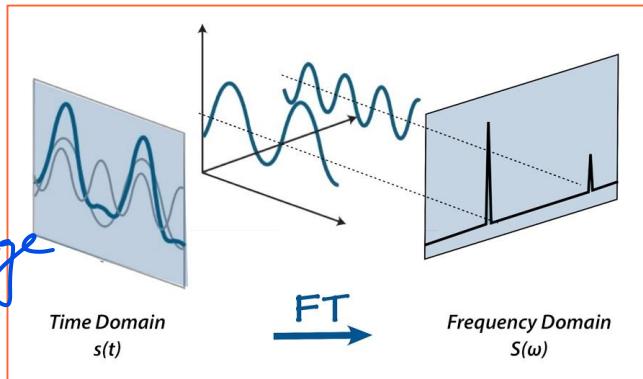


System Theory

or

"If you want to achieve a certain effect,
you have to understand the **cause-effect-principle!**"

25.1. production }
25.1. transfer } of
50.1. application }
knowledge



3rd semester, THWS

WS 23/24

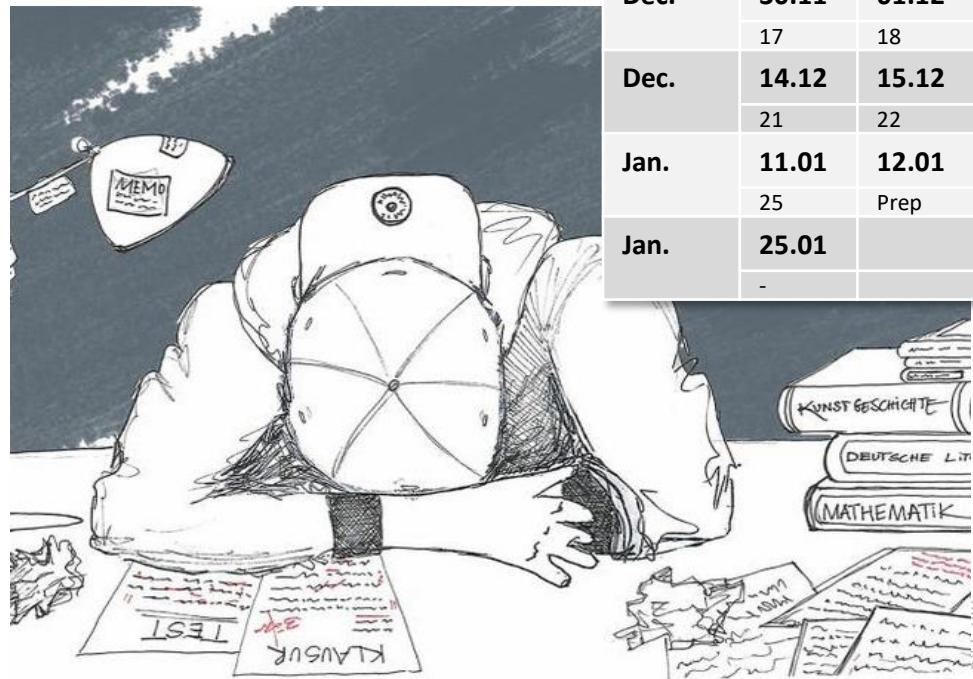
Prof. Dr. Rainer Hirn

Organisation

- **Effort:** 5 ECTS / 4 SWS
- **Langugages:** 1 x German (BRO), 1 x English (IRO)
- **Sequence:** Lectures & exercises alternately
- **Exam:** **90 Min., no auxiliary means, a formula collection will be provided**
- **E-Learning Course:** see Moodle: „Systemtheory“
- **Consultation Hours:** see profile-websites of Faculty Electrical Engineering...

- **Exam Preparation:**

| | THWS | USA (typ.) | Systemt- Theory |
|-----|-------------------|----------------------|------------------------|
| Oct | - | - | - |
| Oct | - | Assessment 1 | Test 1 |
| | | 5 % | 0 % |
| Nov | - | Assessment 2 | - |
| | | 5 % | |
| Nov | - | Mid Term Exam | Test 2 |
| | | 20 % | 0 % |
| Dec | - | Assessment 3 | - |
| | | 5 % | |
| Dec | - | Assessment 4 | Test 3 |
| | | 5 % | 0 % |
| Jan | Final Exam | Final Exam | Final Exam |
| | 100 % | 60 % | 100 % |



Thursday (English, 10:00, 5.1.01)
Thursday (German, 11:45, 9.E.24)
Friday (German, 10:00, 9.E.25)
Friday (English, 11:45, 9.E.26)

| Oct. | 05.10 | 06.10 | 12.10 | 13.10 |
|------|--------------|--------------|--------------|--------------|
| | 1 | 2 | 3 | 4 |
| Oct. | 19.10 | 20.10 | 26.10 | 27.10 |
| | 5 | 6 | 7 | 8 |
| Nov. | 02.11 | 03.12 | 09.11 | 10.11 |
| | 9 | 10 | 11 | 12 |
| Nov. | 16.11 | 17.11 | 23.11 | 24.11 |
| | 13 | 14 | 15 | 16 |
| Dec. | 30.11 | 01.12 | 07.12 | 08.12 |
| | 17 | 18 | 19 | 20 |
| Dec. | 14.12 | 15.12 | 21.12 | 22.12 |
| | 21 | 22 | 23 | 24 |
| Jan. | 11.01 | 12.01 | 18.01 | 19.01 |
| | 25 | Prep | Prep | ? |
| Jan. | 25.01 | | | |
| | - | | | |

OUTLINE

1. Basics of Signals & Systems

▪ 1.1 Properties of signals

- Signal classes
- Test signals
- Functional algebra

▪ 1.2 Properties of systems

- Definition of a system
- Linearity & time invariance
- Causality & stability

2. Analog World

▪ 2.1 System descriptions in the time domain

- Modeling and simulation
- State space representation
- System description through differential equations
- System response in the time domain (four-step method, convolution)

▪ 2.2 System descriptions using the Laplace transformation

- Laplace transform
- Transfer function
- Complex system structures
- Elementary transferring links
- Pole Zero Diagrams & Stability
- Existence of the Laplace transform

▪ 2.3 System descriptions using the Fourier transformation

- Fourier series
- Fourier transform
- Comparison of Laplace & Fourier transformation
- Spectrum and frequency response
- Bode diagram and locus curves
- Phase and group delay,
- Minimum phase systems, All passes, Basics of filter design

3. Digital World

▪ 3.1 Basic properties of time-discrete signals

- Scanning and reconstruction
- Discrete-time test signals

▪ 3.2 System descriptions in the time domain

- Difference equation
- Discrete-time approximation of continuous-time systems
- Solution of a difference equation in the time domain

▪ 3.3 System descriptions by z-Transform

- Z transformation
- Transfer function of time-discrete systems
- Pole Zero Diagram & Stability
- Matched- and Bilinear-Transformation
- Impulse and jump invariant transformation
- Realization of time-discrete systems
- FIR, IIR filters
- Discrete-time state space representation

▪ 3.4 System descriptions by discrete time Fourier Transform

- Discrete Time Fourier Transformation (ZDFT)
- Spectrum and frequency response of time-discrete systems
- Bode diagram and locus curves of time-discrete systems
- Discrete Fourier Transform (DFT)
- Windowing and zeroPadding
- Fast Fourier Transformation (FFT)

| | Analog World | Digital World |
|----------------|--------------|---------------|
| Time Domain | 1 | 4 |
| Laplace Domain | 2 | 5 |
| Fourier Domain | 3 | 6 |

1. Basics of Signals & Systems

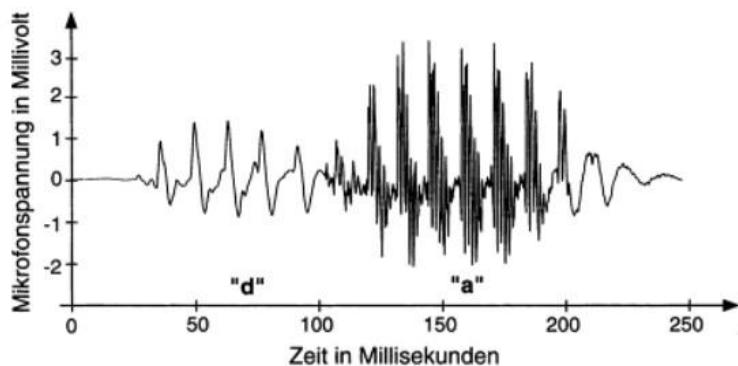
Introduction

The aim of systems theory is to describe and understand systems theoretically (i.e. mathematically).

- Systems theory is the indispensable theory for all further subjects of modern technology and far beyond!

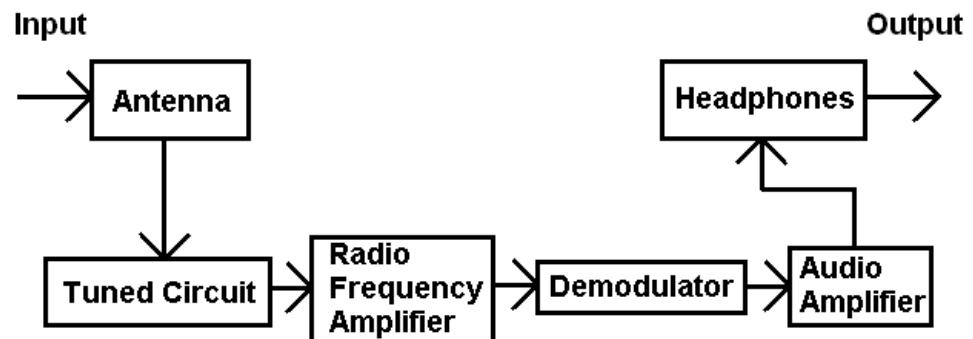
- Examples of signals and systems:

Audio-Signal:



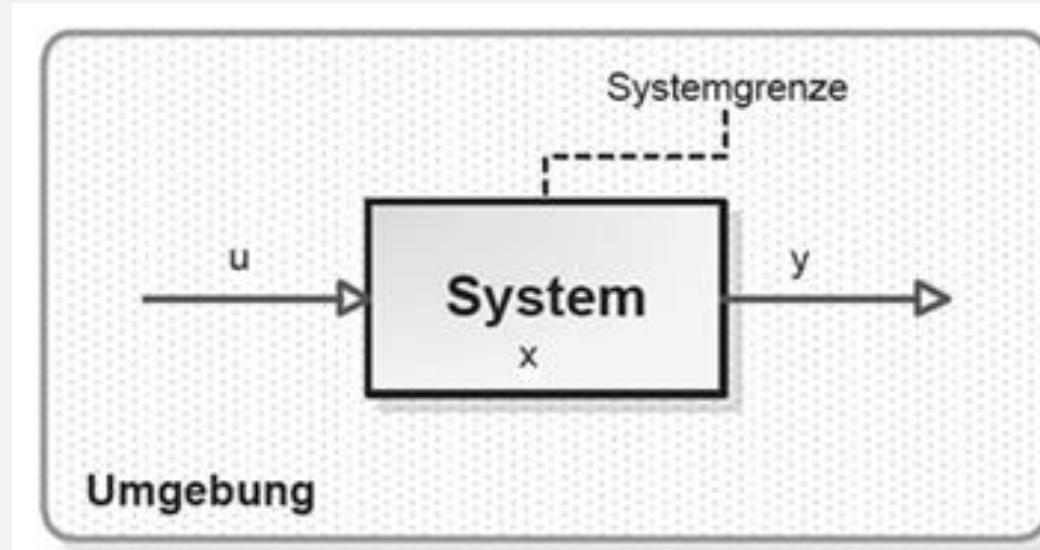
Beispiel eines zeitkontinuierlichen Signals: Sprachsignal der Silbe „da“

Radio-System:



Introduction to Systems Theory

If you want to create a certain effect, you must "only" understand the **cause-effect-principle!**



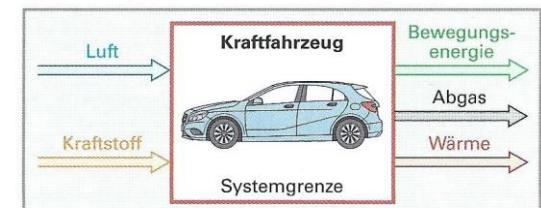
Attention: Systems theory delivers only mathematical descriptions, it does not provide concrete realization of systems!

- **Advantage:**

Not going into details helps keeping the overview of "the big picture" and allows the transferability to completely different areas of science!

- **Disadvantage:**

Unfortunately a certain lack of clarity ...



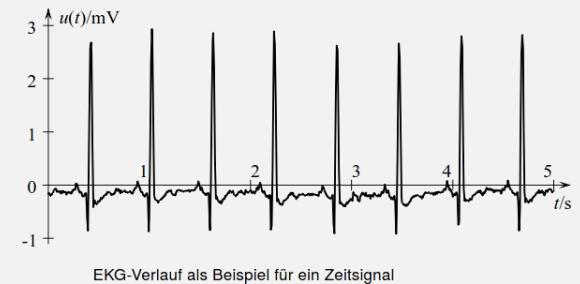
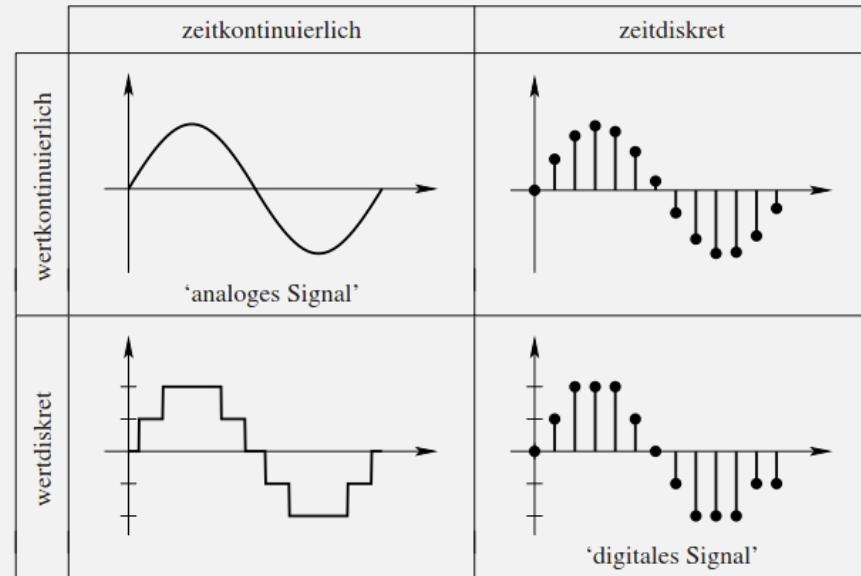
Quelle: Fachkunde Kraftfahrzeugtechnik 2013: 13

1.1 Properties of Signals

Signal Classes

Signals:

- are arbitrary **functions** or **sequences of values** of dependent variables of one or more independent variables (e.g. time, location).
- Four** types can be distinguished:



EKG-Verlauf als Beispiel für ein Zeitsignal

- Continuous time signals** are described as $x(t)$,

Domain of definition $t \in \mathbb{R}$.

diff.

- Discrete-time signals** are described as $x[k]$:

Domain of definition $k \in \mathbb{Z}$.

Signal Classes

Determined Signals:

- the **signal curve is known** for all times.

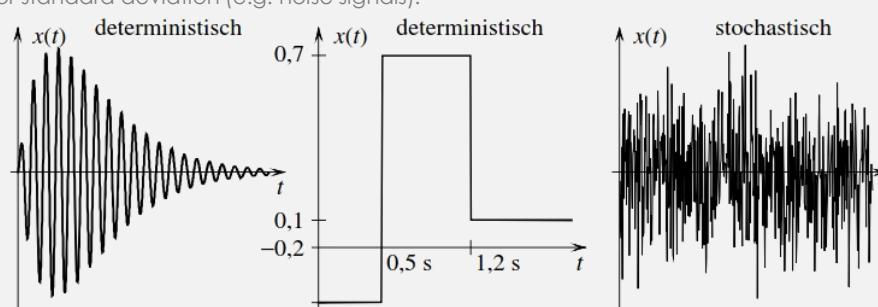
And through a math. Rule or table **exactly writable** (e.g. $\sin(\omega t)$ or fixed sequence of numbers).

Prediktiv
X

Random / Stochastic Signals:

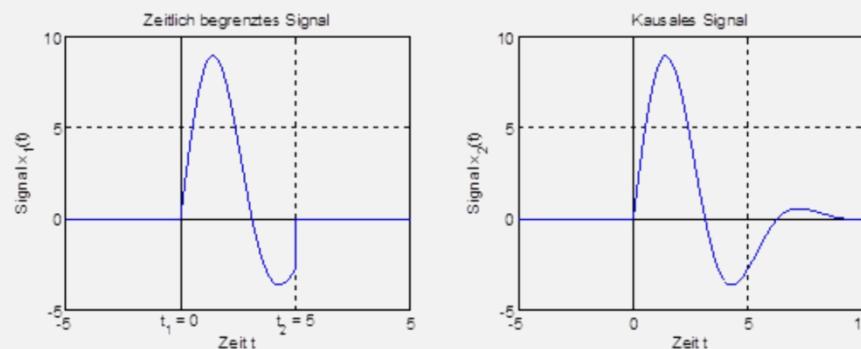
- the **signal curve is not known at all.**

Only statistical signal properties, such as mean value or standard deviation (e.g. noise signals).



Limitation and Causality in signals:

- time-limited**, if a signal lasts a finite time only, i.e. it disappears outside a finite interval:
- causal** if the following applies to the signal: $x(t) = 0$ for $t < 0$. otherwise it is **non-causal**.



Signal Classes (Energy & Power)

- Example of electrical resistance:

The square of the voltage or the current is a measure for the converted power, i.e. the following applies:

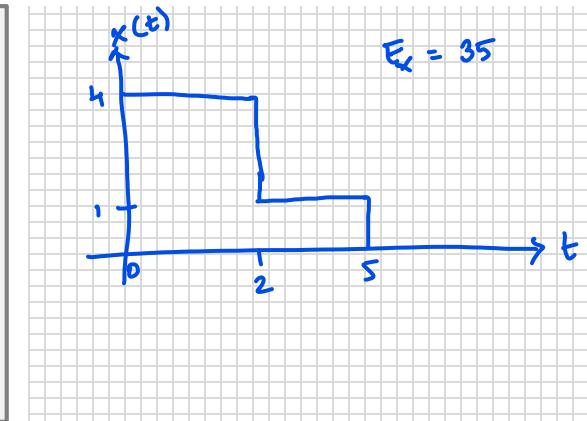
$$\left. \begin{aligned} P = VI &= \frac{U^2}{R} = I^2 R & p(t) = \frac{U^2(t)}{R} = R \cdot i^2(t) \\ P = \frac{\Delta E}{\Delta t} &= \frac{dE}{dt} \Rightarrow \int p dt = \int dE = E \end{aligned} \right\} E = \int_{-\infty}^{+\infty} p(t) dt = \frac{1}{R} \int_{-\infty}^{+\infty} U^2(t) dt$$

Energy & Energy Signals

In systems theory, the energy of a signal is defined (minimalistically) as:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- In words: The **energy** of a signal is "the total area under its square".
- A signal becomes **Energy signal** called if it has a finite energy, ie if: $E_x < \infty$.



Power & Power Signals

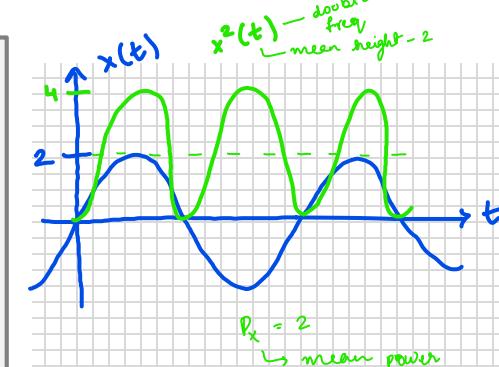
mean

In systems theory, the power of a signal is defined as:

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$

- In words: The **power** of a signal is "the average height of its square".
- A signal becomes **Power signal** called if it has a finite power, ie if: $P_x < \infty$.

The power is used to describe signals using infinite Energy (e.g. the sine signal) is required.



Note: The amount bars are of course only required for complex signals!

Signal Classes

Symmetry

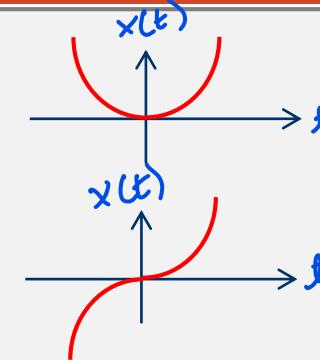
Even Signals:

- A signal is called even if: $x(t) = x(-t)$
(= axis symmetry).
? e.g. cosine line symmetric

Odd Signals:

- A signal is called odd if: $x(t) = -x(-t)$
(= point symmetry).
? e.g. sine, x^3

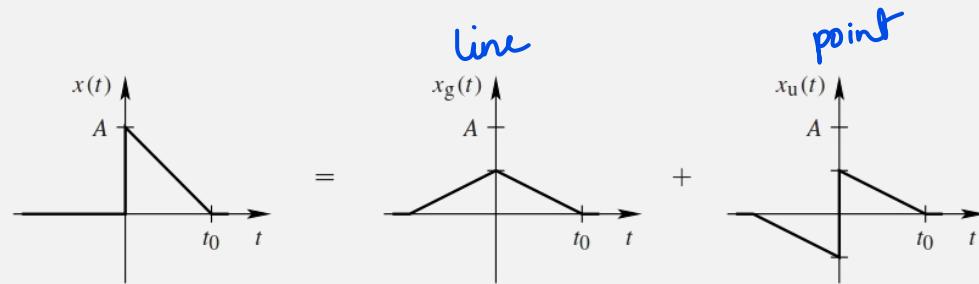
gerade \rightarrow even



Even and odd parts:

- Any signal $x(t)$ can be converted into a even part $x_G(t)$ and an odd part $x_U(t)$, the formula is:

- $x_G(t) = \frac{1}{2} [x(t) + x(-t)]$
- $x_U(t) = \frac{1}{2} [x(t) - x(-t)]$



Zerlegung eines Signals in seinen geraden und ungeraden Anteil

- A complex signal is called:
 - conjugates even if: $x(t) = x^*(-t)$
 - conjugates odd if: $x(t) = -x^*(-t)$.

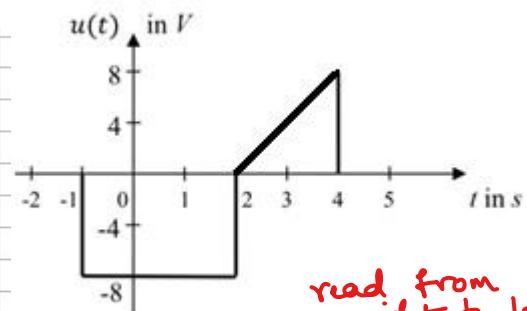
| $x(t)$ | $y(t)$ | $x(t) \pm y(t)$ | $x(t) \cdot y(t)$, $x(t)/y(t)$ | $x(y(t))$ |
|----------|----------|-----------------|---------------------------------|-----------|
| gerade | gerade | gerade | gerade | gerade |
| gerade | ungerade | ungerade | — | ungerade |
| ungerade | ungerade | ungerade | gerade | ungerade |

Symmetrieeigenschaften bei der Verknüpfung von Funktionen

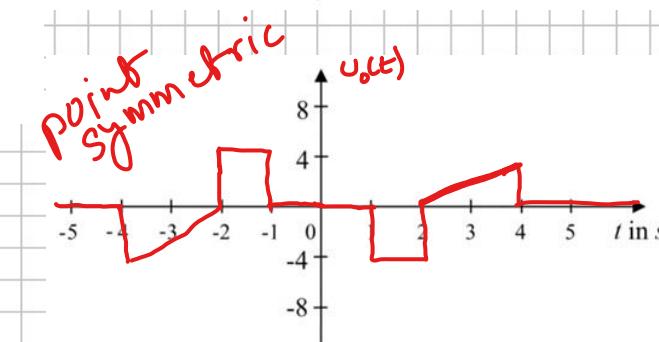
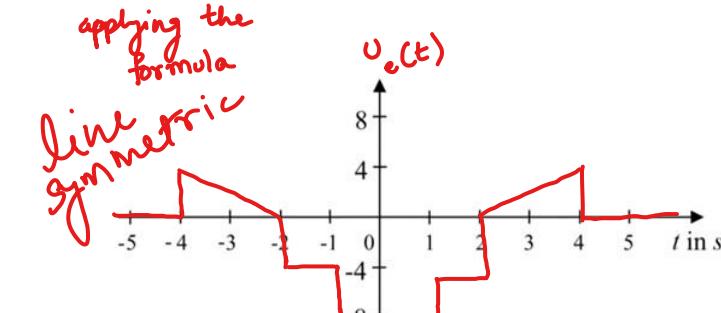
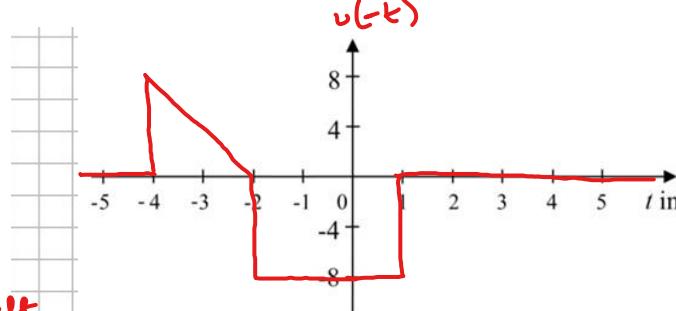
e.g. $x = t^2$
 $y = \cos(t)$
 $x + y = t^2 + \cos(t)$

Ü1.1 Signal Classes

- a) Draw the version $u(-t)$ mirrored to the signal $u(t)$, as well as also its even and odd part $u_G(t)$ and $u_U(t)$.

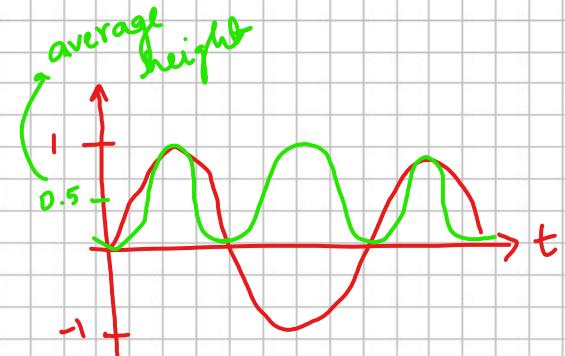


read from right to left
to draw left to right
→ absolute mirror



- b) Calculate the power of a sinusoidal signal sin(ωt) using the integral calculus

$$P_x = \lim_{T_p \rightarrow \infty} \frac{1}{T_p} \int_{-\frac{T_p}{2}}^{+\frac{T_p}{2}} |\sin(\omega t)|^2 dt = \frac{1}{T_p} \int_0^{T_p} \sin^2(\omega t) dt = \frac{1}{T_p} \int_0^{T_p} \left[\frac{1}{2} - \frac{1}{2} \cos(2\omega t) \right] dt = \frac{1}{T_p} \left[\frac{1}{2}t - \frac{1}{2} \frac{\sin(2\omega t)}{2\omega} \right]_0^{T_p} = \frac{1}{T_p} \left[\frac{T_p}{2} - \frac{\sin(2\omega T_p)}{4\omega} \right] \xrightarrow{0 \rightarrow \text{one cycle}} = \frac{1}{T_p} \left[\frac{T_p}{2} - \frac{\sin(2\omega T_p)}{4\omega} \right] = \frac{1}{2} = 0.5$$



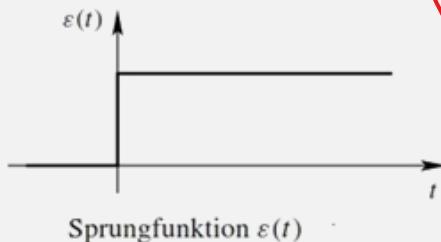
→ in simple → mean of the square of the height

Standard Test Functions

Step function: / switch function

- $\varepsilon(t) = 1, t \geq 0$
- $\varepsilon(t) = 0, t < 0$

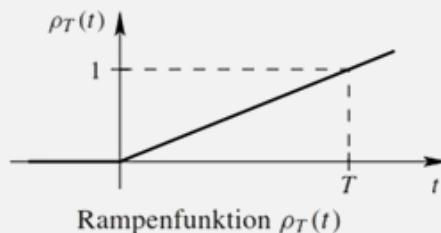
Other terms in the literature is also $\sigma(t)$.



System theory → design & analyze systems mathematically

Ramp function:

- $\rho(t) = t \cdot \varepsilon(t)$

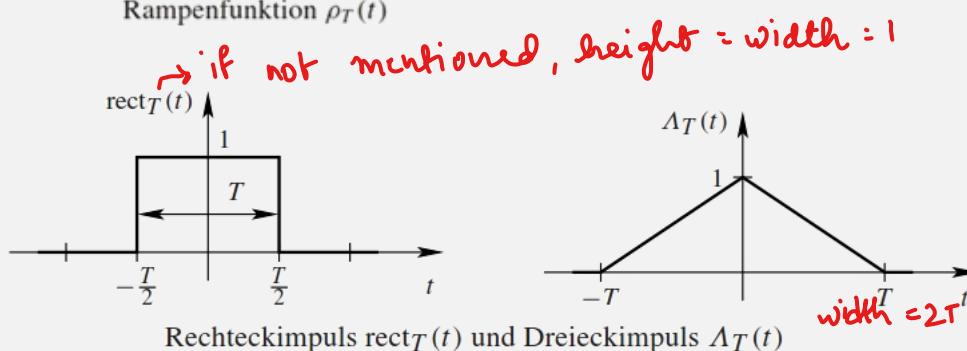


linear increase

Rectangle and triangle function:

- $\text{rect}_T(t) = 1, -T/2 < t < T/2$
 $\text{rect}_T(t) = 0, \text{otherwise}$
- $\Delta_T(t) = 1 - |t/T|, |t| < T$
 $= 0, |t| > T$

For $T = 1$ the basic function are obtained $\text{rect}(t)$ and $\Delta(t)$.

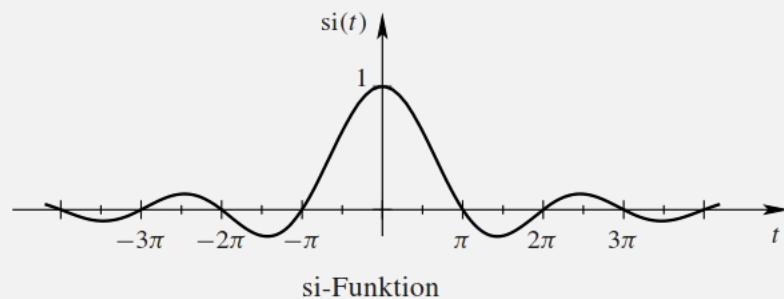


if not mentioned, height = width = 1

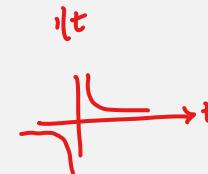
width = $2T$

si-function:

- $\text{si}(t) = \sin(t)/t$



$$\text{si}(t) := \frac{1}{t} \cdot \sin(t)$$



Dirac-Impulse / -Pulse

Dirac Impulse: *perfect pulse*

- It is an infinitely high, infinitely short signal of **area one**,
- It is not a function, but a so-called **distribution**.
It could be seen as an infinitely small rectangle with area one...
- The Dirac pulse is causal and a power signal!

Properties of the Dirac Impulse:

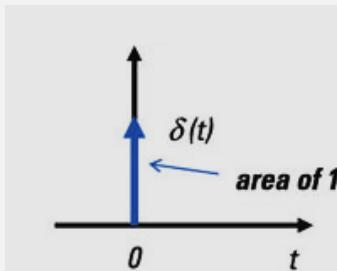
$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$\int_{-\infty}^t \delta(\tau) d\tau = \varepsilon(t)$$

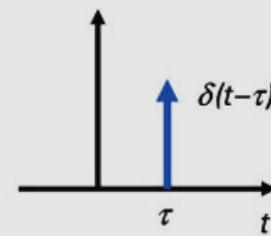
$$\delta(t) = \frac{d\varepsilon(t)}{dt}$$

$$\delta(at) = \frac{1}{|a|} \delta(t)$$

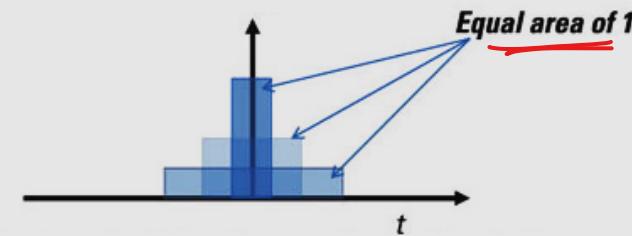
$$\int_{-\infty}^{\infty} \delta(t) \cdot \Phi(t) dt = \Phi(0)$$



Ideal Impulse at $t=0$



Delayed Ideal Impulse at $\tau=0$

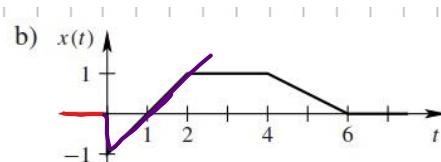
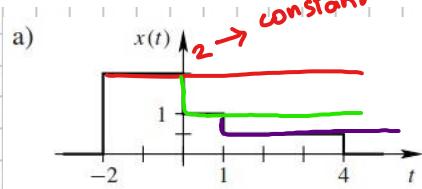


Rectangular Pulses (approach as an impulse)

infinitely thin & high

Ex.1.2 Functional algebra

a) Display the following signals using jump and ramp functions:



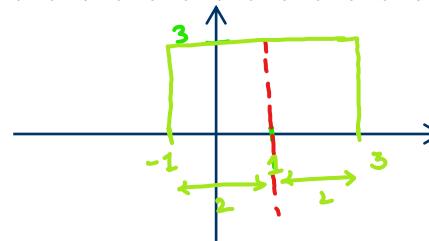
$$a) \quad x(t) = 2 \cdot \varepsilon(t+2) - 1 \cdot \varepsilon(t) \\ - 0.5 \varepsilon(t-1) \\ - 0.5 \varepsilon(t-4)$$

$$b) \quad x(t) = p(t) - \varepsilon(t) - p(t-2) - 0.5 p(t-4) \\ + 0.5 p(t-6)$$

b) Sketch the signal: $x(t) = 3 \operatorname{rect}_{4}(t-1)$

height

width

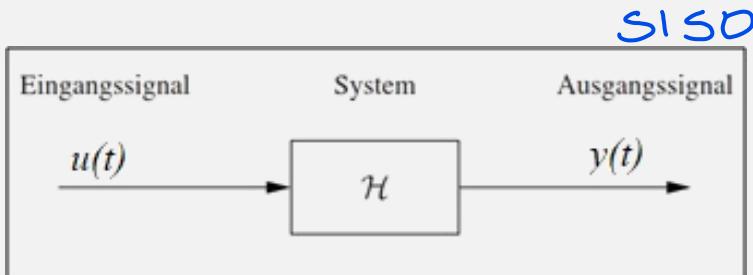


1.2 Properties of Systems

Definition System

System:

- A system is a delimited functional unit that interacts with its **environment only** through **input** and **output signals..**



- Example: Direct setting of the DE of an RC element:

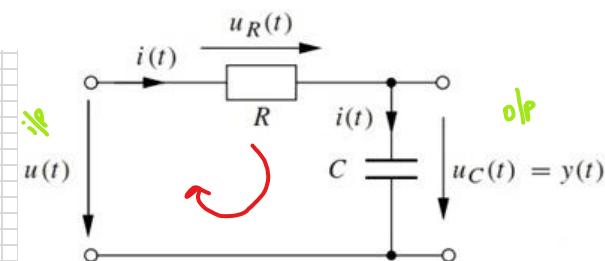
$$\textcircled{1} \quad R = \frac{U_R}{i}$$

$$\textcircled{2} \quad i = C \cdot \frac{du_C}{dt}$$

$$\textcircled{3} \quad U_R + U_C = 0$$

Tipp → DE as a function from U_R U_C i p

$$\frac{du_C}{dt} = \frac{i}{C} = \frac{U_R}{RC} = \frac{(U - U_C)}{RC} \Rightarrow \boxed{U(t) = RC \frac{du_C}{dt} + U_C}$$



Linearity and Time Invariance

Linear system:

For a linear system, the answer to a linear combination of input signals is equal to the corresponding linear combination of the individual system responses (**Superposition principle**).

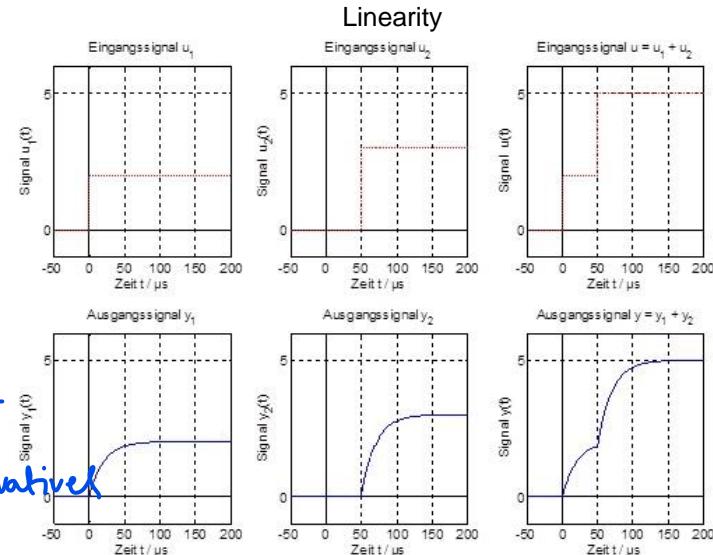
$$f\{a_1 \cdot u_1(t) + a_2 \cdot u_2(t)\} = a_1 \cdot f\{u_1(t)\} + a_2 \cdot f\{u_2(t)\}$$

- Short: "Input and output are proportional to each other".

Linear DGL; Nonlinear DGL:

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_1 u + b_0 u$$

$a_2 \sin(\dot{y}) + a_1 \dot{y}^2 + a_0 y_0 = \dots$
 operations on Derivatives
 → NONLINEAR



Time invariant system: → same op, just delayed

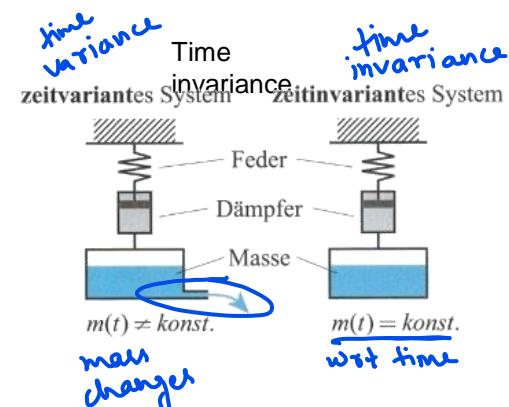
A system reacts to a delayed input signal with a correspondingly delayed output signal, the system is:

$$y(t - t_0) = f\{u(t - t_0)\}.$$

- Short: "The parameter the DE must be constant".

if diff op, time variant

$a_i \in \mathbb{R}$ are const.
 $a_i = f(t) \times$ time



LTI system:

- An LTI system is both linear and time invariant (Linear Time-Invariant system).

Causality and Stability

Causal system:

- „In a causal system, the effect occurs at the earliest at time of the cause, but never before!“
Physically meaningful and feasible systems are because of the **Cause and effect principle** always causal!

Stable system:

- Does a system respond to a limited input signal with a limited output signal, the system is said to be stable.
(BIBO stability, engl. 'Bounded Input - Bounded Output').

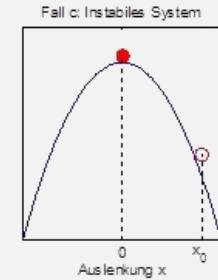
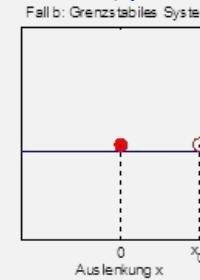
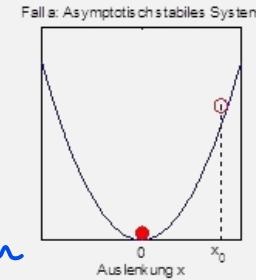
System with / without memory: Same as causal system

- A system whose output depends not only on the current input but also on previous inputs is called a system with Memory, otherwise without.

System with multiple inputs / outputs:

- SISO** (English Single Input Single Output), **MIMO** (Multiple Input Multiple Output).

written as a vector



Without memory:

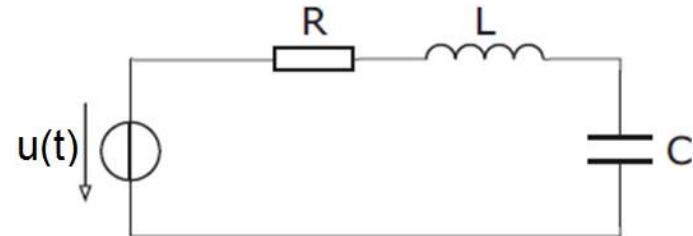
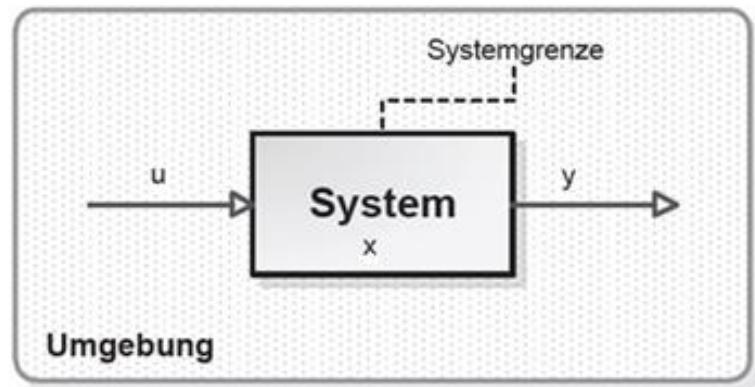


With memory:

2. Analog Signals & Systems

2.1 System Descriptions in the Time Domain

Signals of Modeling:



System inputs u *drives the system ; cannot be influenced*

- are the variables that **excite/stimulate** the system from the **environment** through the system boundary, they **cannot be directly influenced** by the actual model!

System outputs y

- are the model variables of interest, they are **provided by the model** to the environment.

System states x

- States x** describe the **content of storage elements**.

Recipe for Modeling & Simulation *Same as metrology*

1) Define interface

Definition of all inputs u , state variables x and outputs y of the system.

2) Define coordinate system

Introduction of clear coordinates, identification of all forces and moments or currents and voltages.

3) Write down balance equations

Creation of all element and topology equations (e.g. Kirchhoff, Newton, ...).

4) Reshape balance equations

Transform the above equations in such a way that the derivatives of all state variables can be explicitly calculated (i.e. stand alone on one side):

$$\dot{x}_i = f(x, u, p)$$

5) Derive solution

Draw or create the simulation model.

