n_0} \cdot X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{X(z) - X_2(z)}{X(z)} = 1 - a_2 \cdot z^{-2} + a_1 \cdot z^{-1}$$

b) Determine the difference equation $y(n) = \ldots$

$$Y(z) = X(z) \cdot (1 - a_2 z^{-2} - a_1 z^{-1})$$
$$= X(z) - a_2 z^{-2} X(z) - a_1 z^{-1} X(z)$$

$$y(n) = x(n) - a_2 x(n-2) - a_1 x(n-1)$$

Assume : $H(z) = \frac{Y(z)}{X(z)} = \frac{1 - a_1 z^{-1} - a_2 z^{-2}}{1 - b_1 z^{-1}}$

$$Y(z)(1 - b_1 z^{-1}) = X(z)(1 - a_1 z^{-1} - a_2 z^{-2})$$

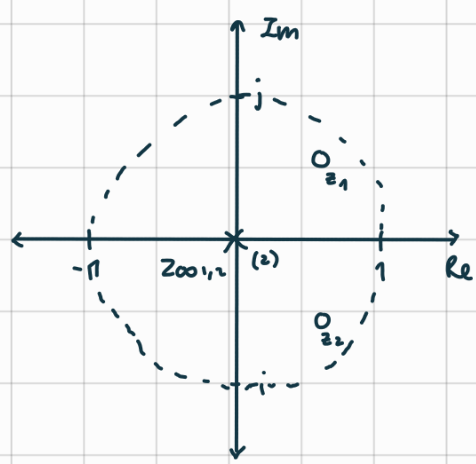
$$y(n) - b_1 y(n-1) = x(n) - a_1 x(n-1) - a_2 x(n-2) + b_1 y(n-1)$$

c) Now set $a_1 = 1$ and $a_2 = -1/2$

Draw the pole-zero-diagram (PN Diagram)

Amplification factor: $H_0 = 1$

$$H(z) = (1 - a_1 z^{-1} - a_2 z^{-2}) \cdot \frac{z^2}{z^2}$$
$$= \frac{z^2 - a_1 z - a_2}{z^2} = \frac{z^2 - z + 1/2}{z^2}$$



Pole : $z_{\infty\,1,2} = 0$

Zeroes: $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm i}{2}$

$$\frac{z^2 - z + 1/2}{z^2} = 1 \cdot \frac{(z - 1/2 - 1/2 i)(z - 1/2 + 1/2 i)}{z \cdot z}$$

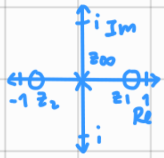
Example for H_0

$$H(z) = \frac{3z^2 - 2}{2z}$$

Poles: 0

zeroes: $z = \pm\sqrt{\frac{2}{3}}$

$$H(z) = \frac{\left(z - \sqrt{\frac{2}{3}}\right)\left(z + \sqrt{\frac{2}{3}}\right)}{z} \cdot H_0 \qquad H_0 = \frac{3}{2}$$



d) Is the system stable?

Yes, because all poles are within the unit circle

Example unit-step

Marginally stable

Stable: Because the amplitude doesn't increase

Non-stable: Because the amplitude never becomes 0

Example $h(n)$ for poles outside of unit circle

System with rising amplitude $|h(n)| \to \infty \Rightarrow$ not stable

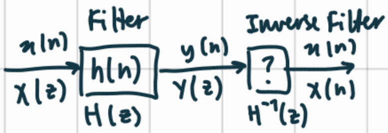
High pitch noise from mic is an unstable system

Example $h(n)$ for poles inside of unit circle

System with damping amplitude $|h(n)| \to 0 \Rightarrow$ stable

BIBO: $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

e)



$$Y(z) = X(z) \cdot H(z) \Rightarrow X(z) = \frac{1}{H(z)} \cdot Y(z) = H^{-1}(z) \cdot Y(z)$$

Determine the inverse filter $H^{-1}(z)$

$$H(z) = \frac{z^2 - z + 1/2}{z^2} \Rightarrow H^{-1}(z) = \frac{z^2}{z^2 - z + 1/2}$$

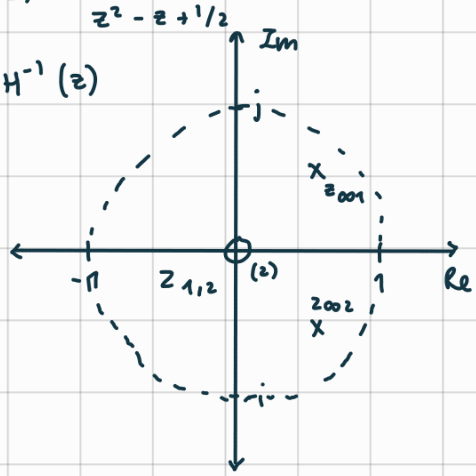
f) Pole-Zero-Diagram of $H^{-1}(z)$

Poles become zeroes

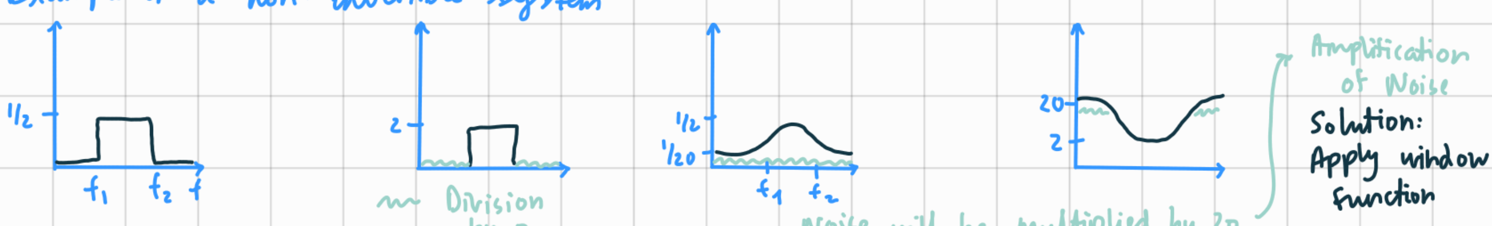
Zeroes become poles

$H_0 = 1$

$H^{-1}(z)$ is stable



Example of a non-invertible system



Solution: Apply window function

Relation between Z transform and the Fourier Transform

$z \to e^{sT} \qquad s = \sigma + j\omega = \sigma + j\,2\pi f$

$z \to e^{j2\pi f T} \qquad T$: sample rate

$$H(z) = \frac{z^2 - z + 1/2}{z^2} = 1 - z^{-1} + \frac{1}{2} z^{-2}$$

$$H(f) = 1 - e^{-j2\pi f T} + \frac{1}{2} e^{-j4\pi f T}$$

Bilinear Transform and Frequency Warping

(z - transform)

Laplace Transform:

R: $U(s) = R \cdot I(s)$

L: $U(s) = s \cdot L \cdot I(s)$

C: $U(s) = \frac{1}{sC} \cdot I(s)$

Steps of the Bilinear Transform:

1) Description of the circuit via Laplace-Transform
2) Substitute $s \approx c\,\frac{z-1}{z+1}$ with $c = \underbrace{\frac{2\pi f_0}{\tan(\pi f_0 T)}}_{\text{Frequency Warping}}$

$G = C$

T: Sampling Time

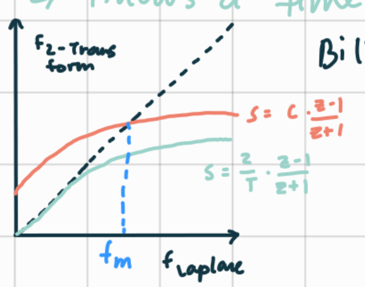
f_0: Frequency at which the appoximation is precise:

Low / High Pass Filter: Cut off frequency

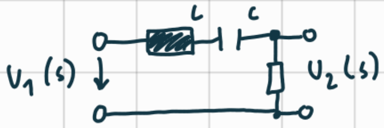
Bandpass: Center Frequency

⇒ Allows a time - discrete realization of a system

Bilinear Transform: $s = \frac{2}{T} \cdot \frac{z-1}{z+1}$



Example: Series resonant circuit



Transfer function:

$$H(s) = \frac{U_2(s)}{U_1(s)} = \frac{R}{R + sL + \frac{1}{sC}}$$

Resonance frequency $f_0 = ?$

Definition of f_0: $H(s)$ is completely real

$$sL + \frac{1}{sC} = 0 \Rightarrow sL = \frac{-1}{sC} \Rightarrow s^2 LC = -1 \Rightarrow s^2 = \frac{-1}{LC}$$

$$\boxed{\begin{array}{l} s = \sigma + j\omega = \sigma + j2\pi f_0 \\ \sigma \overset{!}{=} 0 \end{array}}$$

$$\Rightarrow (j2\pi f_0)^2 = -\frac{1}{LC} \Rightarrow 4\pi^2 f_0^2 = \frac{1}{LC} \Rightarrow f_0 = \frac{1}{2\pi\sqrt{LC}}$$

Determine a time-discrete realization of the filter by using the bilinear transform.