Example: Modeling & simulation of the RC element:

The given input variable u is $u(t)$, the output variable y is the voltage $u_c(t)$.

$$1) u : u(t)$$

$$y : u_c(t) \quad \text{[given]}$$

$$x : u_c(t) \quad \text{internal states}$$

$$2)$$

time dependent

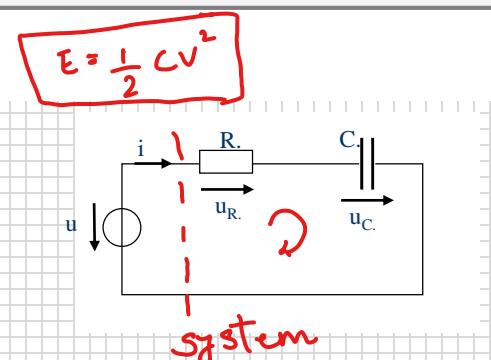
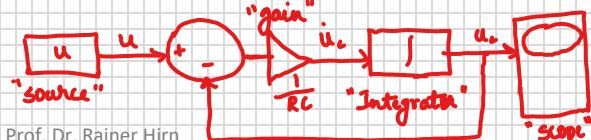
$$3) \text{Element eq.} : R = \frac{u_R}{i} \quad i = C \cdot \frac{du_c}{dt}$$

$$\text{Topology eq.} : u_R + u_c = u \quad \text{KVL}$$

$$4) \dot{x} = f(x, u, p)$$

$$\dot{x} = \dot{u}_c = \frac{i}{C} = \frac{u_R}{RC} = \frac{u - u_c}{RC}$$

MATLAB
Simulink
5>



$$E = \frac{1}{2} CV^2$$

Ex.2.1 Modeling & Simulation (Parallel Resonant Circuit)

A variable voltage $u_E(t)$ is connected to a parallel resonant circuit at time $t = 0$.

a) We are looking for a simulation model for the variation of the voltage across the capacitor over time.

$$\begin{aligned} \text{1)} \quad u &= u_E(t) \quad y = u_c(t) \quad x_1 = i_L(t) \quad x_2 = u_c(t) \end{aligned}$$

2) Labelling, already done

$$3) \quad E: \quad R = \frac{u_E}{i} \quad u_L = L \cdot \frac{di_L}{dt} \quad u_C = C \cdot \frac{du_C}{dt}$$

$$\begin{aligned} \text{Topology eq:} \quad u_E + u_C &= u_E & i_L + i_C &= i \\ \text{KVL} & & \text{KCL} & \end{aligned}$$

$$u_L = u_C$$

$$\begin{aligned} 4) \quad \dot{x} &= f(x, u, p) \quad \xrightarrow{\text{not a state}} \text{replaced by } u_C \\ \dot{x}_1 &= \frac{di_L(t)}{dt} = \frac{u_L}{L} = \frac{u_C}{L} \end{aligned}$$

$$\dot{x}_2 = \frac{du_C(t)}{dt} = \frac{i_C}{C} = \frac{i - i_L}{C} = \frac{u_E - u_C}{R} - i_L = \frac{u_E - u_C}{R} - \dot{x}_1 = \frac{u_E - u_C}{R} - \dot{x}_1$$

Spring damper system *

A variable force $F(t)$ acts on a mass m moving only horizontally. The spring c is ideal and massless.

a) We are looking for a model for the course of the location $s(t)$.

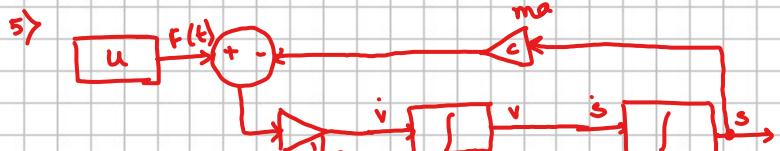
$$1) \quad u = F(t) \quad y = s(t) \quad x_1 = v(t) \quad x_2 = s(t)$$

2) Every variable has a coordinate system, needed for ③

$x_1 = v(t) = \dot{s}(t)$
 ↳ only one not defined (derivative of distance)

3) $\sum F_s = 0$ in the horizontal direction : $F - cs - m\ddot{s} = 0$

$$\begin{aligned} 4) \quad \dot{x} &= f(x, u, p) \\ \dot{x}_1 &= v(t) = \dot{s}(t) = \frac{F - cs}{m} \\ \dot{x}_2 &= \dot{s}(t) = v(t) \end{aligned}$$

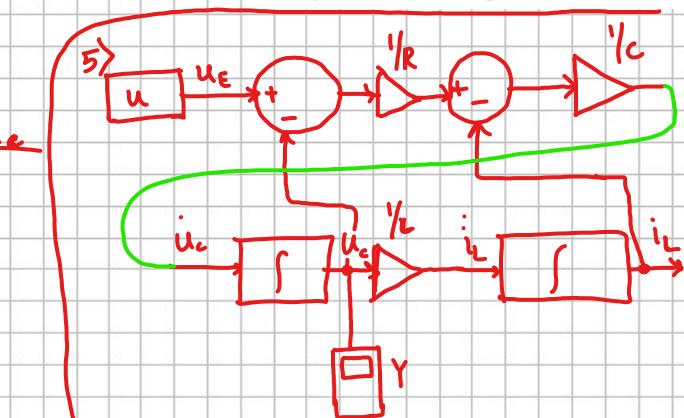
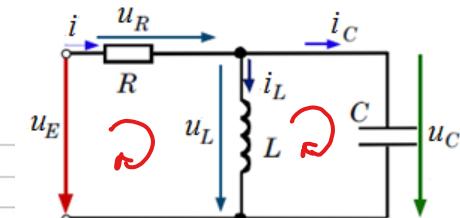
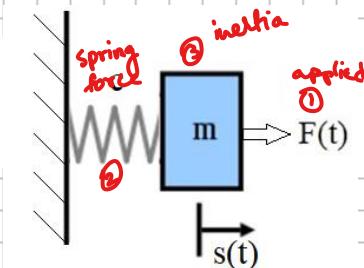


$$\left. \begin{aligned} KE &= \frac{mv^2}{2} \\ \text{state variable} \end{aligned} \right\}$$

$$\text{Spring } E = \frac{cs^2}{2}$$

elongation
compressed length δ
of the spring

$$F = cs \quad \hookrightarrow \text{Hooke's law}$$



State Space Representation of LTI-Systems

- For fast computer input, the standardized **state space representation** is often used.

All **State variables** become one **vector** summarized, the system parameters are thus too **four matrices or vectors**.

State vector:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}}_{\text{System-Matrix}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}}_{\text{Zustd.-Vektor}} + \underbrace{\begin{bmatrix} b_{11} & \dots & b_{1m} \\ b_{21} & \dots & b_{2m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nm} \end{bmatrix}}_{\text{Eingangs-Matrix}} \cdot \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}}_{\text{Eing. Gr.-Vektor}}$$

Equation of state:

$$\dot{x} = A \cdot x + B \cdot u$$

Corresponds to the equations on the right with the restriction to linear systems:

$$\dot{x}_i = f(x, u, p)$$

Output equation:

$$y = C \cdot x + D \cdot u$$

Corresponds to the equations on the right with the restriction to linear systems:

$$\dot{x}_i = f(x, u, p)$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_r(t) \end{bmatrix} = \underbrace{\begin{bmatrix} c_{11} & \dots & c_{1n} \\ c_{21} & \dots & c_{2n} \\ \vdots & \ddots & \vdots \\ c_{r1} & \dots & c_{rn} \end{bmatrix}}_{\text{Ausgangs-Matrix}} \cdot \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}}_{\text{Zustd.-Vektor}} + \underbrace{\begin{bmatrix} d_{11} & \dots & d_{1m} \\ d_{21} & \dots & d_{2m} \\ \vdots & \ddots & \vdots \\ d_{r1} & \dots & d_{rm} \end{bmatrix}}_{\text{Durchgangs-Matrix}} \cdot \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}}_{\text{Eing. Gr.-Vektor}}$$

Ex.2.3 State space representation (parallel resonant circuit)

a) Show the parallel oscillating circuit from the previous exercise in the state space representation.

Step ④ into the state space vector

$$\dot{x}_1 = \dot{i}_1 = \frac{u_c}{L}$$

$$u = u_F \quad y = u_c \quad x_1 = i_L \quad x_2 = u_c$$

$$\dot{x}_2 = \dot{u}_c = \frac{1}{C} \left(\frac{u_E - u_c}{R} + i_L \right)$$

State Space form:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{RC} \end{bmatrix} u$$

$$y = [0 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] u$$

State space representation (spring-mass damper)

a) Represent the spring-mass oscillator from the previous exercise in the state space representation.

$$u = F(t) \quad y = s(t) \quad x_1 = v = \dot{s}(t) \quad x_2 = s(t)$$

$$\begin{aligned} ④ \quad \dot{x}_1 &= \dot{v}(t) = \ddot{s}(t) = \frac{F - cs}{m} & \frac{F}{m} - \frac{c}{m} s(t) \\ \dot{x}_2 &= \dot{s}(t) = v \end{aligned}$$

State Space Form:

$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -c/m \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \end{bmatrix} u$$

$$y = Cx + Du$$

$$y = [0 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] u$$

Description by a Single DE (of higher order)

Mathematical theorem:

- A DE-system of N first-order equations can always be transformed into **a single DE of Nth order!**

$$\left. \begin{array}{l} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{22} \end{bmatrix} \cdot u \\ y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + d \cdot u \end{array} \right\} \Rightarrow a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_2 \ddot{u} + b_1 \dot{u} + b_0 u$$

Result: A single linear DE with constants coefficients

- the **order og differential equation is e.g. $N = 2$** and corresponds the so-called **system order**.

On the left is the output variable y and its derivatives, on the right the input variable u and its derivatives.
Information about the internal states x_i get lost!

Example system of 2nd order:

$$\begin{aligned} \textcircled{4} \quad \dot{x}_1 &= 2x_1 + x_2 & \textcircled{1} \quad y &= x_1 \\ \dot{x}_2 &= x_1 + u \end{aligned}$$

Target: single DE of 2nd order as function of y & u only!

$$\Rightarrow x_2 = \dot{x}_1 - 2x_1 = \dot{y} - 2y$$

$$\Rightarrow \dot{x}_2 = \ddot{y} - 2\dot{y} = \ddot{y} - 2\dot{y} - 2\dot{x}_1$$

$$\therefore \ddot{y} - 2\dot{y} = y + u$$

$$\boxed{\ddot{y} - 2\dot{y} - y = u}$$

Ex.2.4 Converting a DE system into a single DE of a higher order

- a) Describe the input-output behavior of the parallel resonant circuit with input u_E and exit u_C by a only DGL 2nd order.

$$\Rightarrow \dot{i}_1 = \dot{i}_L = \frac{u_C}{L}$$

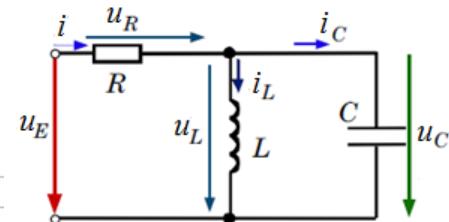
$$\dot{i}_2 = \dot{u}_C = \frac{1}{C} \left(\frac{u_E - u_C}{R} - i_L \right)$$

$$\Rightarrow \ddot{u}_C = \frac{1}{C} \left(\frac{\dot{u}_E - \dot{u}_C}{R} - \dot{i}_L \right)$$

$$\Rightarrow \ddot{u}_C = \frac{1}{C} \left(\frac{\dot{u}_E - \dot{u}_C}{R} - \frac{u_C}{L} \right) | \times LC$$

$$\Rightarrow \boxed{LC \ddot{u}_C + \frac{L}{R} \dot{u}_C + u_C = \frac{L}{R} \dot{u}_E}$$

$$\Rightarrow y = u_C$$



Input-output description by a single DE (higher order)

- a) Describe the input-output behavior of the spring-mass oscillator using a single DGL.

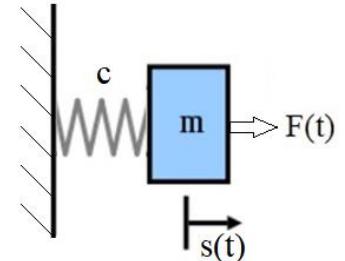
$$\Rightarrow \dot{x}_1 = \dot{v} = \frac{1}{m} (F - cs)$$

$$\dot{x}_2 = \dot{s} = v$$

$$\Rightarrow \ddot{s} = \dot{v} = \frac{1}{m} (F - cs)$$

$$\Rightarrow y = s \quad u = F$$

$$\Rightarrow \boxed{m\ddot{s} + cs = F}$$



System Response in the Time Domain (four-step method)

Four-step method*:

- Calculation of the general homogeneous solutions
- Calculation of a particulate solution
- Superposition of homogeneous and particulate solution
- Determination of the constants via initial conditions

Step and impulse responses are often used to characterize systems. With LTI systems, they can generally be calculated in the time domain using the four-step method, which we do not want to pursue any further here:

solving of DE not there!

Transient Response (homogeneous solution):

- of the form of one or more e-functions

Steady State Response (particulate solution):

- on the form of the stimulus.

Example RC element:

$$\text{DGL: } RC \dot{y} + y = u \quad T = RC$$

A) Jump-like excitation:

$$u(t) = \varepsilon(t)$$

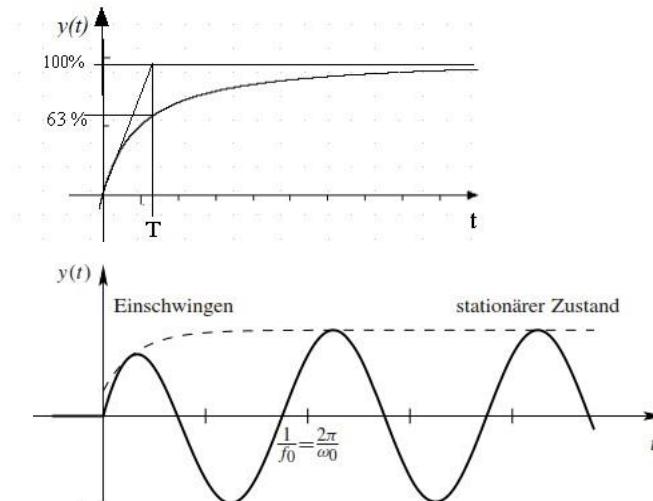
$$y(t) = \underbrace{-e^{-t/T}}_{\text{Einschwinganteil}} + \underbrace{\frac{1}{T}}_{\text{stationärerAnteil}} \cdot \varepsilon(t)$$

B) Oscillatory excitation:

$$u(t) = \cos(\omega t) \cdot \varepsilon(t)$$

$$y(t) = \underbrace{-e^{-t/T}}_{\text{Einschwinganteil}} + \underbrace{\frac{\cos(\omega t + \phi_0)}{\cos(\phi_0)}}_{\text{stationärerAnteil}} \cdot \varepsilon(t)$$

$$\text{mit } \phi_0 = -\arctan(\omega T)$$



Einschwingvorgang und stationärer Zustand am Beispiel RC-Glied

System Responses in the Time Domain (Convolution)

Impulse response $g(t)$:

- Called the response of a system to a pulse $\delta(t)$ at the input.

Step response $h(t)$:

- Called the response of a system to a jump $\epsilon(t)$ at the input.

Convolution:

- It calculates the output signal $y(t)$ of an LTI system for a given impulse response $g(t)$ **for any input signal $u(t)$!**

$$y(t) = \int_{-\infty}^{\infty} u(\tau) \cdot g(t - \tau) d\tau = u(t) * g(t) = g(t) * u(t)$$

Brief derivation of the convolution integral:

An LTI system with the known impulse response $g(t)$ responds to a linear combination of impulses

$$u(t) = c_1 \cdot \delta(t - t_1) + c_2 \cdot \delta(t - t_2)$$

with the same linear combination of impulse responses (superposition principle).

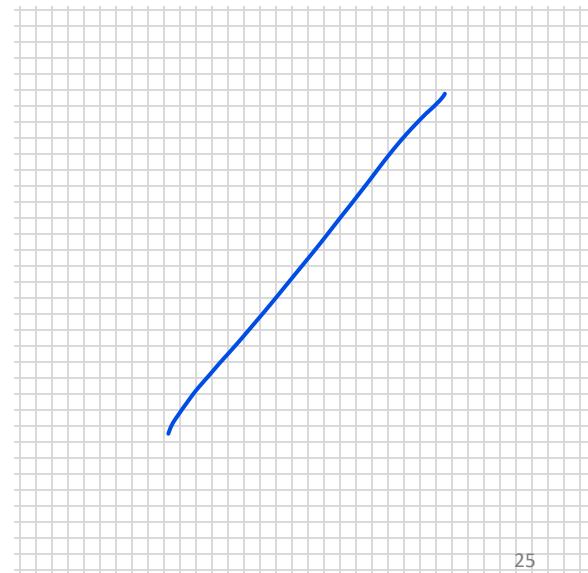
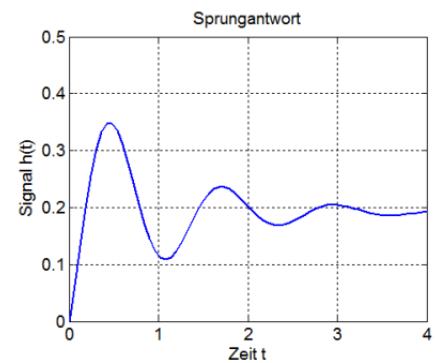
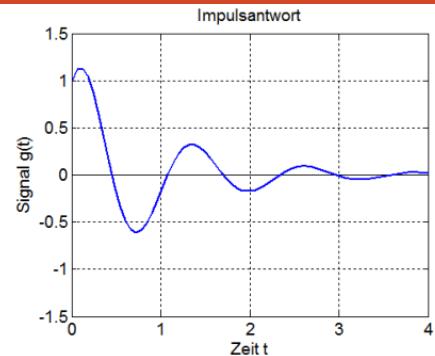
$$y(t) = c_1 \cdot g(t - t_1) + c_2 \cdot g(t - t_2)$$

This relationship can be generalized to any input signals. Because of the hiding property of the Impulse function, any input signal $u(t)$ can be represented as:

$$u(t) = u(t) \cdot \int_{-\infty}^{+\infty} \delta(t - \tau) d\tau = \int_{-\infty}^{+\infty} u(t) \cdot \delta(t - \tau) d\tau = \int_{-\infty}^{+\infty} u(\tau) \cdot \delta(t - \tau) d\tau$$

The above equation can clearly be interpreted as a superposition of an infinite number of impulses $\delta(t - \tau)$ with the weight $u(\tau)$ together represent the signal $u(t)$. Every single impulse $\delta(t - \tau)$ now has the system response $g(t - \tau)$.

The output signal $y(t)$ results from the superposition of an infinite number of system responses $g(t - \tau)$ with the weight $u(\tau)$, what corresponds to the above convolution product.



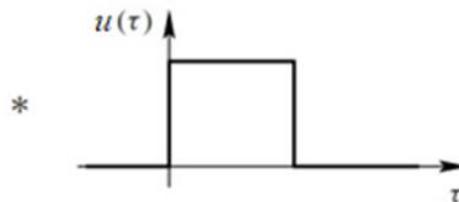
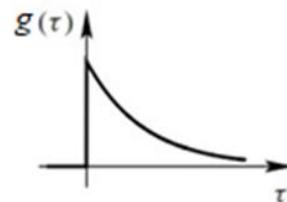
Graphical Estimation of a Convolution

Graphical determination of the convolution integral:

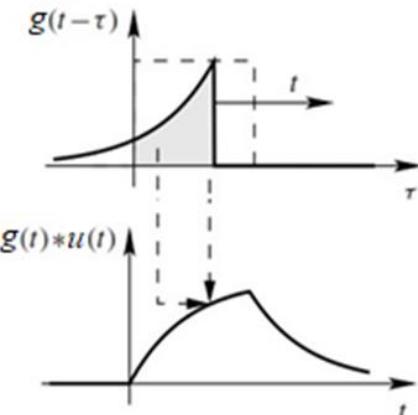
- Sketch the function $u(\tau)$
- Sketch the function $g(t - \tau)$ by mirroring the function $g(\tau)$ and shift by **to the right!**
- Compute the product of the two functions $u(\tau) \cdot g(t - \tau)$ for all τ .
- Determination of the total area under the curve $u(\tau) \cdot g(t - \tau)$.

$$y(t) = u(t) * g(t) = \int_{-\infty}^{\infty} u(\tau) \cdot g(t - \tau) d\tau$$

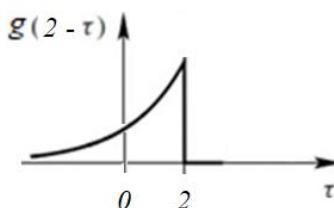
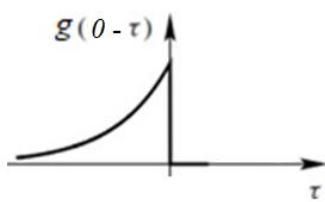
Note: Since the convolution is commutative, it does not matter which function is mirrored and shifted, ie expediently select the simpler function.



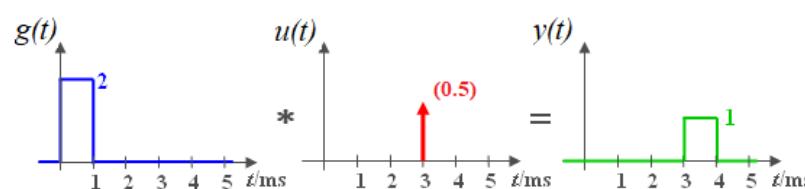
Spiegelung und
Verschiebung



Anschauliche Deutung der (kontinuierlichen) Faltung



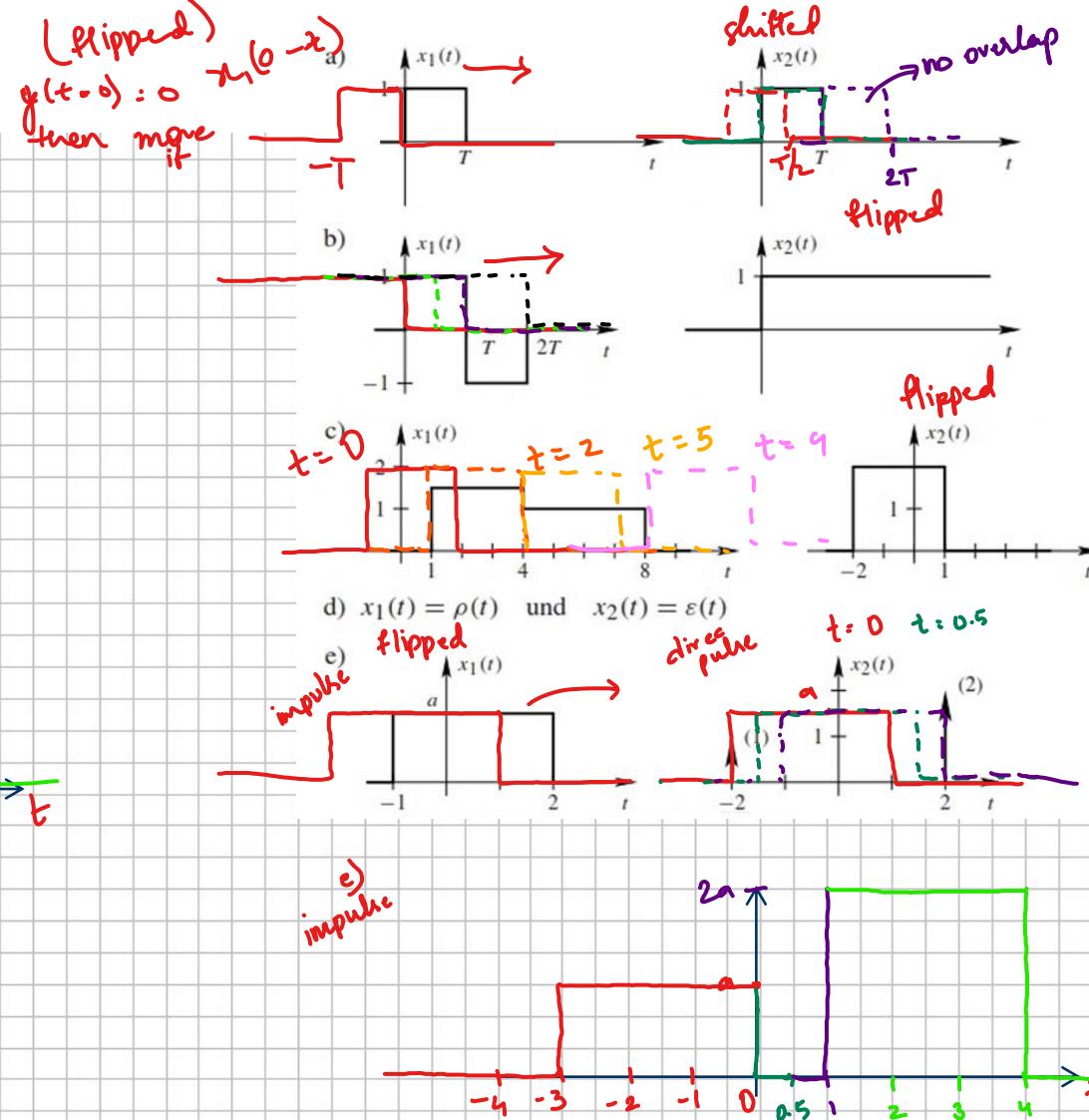
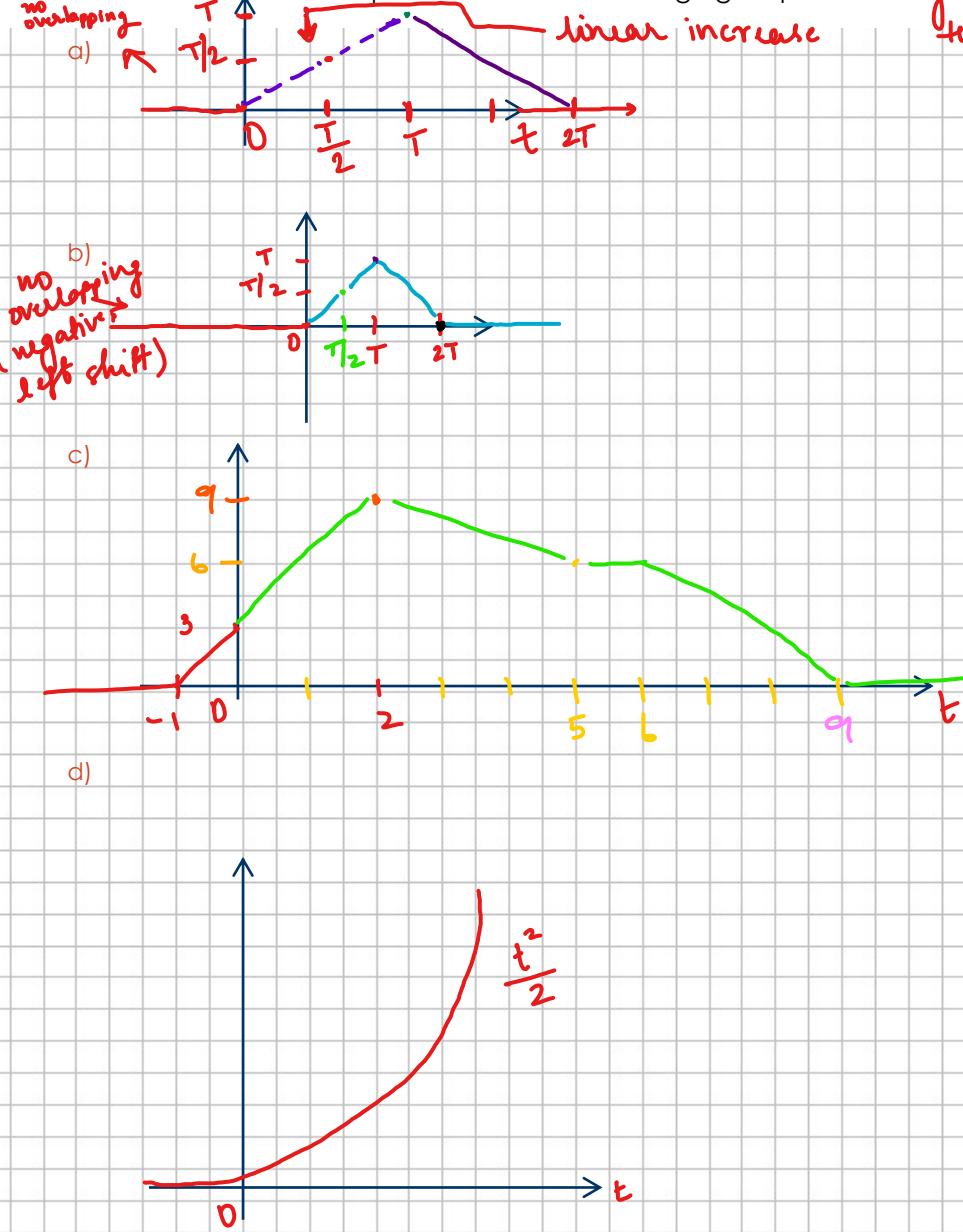
- Convolution with the Dirac pulse:



Ex.2.5 Convolution

flip one of them.

Sketch the convolution products of the following signal pairs:



Dirac pulse \rightarrow neutral function of convolution

Ex.2.6 Discrete 2D-Convolution

typically odd valued functions, to find a center ($3 \times 3, 5 \times 5, 7 \times 7$, etc)

The convolution operation can also be extended to two (location) dimensions. The 2D convolution is the central operation of image processing. A so-called filterKernel $k(x, y)$, e.g. folded into a 3×3 matrix.

- a) Calculate in the two framed areas the convolution of the picture u with the kernels k_i .

$u(x, y)$ gray scale image

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 |

$u(x, y)$

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 |

flipped version
 $k_1(x, y)$

| | | |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 1 | 1 |
| 1 | 1 | 1 |

information available but not used

0's & 1's
(gray transition)

flipped again
(more symmetric)

$k_2(x, y)$

| | | |
|---|----|---|
| 0 | 1 | 0 |
| 1 | -4 | 1 |
| 0 | 1 | 0 |

*

black

analog signal

$$y(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(i, j) k(x-i, y-j) di dj$$

(smoother by grey)

digital signal

$$(x, y) = \sum_{-\infty}^{+\infty} \sum_{-\infty}^{+\infty} u(i, j) k(x-i, y-j)$$

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0 | 3 | 6 | 9 | 9 |
| 0 | 0 | 3 | 6 | 9 | 9 |
| 0 | 0 | 2 | 5 | 8 | 9 |
| 0 | 0 | 1 | 4 | 7 | 9 |
| 0 | 0 | 0 | 3 | 7 | 9 |
| 0 | | | | | |

$y_1(x, y)$

hard transition from
 $y_1(x, y)$ → black to white (grey trans.)
white (highest intensity)

range = 0 to 9

[best contrast: divide by 9]

full white
not considered b/c
of missing info.

⇒ absolute values taken
⇒ divide by 2
⇒ bright slanting line
⇒ edge detector (line
grey light drawing kind
of thing)

| | | | | |
|---|---|----|----|---|
| 0 | 1 | -1 | 0 | 0 |
| 0 | 1 | -1 | 0 | 0 |
| 0 | 1 | -2 | 0 | 0 |
| 0 | 0 | 2 | -1 | 0 |
| 0 | 0 | 1 | -1 | 0 |
| | | | | |

Auto and Cross Correlation Function

Auto-Correlation function ACF (for energy signals):

- Shows within random signals the temporal bindings

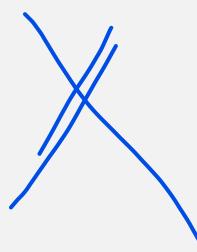
$$\varphi_{xx}(\tau) = \int_{-\infty}^{\infty} x(t) \cdot x(t + \tau) dt$$

The AKF is symmetrical and can be calculated like a convolution operation,

- $\varphi(0)$ is the signal energy E_x .

$$\varphi_{xx}(\tau) = \varphi_{xx}(-\tau)$$

$$\varphi_{xx}(\tau) = x(\tau) * x(-\tau)$$



Cross-Correlation function CCF:

$$\varphi_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) \cdot y(t + \tau) dt$$

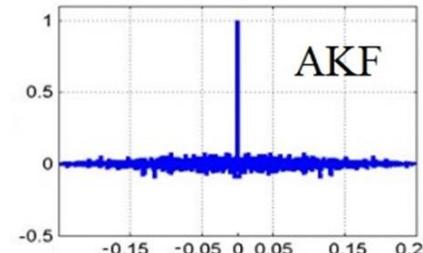
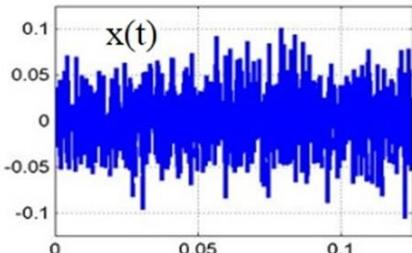
$$\varphi_{xy}(\tau) = \varphi_{yx}(-\tau)$$

- Use:** Finding hidden repetitions in signals:

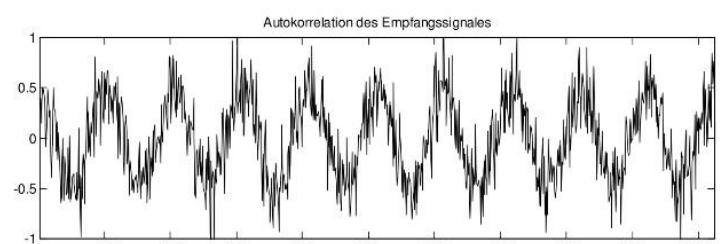
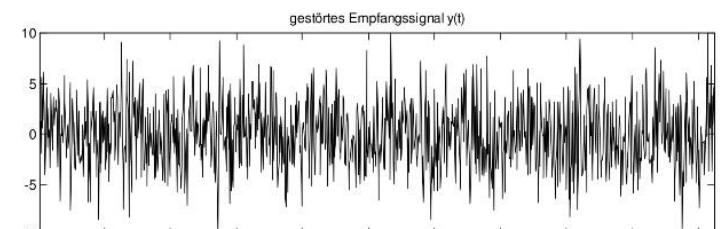
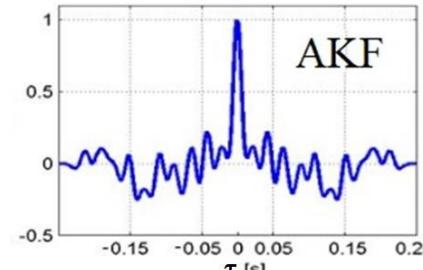
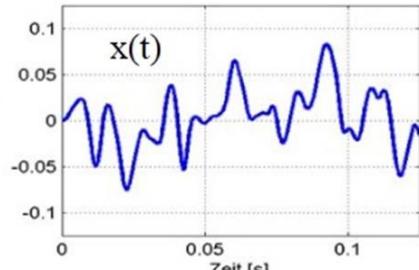
- The AKF of a periodic signal is also periodic with the same period duration
- The AKF shows a maximum at the time an echo occurs (radar ...)

- Examples to the ACF:

Benachbarte Rausch-Sample sind sich *nicht* ähnlich



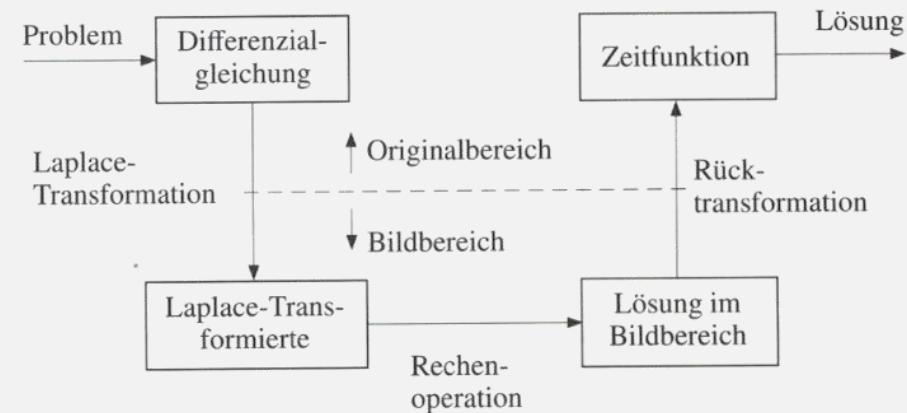
Benachbarte Rausch-Sample sind sich *ziemlich* ähnlich



2.2 System Description by Laplace Transform

Definition of Laplace Transform

The Laplace transform converts a DE in the time domain (original domain) in a simpler algebraic equation in the s-domain (image domain) and is the key to the treatment of LTI systems!



Elementary properties of the ordinary (single-sided) Laplace transform:

- Definition:

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^{\infty} x(t) \cdot e^{-st} dt, \quad \text{mit } x(t) = 0 \quad \text{für } t < 0$$

- Dumbbell icon:

$$\begin{array}{ccc} x(t) & \circ - \bullet & X(s) \\ X(s) & \bullet - \circ & x(t) \end{array}$$

- Differentiation rule*:

$$\dot{x}(t) \quad \circ - \bullet \quad s \cdot X(s)$$

$$\ddot{x}(t) \quad \circ - \bullet \quad s^2 \cdot X(s)$$

- Linearity:

$$a_1 x_1(t) + a_2 x_2(t) \quad \circ - \bullet \quad a_1 X_1(s) + a_2 X_2(s)$$

- s is "only" a complex number, called the Laplace variable $s = \sigma + j\omega$.

$$x(t) = L^{-1}\{X(s)\} = \frac{1}{2\pi j} \int_{\sigma-\infty}^{\sigma+\infty} X(s) e^{st} ds$$

* The above formula only applies if the **left-hand limit** of the function $x(t)$ and all its derivatives (at $t = 0_-$) are zero! In the case of continuous functions, $x(0_-) = x(0) = x(0_+)$. For more details, see the appendix.

$$\begin{aligned} \dot{x}(t) & \circ - \bullet \quad s \cdot X(s) - x(0_-) \\ \text{mit } x(0_-) &= \lim_{h \rightarrow 0} x(0-h) \end{aligned}$$

3.2 Laplace Transform

Calculate the Laplace transform of the following signals.

a) $x(t) = \varepsilon(t)$

d) $x(t) = e^{-4t} \cdot \cos(t) \cdot \varepsilon(t)$

b) $x(t) = e^{-at} \cdot \varepsilon(t)$

e) $x(t) = 3t^3 \cdot \varepsilon(t)$

f) $x(t) = \int_0^t 3e^{-\tau} d\tau$

g) $x(t) = \begin{cases} 2 & \text{für } 0 \leq t < T \\ 0 & \text{sonst} \end{cases}$

h) $x(t) = (t-2) \cdot \varepsilon(t)$

i) $x(t) = (t-2) \cdot \varepsilon(t-2)$

$$a) X(s) = \mathcal{L}\{\varepsilon(t)\} = \int_0^\infty \varepsilon(t) \cdot e^{-st} dt = \int_0^\infty 1 \cdot e^{-st} dt = -\frac{1}{s} \cdot e^{-st} \Big|_0^\infty = -\frac{1}{s}(0-1) = \frac{1}{s} \quad \text{for } s > 0$$

b) $x(s) = \mathcal{L}\{e^{at} \cdot \varepsilon(t)\} = \frac{1}{s-a}$

d) $x'(t) = (t) \cdot \varepsilon(t) \xrightarrow{\mathcal{L}} x'(s) = \frac{s}{s^2+1}$
correspondence

$x(t) = e^{4t} \cdot x'(t) \xrightarrow{\mathcal{L}} x(s) = x'(s+4) = \frac{s+4}{(s+4)^2+1}$

e) $x'(t) = \frac{t}{2} \varepsilon(t) \xrightarrow{\mathcal{L}} x'(s) = \frac{1}{s^2}$

$x(t) = 6 \cdot t \cdot x'(t) \xrightarrow{\mathcal{L}} x(s) = 6(-) \frac{d}{ds} x'(s) = -6(-s) \frac{1}{s^3} = \frac{18}{s^4}$

f) $x(s) = \frac{1}{s} \cdot 3 \cdot \frac{1}{s+1} = \frac{3}{s(s+1)}$

g) $x(t) = 2 \cdot [\varepsilon(t) - \varepsilon(t-T)]$
give the red signal

h) $x(t) = t \cdot \varepsilon(t) - 2 \varepsilon(t)$
 \mathcal{L}

$$x(s) = \frac{1}{s^2} - \frac{2}{s}$$

i) $x'(t) = t \cdot \varepsilon(t) \xrightarrow{\mathcal{L}} x'(s) = \frac{1}{s^2}$

$$x(t) = x'(t-2) \xrightarrow{\mathcal{L}} x(s) = \frac{e^{-2s}}{s^2}$$

[OR]

$$x(t) = t \cdot \varepsilon(t-2) - 2 \varepsilon(t-2)$$

\mathcal{L}

$$x(s) = -\frac{d}{ds} \left(+\frac{1}{s} e^{-2s} \right) - 2 \frac{1}{s} e^{-2s}$$

$$= \frac{2e^{-2s}}{s} - \frac{2e^{-2s}}{s} ?$$

$$= -\frac{e^{-2s}}{s} ??$$

Ex.2.8 Laplace transform for initial value problems

Solve the following two differential equations with initial conditions below using the Laplace transform and sketch the respective solution $x(t)$.

- a) $\dot{x}(t) + x(t) = 0 \quad x(0_-) = 1$
- b) $\dot{x}(t) + x(t) = \varepsilon(t) \quad x(0_-) = 0$
- c) $\dot{x}(t) + x(t) = \delta(t) \quad x(0_-) = 0$

a) DE : $\dot{x}(t) + x(t) = 0 \quad \xrightarrow{\mathcal{L}} \text{AE} : sX(s) - x(0_-) + X(s) = 0$

$$X(s)(s+1) = 1$$

$$X(s) = \frac{1}{s+1}$$

b) DE : $\dot{x}(t) + x(t) = \varepsilon(t) \quad \xrightarrow{\mathcal{L}}$

$$sX(s) - x(0_-) + X(s) = \frac{1}{s}$$

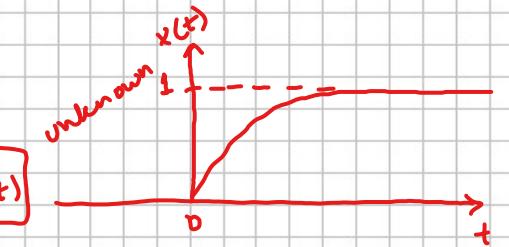
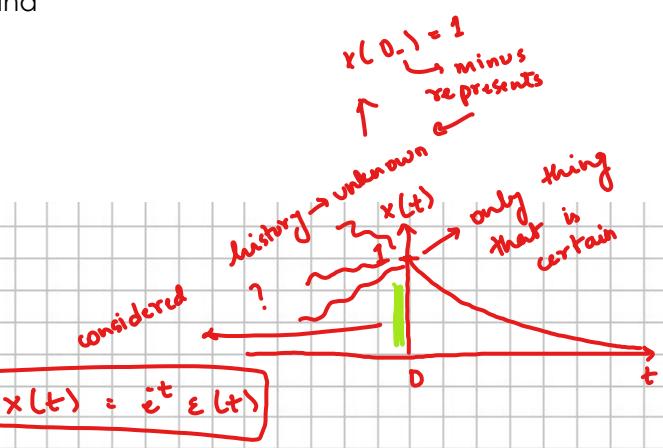
$$X(s)(s+1) = \frac{1}{s}$$

$$X(s) = \frac{1}{s(s+1)}$$

c) DE : $\dot{x}(t) + x(t) = \delta(t) \quad \xrightarrow{\mathcal{L} \text{ direct pulse}}$

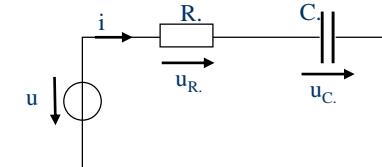
$$X(s)(s+1) = 1$$

$$X(s) = \frac{1}{s+1}$$



Ex.2.9 Laplace transform for initial value problems

Before $t = 0$, the capacitor is charged to $u_C(0)$. At $t = 0$ the circuit is operated with a sinusoidal voltage $u(t)$ of frequency $\omega = 1$ ($R = 1 \text{ Ohm}$, $C = 1 \text{ Farad}$). Calculate the exact course of the voltage $u_C(t)$ using the techniques of the Laplace transform.



DE :

$$\frac{RC}{L} \cdot \dot{u}_c(t) + u_c(t) = u(t)$$

$$\dot{u}_c(t) + u_c(t) = \sin(t) \cdot e(t)$$

$$AE : sU_c(s) - u_c(0-) + U_c(s) = \frac{1}{s^2 + 1}$$

$$U_c(s)(s+1) = \frac{1}{s^2 + 1} + u_c(0-)$$

$$U_c(s) = \frac{1}{(s^2+1)(s+1)} + \frac{u_c(0-)}{s+1}$$

↓ Partial Fraction Differentiation

$$U_c(s) = \frac{0.5}{s+1} + \frac{0.5 - 0.5s}{s^2 + 1} + \frac{u_c(0-)}{s+1}$$

↓

$$u_c(t) = 0.5 [e^t + \sin(t) - \cos(t) + 2u_c(0-)e^t] \cdot e(t)$$

$$u_c(t) = 0.5 [e^t (1 + 2u_c(0-)) + \sqrt{2} \sin(t - \pi/4)] \cdot e(t)$$

Transient
sol.

Static sol.

The Transfer Function

- For LTI systems without prehistory the transfer function is defined as:

$$\dots + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = \dots + b_2 \ddot{u}(t) + b_1 \dot{u}(t) + b_0 u(t)$$

$\Downarrow L$

$$\begin{aligned} y^i(0_-) &= 0 & (= \text{zero state}) \\ u^i(0_-) &= 0 \end{aligned}$$

$$\dots + a_2 s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = \dots + b_2 s^2 U(s) + b_1 s U(s) + b_0 U(s)$$

$$Y(s) \cdot [\dots + a_2 s^2 + a_1 s + a_0] = U(s) \cdot [\dots + b_2 s^2 + b_1 s + b_0]$$

Transfer Function:

- It is (in the image domain) the **output** in relation to the **input**:

$$\underline{G(s)} = \frac{Y(s)}{U(s)} = \frac{\dots + b_2 s^2 + b_1 s + b_0}{\dots + a_2 s^2 + a_1 s + a_0}$$

function of 2 components

System Equation:

$$Y(s) = G(s) \cdot U(s)$$

any LTI function $\xrightarrow[\text{LTF}]{\text{DE}}$

Impulse Response:

(or weight function)

$$g(t) \circ - \bullet G(s)$$

(fraction of 2 functions)

Step response:

$$h(t) \circ - \bullet G(s) \cdot \frac{1}{s}$$

$$\Rightarrow h(t) = \int_0^t g(\tau) d\tau$$

Example Transfer function of the RC element, $R = 2\Omega$, $C = 1F$:

$$\text{DE: } RC \dot{u}_c(t) + u_c(t) = u(t)$$

$\xrightarrow[\text{L}]{}$

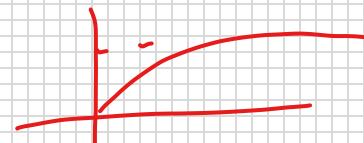
(zero state!)

$$\text{AE: } RC s u_c(s) + u_c(s) = u(s)$$

$$u(s)(RCs + 1) = u(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{U_c(s)}{U(s)} = \frac{1}{RCs + 1} = \frac{1}{2s + 1}$$

exponential curve



Ex.2.10 Transfer function

a) Give the transfer function $G(s)$ of the system.

$$DE: m\ddot{y}(t) + c\dot{y}(t) = F(t)$$

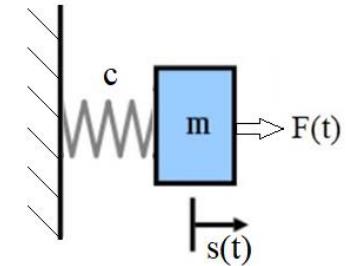
$\int \quad \int$

$$AE: m s^2 Y(s) + c Y(s) = U(s)$$

$$Y(s)(ms^2 + c) = U(s)$$

$$G = \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + c}$$

[only for zero state
nothing has happened yet]



Transfer function (PT2 element)

a) Give the transfer function $G(s)$ of the system.

$$DE: L C \ddot{u}_c(t) + \frac{L}{R} \dot{u}_c(t) + u_c(t) = \frac{L}{R} \dot{u}_e(t)$$

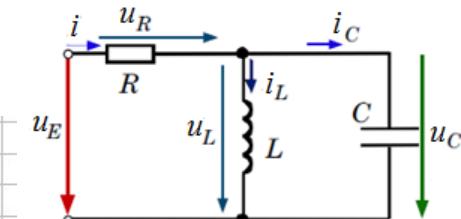
$\int \quad \int$

$$AE: LC s^2 Y(s) + \frac{L}{R} s Y(s) + Y(s) = \frac{L}{R} s U(s)$$

$$Y(s) \left(LC s^2 + \frac{L}{R} s + 1 \right) = \frac{L}{R} s U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\frac{Ls}{R}}{LC s^2 + \frac{Ls}{R} + 1}$$

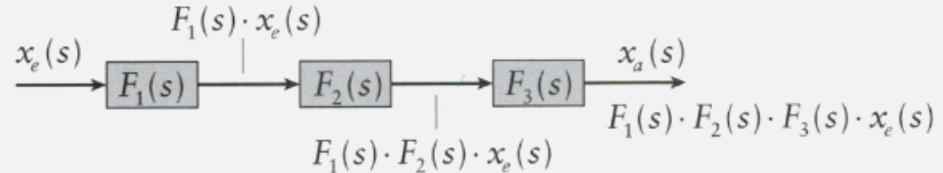
[zero state]



Transfer Functions of Complex Systems

Transfer functions of a

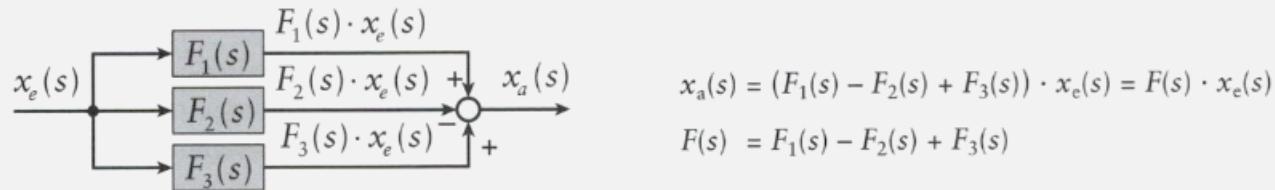
A) Chain structure:



$$x_a(s) = F_1(s) \cdot F_2(s) \cdot F_3(s) \cdot x_e(s) = F(s) \cdot x_e(s)$$

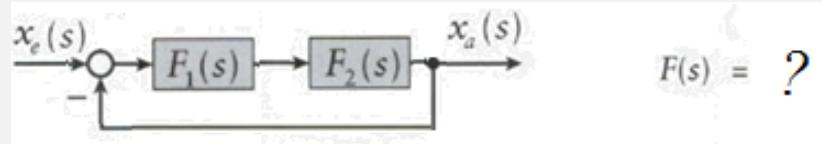
$$F(s) = F_1(s) \cdot F_2(s) \cdot F_3(s)$$

B) Parallel structure:



C) Circle structure:

control loop



$$\begin{aligned}x_a(s) &= F_2(s) f_1(s) (x_e(s) - x_a(s)) \\f(s) &= ?\end{aligned}$$

Ex.2.12 Transfer function of complex systems

- a) Calculate for the two standard control loops ($G_S(s)$ is the process transfer function, $G_R(s)$ the controller transfer function) the total transfer function $G_{WY}(s)$, ie

$$G(s) = \frac{Y(s)}{W(s)}$$

In Laplace form

A) $y = G_S G_R (w - y)$

$$y(1 + G_S G_R) = G_S G_R w$$

$$G(s) = \frac{y(s)}{w(s)} = \frac{G_S(s) G_R(s)}{1 + G_S(s) G_R(s)}$$

Up to
RC circuit

$$= \frac{1}{2s+1}$$

$G_R \rightarrow$ only thing to be found

B) $y = G_S (w - G_R y)$

$$y(1 + G_S G_R) = G_S w$$

$$G(s) = \frac{y(s)}{w(s)} = \frac{G_S(s)}{1 + G_S(s) G_R(s)}$$

A)

- b) For the first standard control loop for a step-like input $w(t)$, calculate the response $y(t)$ at $t = \infty$ using the limit theorem of the Laplace transform.

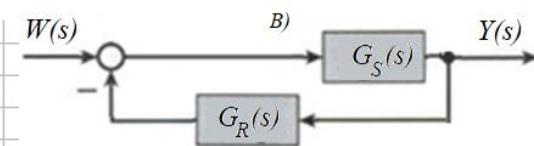
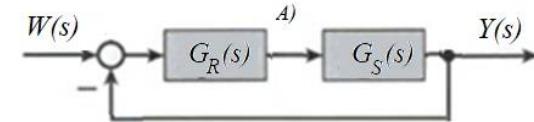
The shown partial transfer functions are given:

$$G_R(s) = K \quad G_S(s) = \frac{s+1}{s^2 + 2s + 2}$$

$$\lim_{t \rightarrow \infty} y(t) \Big|_{w(t) = \text{const.}} = \lim_{s \rightarrow 0} s y(s) \Big|_{w(s) = \frac{1}{s}} = \lim_{s \rightarrow 0} s \cdot G(s) w(s) \Big|_{w(s) = \frac{1}{s}} = \lim_{s \rightarrow 0} \frac{s \cdot G(s)}{s+1} \frac{1}{s} = \lim_{s \rightarrow 0} \frac{1}{s+1} = \frac{1}{2}$$

$$= \lim_{s \rightarrow 0} \frac{K \frac{s+1}{s^2 + 2s + 2}}{1 + K \frac{s+1}{s^2 + 2s + 2}} = \frac{K/2}{1 + K/2} = \frac{2K/2}{2+K} = \boxed{\frac{K}{2+K}}$$

(need huge values of K)



2.9 Transfer Function of complex systems

a) Find the overall transfer function $G_{\text{total}}(s)$ of the system shown.

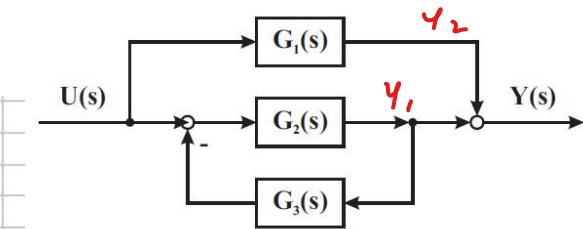
$$y_2 = uG_1$$

$$y_1 = G_2(u - G_3 y_1)$$

$$y_1 = \frac{G_2 u}{1 + G_2 G_3}$$

$$y = y_1 + y_2 = uG_1 + \frac{G_2 u}{1 + G_2 G_3}$$

$$G(s) = \frac{y(s)}{u(s)} = G_1 + \frac{G_2}{1 + G_2 G_3} = \frac{G_1 + G_1 G_2 G_3 + G_2}{1 + G_2 G_3}$$

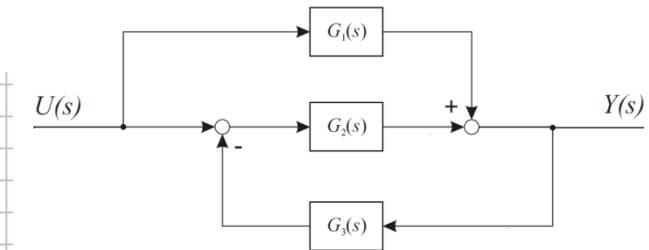


b) Find the overall transfer function $G_{\text{total}}(s)$ of the system shown.

$$y = (G_2(u - G_3 y) + G_1 u)$$

$$y = (1 + G_2 G_3) = u(G_1 + G_2)$$

$$G(s) = \frac{y(s)}{u(s)} = \frac{G_1(s) + G_2(s)}{1 + G_3(s)G_2(s)}$$

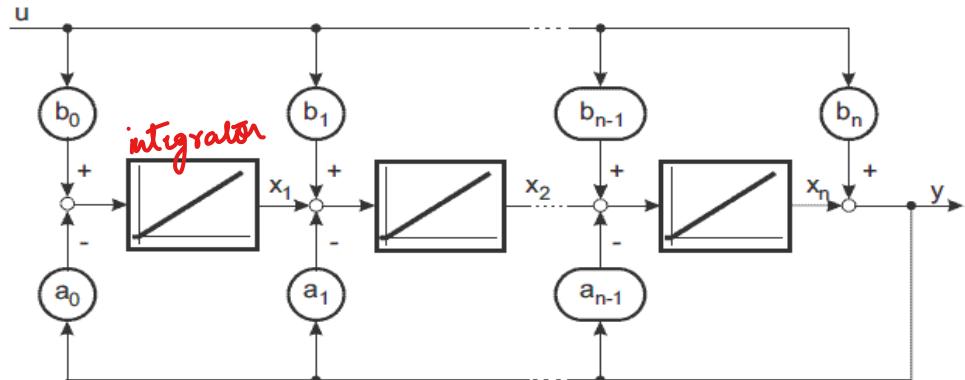


SW Implementation of a Transfer Functions

First canonical structure:

Derivation e.g. in Otto Foellinger; Control technology.

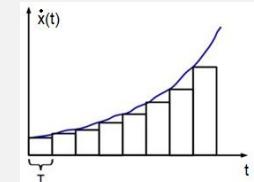
$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + \dots + b_2 s^2 + b_1 s + b_0}{s^n + \dots + a_2 s^2 + a_1 s + a_0}$$



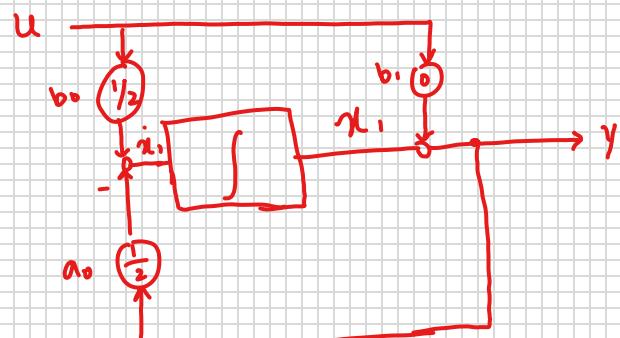
Integration in SW loop:

For example, if you want a filter $G(s)$ in a **SW loop (digital)** realize, the above canonical structures are also used directly as a prototype, if you are doing the integrators for the states x_i replaced by a numerical approximation:
(T is the so-called sampling time, or the time for one loop run.)

$$\boxed{x_i = x_{i_old} + T \cdot \dot{x}_i}$$



- Example: RC element as a digital filter: $G(s) = \frac{Y(s)}{U(s)} = \frac{1}{1+2s} = \frac{\frac{1}{2}}{s + \frac{1}{2}} = \frac{b_0 u}{s + a_0} \tau = RC = 2 \text{ sec}$ $n = 1$



software code

```

x1 = 0, b1 = 0.5, b0 = 0, a0 = 0.5, T = 1 (ms)
while (1) {
    y = x1 + b0 * u;      [ x1 + 0.5 * u ]
    x1p = b1 * u - a0 * y;
    x1 = x1 + T * x1p;
    y
}

```

5.11 Inverse Laplace transform by Partial fraction decomposition

a) Calculate, for example, the inverse Laplace transform of functions a) to c) using partial fraction decomposition and a correspondence table.

PFD

$$a) G(s) = \frac{5s+11}{(s+5) \cdot (s-2)}$$

$$c) G(s) = \frac{2s-3}{s-3}$$

$$b) G(s) = \frac{s^2 + 4s}{(s+1)^2 \cdot (s+2)}$$

a) PFD $5s+11 = A(s-2) + B(s+5)$
 $= As - 2A + Bs + 5B$

$$\begin{aligned} s(A + B) &= 5 \\ A+B &= 5 \\ 5B - 2A &= 11 \end{aligned} \quad \left. \begin{array}{l} A=5-B \\ 5B - 2(5-B) = 11 \\ B=3 \end{array} \right\} \quad \begin{array}{l} A=2 \\ 5B - 2B + 10 = 11 \end{array}$$

Subsystems of Systems

$$F(s) = \frac{\dots + b_2 s^2 + b_1 s + b_0}{\dots + a_2 s^2 + a_1 s + a_0} = c \cdot \frac{\dots (s - \beta_2)(s - \beta_1)}{\dots (s - \alpha_2)(s - \alpha_1)} \xrightarrow{\text{PBZ}} F(s) = F_1(s) + F_2(s) + F_3(s) + \dots$$

Possible partial fractions:

A) Constant:

$$F_1(s) = K$$

B) Simple real pole:

$$F_2(s) = \frac{A}{s - \alpha}$$

C) Multiple real pole:

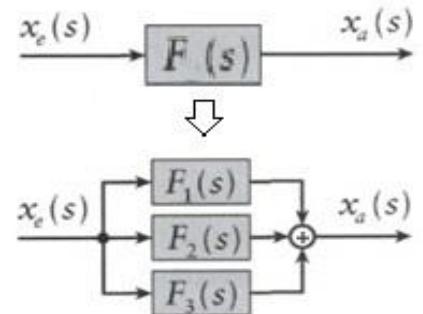
$$F_3(s) = \frac{A_n}{(s - \alpha)^n} + \dots + \frac{A_2}{(s - \alpha)^2} + \frac{A_1}{s - \alpha}$$

D) Complex Pole pair:

- E.g.: $s^2 + 2s + 2$

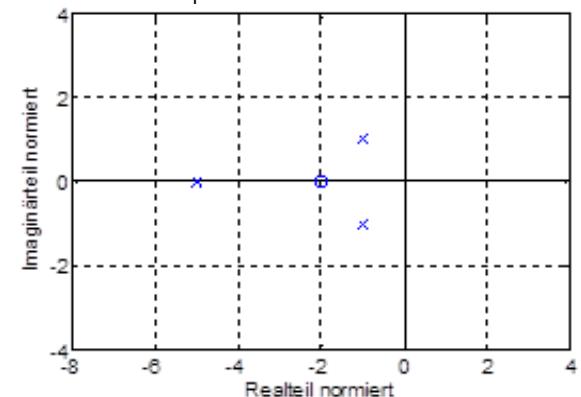
$$s_{\infty 1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm j$$

$\Rightarrow DT_2$ derived PT_2
[not PT_2]



PN diagram

x poles O zeros



Impulse responses:

A) Constant:

$$K \quad \bullet - \circ \quad K \cdot \delta(t)$$

B) Simple real pole:

$$\frac{1}{s - \alpha} \quad \bullet - \circ \quad e^{\alpha t} \cdot \varepsilon(t)$$

C) Multiple real pole:

$$\frac{1}{(s - \alpha)^2} \quad \bullet - \circ \quad t \cdot e^{\alpha t} \cdot \varepsilon(t)$$

D) Complex Pole pair:

$$\frac{1}{s - (\alpha + j\omega)} + \frac{1}{s - (\alpha - j\omega)} \quad \bullet - \circ \quad e^{\alpha t} (e^{j\omega t} + e^{-j\omega t}) \cdot \varepsilon(t) = 2e^{\alpha t} \cos(\omega t) \cdot \varepsilon(t)$$

$$\frac{1}{s - (\alpha + j\omega)} \quad \bullet - \circ \quad e^{(\alpha+j\omega)t} \cdot \varepsilon(t) = e^{\alpha t} e^{j\omega t} \cdot \varepsilon(t)$$

Poles determine stability and dynamics of a system, the zeros only its intensity!

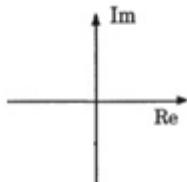
3.3 The P-Element

- Proportional systems are systems in which there is a directly proportional relationship between input and output.

Transfer Function:

$$G(s) = \frac{x_a(s)}{x_e(s)} = K_s$$

PN diagram:



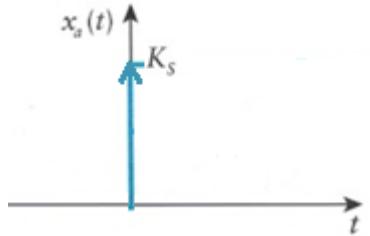
DE:

Due to the proportionality between x_a and x_e The (trivial) DE applies here:

$$x_a(t) = K_s \cdot x_e(t)$$

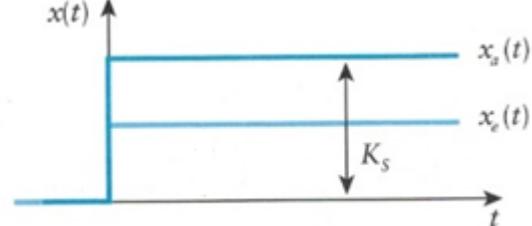
Impulse response:

$$g(t) = K_s \cdot \delta(t)$$

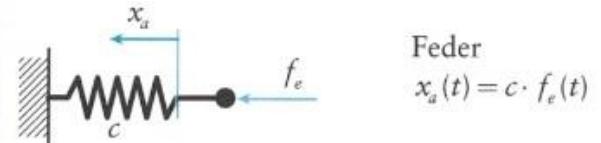


Step response:

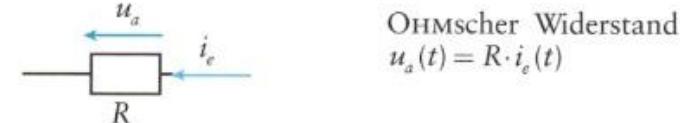
$$h(t) = K_s \cdot \hat{x}_e \cdot \varepsilon(t)$$



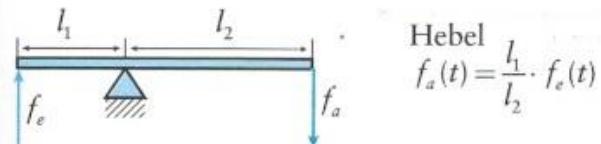
PN diagram:



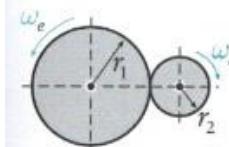
Feder
 $x_a(t) = c \cdot f_e(t)$



OHMSCHER WIDERSTAND
 $u_a(t) = R \cdot i_e(t)$



Hebel
 $f_a(t) = \frac{l_1}{l_2} \cdot f_e(t)$



Getriebe
 $\omega_a(t) = \frac{r_1}{r_2} \cdot \omega_e(t)$

Beispiele für P-Strecken (P-Übertragungsglieder)

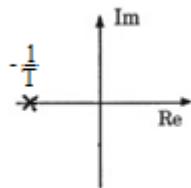
The PT₁-Element

- In the case of 1st order proportional elements, the system does not react immediately but with a **delay**.

Transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{T \cdot s + 1} = \frac{\frac{K}{T}}{s + \frac{1}{T}}$$

PN diagram:

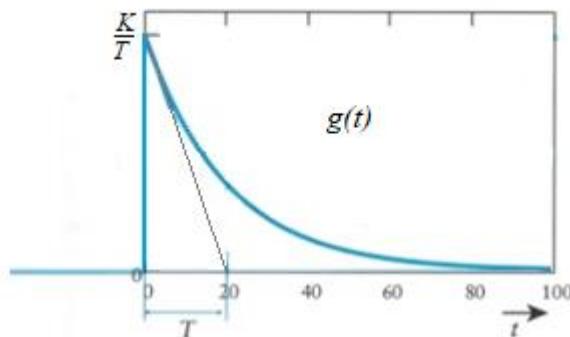


- DE:**

$$T \cdot \dot{y}(t) + y(t) = K \cdot u(t)$$

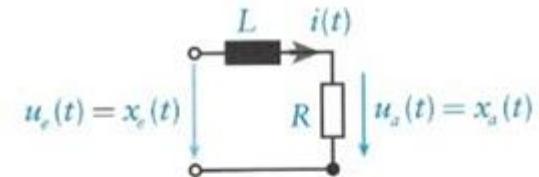
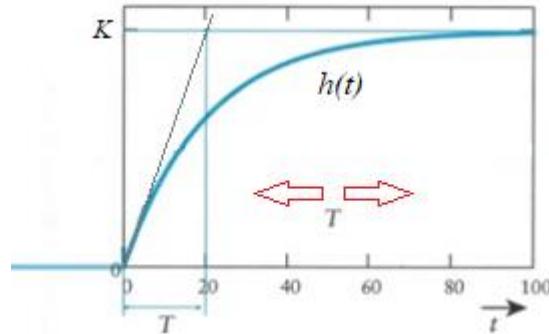
Impulse response:

$$g(t) = \frac{K}{T} \cdot e^{-\frac{t}{T}} \cdot \varepsilon(t)$$

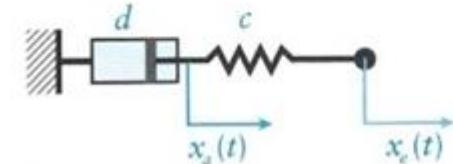


Step response:

$$h(t) = K \cdot (1 - e^{-\frac{t}{T}}) \cdot \varepsilon(t)$$



$$\frac{L}{R} \cdot \dot{u}_d + u_d = u_e \Rightarrow T \cdot \dot{x}_d + x_d = x_e$$



$$d \cdot \dot{x}_d + c \cdot x_d = c \cdot x_e \Rightarrow T \cdot \dot{x}_d + x_d = x_e$$

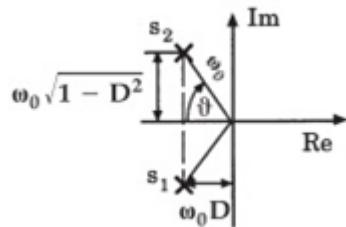
- The more negative a pole is, the faster is the reaction positive poles lead to unstable systems (positive exponent!).

The PT₂-Element

Transfer function:

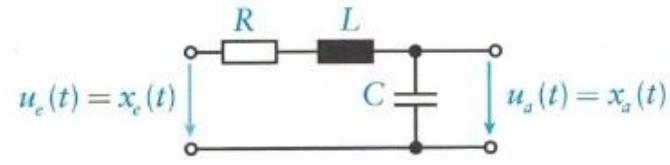
$$G(s) = \frac{K}{a_2 \cdot s^2 + a_1 \cdot s + a_0}$$

PN diagram:



▪ **DE:**

$$a_2 \cdot \ddot{y}(t) + a_1 \cdot \dot{y}(t) + a_0 y(t) = K \cdot u(t)$$



$$LC \frac{d^2 u_a(t)}{dt^2} + RC \frac{du_a(t)}{dt} + u_a(t) = u_e(t)$$

Normalized transfer function:

- To simplify the analysis, it is better to use a standardized representation the new sizes **Degree of damping D** and (undamped) **Natural frequency ω_0** , you get:

$$G_0(s) = \frac{K \cdot \omega_0^2}{s^2 + 2D\omega_0 \cdot s + \omega_0^2}$$

Impulse response (-1 < D < 1):

$$g(t) = K \cdot \frac{\omega_0}{\sqrt{1-D^2}} \cdot e^{-D\omega_0 t} \cdot \sin(\sqrt{1-D^2} \omega_0 t) \cdot \varepsilon(t)$$

Step response (-1 < D < 1):

$$h(t) = K \cdot \left[1 - \frac{1}{\sqrt{1-D^2}} \cdot e^{-D\omega_0 t} \cdot \sin(\sqrt{1-D^2} \omega_0 t + \vartheta) \right] \cdot \varepsilon(t) \quad \text{mit } \vartheta = \arccos(D)$$

For other values of D there are real poles and therefore no sine functions ...

Step Responses of the PT₂-Element (Stability for PT₂)

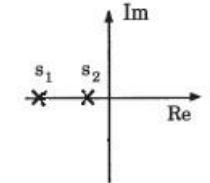
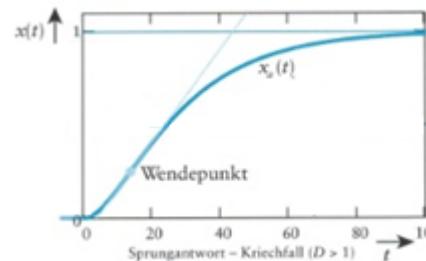
Step responses:

Pole: $s_{\infty 1,2} = -D\omega_0 \pm \omega_0 \sqrt{D^2 - 1}$

- D > 1 (crawling behavior):

PT₂ = Series connection two different PT₁-Links ($T_1 <> T_2$):

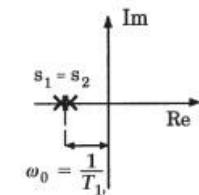
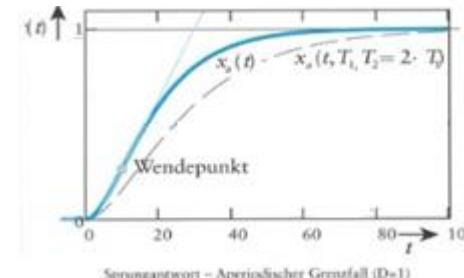
$$x_a(t) = [1 - \frac{1}{T_1 - T_2} \cdot (T_1 e^{-\frac{t}{T_1}} - T_2 e^{-\frac{t}{T_2}})] \cdot \varepsilon(t)$$



- D = 1 (aperiodic limit case):

PT₂ = Series connection two of the same PT₁-Links ($T_2 = 2T_1$):

$$x_a(t) = [1 - (1 + \frac{t}{T_1}) e^{-\frac{t}{T_1}}] \cdot \varepsilon(t)$$

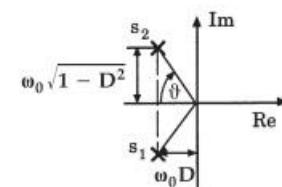
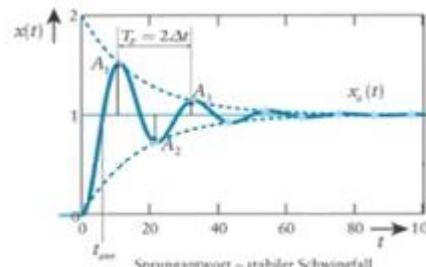


- D < 1 (stable vibration behavior):

At D = 0.707, the shortest possible rise is just achieved without overshooting!

The frequency to be observed ω and the period T_P , are:

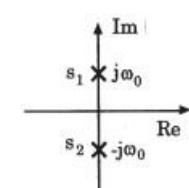
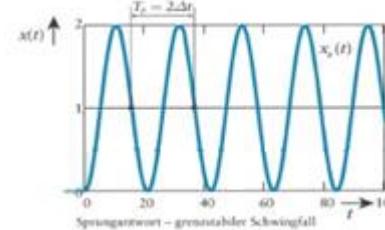
$$\omega = \omega_0 \cdot \sqrt{1 - D^2} \quad ; \quad T_P = \frac{2\pi}{\omega_r}$$



- D = 0 (limit-stable vibration behavior):

Without damping, the step response oscillates permanently with a constant amplitude

$$x_a(t) = [1 - \cos(\omega_0 t)] \cdot \varepsilon(t)$$

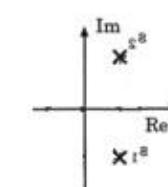
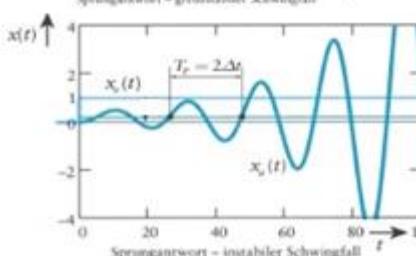


- D < 0 (unstable vibration behavior):

With negative damping, the step response oscillates with increasing amplitude.

$$x_a(t) = [1 - \frac{1}{\sqrt{1 - D^2}} \cdot e^{-D\omega_0 t} \cdot \sin(\sqrt{1 - D^2} \omega_0 t + \vartheta)] \cdot \varepsilon(t)$$

$$\vartheta = \arccos(D) \quad \text{für } -1 < D < 1$$



- Conclusion: Complex conjugate poles oscillate, real ones do not, the imaginary part corresponds to the frequency, the real part of the decay**

Ex.3.3 Step response of the PT2 system

a) Sketch the step responses of the systems:

$$G_0(s) = \frac{K\omega_0^2}{s^2 + 2D\omega_0 \cdot s + \omega_0^2}$$

A) $G(s) = \frac{2}{s^2 + 4s + 1}$

B) $G(s) = \frac{2}{s^2 + 2s + 1}$

C) $G(s) = \frac{2}{s^2 + s + 1}$

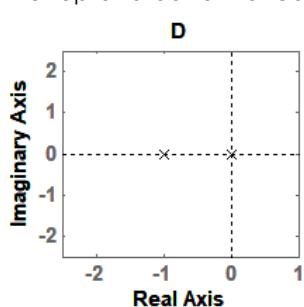
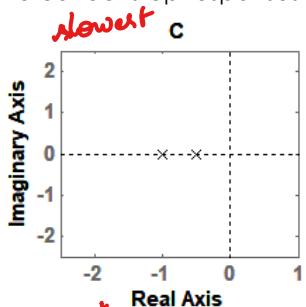
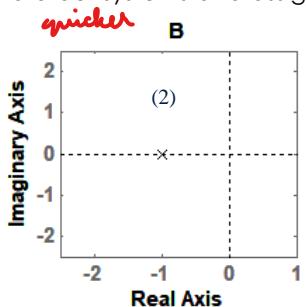
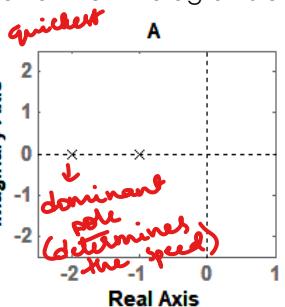
D) $G(s) = \frac{2}{s^2 + 1}$

E) $G(s) = \frac{2}{s^2 - 4s + 1}$

Ex.3.4 Step responses of PT2-systems

a) Look at the PN diagrams of 2nd order systems and assign the correct step responses at the top and bottom of each.

$D \rightarrow b/w$ stable ξ_1 unstable \rightarrow limit stable



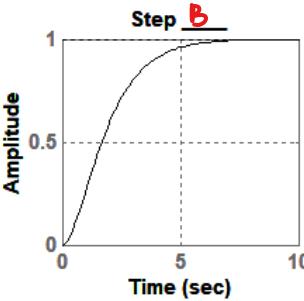
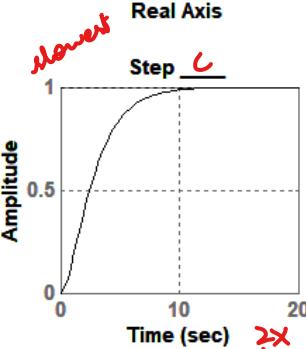
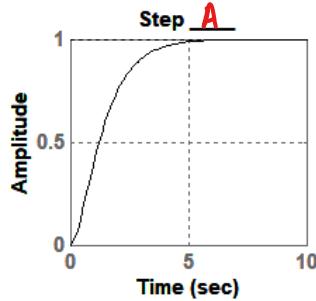
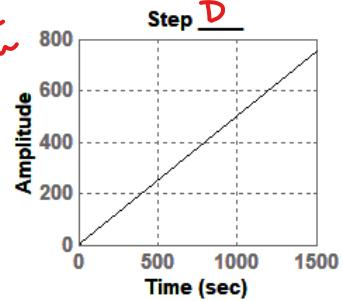
stable

A) $G(s) = \frac{A}{s+2} + \frac{B}{s+1}$

D) $G(s) = \frac{1}{s+1} + \frac{B}{s}$

A) C) B) — BIBO

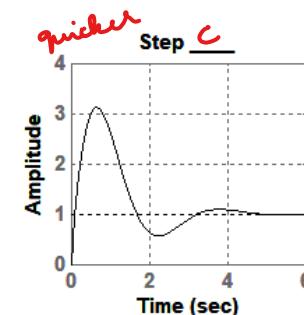
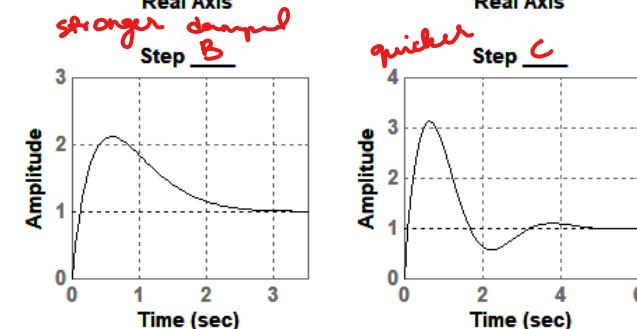
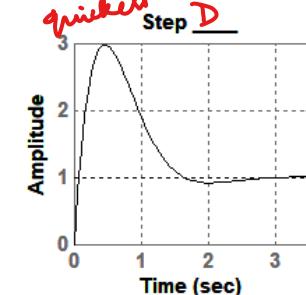
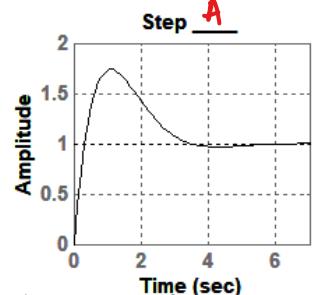
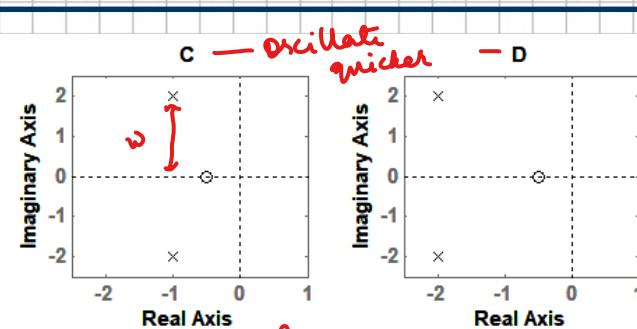
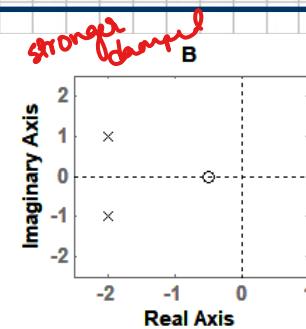
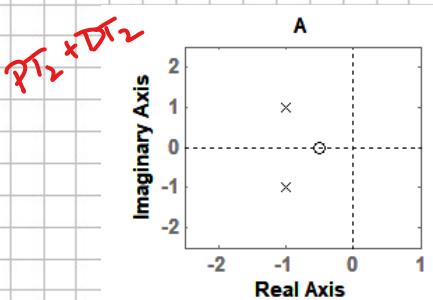
D) — not limited
O/P ∞



c) D) $\omega = 2^{\frac{1}{2}}$ $\omega = \sqrt{1-D} \omega_0$

A) B) $\omega = 1^{\frac{1}{2}}$

Damping \rightarrow more left \rightarrow quicker damping



Realizability of Transfer Functions

$$\begin{aligned} m=n \\ G(s) &= \frac{4s+3}{2s+1} \quad \text{PD} \\ &= 2 + \frac{1}{2s+1} \\ &\sim \underbrace{2}_{P} + \underbrace{\frac{1}{2s+1}}_{\text{PT}_1} \end{aligned}$$

$$\begin{aligned} m>n \\ G(s) &= \frac{4s^2+3s+2}{s+1} \quad \text{PD} \\ &= \underbrace{4s-1}_{\text{D}} + \underbrace{\frac{3}{s+1}}_{\text{PT}_1} \\ &\sim \cancel{s+1} \cancel{4s^2+3s+2} - \cancel{s+1} + \cancel{2} \end{aligned}$$

Derivative System - causes a problem (cannot be realized properly)

Transfer functions with numerator degree M = denominator degree N:

- If $M = N$, there will be a jump in the step response, the system is called jumpable!
- Conversely, systems with $M < N$ are not jumpable!

If $M = N$, after the PBZ there is a P-element and a ÜF with $M < N$. the split-off P-element leads to a Jump in the step response.

Transfer functions with numerator degree M > denominator degree N:

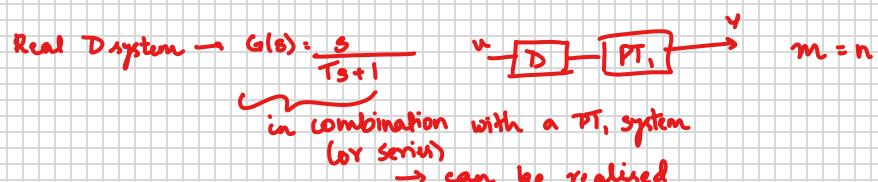
- If $M > N$, the system is not causal and thus not realizable!

If $M > N$, a **Polynomial division** be performed. It results in a sum of **Powers in s**, a **P-term** and one ÜF with $M < N$. The split-off powers in s correspond to derivatives of the Input signal in the time domain. The derivative of a signal $u(t)$ can be math. Are described as:

$$y(t) = \frac{du(t)}{dt} \approx \frac{u(t + \Delta t) - u(t - \Delta t)}{2 \cdot \Delta t}$$

This means that input signals that lie in the future are also used to calculate ideal derivatives. This is not possible!

A perfect D system has a $G(s) = s$, is not causal & cannot be realized → causes problems.



Stability of Systems

Stability criterions in the time domain:

BIBO-Stability:

- A continuous LTI system is exactly BIBO-stable when its impulse response $g(t)$ can be absolutely integrated, i.e. the step response remains finite in any case.

$$\int_0^{\infty} |g(t)| dt < \infty$$

BIBO bounded

$$|y(t)| \cdot \left| \int_{-\infty}^t g(z) u(t-z) dz \right| \leq \int_{-\infty}^t |g(z)| |u(t-z)| dz \leq \int_{-\infty}^{\infty} |g(z)| \cdot u_{\max} dz$$

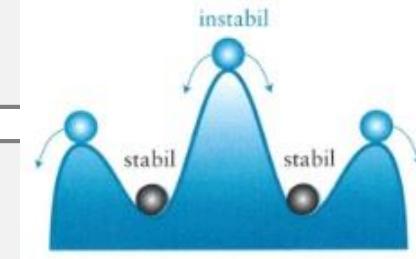
area under curve is not limited, not stable

higher value ever will reach

$$= u_{\max} \int_{-\infty}^{\infty} |g(z)| dz < \infty$$

Asymptotically-Stability:

$$\lim_{t \rightarrow \infty} g(t) = 0$$



Stability criterion in the s-domain: *Laplace domain*

A continuous LTI system is

- stable,**

if all poles of the system function in the **open left s-half plane** lie, i.e. there are no poles on the imaginary axis or in the right s-half plane are located,

- borderline or quasi-stable, *limit stable***

if all poles lie in the left half-plane and only simple poles appear on the imaginary axis,

- unstable,**

as soon as a pole lies in the open right half-plane or a multiple pole lies on the imaginary axis.

Ex.2.16 System identification:

If you are measuring a pure I-element, you will receive the system answer shown here.