$T = \frac{1}{1000}s$, $R = 100\,\Omega$, $L = 100\,mH$, $C = 100\,\mu F$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = 50.3\,Hz$$

$$c = \frac{2\pi f_0}{\tan(\pi f_0 T)} = 1983\,Hz$$

$$H(s) = \frac{R}{R + sL + \frac{1}{sC}} \cdot \frac{sC}{sC} = \frac{sRC}{sRC + s^2LC + 1}$$

Bilinear Transform: $s \approx c \cdot \frac{z-1}{z+1}$

$$H(z) = \frac{c \cdot \frac{z-1}{z+1} \cdot RC}{c\frac{z-1}{z+1} RC + c^2 \frac{(z-1)^2}{(z+1)^2} \cdot LC + 1} \cdot \frac{(z+1)^2}{(z+1)^2} = \frac{c \cdot (z-1)(z+1) \cdot RC}{c \cdot (z-1)(z+1) RC + c^2 (z-1)^2 LC + (z+1)^2}$$

$$\frac{c \cdot RC \cdot (z^2-1)}{c \cdot RC (z^2-1) + c^2 LC (z^2 - 2z + 1) + (z^2 + 2z + 1)}$$

$$= \frac{b_1 z^2 + b_2 z + b_3}{a_1 z^2 + a_2 z + a_3}$$

$b_1 = cRC = 1983\,Hz \cdot 100\,\Omega \cdot 100\,\mu F = 19{,}83$ Unit Check: $\frac{1}{s} \cdot \frac{V}{A} \cdot \frac{As}{V} = 1$

$b_2 = 0$

$b_3 = -cRC = -19{,}83$

$a_1 = cRC + c^2 LC + 1 = 60{,}15$

$a_2 = -2c^2 LC - 2 = -76{,}65$

$a_3 = -cRC + c^2 LC + 1 = 20{,}49$

$$\Rightarrow H(z) = \frac{19{,}83 \cdot z^2 + 0 \cdot z - 19{,}83}{60{,}15 \cdot z^2 - 76{,}65z + 20{,}49}$$

ex: Look at PN Diagram

Difference between Equation and Block Diagram?

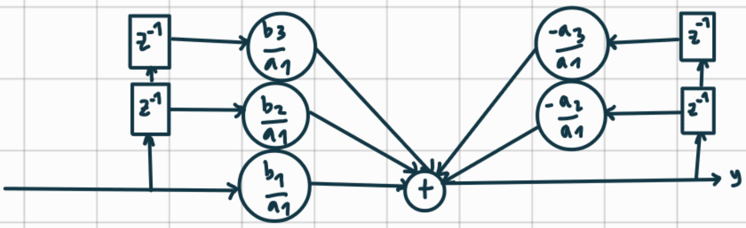
$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_1 z^2 + b_2 z + b_3}{a_1 z^2 + a_2 z + a_3} \cdot \frac{z^{-2}}{z^{-2}} = \frac{b_1 + b_2 z^{-1} + b_3 z^{-2}}{a_1 + b_2 z^{-1} + b_3 z^{-2}}$$

$$Y(z)\cdot(a_1 + a_2 z^{-1} + a_3 z^{-2}) = X(z)\,(b_1 + b_2 z^{-1} + b_3 z^{-2})$$

$$a_1 \cdot y(n) + a_2 \cdot y(n-1) + a_3\, y(n-2) = b_1 \cdot x(n) + b_2\, x(n-1) + b_3\, x(n-2)$$

Time Discrete Form

$$\Rightarrow y(n) = \frac{b_1}{a_1} x(n) + \frac{b_2}{a_1} x(n-1) + \frac{b_3}{a_1} x(n-2) - \frac{a_2}{a_1} y(n-1) - \frac{a_3}{a_1} y(n-2)$$



Example: Discharge Process

MISSING

MISSING (PARTIAL FRACTION DECOMPOSITION)