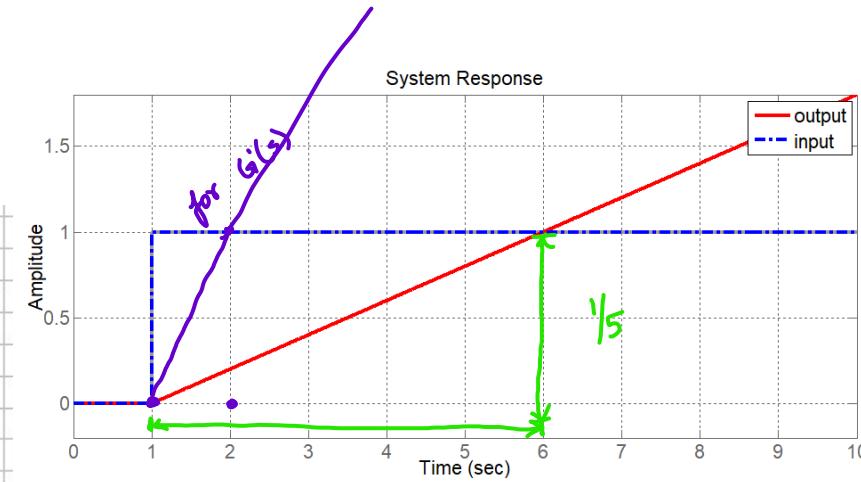
$$G(s) = \frac{1}{T_N s} = \frac{A}{s} ; \stackrel{\text{const}}{A} = ?$$

- a) Determine the integration time constant T from this measurement.

ideal $G(s) = \frac{1}{s}$ (slope - 1)

$$G(s) = \frac{1}{5s}$$

 \rightarrow taken 5 seconds to reach 1



If you are measuring an unknown system, you will receive the following system response.

$$G(s) = ?$$

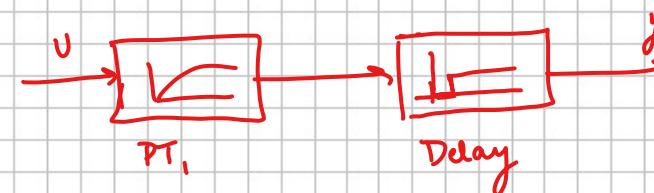
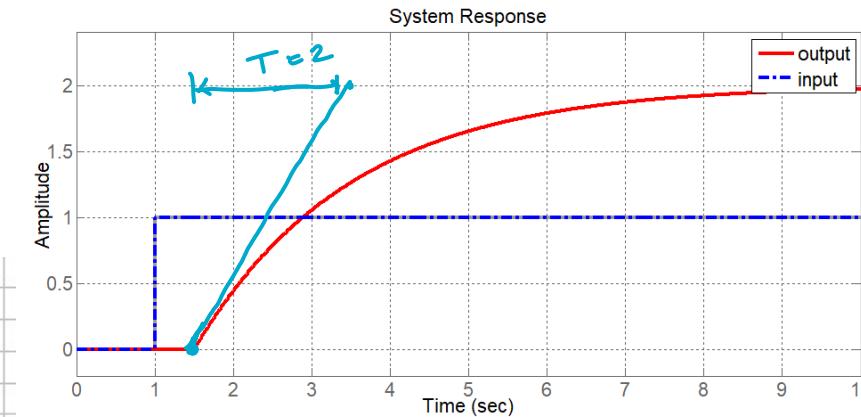
PT₁

- a) Give a transfer function G (s) that describes this system.
 (Note: this time it does not have to consist only of the quotient of two polynomials in s).

$$G(s) = \frac{K}{T_1 s + 1} \cdot e^{s t_0} \quad [K=2]$$

$x(s)$ shows that the $\frac{d}{dt}$ has a delay $t_0 = 0.5$

$$y(t) = u(t - t_0)$$



Existence of the Laplace Transform

- A Laplace transform $X(s)$ exists only if the **Laplace integral converges** (i.e. is finite for certain values of s).

$$X(s) = L\{x(t)\} = \int_0^\infty x(t) \cdot e^{-st} dt, \quad \text{mit } x(t) = 0 \quad \text{für } t < 0$$

- This condition is only fulfilled if $|x(t)|$ for $t \rightarrow \infty$ does **not grow faster than an exponential function!**

In this case the function $x(t)$ can be estimated with $|x(t)| \leq k \cdot e^{at}$.

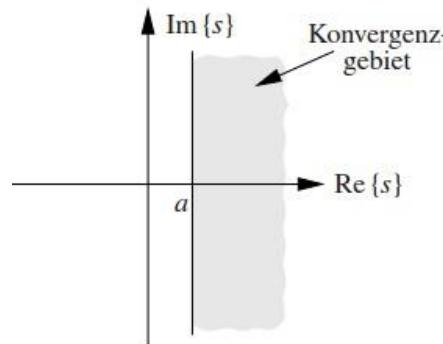
- Only if the **Re(s) > a**, the corresponding Laplace integral converges.

- Explanation:

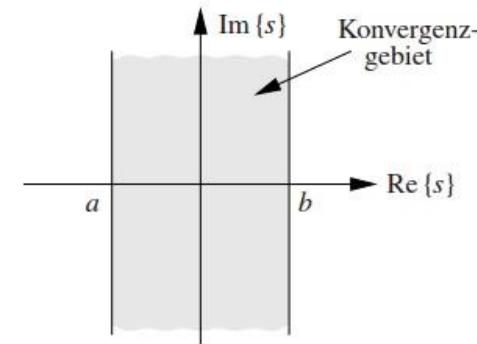
With $x(t) = e^{\lambda t} \cdot \varepsilon(t) = e^{(a+j\omega)t} \cdot \varepsilon(t)$

surrendered:

$$\begin{aligned} X(s) &= \int_0^\infty e^{\lambda t} \cdot \varepsilon(t) \cdot e^{-st} dt = \int_0^\infty e^{(\lambda-s)t} dt = \frac{1}{\lambda-s} \cdot e^{-(s-\lambda)t} \Big|_0^\infty \\ &= \lim_{t \rightarrow \infty} \frac{1}{\lambda-s} \cdot e^{-(s-\lambda)t} - \frac{1}{\lambda-s} \cdot e^{-(s-\lambda)0} \end{aligned}$$



a: einseitige Transformation



b: zweiseitige Transformation

Konvergenzgebiete der Laplace-Transformation

- For $\text{Re}(s - \lambda) > 0$:

strives for the exponential function for $t \rightarrow \infty$ to zero, the integral is therefore convergent, the Laplace transform **exists** for this area of the s-level.

- For $\text{Re}(s - \lambda) < 0$:

if the exponential function tends towards infinity, the integral is accordingly not convergent that Laplace transform **exists** for this area of the s-level **not**.

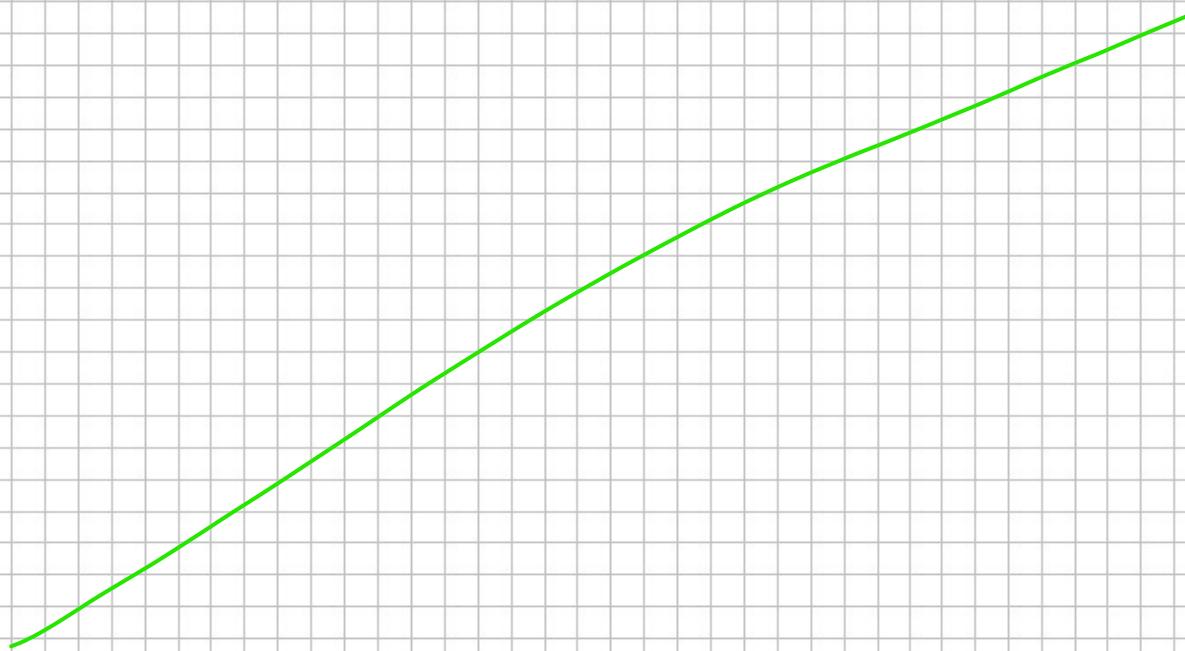
- The Laplace integral is convergent for the entire area of the s-plane with a gray background.

In the cases that are interesting from the point of view of system theory, the convergence of the Laplace integral can be assumed at least in part of the s-plane. Of the **Convergence area** the Laplace transformation is therefore useful for the computation of technically interesting cases of subordinate Meaning.

- Only with the Fourier transformation does the convergence range of the Laplace transformation become important again ...

Ex.2.17 Existence of the Laplace Transform

$$\mathcal{L}_{II}\{x(t)\} = \int_{-\infty}^{+\infty} x(t) \cdot e^{-st} dt$$

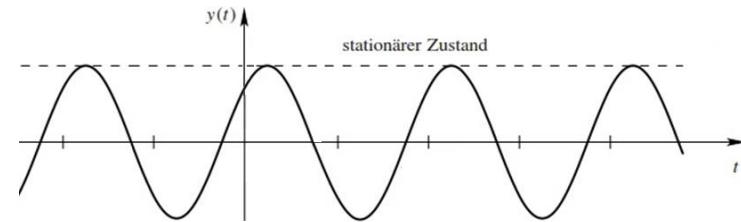
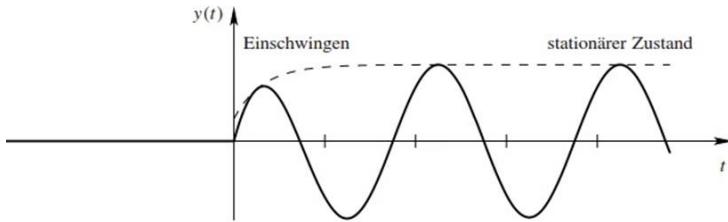


2.3 System Description via the Fourier Transform

Why the Fourier Transform now?

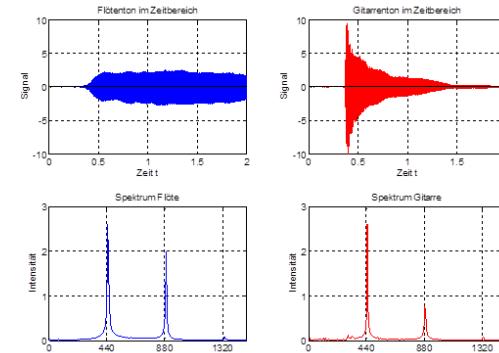
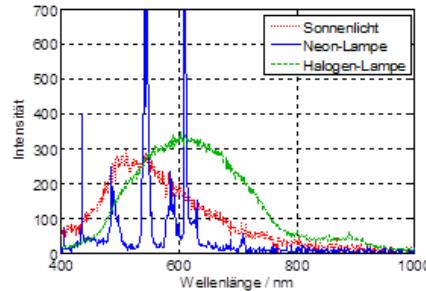
Motivation 1:

- If only **the stationary behavior of a system** (i.e. without transient processes) is of interest, the Fourier transform is much more suitable (= **frequency response**)



Motivation 2:

- The Fourier transformation is the generalization of the Fourier series and thus provides important information about the **spectral composition of a signal** (= **spectrum**).

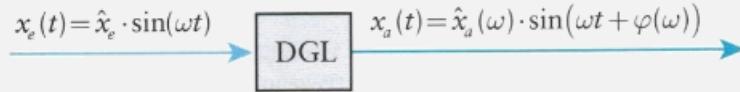


vibrations &
resonance
effect

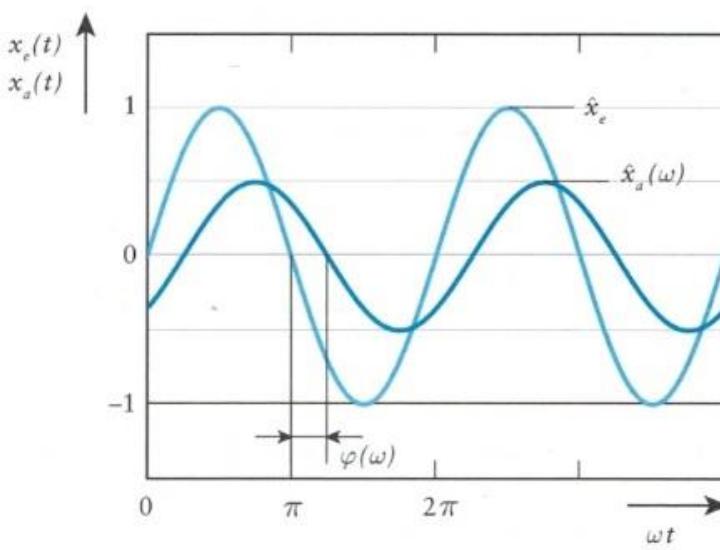
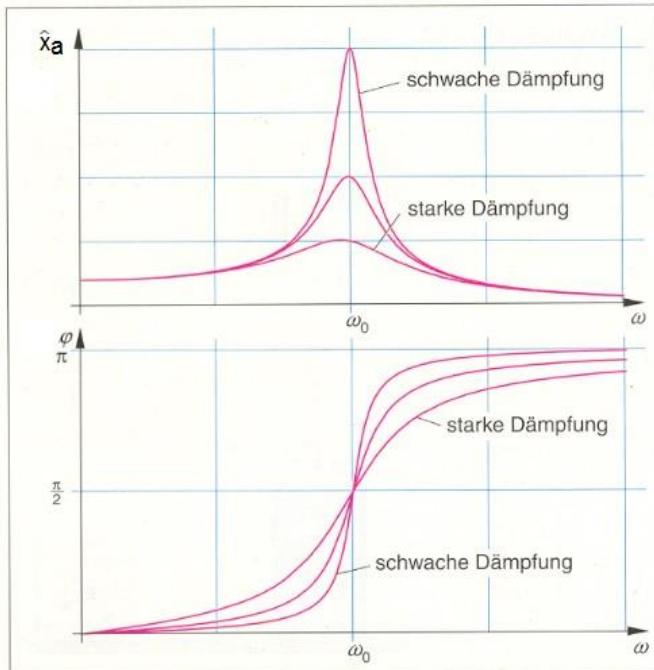
Sinusoidal Excitations

System responses with sinusoidal excitation:

- If a linear system is excited with a constant sinusoidal signal, the output signal oscillates with the **same frequency**, but generally with a **different amplitude and phase!**



Frequency response: Harmonic excitation in the time domain



Fourier Transform & Harmonic Excitation

Definition:

$$X(j\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

not causal

$$x(t) = F^{-1}\{X(j\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Differentiation rule:

$$\dot{x}(t) \circ - \bullet j\omega \cdot X(j\omega)$$

here $\boxed{s = j\omega}$

$$\ddot{x}(t) \circ - \bullet (j\omega)^2 \cdot X(j\omega)$$

Shift:

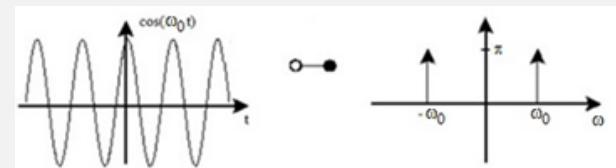
$$x(t + t_0) \circ - \bullet X(j\omega) \cdot e^{j\omega t_0}$$

Harmonic signal:

$$\cos(\omega_0 t) \circ - \bullet \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

Frequency response:

$$G(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = \frac{\dots + b_2(j\omega)^2 + b_1 j\omega + b_0}{\dots + a_2(j\omega)^2 + a_1 j\omega + a_0}$$



System equation:

$$Y(j\omega) = G(j\omega) \cdot U(j\omega)$$

$$Y(j\omega) = |G(j\omega)| \cdot e^{j\angle G(j\omega)} \cdot U(j\omega)$$

*Fourier
Transform gives direct
proportionalities*

Harmonic excitation:

$$\begin{aligned} Y(j\omega) &= |G(j\omega)| \cdot e^{j\angle G(j\omega)} \cdot \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \\ &= |G(j\omega_0)| \cdot \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \cdot e^{j\angle G(j\omega_0)} \end{aligned}$$

Amplitude response:

$$|G(j\omega)| = \frac{|Y(j\omega)|}{|U(j\omega)|}$$

Phase response:

$$\varphi(j\omega) = \angle G(j\omega)$$

Comparison of Laplace and Fourier Transform

If you want to hide the transient processes, you would actually have to apply the two-sided Laplace transform (with all its convergence problems):

Fourier Transform:

$$X(j\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

System equation:

$$Y(j\omega) = G(j\omega) \cdot U(j\omega)$$

Laplace transformation (double-sided):

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$$

$$Y(s) = G(s) \cdot U(s)$$

Limited to $s = j\omega$, both transformations are absolutely identical.

For the important signal class, whose two-sided Laplace convergence region includes the imaginary axis, and which includes all causal and decaying signals, both transforms exist with the substitution:

$$X_{\text{Fourier}}(j\omega) = X_{\text{Laplace}}(s) \Big|_{s=j\omega}$$

only if the $G(s)$ is stable & causal

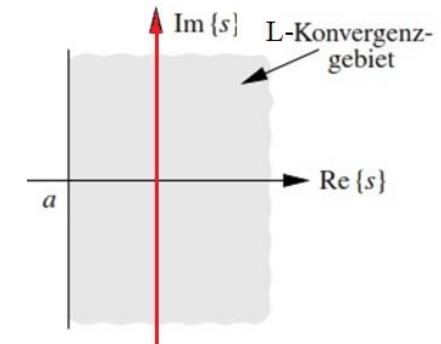
single sided Fourier transform

Frequency Response:

For the important system class whose Laplace convergence region includes the imaginary axis, and to which all stable systems belong, the so-called frequency response is obtained directly from the transfer function with the substitution:

$$G(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = G(s) \Big|_{s=j\omega}$$

$$= \frac{-b_2\omega^2 + b_1j\omega + b_0}{-a_2\omega^2 + a_1j\omega + a_0}$$



Sine-Signals (non-causal):

For the important signal class of (non-causal) constant sine signals actually exist the Fourier transform (with the help of **distributions**) !

$$X_{\text{Fourier}}(j\omega) = \int_{-\infty}^{\infty} \cos(\omega_0 t) e^{-j\omega t} dt = \frac{1}{2} \int_{-\infty}^{\infty} (e^{j\omega_0 t} + e^{-j\omega_0 t}) e^{-j\omega t} dt = \frac{1}{2} \int_{-\infty}^{\infty} (e^{-j(\omega-\omega_0)t} + e^{-j(\omega+\omega_0)t}) dt = \boxed{\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]}$$

Fourier Transform of Standard Signals

Fourier transform of the non-causal constant one:

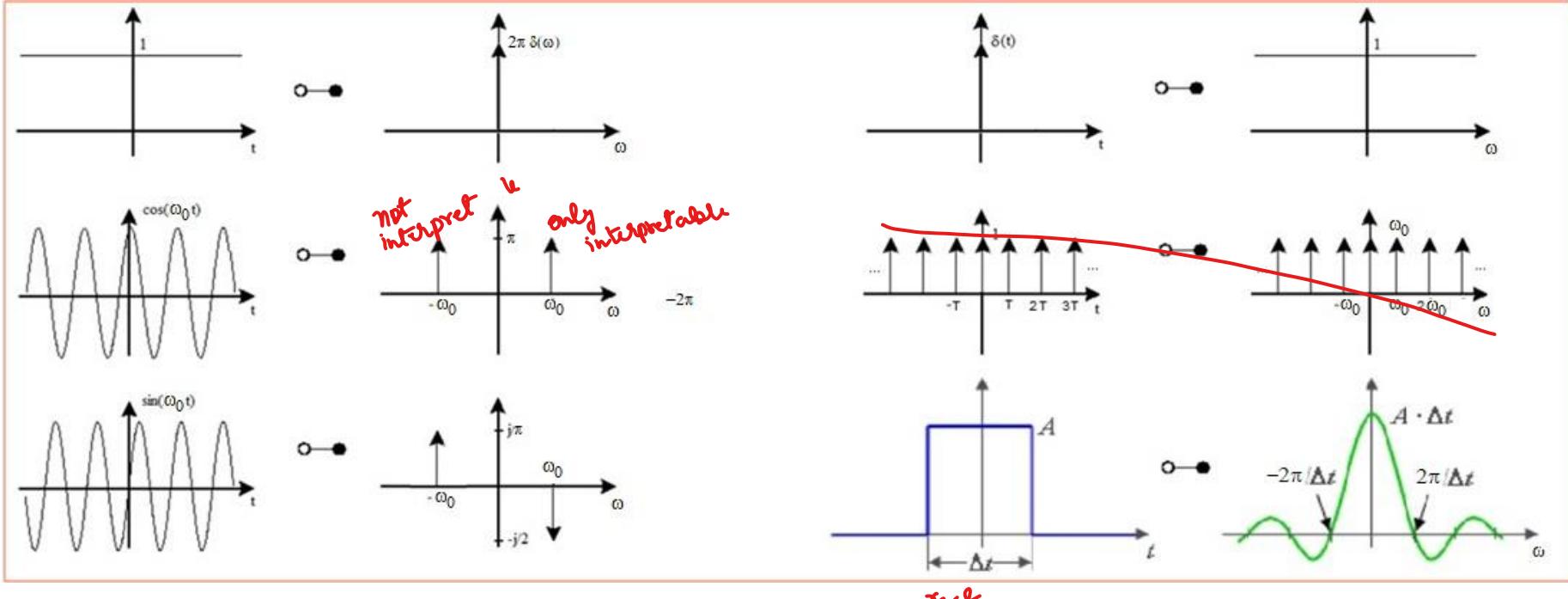
$$F\{1\} = \int_{-\infty}^{\infty} 1 \cdot e^{-j\omega t} dt = \lim_{t_0 \rightarrow \infty} \int_{-t_0}^{t_0} e^{-j\omega t} dt = \lim_{t_0 \rightarrow \infty} \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-t_0}^{t_0} = \lim_{t_0 \rightarrow \infty} \frac{1}{-j\omega} (e^{-j\omega t_0} - e^{j\omega t_0}) = \lim_{t_0 \rightarrow \infty} 2t_0 \frac{\sin(\omega t_0)}{\omega t_0} = 2\pi\delta(\omega)$$

$$L\{1\} = \int_{-\infty}^{\infty} 1 \cdot e^{-st} dt = \lim_{t_0 \rightarrow \infty} \int_{-t_0}^{t_0} e^{-st} dt = \lim_{t_0 \rightarrow \infty} \frac{-1}{s} e^{-st} \Big|_{-t_0}^{t_0} = \lim_{t_0 \rightarrow \infty} \frac{1}{s} (e^{-st_0} - e^{st_0}) \Rightarrow \infty$$


- Conclusion:** The two-sided Laplace transform for $x(t) = 1$ does not exist, the Fourier transform, (introducing distributions) but does!!!

Physically of course there are no negative frequencies, but the Euler equation leads to clearer calculations and therefore also introduces the negative frequency!

learn all



The Frequency Response

Frequency Response:

- For stable systems, the transfer function is used to obtain the **so-called frequency response after the substitution:**

$$G(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = G(s) \Big|_{s=j\omega} = \frac{-b_2\omega^2 + b_1j\omega + b_0}{-a_2\omega^2 + a_1j\omega + a_0}$$

The frequency response is therefore a complex function depending on the frequency variable ω or $j\omega$.

$$G(j\omega) = |G(j\omega)| \cdot e^{j\varphi(j\omega)}$$

Amplitude Response:

- Describes for **each sinusoidal** signal with the frequency ω the ratio of the output to the input amplitude!

$$|G(j\omega)| = \frac{|Y(j\omega)|}{|U(j\omega)|}$$

Phase Response:

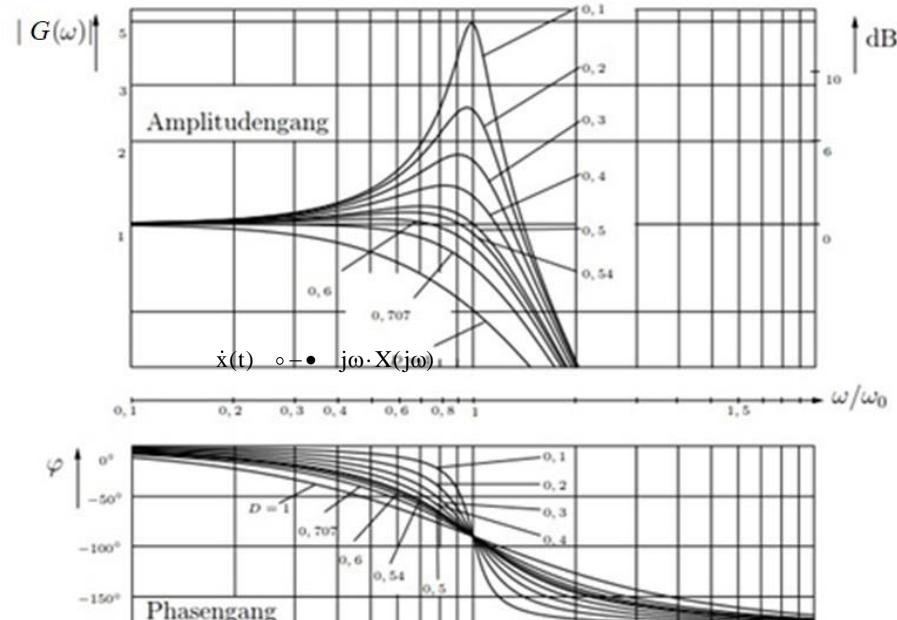
- describes the time shift between the output and input oscillation!

$$\varphi(j\omega) = \angle G(j\omega)$$

The (pseudo) unit Decibel (dB):

$$a(j\omega) = 20 \lg |G(j\omega)| = 10 \lg |G(j\omega)|^2$$

It is only used to rescale extremely large value ranges.



| Pegel | 0 dB | 3 dB | 4.8 dB | 7 dB | 10 dB | 20 dB |
|----------|------|------------|------------|------------|-------------|-------|
| Betrag | 1 | $\sqrt{2}$ | $\sqrt{3}$ | $\sqrt{5}$ | $\sqrt{10}$ | 10 |
| Leistung | 1 | 2 | 3 | 5 | 10 | 100 |

Elementare Umrechnungen von Pegel in Betrags- und Leistungsgrößen

Complex Calculation

Cartesian representation: Polar representation:

$$z = x + iy \quad \text{cartesian form}$$

$$z = r(\cos \varphi + i \sin \varphi) \quad \text{polar form}$$

Addition:

- Rule: real and Imaginary parts add

$$z_1 + z_2 = (x + iy) + (u + iv) = (x + u) + i(y + v)$$

Subtraction:

- Rule: real and Imaginary parts subtract

$$z_1 - z_2 = (x + iy) - (u + iv) = (x - u) + i(y - v)$$

Multiplication:

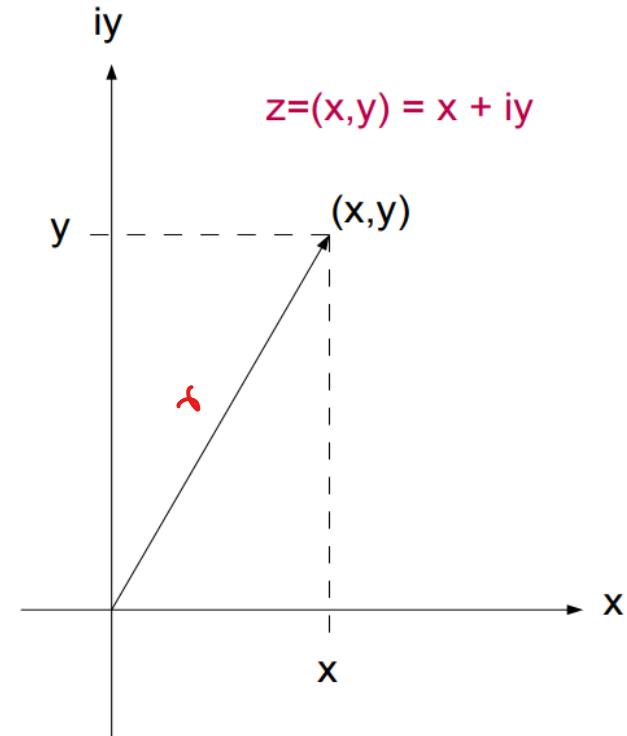
- Rule: multiply amounts and add angles

$$z_1 \cdot z_2 = r_1 r_2 (\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2))$$

Division:

- Rule: divide amounts and subtract angles

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2))$$



$$|z| := \sqrt{x^2 + y^2}$$

Frequency Responses of the Basic Elements

Transfere Funct.

$$P: G(s) = K$$

Frequency response

$$G(j\omega) = K$$

$$PT_1: G(s) = \frac{10}{s+10}$$

low frequency ratio; amplitude = 1;
high frequency ratio; amplitude ≈ 0

low pass filter

$$PT_2: G(s) = \frac{100}{s^2 + 20Ds + 100}$$

[stable when $D > 0$]

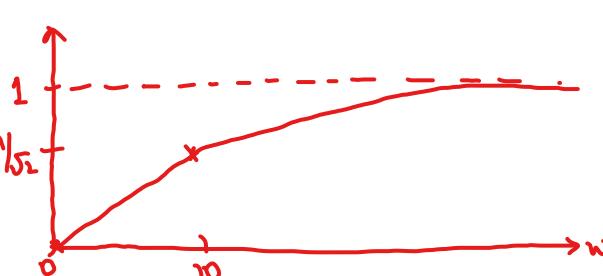
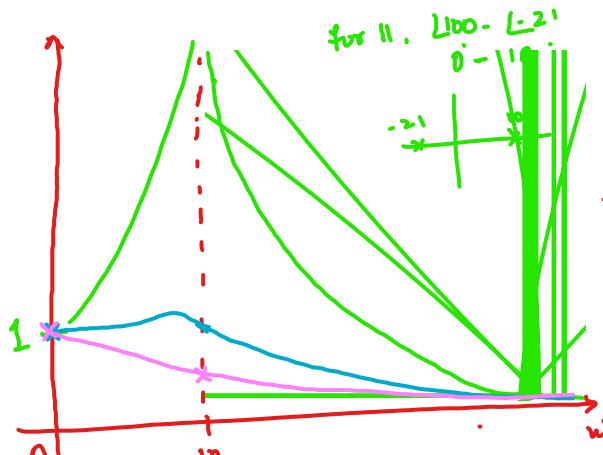
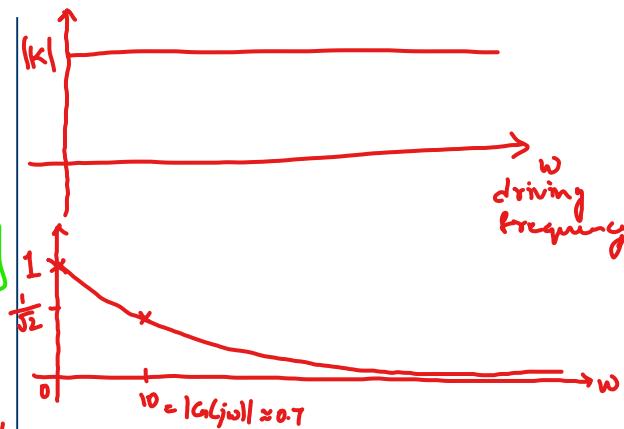
$$\begin{cases} D = 1 \\ \omega = 0 \rightarrow 1 \\ \omega = \infty \rightarrow 0 \\ \omega = 10 \rightarrow 0.5 \end{cases}$$

low pass filter

$$DT_1: G(s) = \frac{s}{s+10}$$

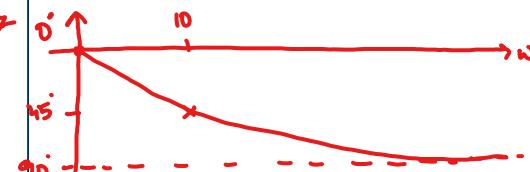
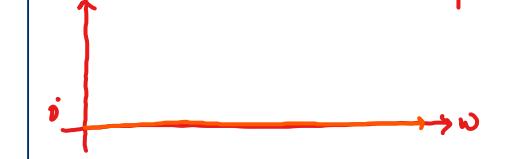
High pass filter

Amplitude response $|G(j\omega)|$



Phase response $\phi(j\omega)$
 $= \angle G(j\omega)$

$0^\circ \rightarrow$ horizontal axis



$$\begin{aligned} \omega = 10 & \quad \frac{10j}{10j + 10} = \frac{10j}{20j} = \frac{1}{2}j \\ 45^\circ & \quad \text{Imaginary part} \\ 10j & \quad \text{Real part} \end{aligned}$$

The Bode-Plot

Bode-Plot (-Diagram): → converting the amplitude factor values into dB scale

- Is a special representation of the **frequency response $G(j\omega)$** in a logarithmic scale.

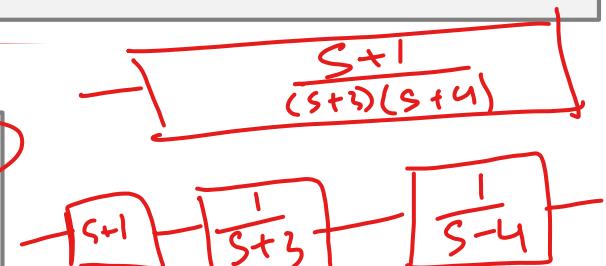
The representation of the logarithmic frequency characteristic has the **great advantage** that the resulting amplitude and phase response **systems connected in series** through (graphic) **addition** the individual amplitude or phase responses can be formed!

$$G_{\text{ges}}(j\omega) = G_1(j\omega) \cdot G_2(j\omega) \quad \rightarrow \quad G_{\text{ges}}(j\omega) = |G_1(j\omega)| \cdot |G_2(j\omega)| \cdot e^{j(\varphi_1(j\omega) + \varphi_2(j\omega))}$$

Amplitude response of a series connection:

$$20 \lg |G(j\omega)| = 20 \lg |G_1(j\omega)| + 20 \lg |G_2(j\omega)|$$

$$\lg(a \cdot b) = \lg a + \lg b$$

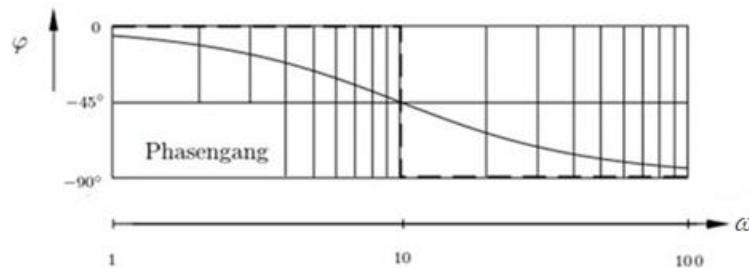
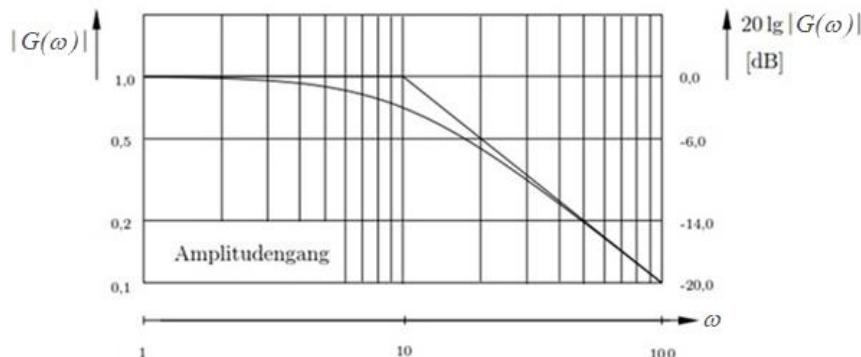


Phase response of a series connection:

$$\varphi(j\omega) = \varphi_1(j\omega) + \varphi_2(j\omega)$$

ExampleE: Bode diagram and its straight line approximation of an RC element (PT₁-System):

$$G(s) = \frac{10}{s+10} \quad \rightarrow \quad G(j\omega) = \frac{10}{j\omega+10}$$

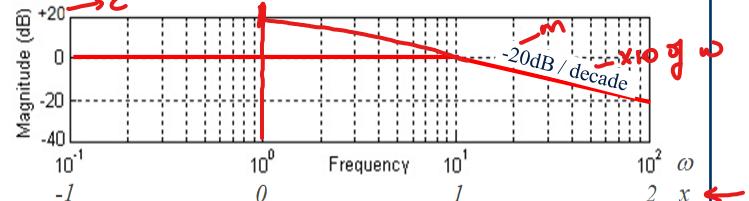
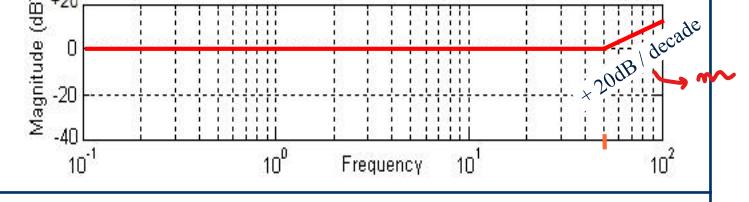
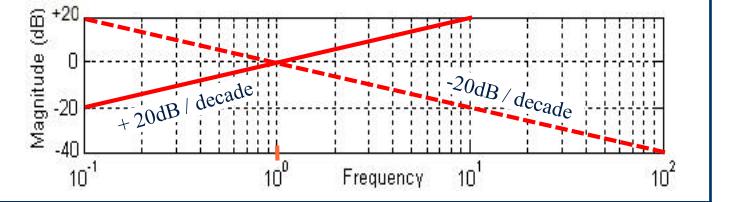
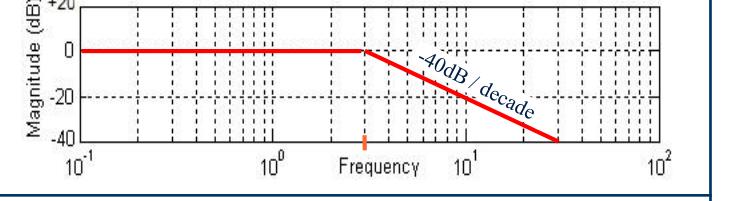
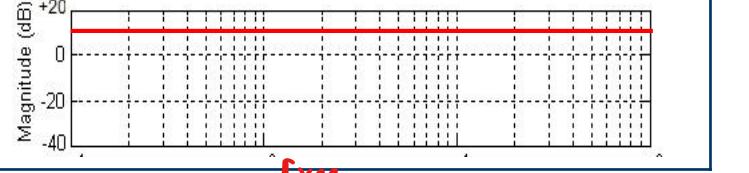


Approximation of the Amplitude Response

$$G_{PT_2}(s) = \frac{1}{\frac{s^2}{\omega_0^2} + \frac{2Ds}{\omega_0} + 1}$$

$$G(s) = \frac{6 \cdot s \cdot (s+50)}{(s+10)(s^2 + 6s + 9)} = \frac{6 \cdot 50 \cdot s \cdot (\frac{s}{50} + 1)}{10 \cdot 9 \cdot (\frac{s}{10} + 1) \cdot (\frac{s^2}{3^2} + \frac{2s}{3} + 1)} \Rightarrow |G(j\omega)| = \frac{6 \cdot 50 \cdot |j\omega| \cdot \left(\frac{|j\omega|}{50} + 1\right)}{10 \cdot 9 \cdot \left(\frac{|j\omega|}{10} + 1\right) \cdot \left(\frac{-\omega^2}{3^2} + \frac{2j\omega}{3} + 1\right)}$$

$$\lg(\frac{a-b}{c}) = \lg a + \lg b - \lg c$$

| G(s) | <i>amplitude</i> $ G(j\omega) $ | <i>conversion</i> $ G(j\omega) _{dB} = 20\lg G(j\omega) $ | Amplitude response |
|---|--|---|---|
| $PT_1 : \frac{1}{(\frac{s}{10} + 1)}$ | $\frac{1}{ \frac{j\omega}{10} + 1 } = \begin{cases} 1 & \omega \ll 10 \\ \frac{1}{\sqrt{2}} & \omega = 10 \\ \frac{10}{\omega} & \omega \gg 10 \end{cases}$ $(10 = \omega_a = s_\infty)$ | $= \begin{cases} 0dB & \omega \ll 10 \\ -3dB & \omega = 10 \\ 20\lg 10 - 20\lg \omega & \omega \gg 10 \end{cases}$ $y = mx + c \quad x \text{ m} = -20 \quad c = 20$ |  |
| $DT_1 : (\frac{s}{50} + 1)$ | $ \frac{j\omega}{50} + 1 = \begin{cases} 1 & \omega \ll 50 \\ \sqrt{2} & \omega = 50 \\ \frac{\omega}{50} & \omega \gg 50 \end{cases}$ $(50 = \omega_b = s_0)$ | $= \begin{cases} 0dB & \omega \ll 50 \\ +3dB & \omega = 50 \\ 20\lg \omega - 20\lg 50 & \omega \gg 50 \end{cases}$ $m \quad n \quad c$ |  |
| $D : s$ $I : \frac{1}{s}$ | $= j\omega = \omega$ $\left(= \left \frac{1}{j\omega} \right = \frac{1}{\omega}\right)$ | $= 20\lg \omega$ $(= -20\lg \omega)$ |  |
| $PT_2 : \frac{1}{(\frac{s^2}{3^2} + \frac{2s}{3} + 1)}$ | $\frac{1}{ \frac{-\omega^2}{3^2} + \frac{2j\omega}{3} + 1 } = \begin{cases} 1 & \omega \ll 3 \\ \approx \pm 1 & \omega = 3 \\ \left(\frac{3}{\omega}\right)^2 & \omega \gg 3 \end{cases}$ $(3 = \omega_0 = s_{\infty,1,2})$ | $= \begin{cases} 0dB & \omega \ll 3 \\ \approx \pm 0dB & \omega = 3 \\ 40\lg 3 - 40\lg \omega & \omega \gg 3 \end{cases}$ |  |
| <i>const.</i> $P : \frac{6 \cdot 50}{10 \cdot 9}$ | $= +3,33$ | $= 20\lg 3,33 = +10,5dB$ <i>conversion</i> <i>const. [indep. of ω]</i> |  |

Approximation of the Phase Response

$$G_{PT_2}(s) = \frac{1}{\frac{s^2}{\omega_0^2} + \frac{2Ds}{\omega_0} + 1}$$

$$G(s) = \frac{6 \cdot s \cdot (s+50)}{(s+10)(s^2 + 6s + 9)} = \frac{6 \cdot 50 \cdot s \cdot (\frac{s}{50} + 1)}{10 \cdot 9 \cdot (\frac{s}{10} + 1) \cdot (\frac{s^2}{3^2} + \frac{2s}{3} + 1)} \Rightarrow \varphi(j\omega) = \angle(\frac{6 \cdot 50}{10 \cdot 9}) + \angle(j\omega) + \angle(\frac{j\omega}{50} + 1) - \angle(\frac{j\omega}{10} + 1) - \angle(\frac{-\omega^2}{3^2} + \frac{2j\omega}{3} + 1)$$

| G(s) | $\phi(j\omega) = \angle G(j\omega) = \arctan\left(\frac{\text{Im}\{G(j\omega)\}}{\text{Re}\{G(j\omega)\}}\right)$ | Phase response |
|---|---|----------------|
| $PT_1 : \frac{1}{(\frac{s}{10} + 1)}$ | $\varphi(j\omega) = \angle\left(\frac{1}{(\frac{j\omega}{10} + 1)}\right) = \begin{cases} 0 & \omega << 10 \\ -45^\circ & \omega = 10 \\ -90^\circ & \omega >> 10 \end{cases}$ | |
| $DT_1 : (\frac{s}{50} + 1)$ | $\varphi(j\omega) = \angle\left(\frac{j\omega}{50} + 1\right) = \begin{cases} 0 & \omega << 50 \\ +45^\circ & \omega = 50 \\ +90^\circ & \omega >> 50 \end{cases}$ | |
| $D : s$ $I : \frac{1}{s}$ | $\varphi(j\omega) = \angle(j\omega) = +90^\circ \quad (\text{immer})$ $\varphi(j\omega) = \angle\left(\frac{1}{j\omega}\right) = -90^\circ \quad (\text{immer})$ | |
| $PT_2 : \frac{1}{(\frac{s^2}{3^2} + \frac{2s}{3} + 1)}$ | $\varphi(j\omega) = \angle\left(\frac{1}{(\frac{-\omega^2}{3^2} + \frac{2j\omega}{3} + 1)}\right) = \begin{cases} 0 & \omega << 3 \\ -\frac{\pi}{2} & \omega = 3 \\ -\pi & \omega >> 3 \end{cases}$ | |
| $P : \frac{6 \cdot 50}{10 \cdot 9}$ | $\varphi(j\omega) = \angle(3, 33) = 0 \quad (\text{immer})$ | |

2.18 Bode Diagram I.

- a) Sketch approximately the Bode diagram for the following system:
 b) The system is excited with the shown signal $u(t)$:
 Which steady-state response $y(t)$ is to be expected?

$$a) G(j\omega) = \frac{1}{(\frac{j\omega}{0.5} + 1)} \cdot \frac{1}{(\frac{j\omega}{8} + 1)} \cdot 10$$

$\underbrace{\quad}_{PT_1}$ $\underbrace{\quad}_{PT_1}$

$$P(10) : 20 \log 10 = 20 \text{ dB} \quad \varphi(j\omega) = 120 \text{ dB} = 0^\circ$$

$$PT_1 \left(\frac{1}{\frac{j\omega}{0.5} + 1} \right) : s_\infty = -0.5 \Rightarrow \omega = 0.5 \text{ rad/s}$$

$\underbrace{\quad}_{\text{zero}}$

$$PT_1 \left(\frac{1}{\frac{j\omega}{8} + 1} \right) : s_\infty = -8 \Rightarrow \omega = 8 \text{ rad/s}$$

Σ : sum of amplitudes and phase responses

↳ approximation of this system

$$b) u(t) = 12 \text{ V} \cdot \sin \left(20 \frac{1}{s} \cdot t \right)$$

steady state $y(t) = A \cdot \sin \left(20 \frac{1}{s} \cdot t + \frac{\rho}{s} \right)$ | $|G(j20)| = -20 \text{ dB}$ ↴ conversion factor

excitation (sin) $0.1 \times 12 \text{ V}$ | $20 \log(0.1) = -20 \text{ dB}$
 $\log(0.1) = -1$
 $s_2 = 10^1$

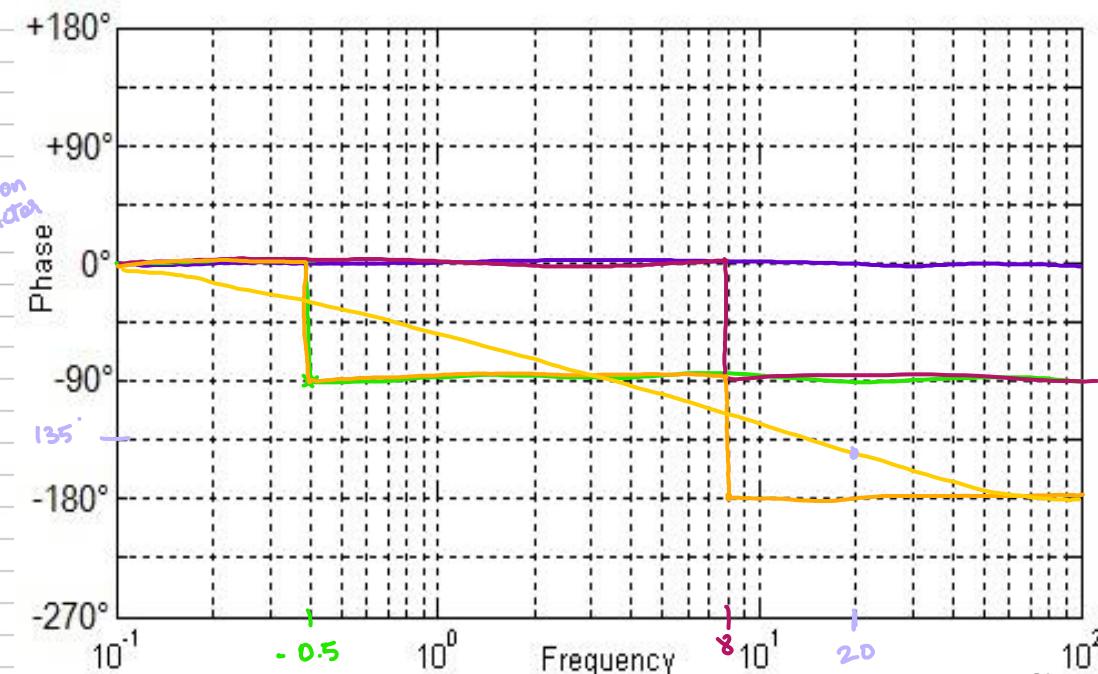
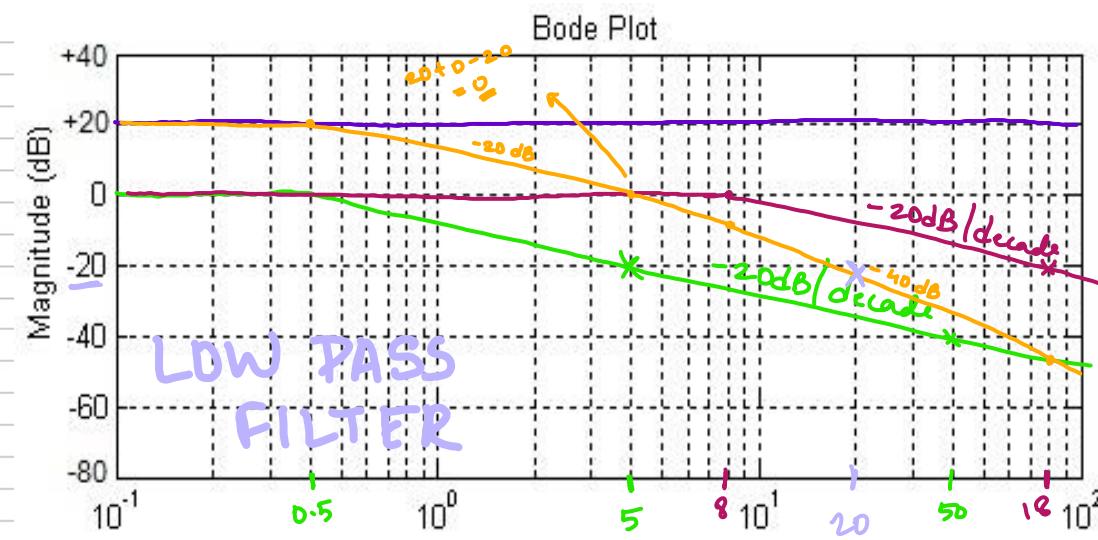
↳ only amplitude & phase shift change [A] [P]

extracted from Amplitude response

$\therefore y = 1.2 \sin \left(20 \frac{1}{s} \cdot t + 150^\circ \right)$

$$G(s) = 10 \cdot \frac{1}{(\frac{s}{0.5} + 1)} \cdot \frac{1}{(\frac{s}{8} + 1)}$$

$$u(t) = 12 \text{ V} \cdot \sin(20 \frac{1}{s} \cdot t)$$



Ex.2.19 Bode Diagram II

Sketch the Bode diagram of the following system:

$$G(s) = \frac{5 \cdot s \cdot (s+2)}{(s+10)^3}$$

$$G(j\omega) = \frac{5 \cdot 2 \cdot s \left(\frac{s}{2} + 1\right)}{10^3 \left(\frac{s}{10} + 1\right)^3}$$

$$P\left(\frac{5 \cdot 2}{10^3}\right) : \frac{1}{10^2} = -20 \log(10^2) = -40 \text{ dB}$$

$\varphi(-40 \text{ dB}) = 0^\circ$
real no.

$$PT_1\left(\frac{s}{2} + 1\right) : s_0 = -2 \Rightarrow \omega = 2 \text{ rad/s}$$

$\varphi(-2) = 90^\circ$

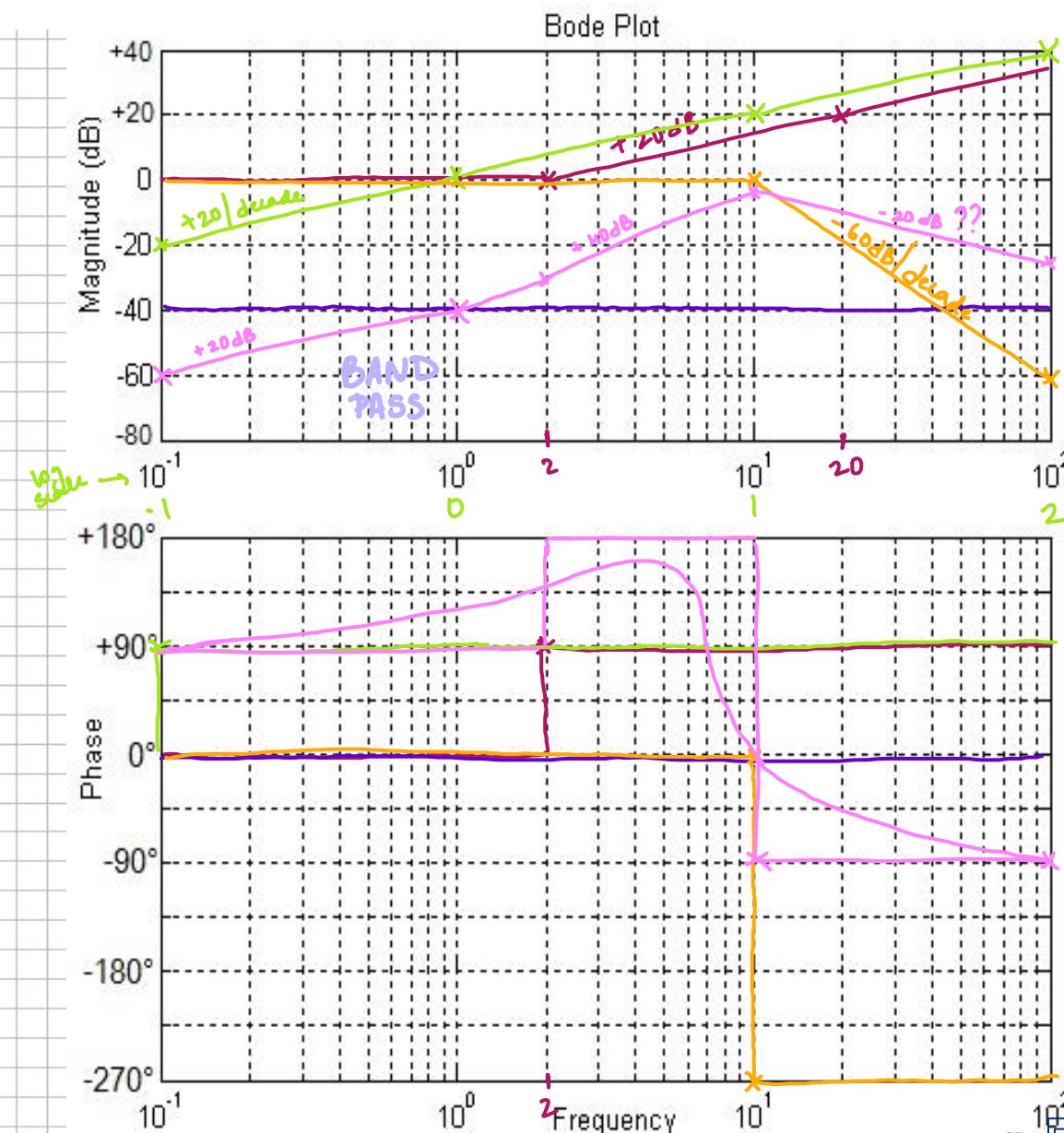
$$PT_1(s) : s_0 = 0 \Rightarrow \omega = 0 \text{ rad/s (out of sketch)}$$

since $\omega=9$
and out \log values
view, \log values
taken

$$PT_1\left(\frac{1}{(s+10)^3}\right) : s_0 = -10 \Rightarrow \omega = 10 \text{ rad/s}$$

$\varphi = (-90)^\circ \text{ per decade} = -270^\circ$

Σ :



Overview PT₂-Element

Übertragungsfunktion

$$\omega_0^2 \ddot{x}_a + 2D\omega_0 \dot{x}_a + x_a = \omega_0^2 * x_e$$

$$G(s) = \frac{1}{1 + \frac{2D}{\omega_0} s + \frac{1}{\omega_0^2} s^2}$$

D: Dämpfungsgrad
 ω_0 : natürliche Kreisfrequenz

oszillatorischer Ausgleichsvorgang
 $0 < D < 1$

schwingfähiges System
 kompl. konj. Lösungen (imaginär)

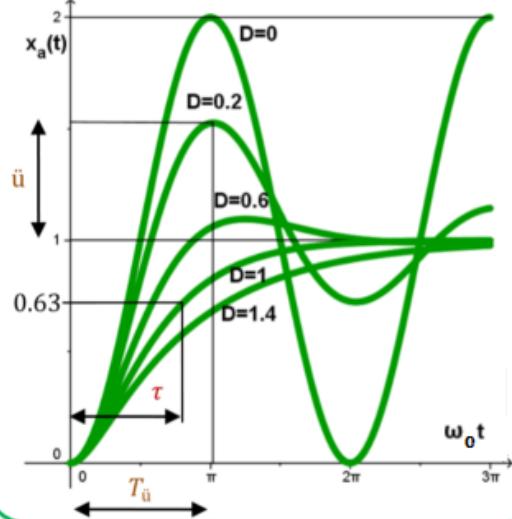
aperiodischer Grenzfall
 $D = 1$

nicht schwingfähiges System
 reelle Lösungen (2 PT1-Glieder)

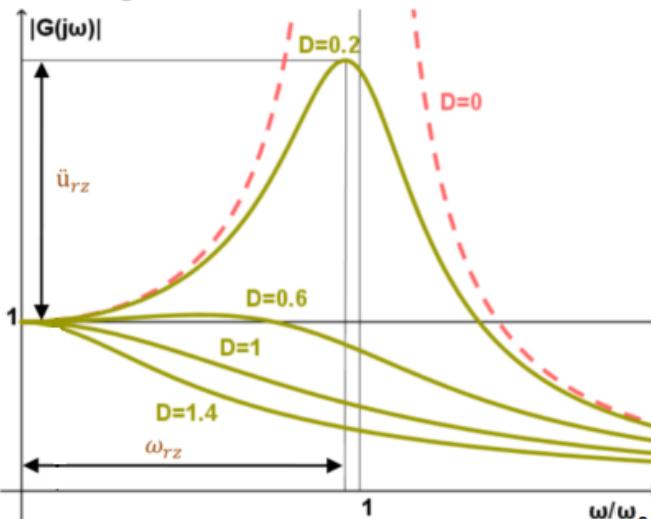
aperiodischer Ausgleichsvorgang
 $D > 1$

| | |
|--------------------|--|
| Resonanzfrequenz | $\omega_{rz} = \omega_0 \sqrt{1 - 2D^2}$ |
| Resonanzüberhöhung | $\ddot{u}_{rz} = \frac{1}{2D\sqrt{1 - D^2}}$ |
| also bei | $ D < \frac{1}{\sqrt{2}}$ |

Sprungantwort



Bodediagramm



Absolute Dämpfung

$$\sigma = D * \omega_0$$

Abklingzeitkonstante
 (Zeit bis auf 37%)

$$\tau = \frac{1}{\sigma}$$

Kreisfrequenz der
 gedämpften Schwingung

$$\omega = \omega_0 \sqrt{1 - D^2} = \frac{\pi}{T_u}$$

Periodendauer der
 gedämpften Schwingung

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0 \sqrt{1 - D^2}}$$

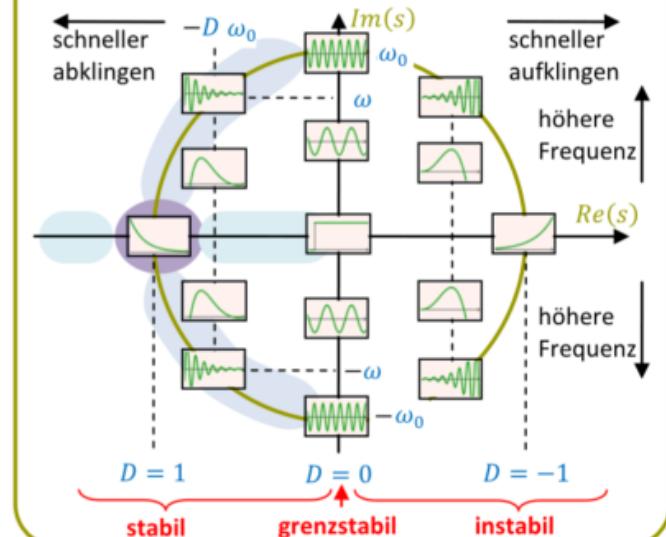
Überschwingzeit T_u

halbe Periodendauer

Überschwingweite

$$\ddot{u} = e^{-\frac{\pi D}{\sqrt{1-D^2}}}$$

Gauss'sche Zahlenebene



Polstellen

$$s_{1,2} = \omega_0 \left(-D \pm \sqrt{D^2 - 1} \right)$$

Betrag

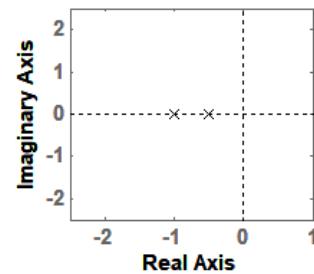
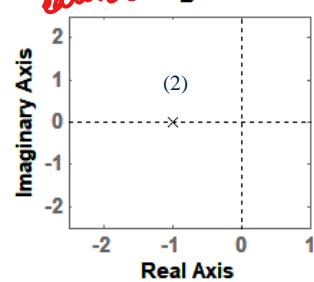
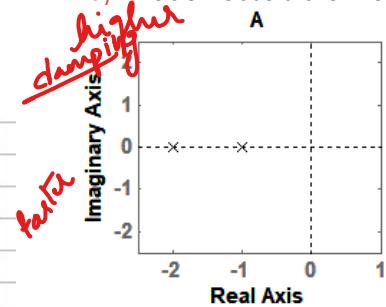
$$|G(j\omega)| = \sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{2D\omega}{\omega_0}\right)^2}$$

Winkel

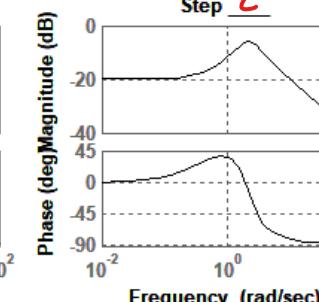
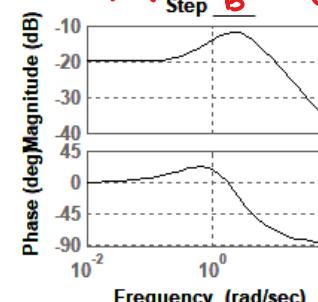
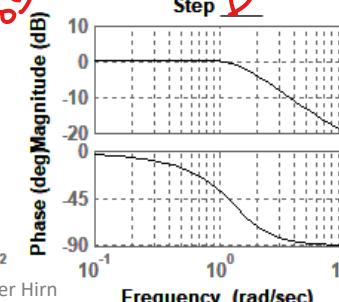
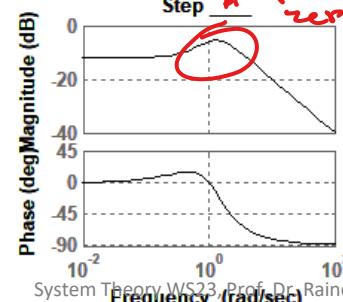
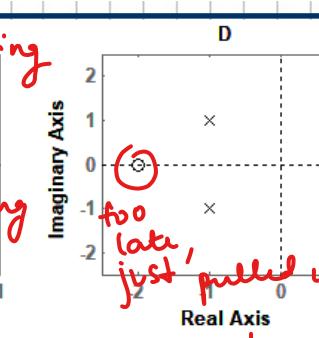
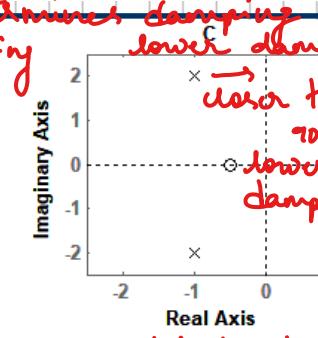
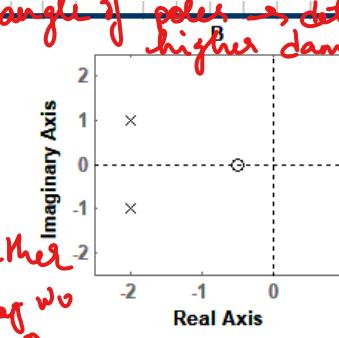
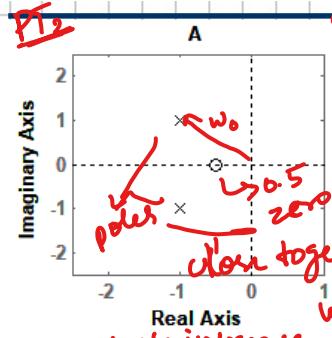
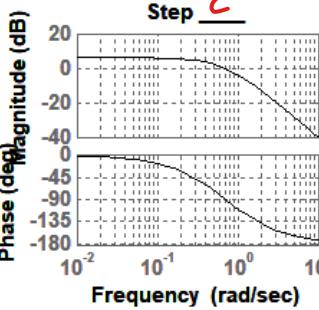
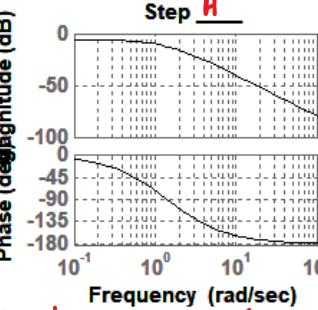
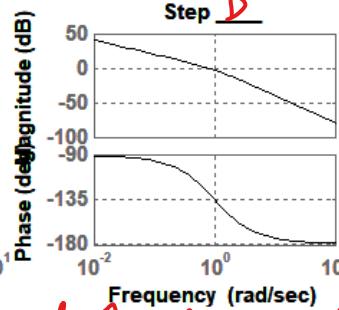
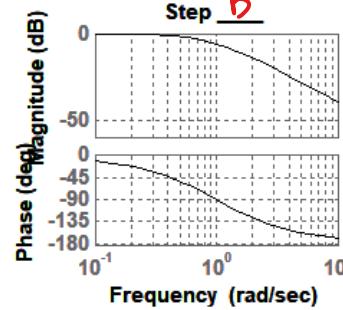
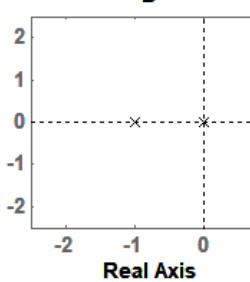
$$\arg(G(j\omega)) = -\tan^{-1}\left(\frac{\frac{2D\omega}{\omega_0}}{1 + \frac{\omega^2}{\omega_0^2}}\right)$$

Ex.2.20 Bode Diagram III

a) In each case state the transfer function ($K = 1$) and assign the correct Bode diagrams.



starts at 0, not visible



$$A] w_0 = \sqrt{1^2 + 1^2} = \sqrt{2}$$

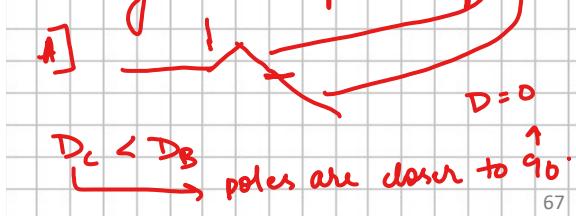
$$B] w_0 = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$C] w_0 = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$D] w_0 = \sqrt{1^2 + 1^2} = \sqrt{2}$$

every pole \rightarrow pull down

every zero \rightarrow pull up



Frequency Response Design

A desired one **amplitude response** can be roughly specified using the pole-zero diagram:

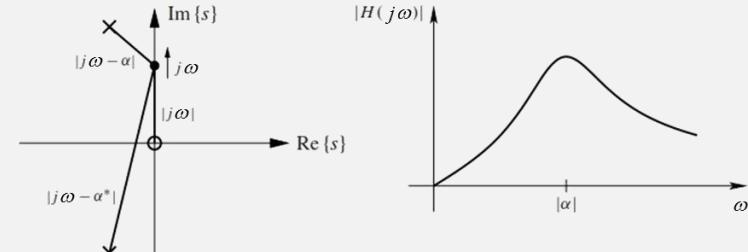
$$|G(j\omega)| = c \cdot \frac{|j\omega - \beta_1| \cdot |j\omega - \beta_2| \cdot \dots}{|j\omega - \alpha_1| \cdot |j\omega - \alpha_2| \cdot \dots}$$

Because the individual terms can **graphically as a distance** of considered frequency point on the imaginary axis of the Poles and zeros are understood ...

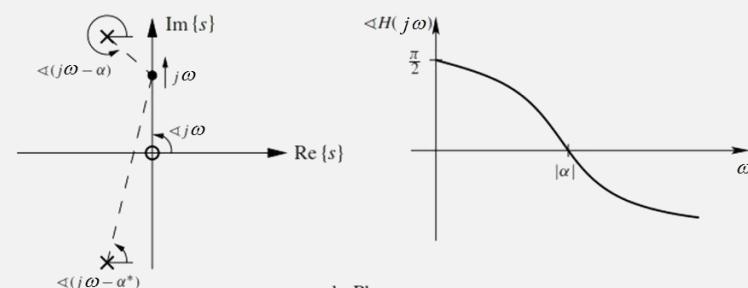
The **Phase response** can be estimated in a similar way:

$$\angle G(j\omega) = \angle c + \sum_{i=1}^M \angle(j\omega - \beta_i) - \sum_{i=1}^N \angle(j\omega - \alpha_i)$$

The respective phases of the individual terms add up here.
The contributions from zeros are positive and the contributions from Poles negative one,



a: Amplitudengang

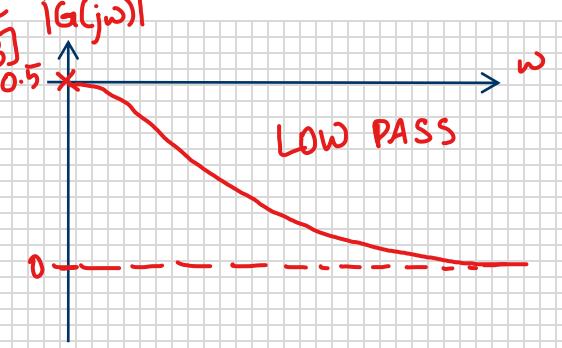
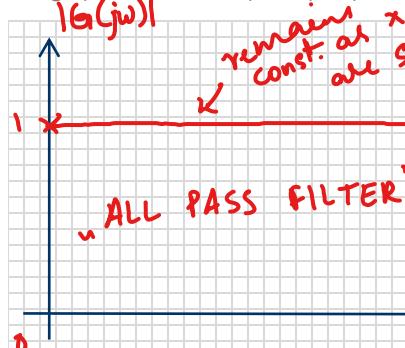


b: Phasengang

Abschätzung des Frequenzganges über die Pol- und Nullstellenlage

Example:

Roughly sketch the frequency response of these two systems



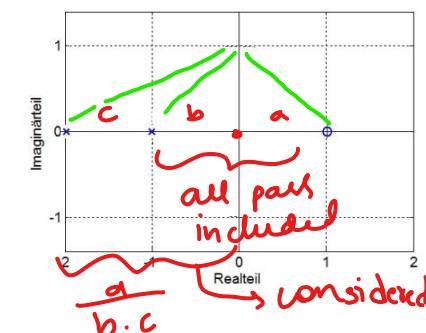
$$G(s) = \frac{s-1}{s+1}$$

pole zero

$$G(s) = \frac{s-1}{(s+2)(s+1)}$$



$$\frac{1}{b} = 1$$



Minimum Phase Systems and Allpasses

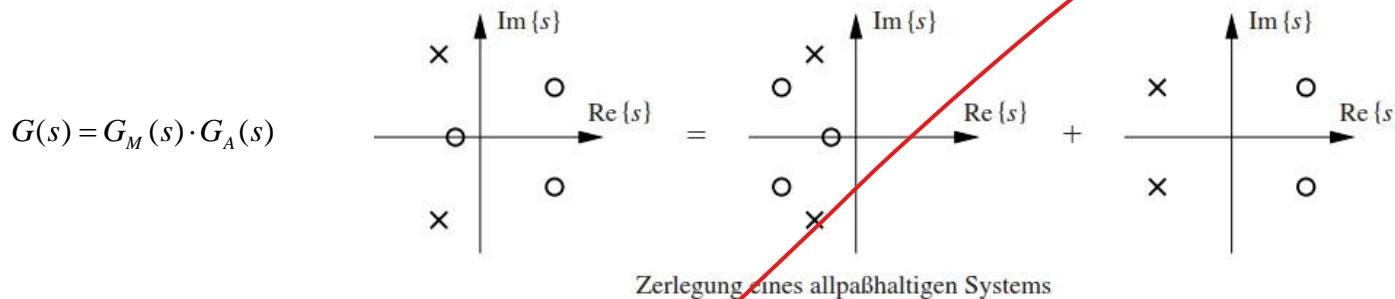
Minimum Phase Systems

- Have no zeros in the right s-half plane.
- Minimum-phase systems have the shortest possible **signal propagation times** (signal delays)!

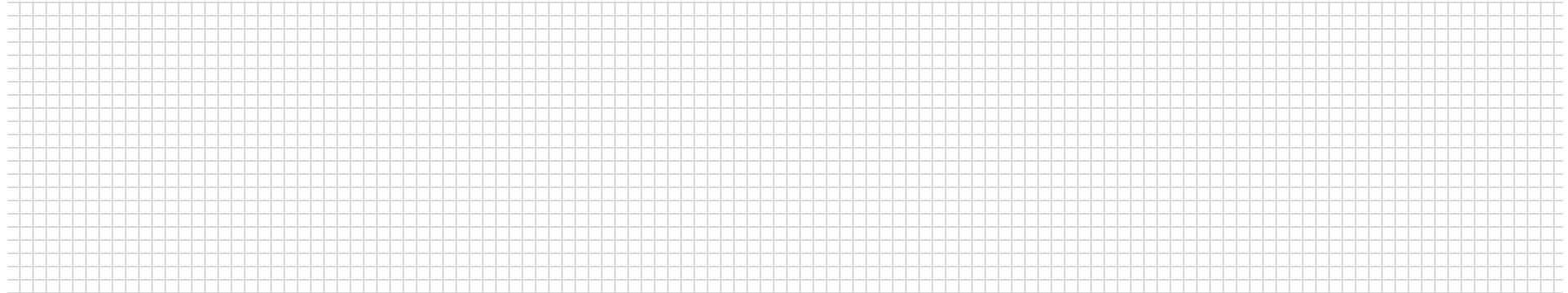
Allpass Systems

- Have a constant amplitude response,
- Their poles and zeros are **mirror images** to each other, they only cause a frequency-dependent phase-shifting effect.
- **Each (all-pass) system** can be **decomposed** into a minimum-phase and an all-pass component!

Example 1:



Example 2:



The Locus Curve (Nyquist-Diagram)

Locus Curve:

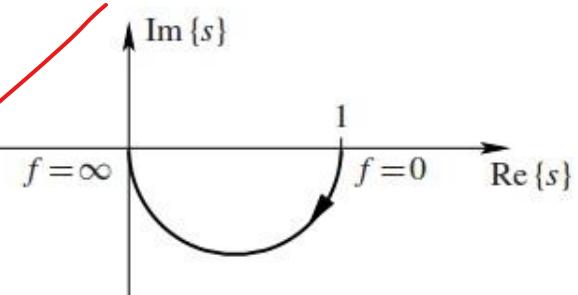
- Here, all values of the frequency response $G(j\omega)$ are plotted directly in the complex plane, while the parameter ω runs through the range from 0 to ∞ .

Very often the consideration of the reciprocal frequency response $1 / H(f)$ helps to determine the locus:

| $\frac{1}{H(f)}$ | $H(f)$ |
|-----------------------------|-----------------------------|
| Gerade durch Ursprung | Gerade durch Ursprung |
| Kreis durch Ursprung | Gerade nicht durch Ursprung |
| Gerade nicht durch Ursprung | Kreis durch Ursprung |
| Kreis nicht durch Ursprung | Kreis nicht durch Ursprung |

Eigenschaften der konformen Abbildung der Inversion

$$G(s) = \frac{1}{s+1}$$

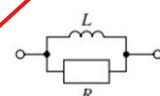


Ortskurve Tiefpaß erster Ordnung

Task:

The admittance of this parallel connection reads: $Y(j\omega) = \frac{1}{Z(j\omega)} = \frac{1}{R} + \frac{1}{j\omega L}$

First draw the locus of the admittance $Y(j\omega)$ and then the locus of the impedance $Z(j\omega)$.



in the

$Y(j\omega)$

in the

$Z(j\omega)$

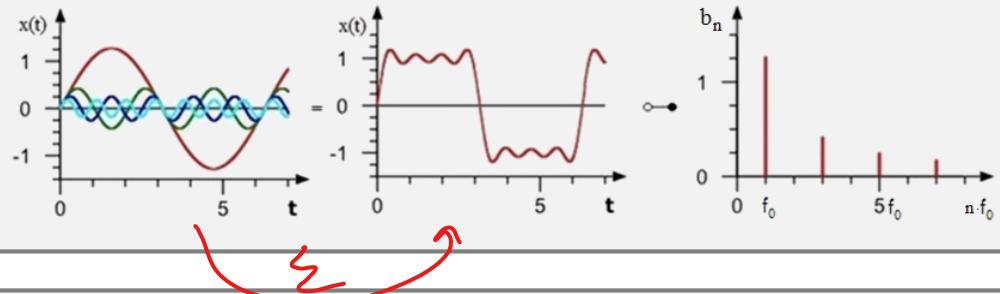
re

re

The Fourier Series Expansion of Periodic Signals

Fourier Series Expansion:

Only integer multiples of the fundamental frequency are needed!



Three forms of representation:

- Sine-cosine representation:

$$x_p(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] , \quad b_0 = 0$$

- Cosine representation:

$$x_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \phi_n) , \quad A_n = \sqrt{a_n^2 + b_n^2} \quad \tan \phi_n = \frac{b_n}{a_n}$$

- Complex representation:

$$x_p(t) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{jn\omega_0 t} , \quad c_n = \frac{a_n - jb_n}{2} = \frac{A_n}{2} \cdot e^{-j\phi_n} \quad c_{-n} = \frac{a_n + jb_n}{2} = \frac{A_n}{2} \cdot e^{j\phi_n}$$

periodic signal coefficients

- Fourier series coefficients:

$$c_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cdot e^{-jn\omega_0 t} dt$$

$$a_n = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cos(n\omega_0 t) dt$$

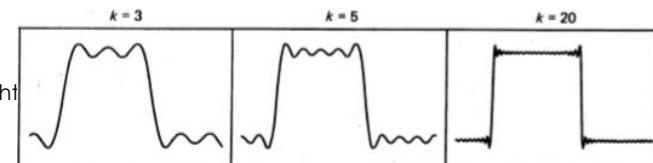
$$b_n = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \sin(n\omega_0 t) dt$$

The coefficients indicate with what weight and with what displacement the **Fundamental wave** and the individual **Harmonics** occur in the approximation.
The Fourier series can be used as a **discrete spectrum** be interpreted.

Helpful: As you can see, the are complex Fourier coefficients c_n and c_{-n} of a real signal always complex conjugate to each other!

Gibb's Phenomenon:

In the case of jumps, one always remains even with an infinite number of harmonics **Error of approx. 9%** the jump height



Fourier Transform as Spectrum of a Signal

- The Fourier transform can be understood as a **boundary transition of the complex Fourier series**.

Because a non-periodic function $x(t)$ can be used as a periodic function with an infinitely long period $T_0 \rightarrow \infty$ can be understood:

Definition of Fourier Transformation:

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Inverse Fourier transformation:

$$x(t) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$$

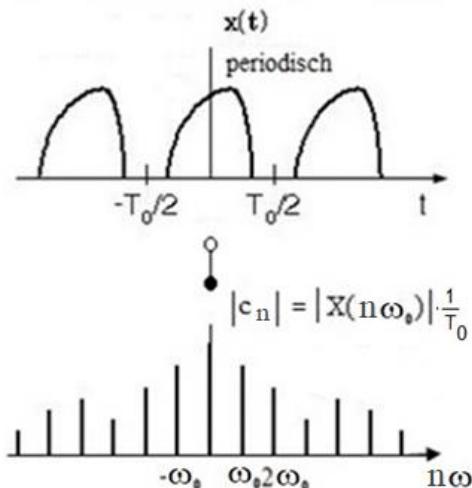
Spectrum:

- is called the Fourier transform of a signal $x(t)$.
- The spectrum gives the **spectral composition** of the signal, i.e. how strong and with what phase shift each frequency is contained.
- The spectrum of one non-periodic signal is **continuous**!
- The spectrum of one periodic Signal is **discrete** (it consists only of Dirac impulses)!

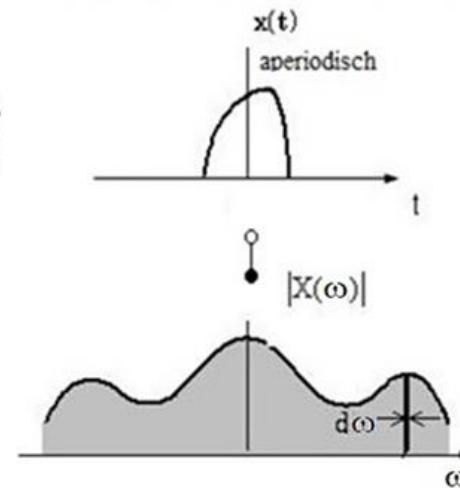
Since the spectrum is a complex function, one needs for the representation **Amount and phase**.

Note: Exist in the literature **version with f** as a frequency variable!

Fourier-Reihenentwicklung



Fourier-Transformation



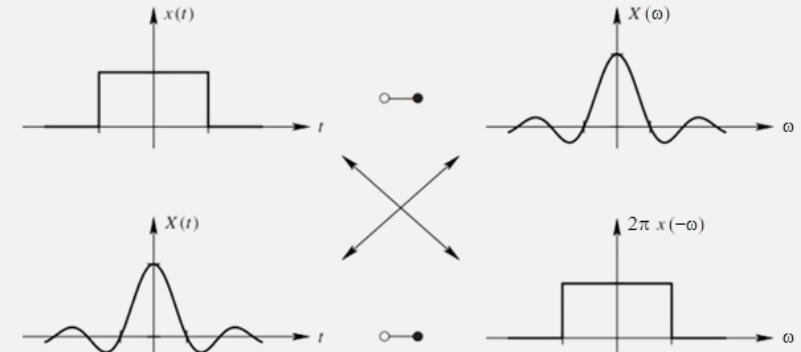
Properties of the Fourier Transform

Duality:

$$x(t) \circ - \bullet X(\omega)$$

$$X(t) \circ - \bullet 2\pi \cdot x(-\omega)$$

This takes place Return transfer. the FT generally in the same way as the outward transformation.



Symmetry:

$$\begin{aligned} x(t) &= \underbrace{x_g(t)}_{\text{even}} + \underbrace{x_u(t)}_{\text{odd}} + jx'_g(t) + jx'_u(t) \\ X(j\omega) &= \underbrace{X_g(j\omega) + X'_u(j\omega)}_{X(j\omega)} + \underbrace{jX'_g(j\omega) + jX_u(j\omega)}_{jX(j\omega)} \end{aligned}$$

Band Limitation:

- A signal $x(t)$ is called **band limited** if its spectrum $X(\omega)$ from a certain cut-off frequency ω_G disappears:
 $X(\omega) = 0$ for $|\omega| \geq \omega_G$.
- Because of the **duality**, a **time-limited Signal is not band limited** and a **band limited signal is not time-limited!**

Parseval's Theorem (this formula only applies to energy signals!):

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = E_x$$

i.e. the signal **energy** of a signal can be **both in the time domain and in the frequency domain** to calculate.

Ex.2.21 Spectrum I.

a) Determine and sketch the spectrum of the decaying causal e-function $x(t)$ as magnitude and phase.

$$x(t) = e^{-at} \cdot \varepsilon(t), \quad a > 0$$

decaying exponential function

As $x(t)$ is causal and decaying:

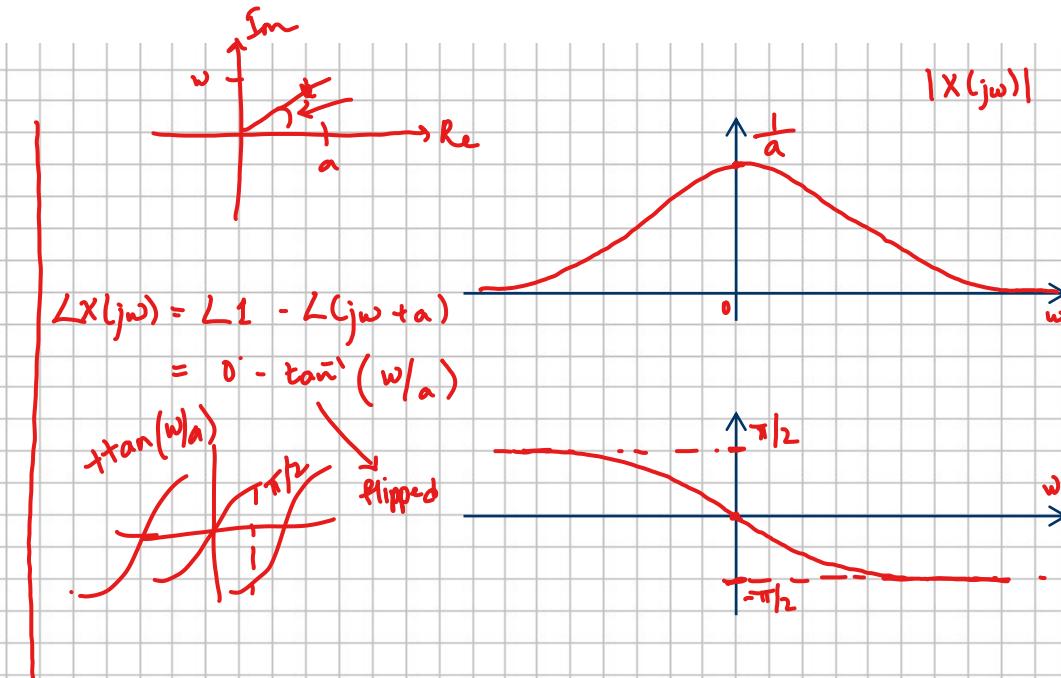
$$X(j\omega) = X(s)|_{s=j\omega} = \frac{1}{s+a}|_{s=j\omega} = \boxed{\frac{1}{j\omega + a}}$$

Laplace Transform

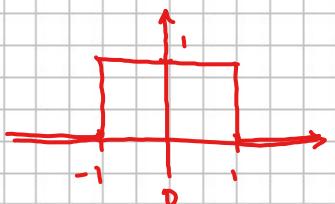
$$\begin{aligned} \text{abs. value of spectrum} \\ |X(j\omega)| &= \frac{|1|}{|j\omega + a|} = \boxed{\frac{1}{\sqrt{\omega^2 + a^2}}} \end{aligned}$$

← real part
← imag part

↓
sq. root of real parts



b) Calculate and sketch the spectrum of the rectangular function of width 2 and height 1 as magnitude and phase.