SIGNAL CORRELATION (CHAPTER 5)

Signals with finite and infinite energy

Parseval theorem:

Energy of a signal can be calculated in time and frequency domain:

$$E_s = \int_{-\infty}^{\infty} |s(t)|^2\,dt = \int_{-\infty}^{\infty} |S(f)|^2\,df$$

electric power: $P = U \cdot I = \frac{U^2}{R} = I^2 R$

energy: $E = \int P\,dt = \int \frac{1}{R} U^2\,dt$

$R = 1\Omega$

$E_s < \infty \rightarrow$ Energy Signal

- $s(t) = \text{rect}(t)$
- Piano
- Discharging Processes

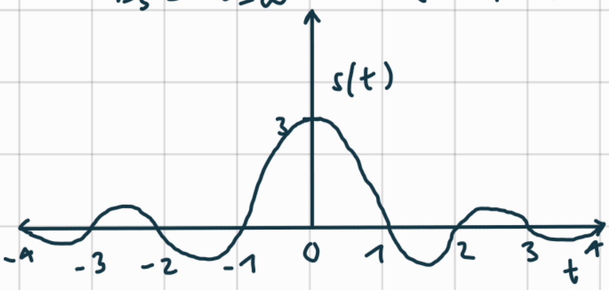
$E_s \to \infty \rightarrow$ Power Signal

- $s(t) = \sin(2\pi f t)$
- PWM
- Power Plug 230V

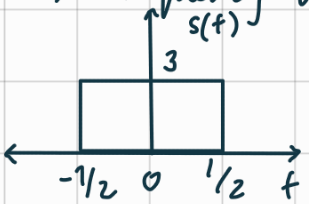
Example: Determine the energy E_s of $s(t) = 3\,si(\pi t)$

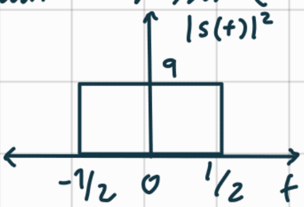
$E_s = \int_{-\infty}^{\infty} |3 \cdot si(\pi t)|^2 \, dt$

$si(\pi t) = \frac{\sin(\pi t)}{\pi t}$



→ Frequency domain $3 \cdot \sin(\pi t)$ o—● $3\,\mathrm{rect}(f)$





Parseval: $E_s = \underbrace{\int_{-\infty}^{\infty} |S(f)|^2 \, df}_{\text{Surface / Area under the Signal}} = 9$

Cross - Correlation for $E_s < \infty$

$\varphi_{sg}^{E}(\tau) = \int_{-\infty}^{\infty} s(\tau) \cdot g(t-\tau)\,dt = s(-\tau) * g(\tau)$

- Measure the similarity between two signals $s(t)$ and $g(t)$
- Find $g(t)$ in $s(t)$
- How much we have to shift $g(t)$, so that the similarity to $s(t)$ is maximum?

MISSING

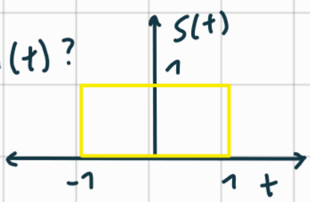
$\xrightarrow{x(t)} \boxed{h(t)} \xrightarrow{y(t)}$

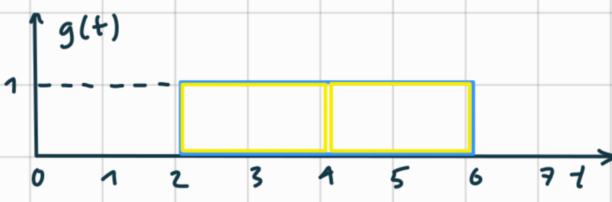
$\varphi_{sg}^{E}(\tau)$ is able to automatically determine the time-delay and attenuation between $x(t)$ and $y(t)$.

Example: $s(t) = \mathrm{rect}\left(\frac{t}{2}\right)$ $\xrightarrow{s(t)} \boxed{h(t)} \xrightarrow{g(t)}$

$g(t) = \mathrm{rect}\left(\frac{t-4}{4}\right)$

Which delay is caused by $h(t)$?