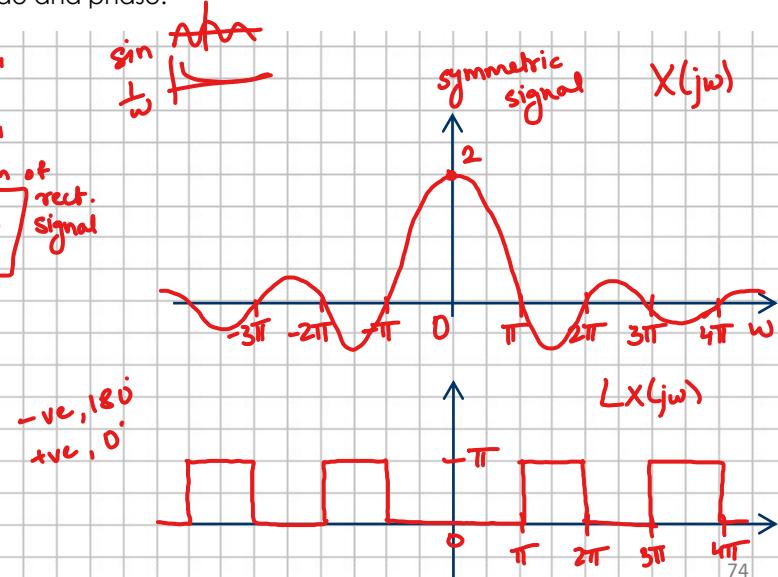


$$\begin{aligned} X(j\omega) &= \int_{-1}^{+1} x(t) \cdot e^{-j\omega t} dt = \int_{-1}^{+1} 1 \cdot e^{-j\omega t} dt = \left[-\frac{1}{j\omega} e^{-j\omega t} \right]_{-1}^{+1} \\ &= -\frac{1}{j\omega} (e^{j\omega} - e^{-j\omega}) = \boxed{\frac{2j \sin(\omega)}{j\omega}} \end{aligned}$$

↑ amplitude

difference
so sinus

$$\angle \left(2 \cdot \frac{\sin(\omega)}{\omega} \right) =$$

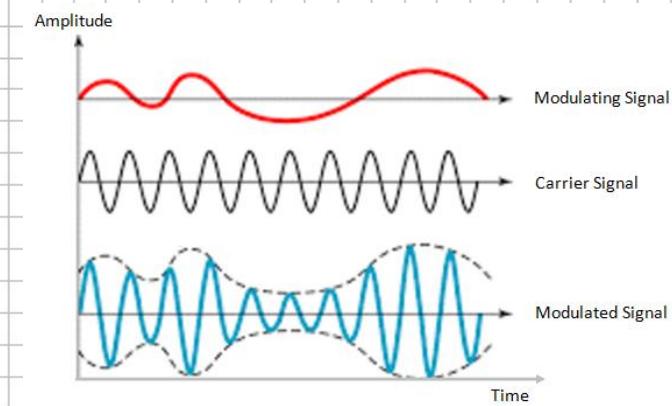
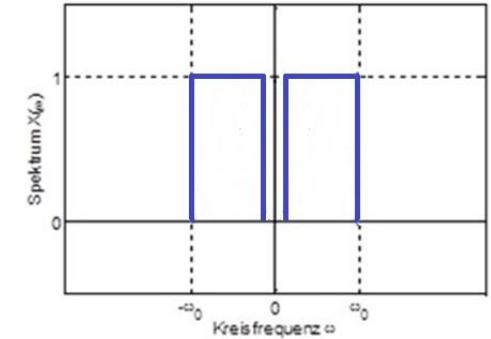


Ex 2.22 Spectrum II (AM broadcasting)

- a) In an AM transmitter, in principle, a low-frequency audio signal $x(t)$, whose spectrum $X(j\omega)$ is shown is multiplied by a high frequency carrier signal (this is called modulation).
I.e. the antenna signal is e.g.:

$$y(t) = x(t) \cdot \cos(5\omega_0 t)$$

Calculate the spectrum $Y(j\omega)$ of the antenna signal and sketch it.

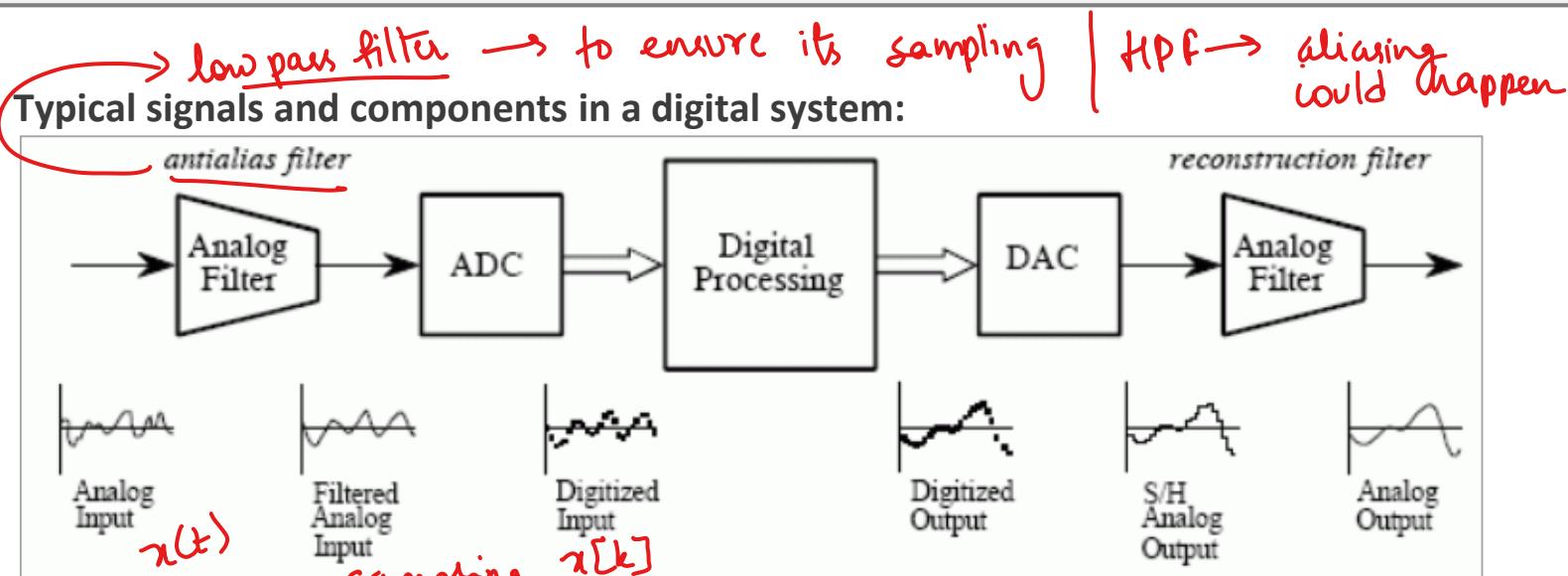


3) Digital Signals and Systems

3.1 Basic Properties of Time-Discrete Signals

Digital Signal Processing

- Deals with the generation and processing of digital signals using digital systems.
- A digital signal is **discrete in time and values!**



- The design of time-discrete algorithms for the easy implementation of systems in software is the subject of the following chapters.
(info lost)

The Transformations of Systems Theory

An overview of the most important transformations („tools“) of systems theory:

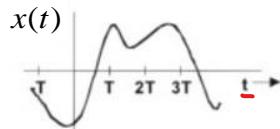
- Hint: The **z-transformation** is only a Laplace transformation version adapted to the notation of discrete systems (integrals become sums)!

| | | Zeitbereich | | |
|---|------------------------------|--|--|--|
| | | analog signals kontinuierliches Signal $x(t)$ | digital signals [summation] diskretes Signal $x[k]$ | |
| Bildbereich | komplex | Laplace-Transformation | \rightarrow z-Transformation | |
| | | | | |
| Frequenzbereich | kontinuierlich any signal | Fourier-Transformation | Zeitdiskrete Fourier-Transformation Time | |
| | | | | |
| Frequenzbereich | diskret periodic signals | Fourier-Reihe Series | Diskrete Fourier-Transformation | |
| | | | | |
| \times Some transformations, only different notations converts numbers to numbers (only Transformation) | | | | |
| Discrete Signal $\circ - \bullet$ Periodically Spektrum Periodically Signal $\circ - \bullet$ Discrete Spektrum | | | | |
| <p>Zeitbereich</p> <p>Frequenzbereich</p> <p>Fourierreihe</p> <p>Zeitkontinuierliche Fouriertransformation</p> <p>DFT</p> <p>DTFT</p> <p>$x_p(t)$, $x(t)$, $x_p(n)$, $x(n)$, α_k, $X(j\omega)$, $X_p(k)$, $X(j\Omega)$</p> <p>t, k, ω, n, Ω</p> <p>\rightarrow periodic discrete</p> <p>\rightarrow aperiodic discrete</p> <p>\rightarrow no. \rightarrow no.</p> <p>\rightarrow periodic spectrum cont.</p> | | | | |

Overview Continuous vs. Discrete Systems

Continuous:

- Signal:



- Differential-Eq.: $a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = b_1\dot{u}(t) + b_0u(t)$

$$\Updownarrow L$$

- TF:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$$

- LaplaceTransf.:

$$X(s) = \int_0^\infty x(t) e^{-st} dt$$

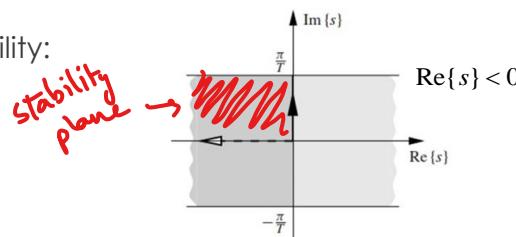
- Convolution:

$$y(t) = g(t) * u(t) = \int_{-\infty}^{\infty} g(\tau) \cdot u(t - \tau) d\tau$$

- Frequency resp.:

$$G(j\omega) = G(s) \Big|_{s=j\omega}$$

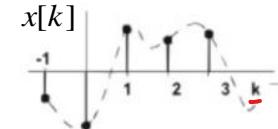
- Stability:



Sampling done

- Signal:

Discrete:



Abkü.: $x[k] = x(kT)$

sampling time omitted
Sampling time including

- Differences-Eq.:

no time dependencies

$$\tilde{a}_2 y[k+2] + \tilde{a}_1 y[k+1] + \tilde{a}_0 y[k] = \tilde{b}_1 u[k+1] + \tilde{b}_0 u[k]$$

indexes shifted

$$\Updownarrow Z$$

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\tilde{b}_1 z + \tilde{b}_0}{\tilde{a}_2 z^2 + \tilde{a}_1 z + \tilde{a}_0}$$

- z-Transform.:

$$X(z) = \sum_{k=0}^{\infty} x[k] \cdot z^{-k}$$

Abkü.: $z = e^{j\Omega}$

- Convolution:

$$y[k] = g[k] * u[k] = \sum_{n=-\infty}^{\infty} g(n) \cdot u(k-n)$$

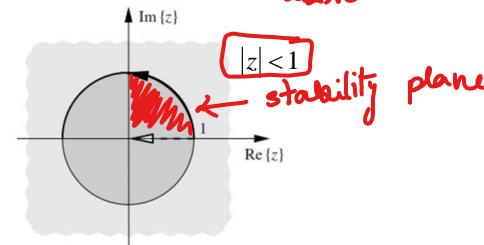
- Frequency resp.:

$$G(e^{j\Omega}) = G(z) \Big|_{z=e^{j\Omega}}$$

Abkü.: $\Omega = \omega T$
abbreviation

- Stability:

$$z = e^{j\Omega}$$



e.g.: (forward) difference quotient: e.g. bilinear transformation: e.g. recursive solution of a difference-q.:

$$\frac{dy(t)}{dt} \Big|_{t=kT} \approx \frac{y((k+1)T) - y(kT)}{T} = \frac{y[k+1] - y[k]}{T}$$

$$G(z) = G(s) \Big|_{s=\frac{2}{T} \cdot \frac{z-1}{z+1}}$$

$$y[k] = \frac{1}{\tilde{a}_2} \left(-\tilde{a}_1 y[k-1] - a_0 y[k-2] + \tilde{b}_1 u[k-1] + \tilde{b}_0 u[k-2] \right)$$

The Sampling Process

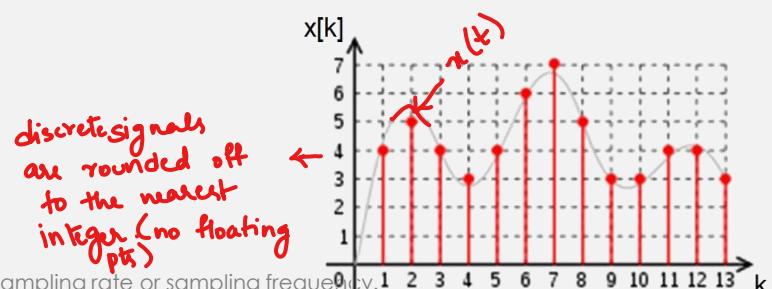
Sampling

- is taking signal samples of a continuous signal $x(t)$ at a time interval T , the following signal is generated:

$$x[k] = \underbrace{x(t)}_{t=kT} \quad k \in \mathbb{Z}$$

- $x[k]$ is called a **time-discrete signal, signal sequence or short samples**.

the time interval T between two samples as the sampling interval, the reciprocal value $f_s = 1 / T$ as the sampling rate or sampling frequency.

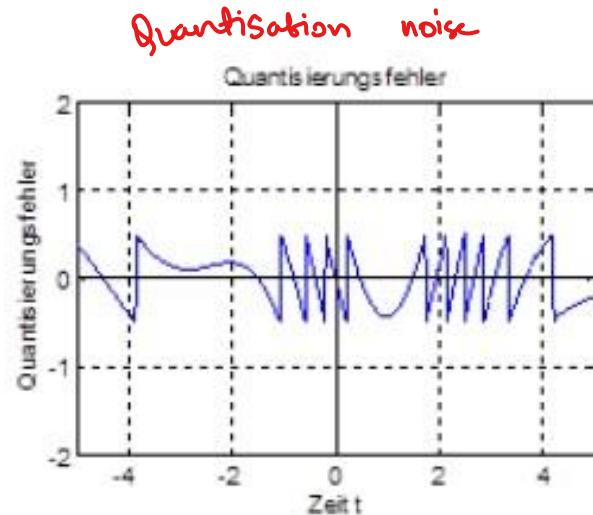
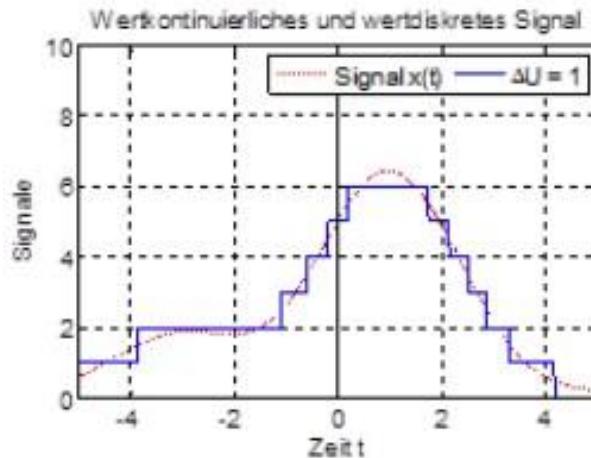


Quantization error

- is the rounding errors, which arises when the values are assigned in the AD converter, as this only works with a limited resolution.
It will usually be neglected or only viewed as a weak noise signal.

For example, a 10-bit AD converter can only handle $2^{10} = 1024$ different values. At 5V Measuring range is the minimum possible resolution, for example:

$$\Delta U = \frac{5V}{2^{10}} = \frac{5V}{1024} \approx 5mV \Rightarrow \text{amplitude of noise}$$



The Ideal AD-Conversion

Spectrum of a sampled signal:

- Math. the simplest way to represent the sampling process is a **multiplication** of the time signal $x(t)$ with a **impulse train** (time interval T):

$$x_a(t) = x(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

- This multiplication in the time domain corresponds to a convolution with the spectrum of this pulse comb in the spectral domain. This is again an impulse comb (only with the factor ω_a multiplied)! - it follows:

$$X_a(j\omega) = \frac{1}{2\pi} X(j\omega) * \left(\omega_a \cdot \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_a) \right) = \frac{1}{T} \cdot \sum_{n=-\infty}^{\infty} X(j(\omega - n\omega_a))$$

“Sampling repeats a signal spectrum infinitely periodically in ω_a !”

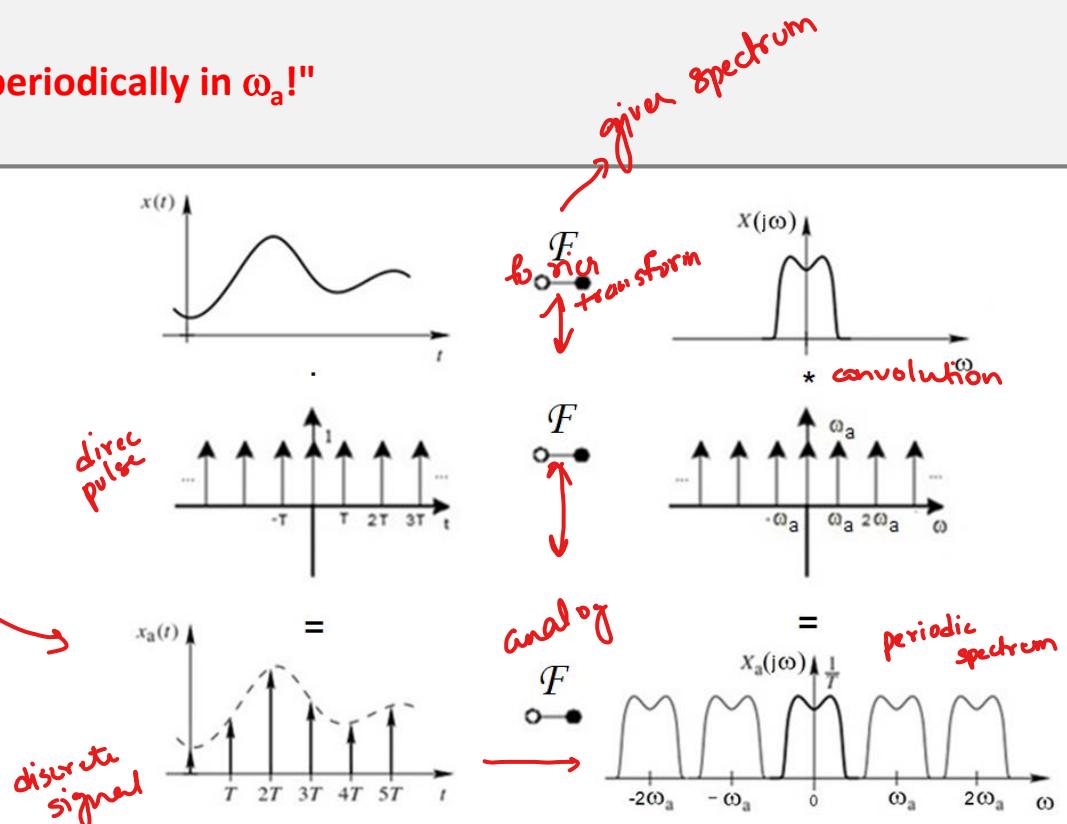
and weighted with $\underline{1/T}$. amplitude

- Description of sampling by multiplication with a Dirac pulse train:

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) \quad \circ-\bullet \quad \omega_a \cdot \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_a)$$

$$\omega_a = 2\pi f_a = \frac{2\pi}{T}$$

Sampling \Rightarrow analog \cdot direct pulse

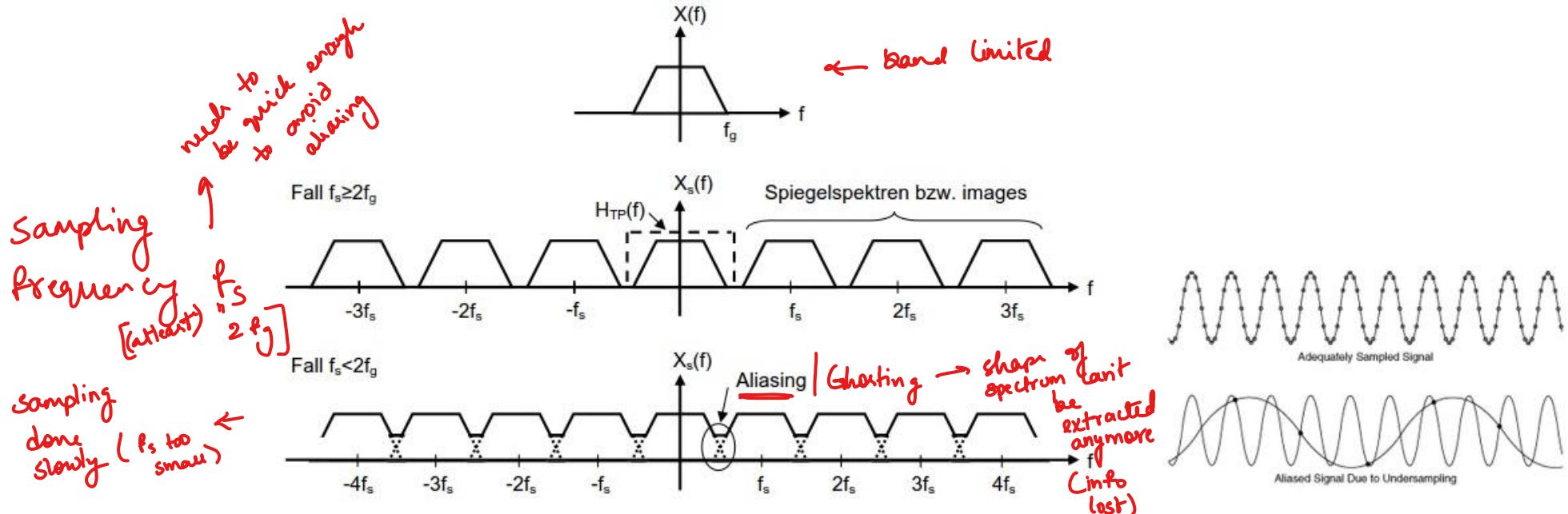


Nyquist-Shannon's Sampling Theorem

Undersampling (Aliasing)

- occurs when a signal is sampled **too slowly**.

This results in an overlap of the new sidebands, which leads to a loss of information content in the signal - this cannot be reversed!



Nyquist-Shannon Sampling Theorem:

- Every band-limited signal $x(t)$ can be clearly represented by means of samples $x[k] = x(kT)$, the **sampling frequency ω_a** must be greater than twice the **cutoff frequency ω_g (maximum frequency)** of the signal

$$\omega_a \geq 2\omega_g$$

- Headquarters **condition** for the 'Scannability' of a signal is the **Band limitation**, therefore an analog so-called. **Anti-aliasing Low pass** upstream!

The Ideal Signal Reconstruction (DA-conversion)

Signal reconstruction

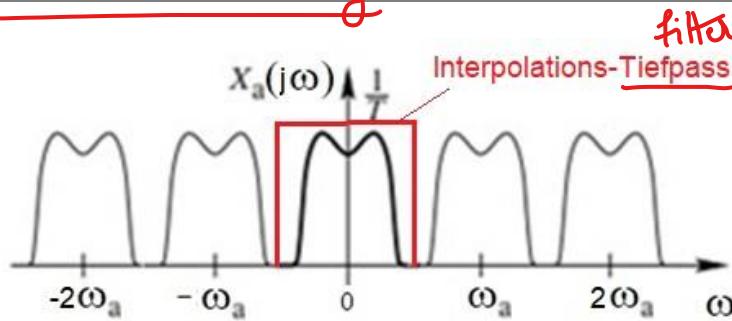
- from the weighted pulse train $x_a(t)$ to the analog signal $x(t)$ is achieved by eliminating the **periodic continuations** of the spectrum with the aid of an analog **low pass filter** (Interpolation low pass).

An ideal would be suitable Low pass with the cutoff frequency equal to half the sampling frequency.

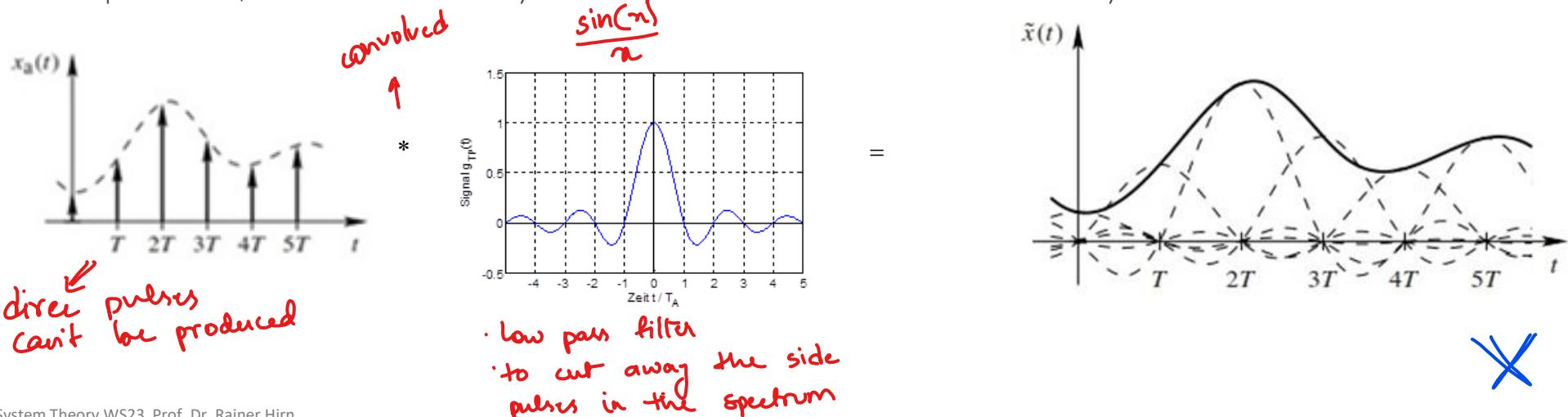
His Impulse response would be an si function:

$$G_{TP}(j\omega) = T \cdot \text{rect}\left(\frac{\omega T}{2\pi}\right) \quad \bullet - \circ \quad g_{TP}(t) = \text{si}(\pi \frac{t}{T})$$

Discrete to Analog



- The ideally reconstructed time signal $x(t)$ is therefore simply a superposition of impulse responses of this filter weighted with the sampled values, i.e. si-functions! In reality this is **not possible** because of the lack of causality!



Spectral Consideration of the Real DA-Conversion

1. A Zero-Hold-Element (ZOH)

- first creates an analog **step signal** x_H in the real DA-converter,

Consideration in the time domain:

The holding element thus generates a rectangle of width T from an impulse, ie its impulse response =

$$g(t) = \varepsilon(t) - \varepsilon(t-T)$$



The signal at the holding element output $x_H(t)$ is thus the **folding** the one with the $x(kT)$ weighted Dirac impulse comb and the above impulse response:

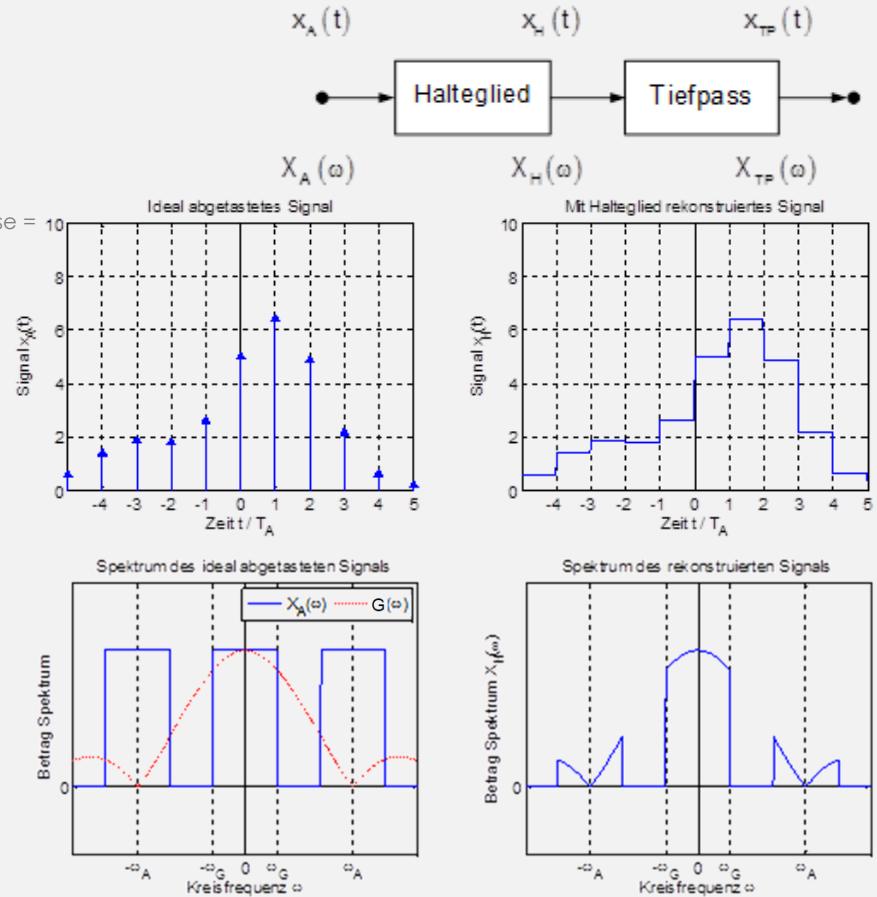
$$x_H(t) = \left(\sum_{k=-\infty}^{\infty} x(kT) \delta(t - kT) \right) * (\varepsilon(t) - \varepsilon(t-T))$$

Consideration in the spectral range:

The spectrum of the staircase signal $x_H(t)$ is therefore the ideal **periodic signal spectrum** $X_a(j\omega)$ multiplied by the **si spectrum** $G(j\omega)$ of the sample and hold element:

$$X_H(j\omega) = X_a(j\omega) \cdot G(j\omega) \quad \text{mit} \quad G(j\omega) = \frac{2}{\pi} \operatorname{si}\left(\frac{\omega T}{2}\right) \cdot e^{-j\frac{\omega T}{2}}$$

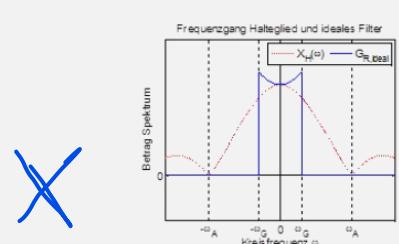
This slightly attenuates higher frequencies in the baseband, the sidebands are already more strongly suppressed, which makes the subsequent low-pass filtering easier!



2. Analog Interpolation Low pass:

- Is required to remove the remaining (already weakened) sidebands of the staircase signal.
- ZOH compensation:**

The above-mentioned ZOH can be reduced in the higher frequency components either by additional digital or analog inverse filtering can be undone in the baseband!

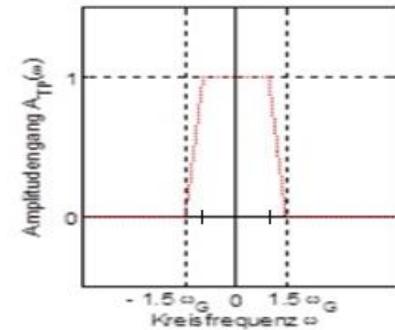
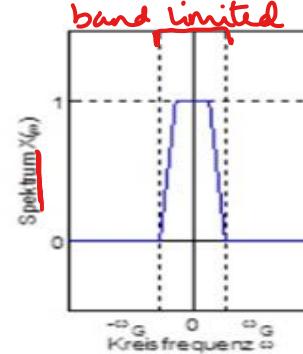
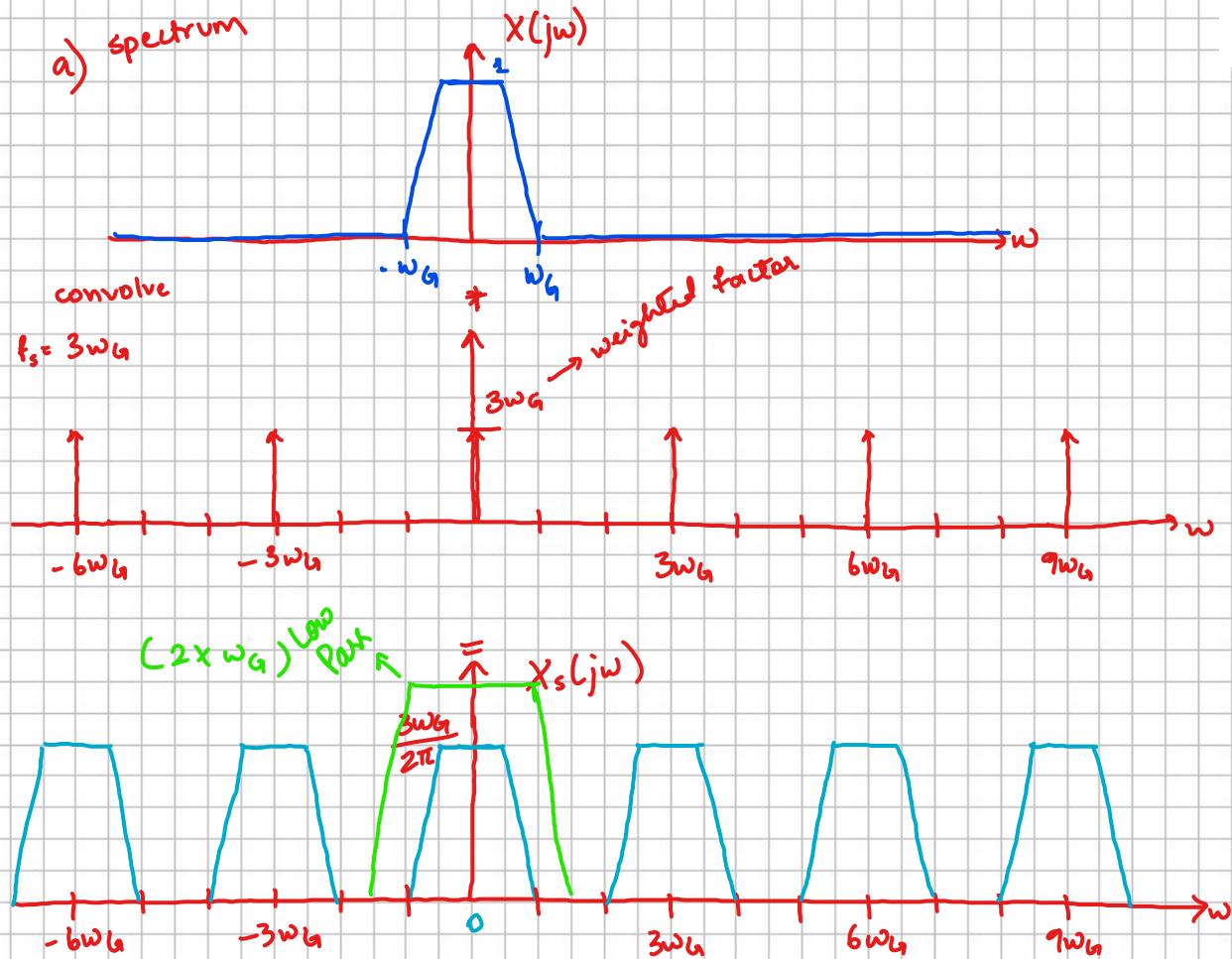


Ex.3.1 Sampling with a filter with finite slopes

Given is the spectrum $X(j\omega)$ an analog signal $x(t)$ which is to be sampled.
For later reconstruction, an analogue low pass filter is to be used, which has the amplitude response $A_{LP}(j\omega)$ as shown.

- First, sketch the resulting spectrum when $x(t)$ at a sampling frequency of $3\omega_G$ ab is keyed.
- What is the minimum sampling frequency that must be selected for this Reconstruction low pass no aliasing occurs at the output?

a) spectrum



frequency response
of a real
signal
with falling
area

1.5?

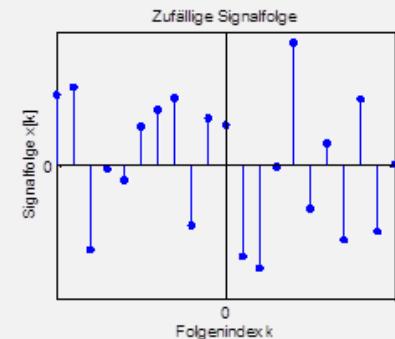
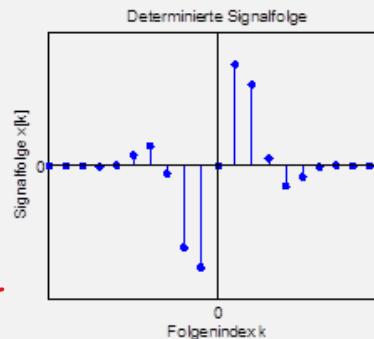
b) $w_s \geq 2.5\omega_G$

blue would
touch
green,
lowest
limit

Time Discrete Signals

Determined signals

- i.e. math. Either explicitly described
 $x[k] = 10 \cdot r^k \cdot \sin(\Omega_0 k)$ *direct description of func.*
- or just implicitly as so-called **difference equation** e.g. with the Initial condition $x[0] = x_0$:
 $x[k] = 2 \cdot x[k-1] - 4 \cdot x[k-2]$ *func. based on old value converted to explicit*



Random signals / Stochastic

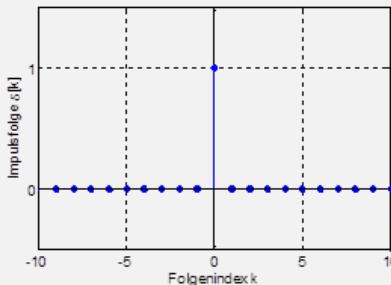
- cannot be specified exactly, only statistical properties are known for them.

Pulse Signal: / direct pulse

- Is not defined as a distribution, only as

$$\delta[k] = \begin{cases} 1 & \text{für } k = 0 \\ 0 & \text{für } \text{ganzahlige } k \neq 0 \end{cases}$$

It is zero for $k < 0$ and is therefore a causal consequence.

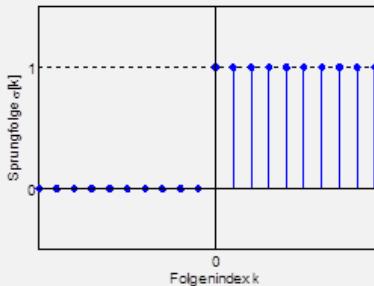


Step Signal:

- Is not defined as a distribution but just as

$$\sigma[k] = \varepsilon[k] = \begin{cases} 0 & \text{für } k < 0 \\ 1 & \text{für } k \geq 0 \end{cases}$$

It is zero for $k < 0$ and is therefore a causal consequence.



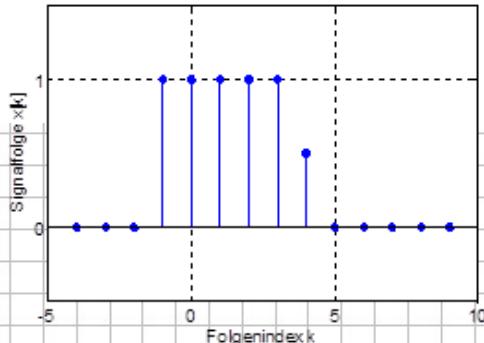
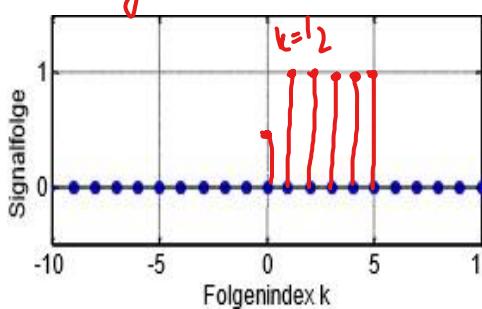
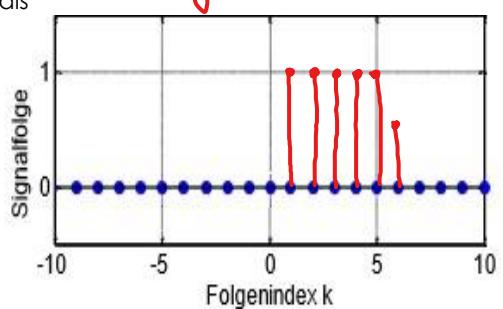
Ex.3.2 Discrete-time Signals

Given is a time-discrete signal $x[k]$.

a) Sketch the two signals

$$y_1[k] = x[k-2]$$

$$y_2[k] = x[4-k]$$



b) Give a math. explicit expression of the following signals.

Are they causal?

Are they energy or power signals?

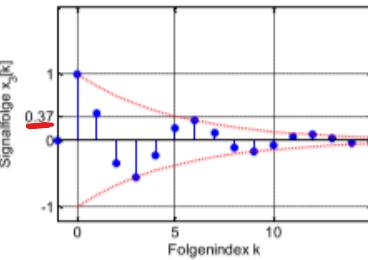
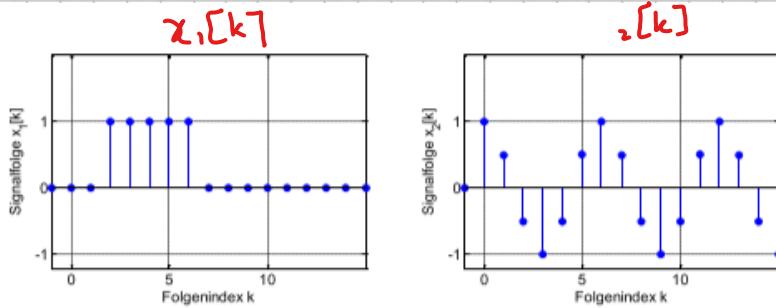
direct pulse jumps $x_1[k] = \delta[k-2] + \delta[k-3] + \delta[k-4] + \delta[k-5] + \delta[k-6]$
 $= \varepsilon[k-2] - \varepsilon[k-7]$ $E_{x_1} = 5$ [sum of sq.]
 ↓ one later

amplitude $x_2[k] = 1 \cdot \cos[2\pi/6 \cdot k] \cdot \varepsilon[k]$ to make it causal

its falling

Power signal \rightarrow area under $= \infty$

exponential decay $x_3[k] = e^{-\frac{1}{5}k} \cos[2\pi/6 \cdot k] \cdot \varepsilon[k]$



$$x_3[k] \text{ at } 5, 0.57$$

$$\bar{e}^1 = 0.37$$

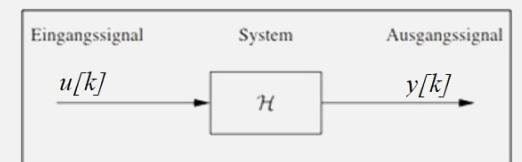
3.2 System Descriptions in the Time Domain

The Difference Equation

Description of time-discrete systems in the time domain:

- in the **continuous time** one describes systems with **differential equations**.
- in the **diskrete time** one describes systems with **difference equations**.

Algorithms for digital signal processing usually work with a clock with fixed sampling time T.



Example Moving average filter for signal smoothing

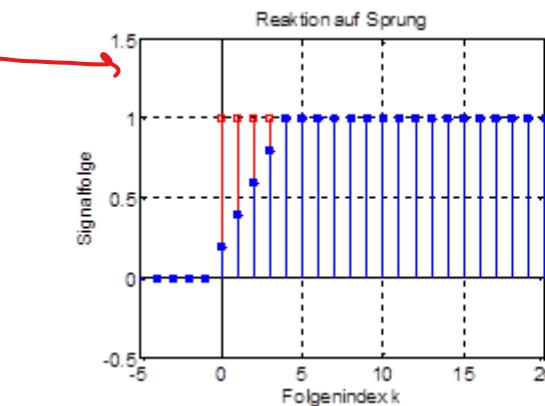
Moving avg window

$$y[k] = \frac{1}{5} \cdot (u[k] + u[k-1] + u[k-2] + u[k-3] + u[k-4])$$

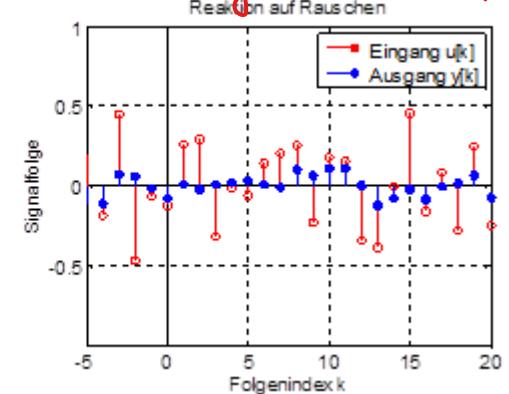
actual ip signal

The following noise suppressor is more memory-saving
(= digital low pass filter)

$$y[k] = \frac{4}{5} \cdot y[k-1] + \frac{1}{5} \cdot u[k]$$



*- low pass filter
- damping down the ip*



Impulse and step responses:

- Provide both (filter) algorithms **linear difference equations** which, as in continuous time, must first settle in and are already clearly described by their impulse and step responses!

Approximation by Difference Quotients

Discrete-time approximation of a DE:

$$a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_2 \ddot{u}(t) + b_1 \dot{u}(t) + b_0 u(t)$$

system

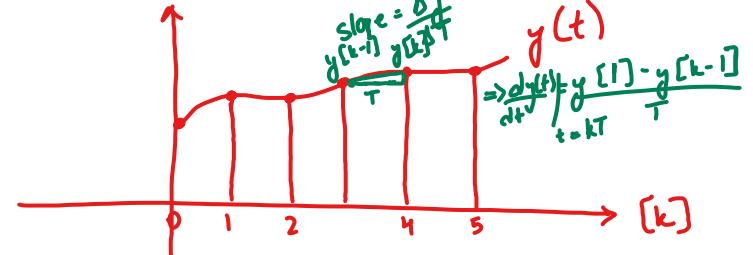
not relevant

$T=1 \text{ ms}$

- is obtained by replacing the differentials for example by (backward) **-difference quotients** over a sampling period T:

$$\left. \frac{dy(t)}{dt} \right|_{t=kT} \approx \frac{y[k] - y[k-1]}{T}$$

$$\left. \frac{d^2 y(t)}{dt^2} \right|_{t=kT} \approx \frac{y[k] - y[k-1] - y[k-1] - y[k-2]}{T} = \frac{y[k] - 2y[k-1] + y[k-2]}{T^2}$$



The result is a sum of sequence values which are each shifted by one sample value. In general, the current output signal is a function of the past input and output values and the current input signal.

Difference Equation:

- Analogous to the continuous time, the description of linear time-invariant **systems in discrete time** leads to linear **difference equations of Nth order with constant coefficients**:

$$\tilde{a}_2 y[k-2] + \tilde{a}_1 y[k-1] + \tilde{a}_0 y[k] = \tilde{b}_2 u[k-2] + \tilde{b}_1 u[k-1] + \tilde{b}_0 u[k]$$

or:

$$y[k] = -\tilde{a}_1 y[k-1] - \tilde{a}_2 y[k-2] + \tilde{b}_0 u[k] + \tilde{b}_1 u[k-1] + \tilde{b}_2 u[k-2] \quad (\tilde{a}_0 = 1)$$

System Response in the Time Domain

1) Direct recursive solution of the difference equation: *general form*

$$y[k] = \sum_{m=0}^M d_m \cdot u[k-m] - \sum_{n=1}^N c_n \cdot y[k-n]$$

This recursive solution method is completely sufficient for computer implementations. However, it is not a closed solution.



Discrete Convolution (aperiodic):

- the **Impulse response $g[k]$** of a time-discrete system is again the response to an impulse.
Any input signal $u[k]$ can now be described as a weighted sum of pulses:
- The system response $y[k]$ is thus simply the superposition of impulse responses weighted with $u[k]$ = **convolution**

$$u[k] = \sum_{\kappa=-\infty}^{\infty} u[\kappa] \cdot \delta[k-\kappa]$$

$$y[k] = g[k] * u[k] = \sum_{n=-\infty}^{\infty} g[n] \cdot u[k-n]$$

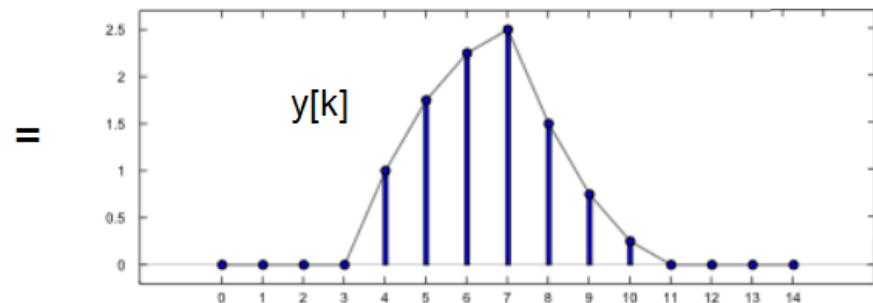
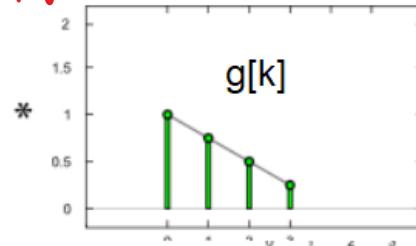
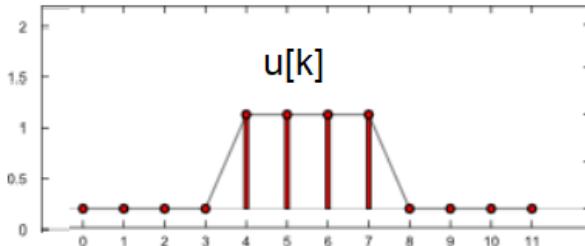
*impulse response
input signal
for pulses*

time discrete

$$y(t) = \int_{-\infty}^{\infty} g(\tau) \cdot u(t-\tau) d\tau$$

analog world

*flip it & sum it up &
shift it to the right & get the qp*



3.3 System Description via z-Transform

The z-Transform

(the Laplace transform equivalent)

The equivalent to the Laplace transform in discrete time is the z transformation.

- Derivation:

$$x_A(t) = \sum_{k=0}^{\infty} x(kT) \cdot \delta(t - kT) \quad \text{model of a discrete signal}$$

sample direct pulse at all positions

$$X_A(s) = L\left\{\sum_{k=0}^{\infty} x(kT) \cdot \delta(t - kT)\right\} = \sum_{k=0}^{\infty} x(kT) \cdot L\{\delta(t - kT)\} = \sum_{k=0}^{\infty} x(kT) \cdot e^{-skT} = \sum_{k=0}^{\infty} x[k] \cdot (e^{sT})^{-k} \quad q.e.d.$$

shifted direct pulse

as before sum after sum

just a no. brought in w.r.t

Definition of the ordinary (one-sided) z-Transform:

$$X(z) = Z\{x[k]\} = \sum_{k=0}^{\infty} x[k] \cdot z^{-k} \quad \text{mit } x[k] = 0 \quad \text{für } k < 0$$

$$z = e^{sT} \in C$$

$$x[k] = \frac{1}{2\pi j} \oint_C X(z) \cdot z^{k-1} dz$$

- Dumbbell symbol: $x[k] \circ - \bullet X(z)$

- Correspondence: $\delta[k] \circ - \bullet 1 \quad a^k \cdot \varepsilon[k] \circ - \bullet \frac{z}{z-a}$

- Shift rule: $x[k - k_0] \circ - \bullet z^{-k_0} \cdot X(z)$

- Further correspondence: See Annex:

Ex.3.3 z-Transform (calculation rules)

Compute the z-transforms of the following sequences using the properties of the z-transform:

- $x[k] = k^2 \cdot \varepsilon[k]$
- $x[k] = k \cdot a^k \cdot \varepsilon[k]$
- $x[k] = (k-k_0) \cdot a^{k-k_0} \cdot \varepsilon[k-k_0]$

$$a) x[k] = k^2 \cdot \varepsilon[k]$$

$$x[k] = \underbrace{k}_z \cdot x'[k]$$

$$x(z) = -z \frac{d}{dz} \frac{z}{(z-1)^2}$$

$$= \frac{-z(z+1)}{(z-1)^3}$$

$$x'[k] = \underbrace{k\varepsilon[k]}_z$$

$$x'(z) = \frac{z}{(z-1)^2}$$

$$b) x'[k] = k \varepsilon[k] \xrightarrow{z} x'(z) = \frac{z}{(z-1)^2}$$

$$x[k] = a^k x'[k] \xrightarrow{z} X(z) = x'(z/a)$$

$$= \frac{z/a}{(z/a-1)^2}$$

$$= \frac{za}{(z-a)^2}$$

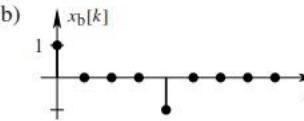
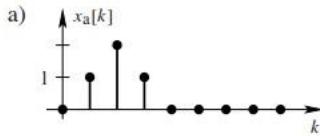
c) shifted version of b)
by k_0

$$\therefore x[k] = \underbrace{(k-k_0)}_z a^{k-k_0} \varepsilon(k-k_0)$$

$$X(z) = z^{k_0} \frac{za}{(z-a)^2} \quad (\text{for } k_0 > 0)$$

z-Transform (Signal values)

Calculate the z-transform of the following signals.



$$a) x_a[k] = \delta[k-1] + 2\delta[k-2] + \delta[k-3]$$

$$\underbrace{}_z$$

$$X_a(z) = z^{-1} + 2z^{-2} + z^{-3}$$

$$b) x_b[k] = -\delta[k-4] + \delta[k]$$

$$= -z^{-4} + 1$$

The z-Transfer Function

Difference equation:

$$\dots + c_2 y[k-2] + c_1 y[k-1] + c_0 y[k] = \dots + d_2 u[k-2] + d_1 u[k-1] + d_0 u[k]$$

$\Downarrow Z$

$$\dots + c_2 z^{-2} Y(z) + c_1 z^{-1} Y(z) + c_0 Y(z) = \dots + d_2 z^{-2} U(z) + d_1 z^{-1} U(z) + d_0 U(z)$$

z-Transfer Function:

- is the ratio of output to input in the image area:

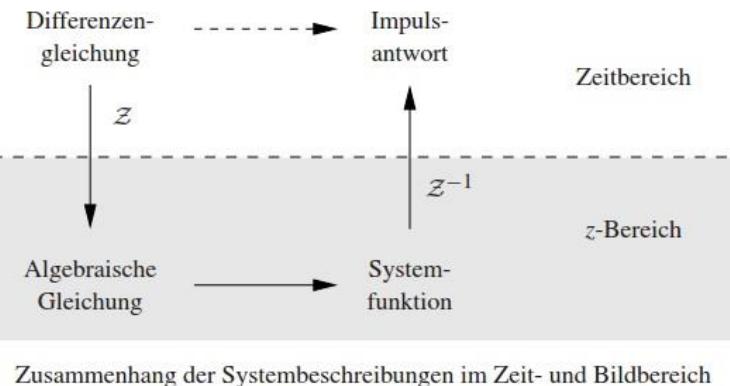
$$G(z) = \frac{Y(z)}{U(z)} = \frac{\dots + d_2 z^{-2} + d_1 z^{-1} + d_0}{\dots + c_2 z^{-2} + c_1 z^{-1} + c_0} \cdot \frac{z^N}{z^N} = \frac{\dots + b_2 z^2 + b_1 z + b_0}{\dots + a_2 z^2 + a_1 z + a_0} \quad \text{mit} \quad a_i = c_{N-i} \quad \text{und} \quad b_i = d_{N-i}$$

System equation:

$$Y(z) = G(z) \cdot U(z)$$

Impulse response: $g[k] \circ - \bullet G(z)$

Convolution: $y[k] = g[k] * u[k] = \sum_{n=-\infty}^{\infty} g[n] \cdot u[k-n]$



Attention:

- As you can see, the z-Transfer Function exists in **two forms** (once in **z** and **z^{-1}**) with mirrored coefficients!

Stability of Time-Discrete Systems

Between the s-area and z-area the formal relationship holds (is only a substitution):

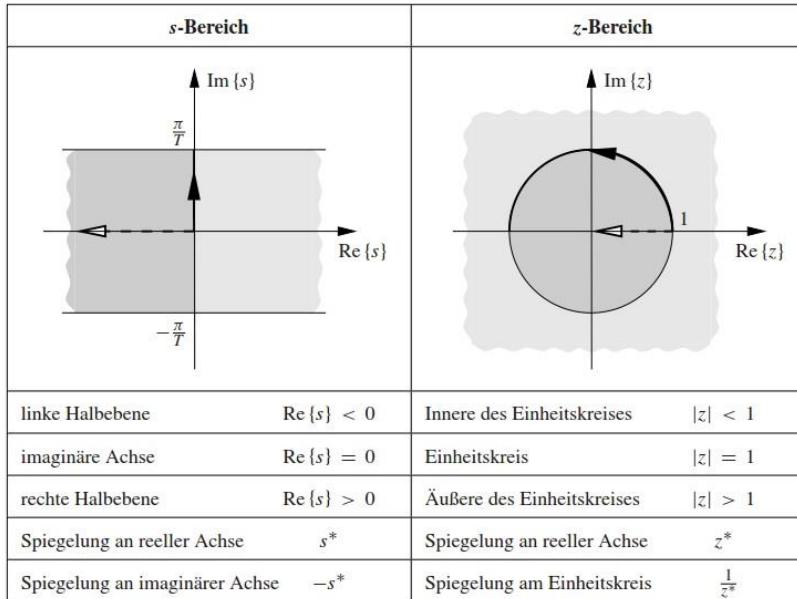
$$z = e^{sT} = e^{T \cdot \operatorname{Re}\{s\}} \cdot e^{jT \cdot \operatorname{Im}\{s\}}$$

$$0_s \xrightarrow{z=e^{sT}} 1_z \quad -\infty_s \xrightarrow{z=e^{sT}} 0_z \quad \pm \frac{j\pi}{T}_s \xrightarrow{z=e^{sT}} -1_z$$

- The above relationship can be interpreted mathematically as mapping of the complex s-plane into the complex z-plane.

Stability

- Of a time-discrete exists when **all poles of the z-transfer function lie inside the unit circle!**



Zusammenhang zwischen s- und z-Bereich

Convergence area of the one-sided z-Transform:

- is the area **outside a circle** enclosing the largest pole in magnitude (completely analogous to the Laplace transformation)

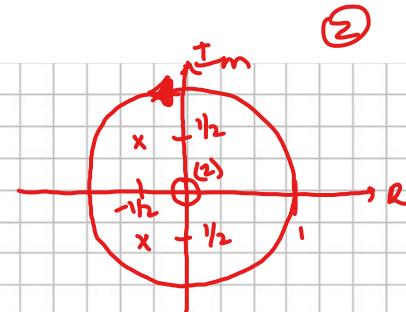
Ex.3.4 Stability analysis of time-discrete systems

Check the following systems for stability:

a) $G(z) = \frac{z^2}{z^2 + z + 0.5}$ b) $y[k] = y[k-1] + y[k-2] + \cancel{x}[k]$ c) $g[k] = \varepsilon[k] - \varepsilon[k-3]$

a) $z_{\alpha,1,2} = \frac{-1 \pm \sqrt{1-2}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$

$|z_{\alpha,1,2}| < 1 \rightarrow \text{Stable}$

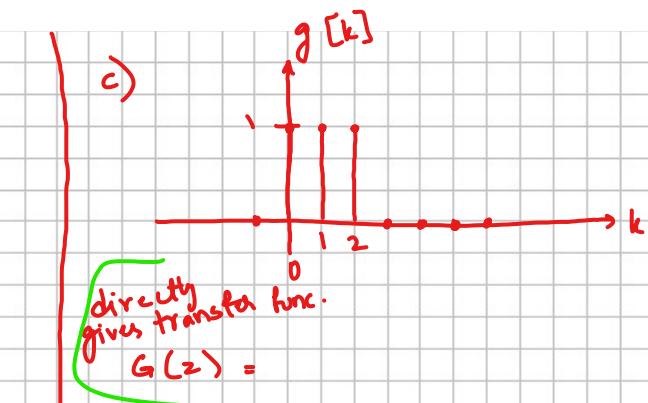
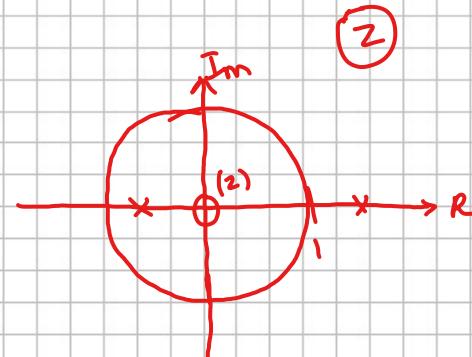


b) $Y(z) = \bar{z}^1 Y(z) + \bar{z}^2 Y(z) + U(z)$

$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{1 - \bar{z}^1 - \bar{z}^2} = \frac{z^2}{z^2 - z - 1}$$

$$z_{\alpha,1,2} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2} \\ = \frac{1}{2} \pm \sqrt{1.25}$$

$|z_{\alpha,1}| > 1 \rightarrow \text{Unstable}$



alternatively, in the time domain,
BIBO stable.

$$\int_0^\infty |g(t)| dt < \infty \rightarrow \text{stable}$$

in the discrete world,

BIBO stability:
area under curve. $< \infty$

$$\boxed{\sum_{k=0}^{\infty} |g[k]| < \infty} \times$$

here,

$$\sum_{k=0}^{\infty} 1 + 1 + 1 = 3 < \infty \rightarrow \text{stable}$$

Ex 3.5 Inverse z-Transform through partial fraction decomposition

a) Determine the time signal $x[k]$ using the partial fraction decomposition:

$$X(z) = \frac{3z^2 - 4z}{(z-1)(z-2)} \quad \text{here, } m=n$$

$$X(z) = \frac{3z^2 - 4z}{(z-1)(z-2)}$$

to be able to do it, $\div z$ &
split ε , then $x \cdot z$.

$$\begin{aligned} X'(z) &= \frac{X(z)}{z} = \frac{3z-4}{(z-1)(z-2)} \\ &= \frac{A}{z-1} + \frac{B}{z-2} \\ &= \frac{1}{z-1} + \frac{2}{z-2} \end{aligned}$$

$n[k]$ not possible as the PFD is not possible since you would ideally get

$\frac{A}{z-1} + \frac{B}{z-2} \rightarrow A$ is a const. \rightarrow doesn't give inverse transformation

$$\begin{aligned} X(z) &= x'(z) \cdot z \\ &= \frac{z}{z-1} + \frac{2z}{z-2} \\ &\quad \text{g} z' \end{aligned}$$

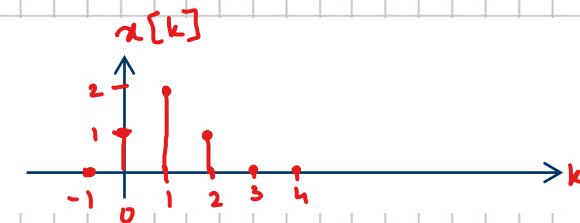
$$x[k] = \varepsilon[k] + 2z^k \varepsilon[k] = \varepsilon[k](1 + 2z^k)$$

Inverse z-transformation by Polynomial division

a) Calculate and plot the time signal $x[k]$ using the Polynomial division:

$$X(z) = \frac{z^2 + 2z + 1}{z^2} = 1 + \frac{2}{z} + \frac{1}{z^2}$$

$$x[k] = \delta[k] + 2\delta[k-1] + \delta[k-2]$$



Discrete-time systems

a) What is the difference equation of this system?

b) How must M be chosen so that the system is causal? if $n > m \rightarrow$ non-causal ; $M \leq 2 \rightarrow$ causal

$$a) Y(z) \cdot (z^2 + 2z + 1) = U(z) \cdot (z^M + 1) \quad | \div z^2$$

$$Y(z) \cdot (1 + 2z^{-1} + z^{-2}) = U(z) \cdot (z^{M-2} + z^{-2})$$

$\underbrace{z^{-1}}$

$$y[k] = -2y[k-1] + y[k-2] + u[k+M-2] + u[k-2]$$

$$G(z) = \frac{z^M + 1}{z^2 + 2z - 1}$$

Design over Matched- and Bilinear Transform

Exact (Matched)-Transform

$$z = e^{sT} \quad b_z w. \quad s = \frac{1}{T} \cdot \ln z \quad \rightarrow \text{non linear not possible}$$

Since the transformation between s and z areas is an e-function, polynomials in s and z are unfortunately only mapped into so-called transcendent functions.

- However, you can do the exact transformation, for example, by a **Taylor series** development with cut-off after the linear term:

$$s = \frac{1}{T} \cdot \ln z \approx \frac{2}{T} \cdot \left\{ \frac{z-1}{z+1} + \frac{1}{3} \left(\frac{z-1}{z+1} \right)^3 + \frac{1}{5} \left(\frac{z-1}{z+1} \right)^5 + \dots \right\}$$

Bilinear (Tustin) -Transform / Trapezoidal Rule:

$$s = \frac{2}{T} \cdot \frac{z-1}{z+1} \quad b_z w. \quad z = \frac{1+s\frac{T}{2}}{1-s\frac{T}{2}}$$

Because of its simplicity, the bilinear transformation is the **most widely used method** to obtain a digital algorithm (Euler methods are less precise)! With the filter design, however, there is a slight frequency distortion (so-called. wraping). For more details see literature ...

example ($T = 1$):

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{4s+1} \Rightarrow G(z) = \frac{1}{4s+1} \Big|_{s=\frac{2}{T} \frac{z-1}{z+1}} = \frac{1}{4 \frac{2}{T} \cdot \frac{z-1}{z+1} + 1} = \frac{z+1}{8z-8+z+1} = \frac{z+1}{9z-7} \cdot \frac{z^{-1}}{z^{-1}} = \frac{1+z^{-1}}{9-7z^{-1}}$$

$$Y(z) \cdot (9-7z^{-1}) = U(z) \cdot (1+z^{-1}) \quad \bullet - \circ \quad 9y[k] - 7y[k-1] = u[k] + u[k-1]$$

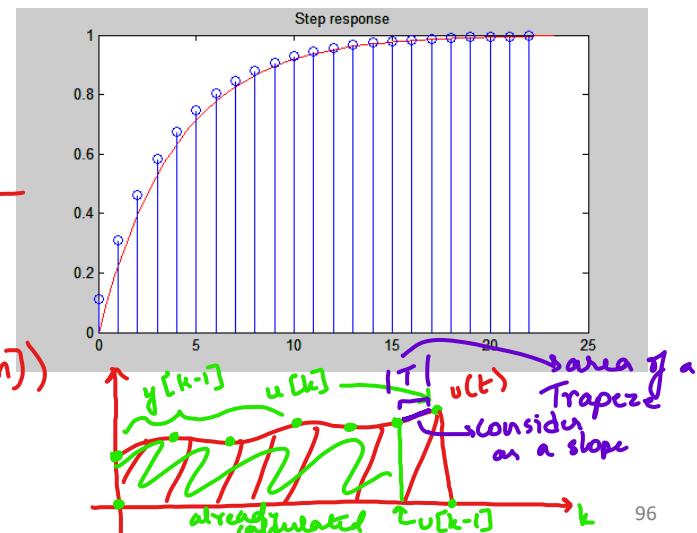
$$y[k] = \frac{1}{9} y[k-1] + \frac{1}{9} u[k] + \frac{1}{9} u[k-1]$$

Ex. 2
 $G(s) = \frac{1}{s}$

$$G(z) = \frac{1}{s} \Big|_{s=\frac{2}{T} \frac{z-1}{z+1}} = \frac{1}{2} \frac{z+1}{z-1} \quad T \Rightarrow \text{sampling time}$$

$$y(z)(z-1) = u(z) \frac{1}{2} (z+1) \quad | \div z \quad (\text{never want } [k,n])$$

$$y[k] = y[k-1] + \frac{1}{2} u[k] + \frac{1}{2} u[k-1]$$

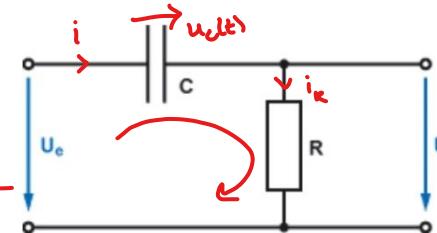


Ex.3.6 Filter Realization by using Bilinear Transform

includes most from before
 $DE \rightarrow AE \xrightarrow{G(s)} G(z) \rightarrow y[k]$

A digital filter is to be developed:

- First determine the DE and the transfer function $G(s)$ of this analog high-pass filter.
 (Assume: $C = 2\mu F$ and $R = 1 M\Omega$, ie $\tau = RC = 2$ sec.)
- Convert this analog filter into a digital filter using the bilinear transformation (for $T = 0.4$ sec).
- Finally, give the difference equation to be coded.



a) ① $u = u_e(t) \quad y = u_a(t) \quad n = u_c(t)$
 ② Done on diagram
 ③ $i = C \frac{du_c}{dt} \quad R = \frac{U_a(t)}{i} \quad u_c + u_a = u_e$

$$④ i(t, n, u) = \frac{du_c}{dt} = \frac{i}{C} = \frac{u_a(t)}{RC} = \frac{u_e - u_c}{RC}$$

* ⑤ DE : $f(y_a, e, u_e)$

$$i_c = \frac{1}{RC} (u_e + u_a - u_c) = \frac{u_a}{RC}$$

$$\begin{aligned} u_a &= RC i_c \\ u_a &= RC (i_c - i_a) \\ [RC y + y = RC i] \Rightarrow 2y + y &= 2i \end{aligned}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2sY(s) + Y(s)}{2sU(s)} = \frac{2s}{2s+1} \rightarrow DT, \text{ High Pass filter}$$

b) $T = 0.4 s$

$$G(z) = G(s) \Big|_{s=\frac{z-1}{T}} = \frac{\frac{4}{0.4} \frac{z-1}{z+1}}{\frac{4}{0.4} \frac{z-1}{z+1} + 1} = \frac{10(z-1)}{10(z-1)+(z+1)} = \frac{z-1}{(z-1)+0.1(z+1)} = \frac{z-1}{1.1z-0.9} \rightarrow \text{Discrete HPF}$$

$$Y(z)(1.1z - 0.9) = U(z)(z-1) \quad | \div z$$

$$Y(z)(1.1 - 0.9z^{-1}) = U(z)(1 - z^{-1})$$

c) $1.1y[k] - 0.9y[k-1] = u[k] - u[k-1]$

$$y[k] = \frac{0.9}{1.1} y[k-1] + \frac{1}{1.1} u[k] - \frac{u[k-1]}{1.1}$$

Software Code

Pulse & Step Invariant Transform

Pulse Invariant Transform:

If the impulse response is to be simulated exactly, one must sample the impulse response of the continuous-time system and subject this sequence to the z-transformation, ie

$$G(z) = \cdot Z \left\{ L^{-1} \left\{ G(s) \right\} \Big|_{t=kT} \right\}$$

→ z transform of the discrete signal

Step Invariant Transform:

If the step response is to be reproduced exactly, the step response of the time-continuous system must be sampled, this derivation and this sequence must be subjected to the z-transformation. The derivation can expediently be carried out in the z range (a multiplication by $(z-1)/z$), ie one obtains

$$G_{HS}(z) = \frac{z-1}{z} \cdot Z \left\{ L^{-1} \left\{ \frac{G(s)}{s} \right\} \Big|_{t=kT} \right\}$$

Example: Step invariant transform

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{4s+1}$$

$$\textcircled{1} \Rightarrow H(s) = \frac{G(s)}{s} = \frac{1}{s(4s+1)}$$

$$\bullet \textcircled{2} \quad h(t) = (1 - e^{-\frac{t}{4}}) \cdot \varepsilon(t)$$

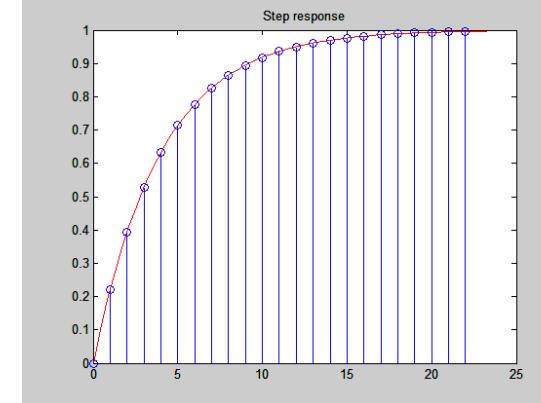
$$\textcircled{3} \Rightarrow h(t)|_{t=kT} = (1 - e^{-\frac{kT}{4}}) \cdot \varepsilon(kT) = (1 - (e^{-\frac{T}{4}})^k) \cdot \varepsilon(kT)$$

$$\Rightarrow h[k] = (1 - c^k) \cdot \varepsilon[k] \quad \textcircled{4} \quad H(z) = \frac{z}{z-1} - \frac{z}{z-c} \Rightarrow$$

$$c = e^{-\frac{T}{4}}$$

$$\textcircled{5} \quad G(z) = \frac{z-1}{z} \cdot H(z) = \frac{1-c}{z-c}$$

$$\text{DE} \Rightarrow y[k] = (1 - c) \cdot u[k-1] + c \cdot y[k-1]$$



Ex.3.7 Step-Invariant Transform

A controller development led to the following time-continuous I-controller:

- a) Perform the discretization by using the step invariant transform and again give the difference equation of the time-discrete controller. This time the sampling time is $T = 500\text{ms}$.

$$G(s) = \frac{2}{s}$$

$$\textcircled{1} \quad h(s) = \frac{G(s)}{s} = \frac{2}{s^2} \xrightarrow{\text{z}^{-1}} \textcircled{2} \quad h(t) = 2t \varepsilon(t)$$

$$\textcircled{3} \quad h(t)|_{t=kT} = 2kT \varepsilon(kT) \quad \textcircled{4} \quad T = 500\text{ms} \\ = 1000 k \varepsilon[k] \xrightarrow{\text{z}^{-1}} H(z) = 1000 \frac{z}{(z-1)^2}$$

$$\textcircled{5} \quad G(z) = \frac{z-1}{z} H(z) \\ = \frac{z-1}{z} 1000 \cdot \frac{z}{(z-1)^2} \\ = \frac{1000}{z-1} \quad = \frac{Y(z)}{U(z)}$$

$$\textcircled{6} \quad zY(z) - Y(z) = 1000 U(z) \quad | \div z$$

$$Y(z) - z^1 Y(z) = 1000 z^1 U(z)$$

$$\text{DE: } \boxed{y[k] = y[k-1] + 1000 u[k-1]}$$

SW code

Implementation of Time-Discrete Systems

Direct Form I:

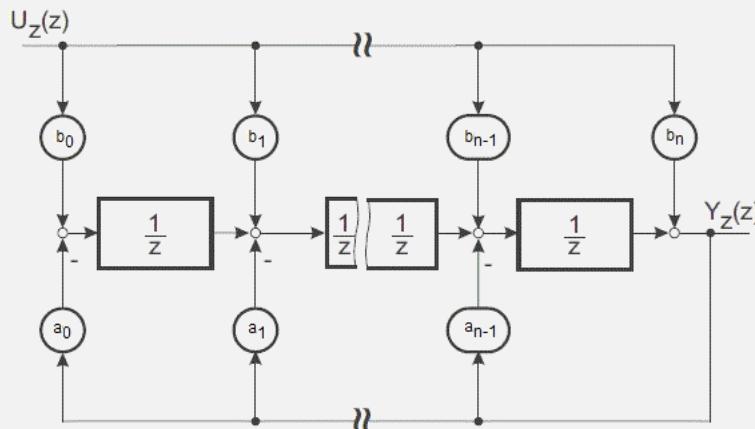
$$G(z) = \frac{Y(z)}{U(z)} = \frac{\dots + b_2 z^2 + b_1 z + b_0}{\dots + a_2 z^2 + a_1 z + a_0} \cdot \frac{z^{-N}}{z^{-N}} = \frac{\dots + d_2 z^{-2} + d_1 z^{-1} + d_0}{\dots + c_2 z^{-2} + c_1 z^{-1} + c_0}$$

$$Y(z) \cdot (\dots + c_2 z^{-2} + c_1 z^{-1} + c_0) = U(z) \cdot (\dots + d_2 z^{-2} + d_1 z^{-1} + d_0) \quad \bullet - \circ \quad c_0 y[k] = d_0 u[k] + d_1 u[k-1] + d_2 u[k-2] \dots - c_1 y[k-1] - c_2 y[k-2] \dots$$

Analogous to continuous time, there are canonical ones **Block diagrams for simulation or as implementationPrototypes:** (Derivation see literature)

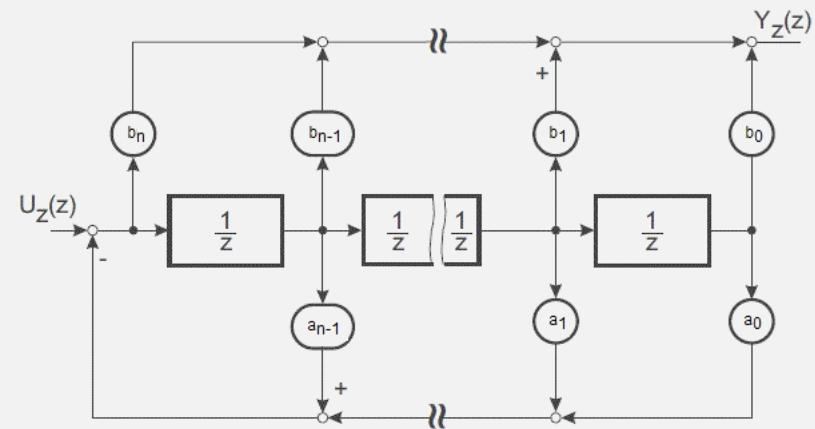
Canonical Transposed Direct Form II:

- (Observation normal form)



Direct Form II:

- (Controll normal form):



- The **block z^{-1}** describes a **delay (Memory) by one clock pulse**.

The direct forms II come with the **Minimum number** to save, but are not the first choice, especially for fixed point arithmetic (better direct form I or other cascade structures ...)

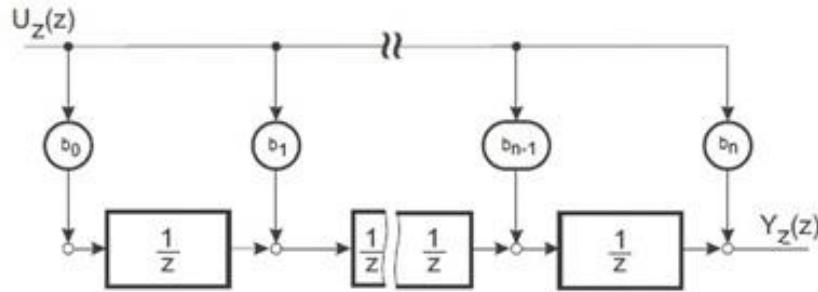
FIR Filter vs. IIR Filter

FIR Filter

- have no feedbacks and therefore have a **finite** impulse response (**finite impulse response**).
- they are always stable (because they have only poles at the origin or no feedback) and have a linear phase.

$$G_{FIR}(z) = \frac{\dots + b_2 z^2 + b_1 z + b_0}{z^N}$$

Because of their linear phase, they always have one constant group delay and are therefore very popular, especially when designing digital filters ...

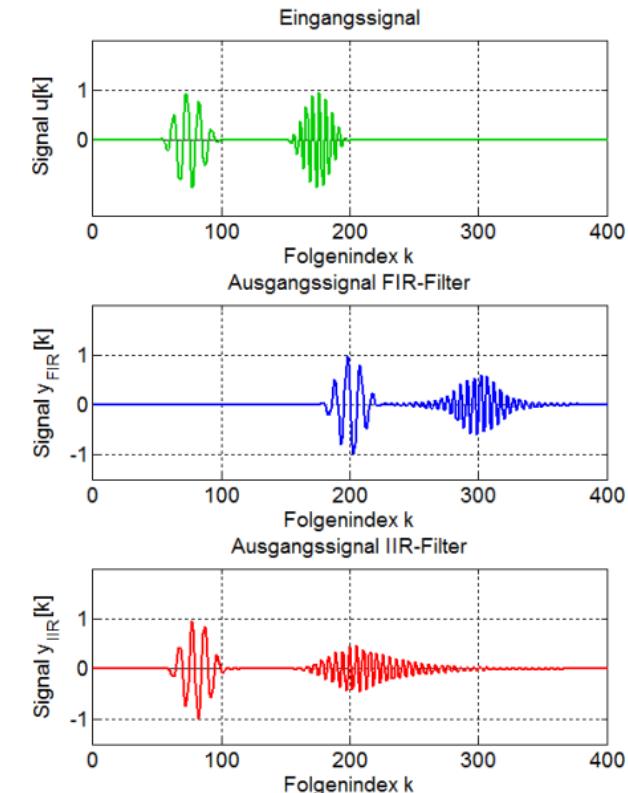


IIR Filter

- have feedback and therefore have a **infinite** long impulse response (**Infinite Impulse Response**).
- They **can be unstable** and **do not** have a linear phase.

$$G_{IIR}(z) = \frac{\dots + b_2 z^2 + b_1 z + b_0}{\dots + a_2 z^2 + a_1 z + a_0}$$

With the same filter order, however, they have a higher slope than FIR filters ...



3.4 System Descriptions in the Frequency Domain

The Discrete Time Fourier Transformation (DTFT)

The for discrete signals adapted **Fourier Transform** is the **Discrete Time Fourier Transform**

o Derivation:

$$x_a(t) = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta(t - kT)$$

$$X_a(j\omega) = F\{x_a(t)\} = \int_{-\infty}^{+\infty} \left(\sum_{k=-\infty}^{\infty} x[k] \cdot \delta(t - kT) \right) e^{-i\omega t} dt = \sum_{k=-\infty}^{\infty} x[k] \cdot \int_{-\infty}^{+\infty} \delta(t - kT) e^{-i\omega t} dt = \sum_{k=-\infty}^{\infty} x[k] e^{-i\omega kT} \quad q.e.d.$$

Definition of the discrete-time Fourier transform:

$$X(j\Omega) = \sum_{k=-\infty}^{\infty} x[k] e^{-i\Omega k}$$

$$\Omega = \omega T = \omega \frac{2\pi}{\omega_A}$$

given periodic spectrum

$$X(z) = \sum_{k=0}^{\infty} x[k] z^k$$

z-transform

only causal signals

$$\therefore z = e^{j\Omega}$$

Inverse:

$$x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(j\Omega) e^{j\Omega k} d\Omega$$

▪ Ω = normalized angular frequency (to become independent of T !)

▪ I.e. that the spectrum of a time-discrete signal is continuous and periodically in 2π .

If $x[k]$ is the time-discrete signal sampled by a signal $x(t)$, then the spectrum of $x[k]$ (except for the factor $1/T$) is the superposition of infinite number of periodic continuations of the Fourier spectrum $X_F(j\omega)$ from $x(t)$!