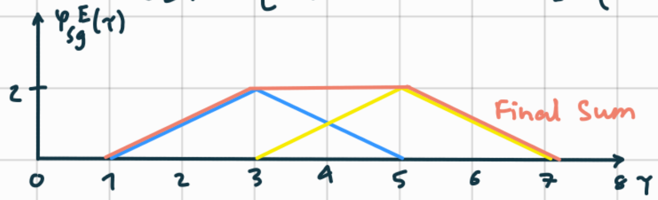


$\varphi^E_{sg}(\tau) = s(-\tau) * g(\tau)$

$= s(\tau) * g(\tau)$ (Because of symmetry)

$= rect\left(\frac{\tau}{2}\right) * rect\left(\frac{\tau}{2}\right) * [\delta(\tau-3) + \delta(\tau-5)]$

$= 2 \cdot \Delta\left(\frac{\tau}{2}\right) * [\delta(\tau-3) + \delta(\tau-5)]$ (Formula Sheet Table 11)



Note $\varphi^E_{gs}(\tau) = \varphi^E_{gs}(-\tau)$ due to symmetry

$\varphi^E_{gs}(\tau)$ has its maximum within $3 \leq \tau \leq 5$

when we shift $s(t)$ by $t_0 = 3 \ldots 5$ towards the right hand side then both signals have maximum similarity.

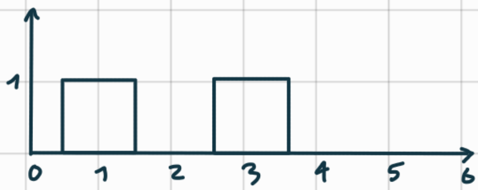
Autocorrelation

$\varphi^E_{ss}(\tau) = \int_{-\infty}^{\infty} s(\tau) \cdot s(t+\tau)\, dt = s(-\tau) * s(\tau)$

How similar is a signal $s(t)$ to itself when we shift it by τ? (Used to find period)

Example :

$s(t) = rect(t) * [\delta(t-1) + \delta(t-3)]$



$\varphi^E_{ss}(\tau) = rect(-\tau) * [\delta(-\tau-1) + \delta(-\tau-3)] * rect(\tau) * [\delta(\tau-1) + \delta(\tau-3)]$

$= rect(\tau) * [\delta(\tau+1) + \delta(\tau+3)] * rect(\tau) * [\delta(\tau-1) + \delta(\tau-3)]$

$= rect(\tau) * rect(\tau) * [\delta(\tau+1) * (\delta(\tau-1) + \delta(\tau-3)) + \delta(\tau+3) * (\delta(t-1) + \delta(t-3))]$

$= \Delta(\tau) * [\delta(\tau) + \delta(\tau-2) + \delta(\tau+2) + \delta(t)]$

$= \Delta(\tau) * [\delta(\tau+2) + 2\delta(\tau) + \delta(\tau+2)]$

Correlation functions - Fourier Transform

CCF: $\varphi^E_{sg}(\tau) = s(-\tau) * g(\tau)$

$$\Phi^E_{sG}(f) = S^*(f) \cdot G(f)$$

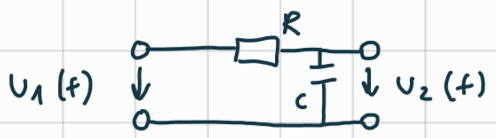
ACF : $\varphi^E_{ss}(\tau) = s(-\tau) * s(\tau)$

$$\varphi^E_{ss}(f) = S^*(f) \cdot S(f) = |S(f)|^2$$

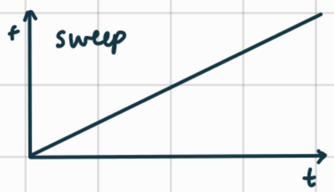
- If the energy is finite: energy density (can find?)
- For energy intinite: power density spectrum (can find?)

$s(t) \to \boxed{h(t)} \to g(t)$ How much energy do we have at the output?

Example: RL Low Pass (Signal model of a cable)



- $s(t)$ has constant spectrum : $s(f) = 1$
  - Dirac Impulse
  - White noise
  - Sweep (frequency)



$$U_2(f) = U_1(f) \cdot \frac{\frac{1}{j\omega C}}{R + j\omega C} = U_1(f) \cdot \frac{1}{1 + j\omega RC}$$

$$H(f) = \frac{U_2(f)}{U_1(f)} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j2\pi f RC}$$

$$s(t) \rightarrow \boxed{h(t)} \xrightarrow{} g(t) \quad \hat{=} \quad v_1(t) \rightarrow \boxed{h(t)} \rightarrow v_2(t)$$

$$G(f) = H(f) \cdot \underbrace{S(f)}_{1}$$

$$G(f) = \frac{1}{1 + j\,2\pi f RC}$$

Determine the Energy Spectral Density of $G(f)$:

$$|G(f)|^2 = \frac{1}{\left(\sqrt{1^2 + (2\pi f RC)^2}\right)^2} = \frac{1}{1 + (2\pi f RC)^2}$$

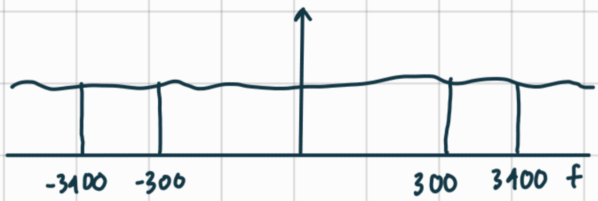
How much energy do we have at the output?

Parseval: $E_g = \int_{-\infty}^{\infty} |g(t)|^2 \, dt = \int_{-\infty}^{\infty} |G(f)|^2 \, df$

Energy within 300 Hz and 3400 Hz (ISDN)

$$E = \int_{300\,Hz}^{3400\,Hz} |G(f)|^2 \, df + \int_{-3400\,Hz}^{-300\,Hz} |G(f)|^2 \, df$$

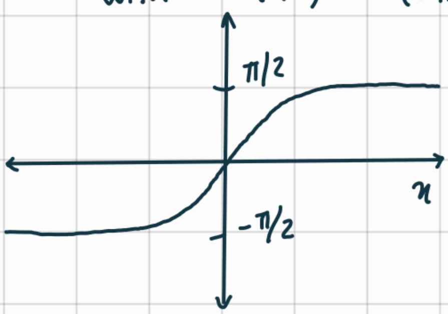
$$= 2 \cdot \int_{300\,Hz}^{3400\,Hz} |G(f)|^2 \, df$$



e.g. Bronstein (famous Book)

Here: $E_g = \int_{-\infty}^{\infty} \frac{1}{1 + (2\pi f RC)^2} \, df = \left[\frac{2}{\sqrt{\Delta}} \cdot \arctan\left(\frac{2af}{\sqrt{\Delta}}\right) \right]_{-\infty}^{\infty}$

with $\Delta = 4a$, $a = (2\pi RC)^2$



$$\Rightarrow E_g = \frac{2}{\sqrt{\Delta}} \cdot \frac{\pi}{2} - \frac{2}{\sqrt{\Delta}}\left(-\frac{\pi}{2}\right) = 2 \cdot \frac{2\pi}{\sqrt{\Delta} \cdot 2} = \frac{2\pi}{1 \cdot \sqrt{4\pi^2 R^2 C^2}} = \frac{1}{2RC}$$

Correlation Functions and LTI Systems

$$s(t) \rightarrow \boxed{h(t)} \rightarrow g(t) \qquad h(t) * s(t)$$

CCF: $\varphi^E_{sg}(\tau) = s(-\tau) * g(\tau)$

$$= s(-\tau) * h(\tau) * s(\tau)$$
$$= s(-\tau) * s(\tau) * h(\tau)$$
$$= \varphi^E_{ss}(\tau) * h(\tau)$$

$\circ\!-\!\bullet$

$$\Phi^E_{SG}(f) = |S(f)|^2 \cdot H(f)$$

ACF: $\varphi^E_{gg}(\tau) = g(-\tau) * g(\tau)$

$$= h(-\tau) * s(-\tau) * h(\tau) * s(\tau)$$
$$= h(-\tau) * h(\tau) * s(-\tau) * s(\tau)$$

$$\varphi^E_{gg}(\tau) = \varphi^E_{hh}(\tau) * \varphi^E_{ss}(\tau)$$

Wiener - Khinchin - Therrem / Wiener - Lee

$\circ\!-\!\bullet$

$$\Phi^E_{GG}(f) = \Phi^E_{HH}(f) \cdot \Phi^E_{SS}(f)$$
$$|G(f)|^2 = |H(f)|^2 \cdot |S(f)|^2$$

Symmetry of the Correlation Functions

ACF: $\varphi^E_{gg}(\tau) = \varphi_{gg}(-\tau)$

$\Phi^E_{GG}(f) = \Phi^E_{GG}(-f)$

CFF: $\varphi^E_{gs}(\tau) = \varphi^E_{sg}(-\tau)$

$\Phi^E_{GS}(\tau) = \Phi^E_{SS}(f) \cdot H(-f)$

Orthogonality:

Are the signals similar to each other?

2 Prototype signal:

$s_1(t)$: intact bearing of motor

$s_2(t)$: defect bearing

MISSING

Time Domain: $\varphi^E_{sg}(0) = 0$

$$\int_{-\infty}^{\infty} s(t) \cdot g(t)\, dt = 0$$

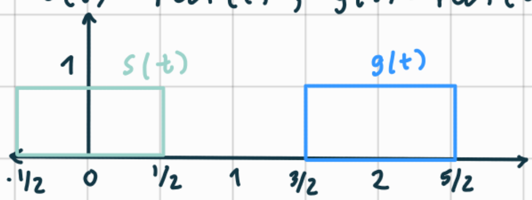
Frequency Domain: $\varphi^E_{sg}(0) = \int_{-\infty}^{\infty} S^*(f) \cdot G(f)\, df = 0$

Test of Orthogonality:

- Test of Overlap (If it doesn't overlap in either $t \to f$, no need to check other directly orthogonal)
- Use symmetry of the signals

Example 1:

$s(t) = rect(t)$, $g(t) = rect(t-2)$



$\varphi^E_{sg}(0) = \int_{-\infty}^{\infty} s(t)\, g(t)\, dt = 0$

$\Rightarrow$ orthogonal because both signals don't overlap in time domain

With real functions you have symmetry after Fourier

With complex you can do anything

Example 2: $s(t) = si(\pi t)$, $g(t) = si(\pi t) \cdot \cos(2\pi \cdot 3t)$

$\cos(2\pi F t) ○—● \frac{1}{2}[\delta(f+F) + \delta(f-F)]$

MISSING

$S(f) = rect(f)$, $G(f) = rect(f) * \frac{1}{2}[\delta(f+3) + \delta(f-3)]$

Bat is doing whatever a bat does

$\varphi^E_{sg}(\tau = 0) \stackrel{!}{=} 0 = \int_{-\infty}^{\infty} s(t) \cdot g(t)\, dt = \int_{-\infty}^{\infty} S^*(f) \cdot G(f)\, df$

$\Rightarrow$ Atleast one of the spectra is always 0

$\Rightarrow$ Orthogonal because no overlap in frequency domain

Example 3: $s(t) = rect(t) \cdot \sin(2\pi t)$, $g(t) = rect(t) \cdot \cos(2\pi t)$

MISSING

- Overlap in time-domain
- Overlap in frequency-domain

$\int_{-\infty}^{\infty} s(t) \cdot g(t)\, dt$

$s(t)$: odd signal/function

$g(t)$: even signal/function

Odd function: $s(t) = -s(-t)$

Even function: $g(t) = g(-t)$

Odd · Even = odd $\Rightarrow \int_{-\infty}^{\infty}$ "odd fct" $dt = 0$

MISSING $\Rightarrow s(t)$ & $g(t)$ are orthogonal

If one is completely real and the other is completely real then they don't overlap

MISSING

Example 4: $s(t) = \Delta(t)$, $g(t) = \frac{1}{2} \cdot e^{-|t|}$

MISSING $\varphi_{sg}^{E}(0) = \int_{-\infty}^{\infty} s(t)\, g(t)\, dt$

$s(t)$: even, $g(t)$: even

$s(t) \geq 0 \quad g(t) > 0$

Both signals are completely positive and thus the integral over their surface can't cancel out $\Rightarrow$ not orthogonal

Time Discrete Correlation Functions

$\int \ldots dt \rightarrow \Sigma$

$$\varphi_{xy}^{E}(m) = x(-m) * y(m) = \sum_{n=-\infty}^{\infty} x(n) \cdot y(n-m)$$

Orthogonality of Time Discrete Functions:

$$\varphi_{xy}^{E}(0) = 0 = \sum_{n=-\infty}^{\infty} x(n) \cdot y(n+0) \qquad \text{Time Domain}$$

$$= \int_{-\infty}^{\infty} X(f)^{*} \cdot Y(f)\, df \qquad \text{Frequency Do}$$