This also applies if the sampling theorem is disregarded, but then again leads to spectral overlap of the sidebands ...

$$X(j\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_F(j(\omega - n \cdot \omega_A)) \Big|_{\omega = \frac{\Omega}{T}}$$

periodic repetition of the cont. fourier transform

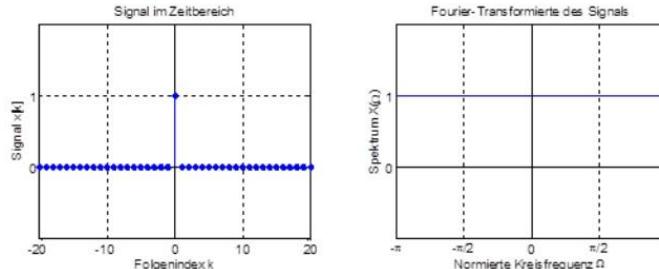
(every 2π) sampling freq.)

| Normierte Kreisfrequenz Ω | Kreisfrequenz ω |
|----------------------------------|------------------------|
| $-\pi$ | $-\omega_A/2$ |
| $-\pi/2$ | $-\omega_A/4$ |
| 0 | 0 |
| $\pi/2$ | $\omega_A/4$ |
| π | $\omega_A/2$ |

Spectra of important discrete Signals

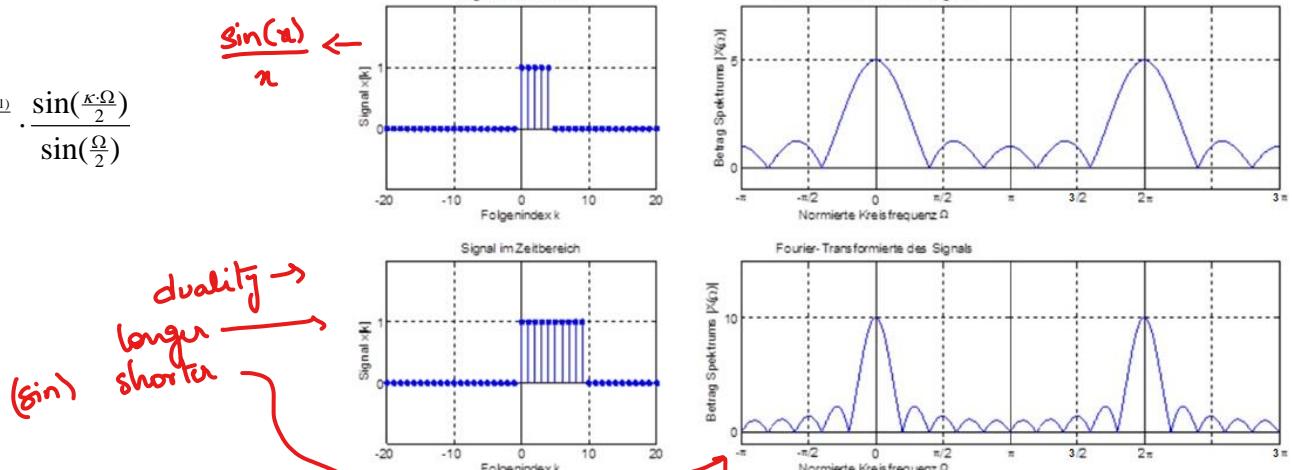
Discrete time Impulse function:

$$x[k] = \delta[k] \quad \circ - \bullet \quad X(j\Omega) = 1$$



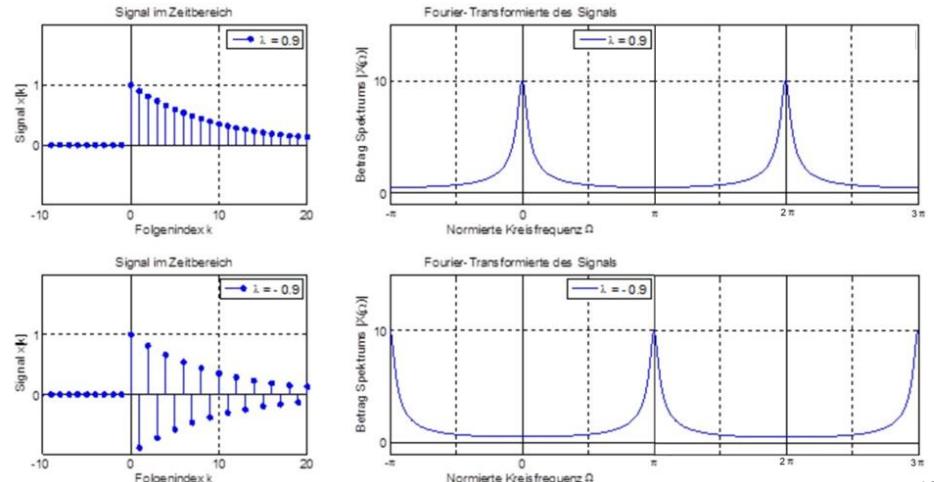
Discrete-time rectangular pulse:

$$x[k] = \varepsilon[k] - \varepsilon[k - \kappa] \quad \circ - \bullet \quad X(j\Omega) = e^{-j\frac{\Omega(\kappa-1)}{2}} \cdot \frac{\sin(\frac{\kappa\cdot\Omega}{2})}{\sin(\frac{\Omega}{2})}$$



Discrete-time exponential sequence:

$$x[k] = \lambda^k \cdot \varepsilon[k] \quad \circ - \bullet \quad X(j\Omega) = \frac{1}{1 - \lambda \cdot e^{-j\Omega}}$$



Ex.3.9 Discrete-Time Fourier Transformation

A time-discrete signal $x[k]$ has the following spectrum.

a) Determine and sketch the time signal $x[k]$.

b) Determine the spectra of the following signals using the theorems of the discrete-time Fourier transform:

$$x_1[k] = x[k-5], x_2[k] = x[-k], x_3[k] = k \cdot x[k], x_4[k] = x[k] * x[k], x_5[k] = x[k] - x[k-1]$$

spectrum given

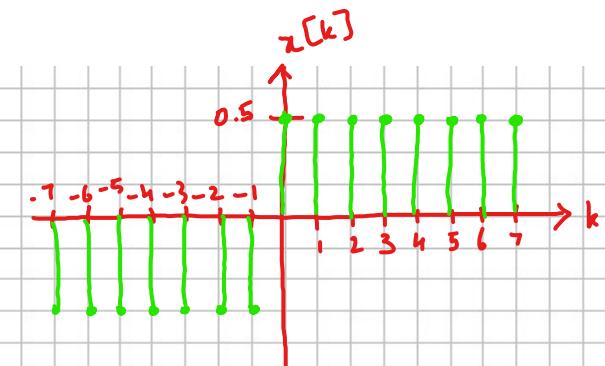
$$X(j\Omega) = \frac{1}{1 - e^{j\Omega}}$$

$$a) X'(j\Omega) = \pi \delta(\Omega) + \frac{1}{1 - e^{j\Omega}}$$

$$\xrightarrow{\text{DTFT}^{-1}} x'[k] = \varepsilon[k]$$

$$X(j\Omega) = X'(j\Omega) - \pi \delta(\Omega)$$

$$\xrightarrow{\text{DTFT}^{-1}} \boxed{x[k] = \varepsilon[k] - \frac{1}{2}}$$



$$b) X_1(j\Omega) = X(j\Omega) e^{j\omega_5}$$

$$= \frac{1}{1 - e^{j\Omega}} e^{j\omega_5}$$

flipped $X_2(j\Omega) = \frac{1}{1 - e^{j\Omega}}$ (biggest)

$$X_3(j\Omega) = j \frac{d}{d\Omega} \frac{1}{1 - e^{j\Omega}} = : \frac{e^{j\Omega}}{(1 - e^{j\Omega})^2}$$

$$X_4(j\Omega) = (X(j\Omega))^2 = \frac{1}{(1 - e^{j\Omega})^2}$$

$$X_5(j\Omega) = \frac{1}{1 - e^{j\Omega}} - \frac{1}{1 - e^{j\Omega}} e^{j\Omega} = \frac{1}{1 - e^{j\Omega}} [1 - e^{j\Omega}] = \underbrace{1}_{\text{const. spectrum}} \xrightarrow{\text{DTFT}^{-1}} x_5[k] = \delta[k]$$

Convolution in one domain
is multiplication in the
other domain

The Frequency Response of Discrete Systems

Frequency response:

$$G(j\Omega) = \frac{Y(j\Omega)}{U(j\Omega)}$$

z transform to discrete fourier transform applied on the difference eq.

- It is a complex function of the frequency and the discrete time Fourier transform of the impulse response $g[k]$ of a system.

Calculation from the z-transfer function:

- For the important signal and system class whose z-convergence area includes the unit circle, and which includes all causal and decaying signals and all stable systems both transforms exist with the substitution:

$$G(j\Omega) = G(z)|_{z=e^{j\Omega}}$$

← discrete fourier transform

Amplitude response:

- describes the ratio of the output to the input amplitude for every sinusoidal signal with the normalized frequency Ω :

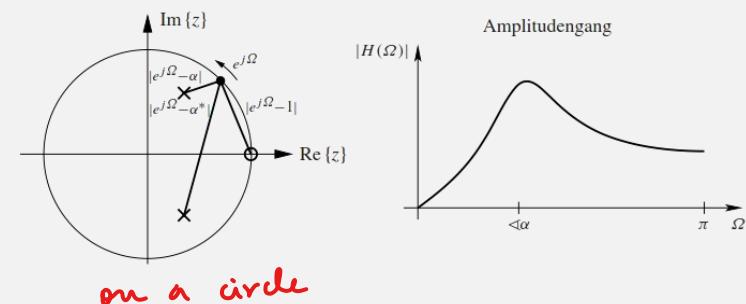
$$|G(j\Omega)| = \frac{|Y(j\Omega)|}{|U(j\Omega)|}$$

Phase response:

- describes the temporal phase shift between the output and the input oscillation.

$$\varphi(j\Omega) = \angle G(j\Omega)$$

In the s-plane the frequency response corresponded to the course of the s-TF along the imaginary axis,
In the z-plane the frequency response corresponds to the **course of the z-TF along the unit circle!**



Ex.3.10 Frequency response of a digital filter

The difference equation of a digital filter for forming the mean value is given:

- Give its transfer function.
- Sketch the PZ diagram of the filter.
- Give its frequency response.
- Calculate and sketch the magnitude of the frequency response.

$$a) \quad y[k] = \frac{1}{3}(u[k] + u[k-1] + u[k-2]) : DE$$

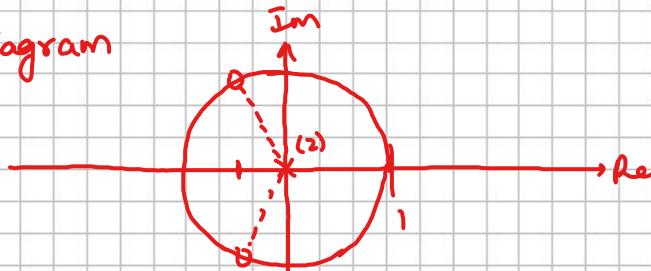
$$AE: \quad 3y(z) = u(z) + z^{-1}u(z) + z^{-2}u(z)$$

$$3y(z) = u(z)[1 + z^{-1} + z^{-2}]$$

$$G(z) = \frac{y(z)}{u(z)} = \frac{1 + z^{-1} + z^{-2}}{3}$$

$$\Rightarrow G(z) = \frac{z^2 + z + 1}{3z^2} \Rightarrow FIR \rightarrow \text{no feedback}$$

b) PZ diagram



$$z_{\infty,1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$\sin(60^\circ)$
exactly on
the circle

$$y[k] = \frac{1}{3}(u[k] + u[k-1] + u[k-2])$$

poles \rightarrow denominator
zeros \rightarrow numerator

c) as $G(z)$ is stable (PZ diagram):

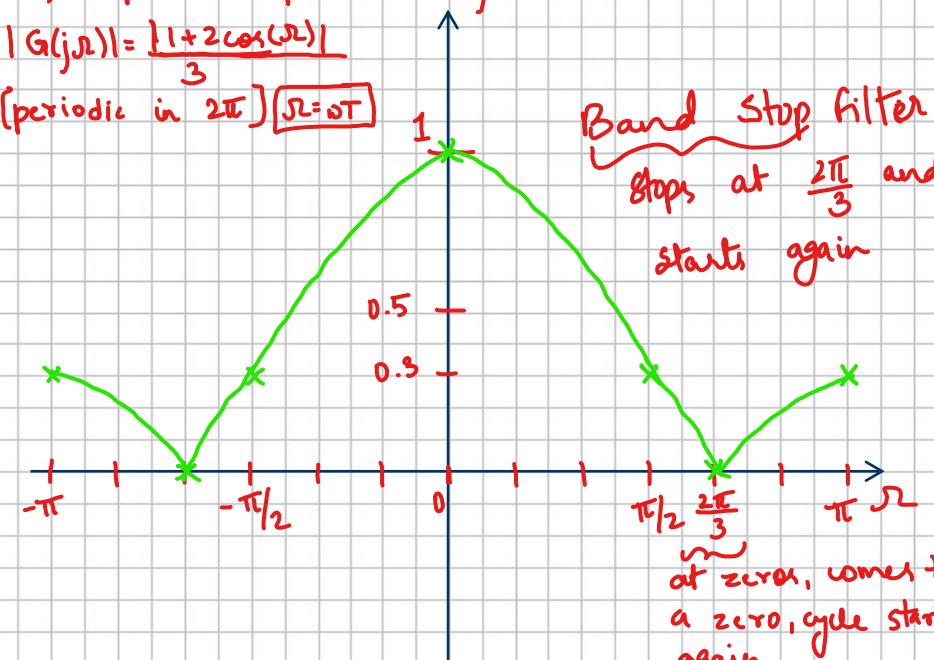
$$G(j\omega) = G(z) \Big|_{z=j\omega} = \frac{\frac{1}{3}(j\omega + \bar{j})}{e^{j\omega} + e^{-j\omega} + 1} = \frac{\frac{1}{3}(j\omega + \bar{j})}{3e^{j\omega}}$$

$$G(j\omega) = \frac{1 + 2\cos(\omega)}{3e^{j\omega}}$$

d) amplitude response $\rightarrow |G(j\omega)|$

$$|G(j\omega)| = \frac{|1 + 2\cos(\omega)|}{3}$$

(periodic in 2π) $[j\omega = \omega T]$



Discrete Fourier Transform (DFT)

- It directly calculates (except for a prefactor) the **Fourier coefficients of the discrete-time Fourier series**:

$$c_n = \frac{1}{N} \sum_{k=0}^{N-1} x[k] \cdot e^{-j(2\pi \frac{n}{N} k)}$$

Discrete Fourier Transform (DFT):

$$X[n] = DFT\{x[k]\} = \sum_{k=0}^{N-1} x[k] \cdot e^{-j(2\pi \frac{n}{N} k)} \quad \text{für } k = 0, \dots, N-1$$

- The DFT computes „from **N Samples** $x[k]$, **N complex Fourier coefficients** $X[n]$ “

Or:

even rect. func \rightarrow $\frac{\sin x}{n}$ func \rightarrow real

“It provides **N values** (of a period) of the spectrum of a periodically continued time signal of length **N**. ”

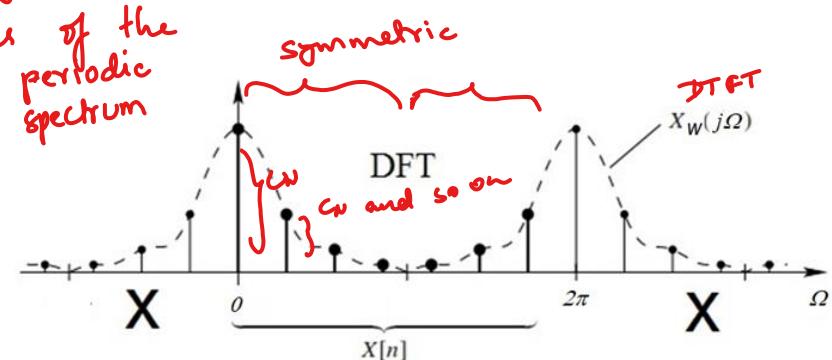
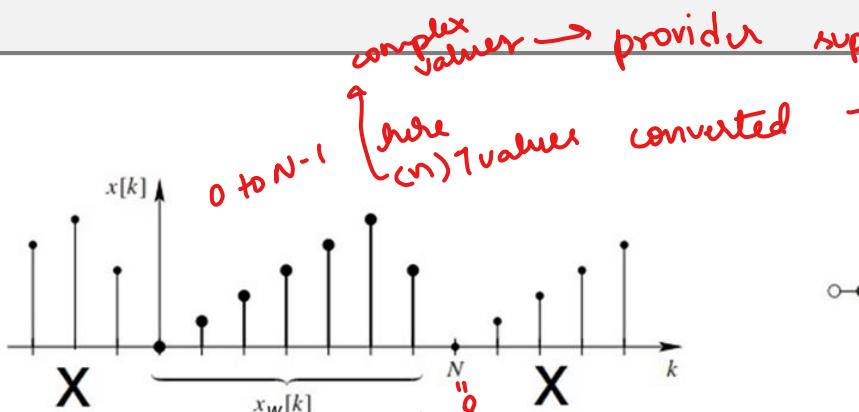
- The following applies:

$$X[n] = X_w(j\Omega) \Big|_{\Omega=\frac{2\pi}{N} \cdot n}$$

← DFT to DFT

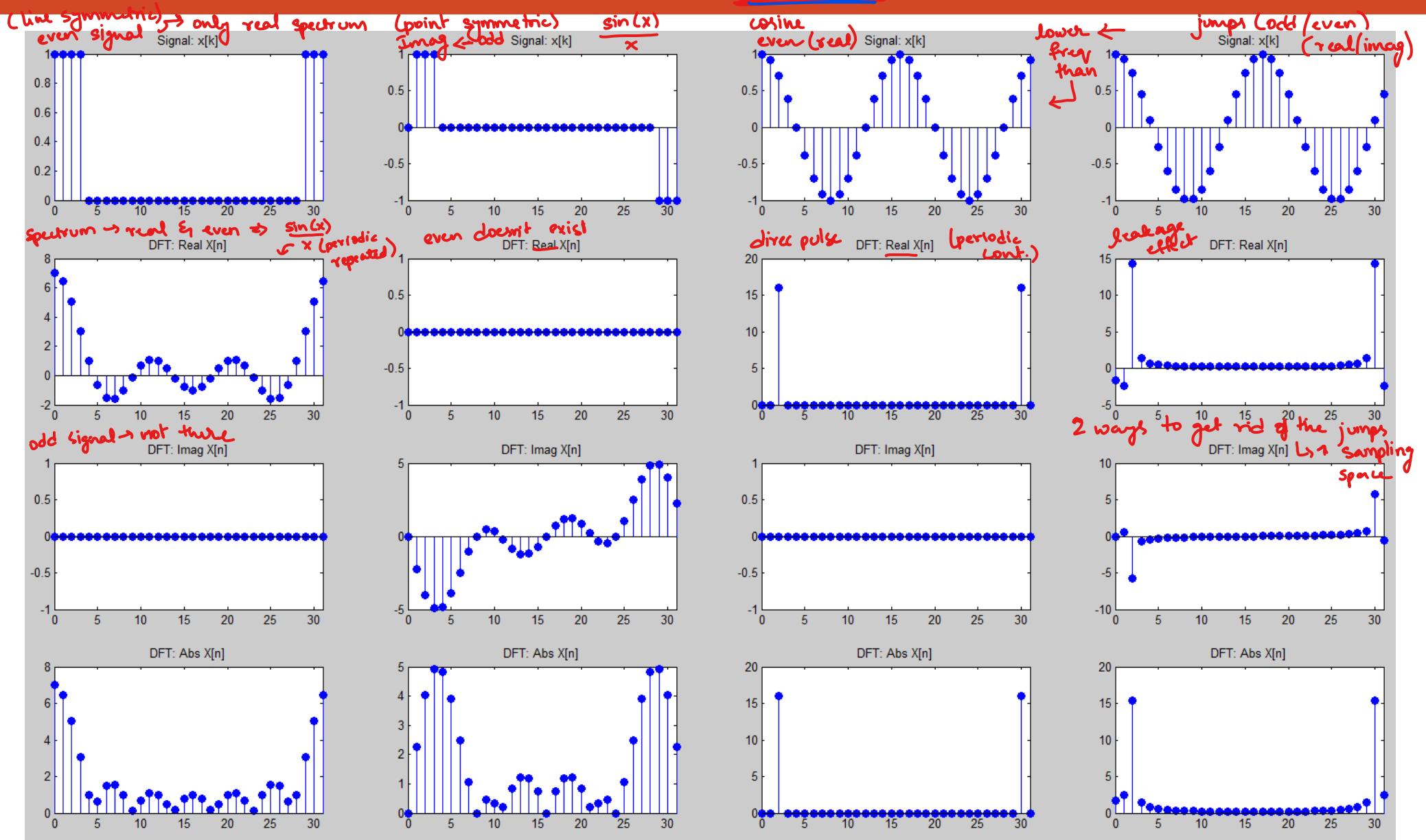
Hints:

- The DFT “only” works with a signal section $x[k]$, but the results correspond to those of the **infinitely periodically continued time signal**!
- Tip: For real signals $x[k]$, due to the symmetry properties of the Fourier series, the calculation of $N / 2$ DFT values is already sufficient!



Diskretes periodisches Zeitsignal mit zugehörigem diskreten Spektrum

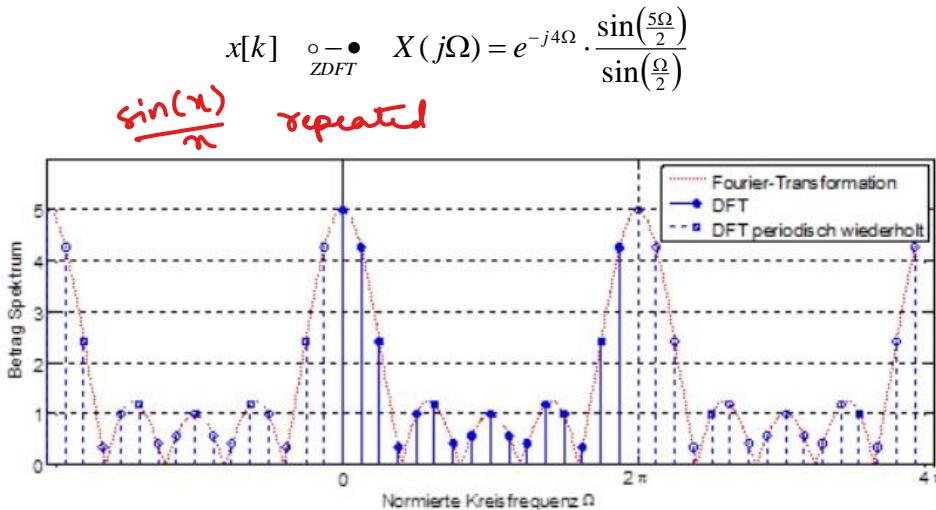
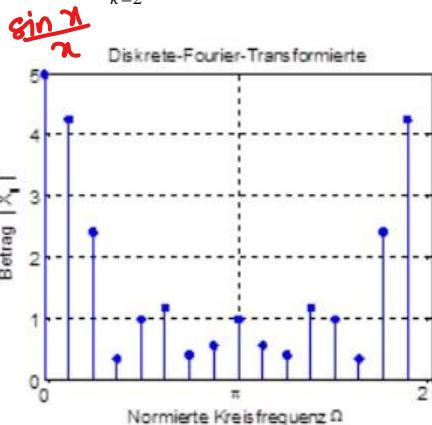
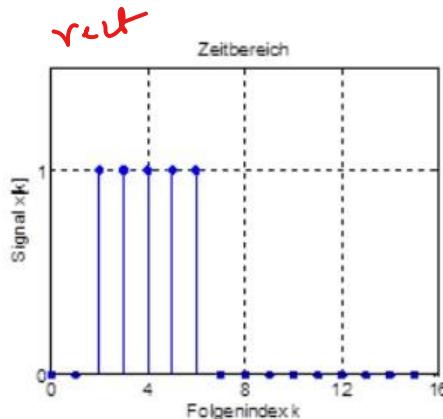
Some Examples of DFT / FFT ($N = 32$) \rightarrow 0 to 31 (32)



The DFT as Spectral Analysis Tool

Example: Rectangular sequence $x[k]$ with $N = 16$ signal values.

$$x[k] = \varepsilon[k-2] - \varepsilon[k-7] \quad \text{DFT} \quad X[n] = \sum_{k=2}^6 1 \cdot e^{-j(2\pi \frac{n}{N} k)}$$



Limits of DFT:

- 1) The signal $x[k]$ is just a sample of $x(t)$. Signal frequencies that are too high lead to Aliasing.
 - 2) The signal section $x_w[k] = x[k] \cdot w[k]$, which is used for the DFT, arises in the time domain by multiplication with a window function $w(t)$. This leads to the so-called Leakage-Effect.
 - 3) Only N values $X[n]$ of the spectrum are available, but $X_w(j\omega)$ is a continuous function, i.e. the spectral resolution is limited.
- avoided by LOW PASS filtering

Fast Fourier Transform (FFT):

- Is an algorithm for extremely efficient calculation of the Discrete Fourier Transform. N must always be a **power of two!**
- The computational effort of the **FFT algorithm**, for example with $N = 1024$, is only **1% of the effort of the direct DFT!**

This is achieved through consistent multiple use of recurring intermediate results ... (for further details see literature)

Notes on the Practical Application of DFT / FFT

- 1) Consistent use of anti-aliasing filters!
- 2) Observation period N as long as possible

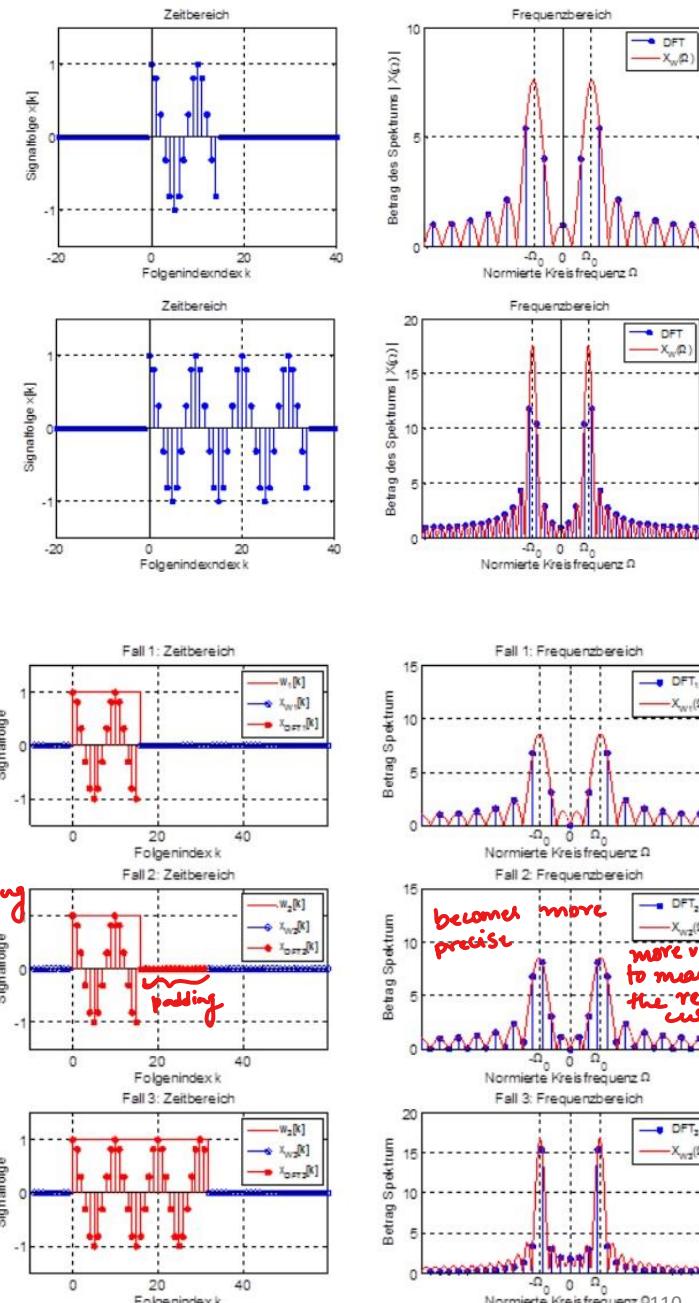
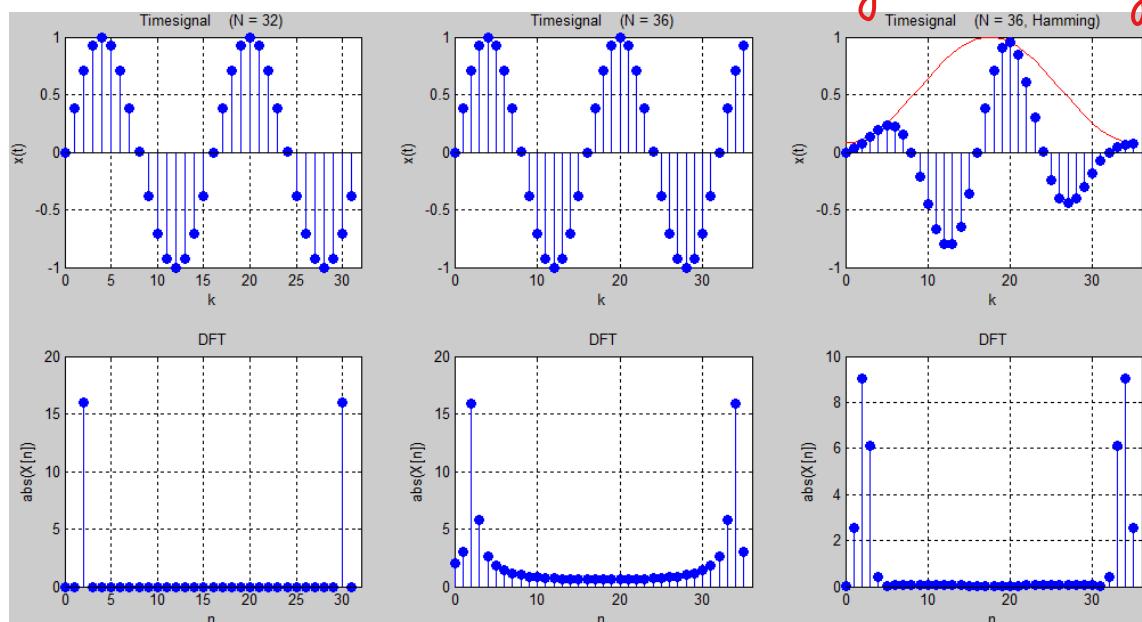
▪ Basically this increases the spectral resolution.
 ▪ Of the Leakage-Effect at the edges is reduced, because only if the signal $x[k]$ is inherent shorter than the observation window, or the window lies exactly in such a way that it is periodic by itself the continued signal is created Leakage-Effect less strong (almost impossible in practice).

~~3) Use of edge-smoothing window functions~~

Multiplication with bell-shaped window functions leads to a reduction in the LeakageEffect
 (e.g. Hamming-Window, HannWindows, Blackman windows ...)

4) Zero Padding (Zero-Filling)

To increase the frequency resolution, additional zeros can be appended to $x[k]$
 The spectrum can thus be drawn more finely (= interpolation in the spectral range), the real one
 However, this increases the information content not. Of the Leakage-Effect can thereby not get better!



Ü 3.11 Discrete Fourier Transformation (DFT or FFT)

As an exercise, calculate the DFT of length $N = 4$ of the following signal:

$$x[k] = \delta[k] + \delta[k-1] + \delta[k-3]$$

$$X[n] = DFT\{x[k]\} = \sum_{k=0}^{N-1} x[k] \cdot e^{-j(2\pi \frac{n}{N} k)}$$

$$x[k] = \delta[k] + \delta[k-1] + \delta[k-3]$$

DTFT

$$X(j\omega) = 1 + e^{j\omega} + e^{j3\omega} \rightarrow \text{gives the basic wave}$$

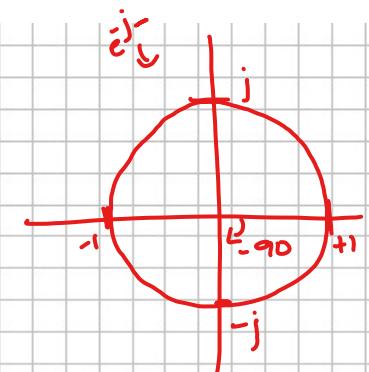
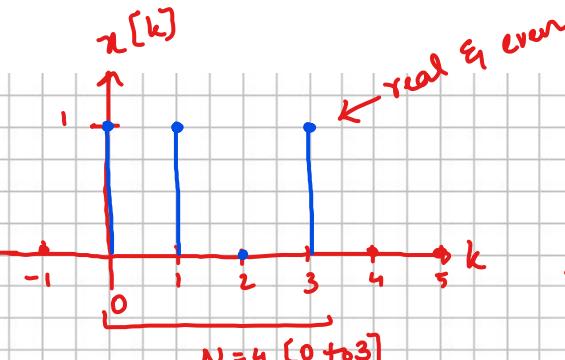
(red curve in prev graphs)

$$\omega_0(\omega = \frac{\pi}{2}) = \frac{e^{j\pi} + e^{j3\pi}}{2} \quad \text{cosine}$$

$$X(j\omega) = 1 + e^{j2\omega} + 2 \cos(\omega)$$

← not even, complex

only DFT is even & real



DFT:

$$X[n] = DFT\{x[k]\} = \sum_{k=0}^{N-1} x[k] e^{-j(2\pi \frac{n}{N} k)}$$

$e^{j0}=1$

$$X[0] = \sum_{k=0}^3 x[k] e^{j0} = \sum_{k=0}^3 x[k] = 1 + 1 + 0 + 1 = 3$$

$$X[1] = \sum_{k=0}^3 x[k] e^{j(\frac{2\pi}{4}k)} = (1 \times 1) + (1 \times -j) + (0 \times -) + (1 \times j) = 1$$

$$X[2] = \sum_{k=0}^3 x[k] e^{j(\frac{4\pi}{4}k)} = (1 \times 1) + (1 \times -1) + (0 \times -) + (1 \times -1) = -1$$

$$X[3] = \sum_{k=0}^3 x[k] e^{j(\frac{6\pi}{4}k)} = (1 \times 1) + (1 \times j) + (0 \times -) + (1 \times -j) = 1$$

$$\Rightarrow X(j\frac{\pi}{4} \cdot 0) = 1 + e^0 \cdot 2 \cos(0) = 3$$

$$\Rightarrow X(j\frac{\pi}{4} \cdot 1) = 1 + e^{j\frac{\pi}{4}} \cdot 2 \cos(\frac{\pi}{4}) = 1$$

$$\Rightarrow X(j\frac{\pi}{4} \cdot 2) = 1 + e^{j\frac{2\pi}{4}} \cdot 2 \cos(\frac{\pi}{2}) = -1$$

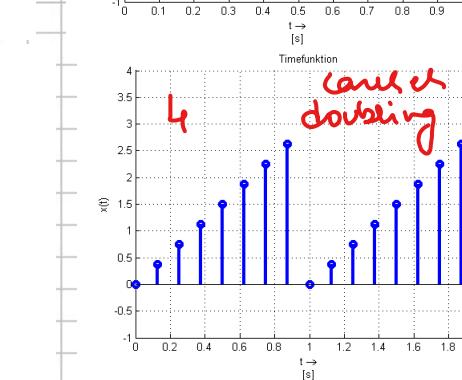
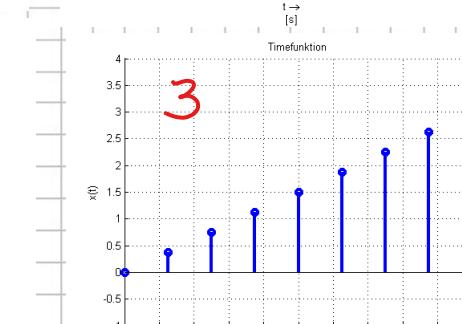
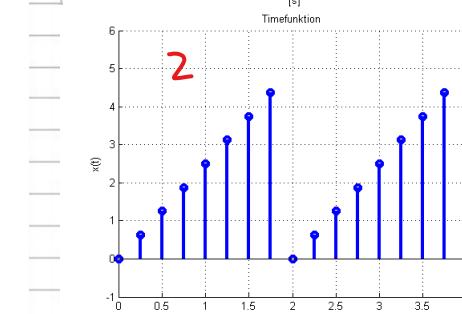
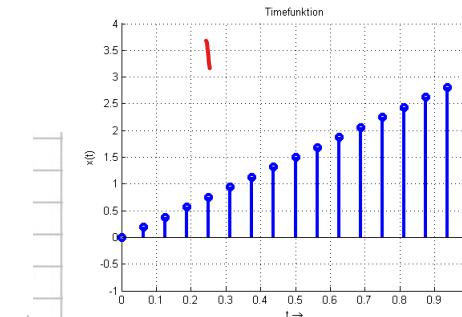
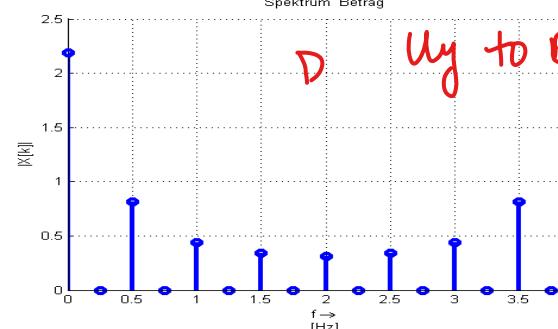
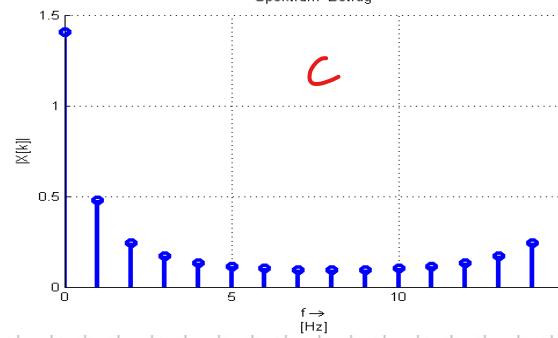
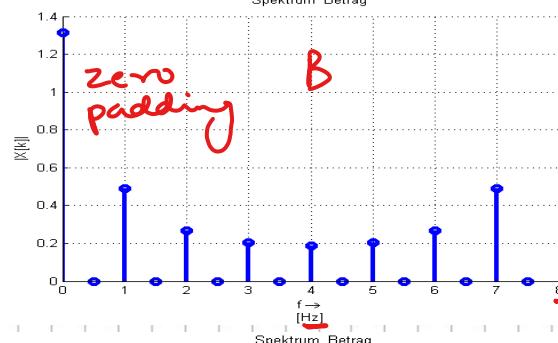
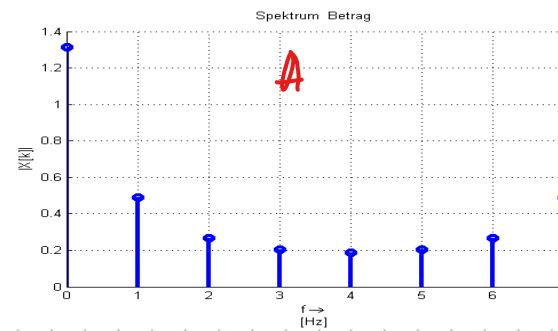
$$\Rightarrow X(j\frac{\pi}{4} \cdot 3) = 1 + e^{j\frac{3\pi}{4}} \cdot 2 \cos(\frac{3\pi}{2}) = 1$$



Ex 3.12 Discrete Fourier Transformation (DFT or FFT)

- a) The spectra $X_a[k]$ to $X_e[k]$ were calculated using a DFT.
Assign to each spectrum the time function and complete the table below.

| Spektrum | Zeitfunktion | Warum passen diese Funktionen zusammen? | N | T_s [s] | F_s [Hz] |
|----------|--------------|---|----|--------------|------------|
| (5) | (1) | | | | |
| (6) | (2) | | | | |
| (7) | (3) | | | | |
| (8) | (4) | | | | |
| C | 1 | | 16 | $1/16$ | 16 |
| D | 2 | | 16 | $4/16 = 1/4$ | 4 |
| A | 3 | | 8 | $1/8$ | 8 |
| B | 4 | | 16 | $2/16 = 1/8$ | 8 |



Appendix:

Have fun and good luck!

Construction and the essential contents of this lecture were taken from the standard works:

- [1] T. Frey, M. Bossert, "Signal and System Theory", Vieweg + publishing house Teubner, 2nd edition, 2008
- [2] B. Girod, R. Rabenstein, A. Stenger, "Introduction to Systems Theory", publisher Teubner,, 2007
- [3] M. Strohmann et al .: "System Theory Online", I-Net of the Karlsruhe University of Applied Sciences - Technology and Economics, 2016

Other literature used and recommended:

- [3] S. Wyrsch, "Signals and Systems I & II", Teaching materials, HSR, 2011.
- [4] M. Meyer, "Signalverarbeitung", Vieweg, 2000.
- [5] M. Rupf, "Signals and Systems SiSy", Script ZHAW, 2011.
- [6] H. Schmidt-Walter, "Taschenbuch der Elektrotechnik", Verlag Harri Deutsch, 9th edition. 2
- [7] I. Rennert, B. Bundschuh, "Signals and Systems", Verlag Hanser, 2013
- [8] BP Lathi, "Linear Systems other Signals ", 2nd edition, Oxford University Press, 2005.
- [9] H.-W. Schüssler, "Networks, Signals and Systems", Springer Verlag, 1990
- ...



Important MATLAB commands for Systems Theory

| Mathematische Darstellung | Matlab – Darstellung |
|--|--|
| $a(x) = a_N x^N + \dots + a_1 x + a_0$ | $a = [a_N \ a_{N-1} \ \dots \ a_1 \ a_0]$ |
| $x[k], \ k = k_1, k_1 + 1, \dots, k_2$ | $x = [x[k_1] \ x[k_1+1] \ \dots]$ $k = [k_1 : k_2]$ |
| $x(t), \ t_1 \leq t \leq t_2$ | $x = [x(t_1) \ x(t_1+\Delta T) \ \dots]$ $t = [t_1 : \Delta T : t_2]$ |

Berechnungen kontinuierlicher Systeme

| | | |
|--|--------------------------------------|---|
| Systemdarstellung | $H(s) = \frac{b(s)}{a(s)}$ | $H = \text{tf}(b, a)^{\text{CS}, \text{I}}$ |
| numerische Berechnung Systemantwort | $y(t) \leftarrow H(s) \cdot X(s)$ | $[y, ty] = \text{lsim}(H, x, tx)^{\text{CS}}$ |
| Bestimmung Impulsantwort | $h(t) \leftarrow H(s)$ | $[h, t] = \text{impulse}(H)^{\text{CS}}$ |
| Bestimmung Sprungantwort | $h_e(t) \leftarrow \frac{1}{s} H(s)$ | $[he, t] = \text{step}(H)^{\text{CS}}$ |

Grundlegende Berechnungen

| | | |
|---|---|--|
| Polynommultiplikation, Faltung | $c(x) = a(x) \cdot b(x)$ | $c = \text{conv}(a, b)$ |
| Polynomdivision, Entfaltung | $\frac{c(x)}{a(x)} = q(x) + \frac{r(x)}{a(x)}$ | $[q, r] = \text{deconv}(c, a)$ |
| Nullstellenbestimmung von Polynomen | $a(x) = \prod_i (x - r_i)$ | $r = \text{roots}(a)$ $a = \text{poly}(r)$ |
| Umrechnung von Polynom- und Produktdarstellung | $\frac{b(x)}{a(x)} = k \frac{\prod_i (x - z_i)}{\prod_i (x - p_i)}$ | $[b, a] = \text{zp2tf}(z, p, k)^{\text{SP}}$ $[z, p, k] = \text{tf2zp}(b, a)^{\text{SP}}$ |
| Partialbruchzerlegung | $\frac{b(x)}{a(x)} = \sum_i \frac{r_i}{x - p_i} + \sum_j k_j x^j$ | $[r, p, k] = \text{residue}(b, a)$ $[b, a] = \text{residue}(r, p, k)$ |
| Polynomauswertung an gegebener Stelle | $y_0 = a(x_0)$ | $y_0 = \text{polyval}(a, x_0)$ |

Berechnungen diskreter Systeme

| | | |
|--|--|--|
| Systemdarstellung | $H(z) = \frac{b(z)}{a(z)}$ | b, a |
| Berechnung Systemantwort (Differenzengleichung) | $y[k] \leftarrow \frac{b(z)}{a(z)} \cdot X(z)$ | $y = \text{filter}(b, a, x)$ |
| Bestimmung Impulsantwort (numerische Rücktransf.) | $h[k] \leftarrow \frac{b(z)}{a(z)}$ | $h = \text{impz}(b, a)^{\text{SP}}$ |
| Partialbruchzerlegung für z -Transformierte | $\frac{b(z)}{a(z)} = \sum_i \frac{r_i \cdot z}{z - p_i} + \sum_j k_j z^{-j}$ | $[r, p, k] = \text{residue}(b, a)^{\text{SP}}$ $[b, a] = \text{residue}(r, p, k)^{\text{SP}}$ |
| Korrelation | $z[k] = \sum_i x^*[i] \cdot y[i+k]$ | $z = \text{xcorr}(x, y)^{\text{SP}}$ |

Zusammenschaltung kontinuierlicher Systeme

| | Matlab | Octave |
|-------------------|--|---|
| Reihenschaltung | $H_1(s) \cdot H_2(s)$ | $\text{series}(H_1, H_2)^{\text{CS}}$ |
| Parallelschaltung | $H_1(s) + H_2(s)$ | $\text{parallel}(H_1, H_2)^{\text{CS}}$ |
| Rückkopplung | $\frac{H_1(s)}{1 + H_1(s) \cdot H_2(s)}$ | $\text{feedback}(H_1, H_2)^{\text{CS}}$ |

Graphische Darstellung

| | | |
|------------------------------|------------------|---------------------|
| reelles Signal | diskret | kontinuierlich |
| Ortskurve (komplexes Signal) | $\text{stem}(x)$ | $\text{plot}(t, x)$ |
| | – | $\text{plot}(x)$ |

Berechnung und Darstellung

| | | |
|--------------------------|-------------------------------------|----------------------------------|
| Pol-Nullstellen-Diagramm | $\text{zplane}(b, a)^{\text{SP}}$ | $\text{pzmap}(H)^{\text{CS}}$ |
| Frequenzgang | $\text{freqz}(b, a)^{\text{SP}}$ | $\text{freqs}(b, a)^{\text{SP}}$ |
| Gruppenlaufzeit | $\text{grpdelay}(b, a)^{\text{SP}}$ | – |
| Bodediagramm | – | $\text{bode}(H)^{\text{CS}}$ |
| Impulsantwort | $\text{impz}(b, a)^{\text{SP}}$ | $\text{impulse}(H)^{\text{CS}}$ |
| Sprungantwort | – | $\text{step}(H)^{\text{CS}}$ |

Weitere Befehle:

| | |
|--------------------------------|-------------------------------|
| Tiefpaß-Bandpaß-Transformation | lp2bp^{SP} |
| Bilineare Transformation | $\text{bilinear}^{\text{SP}}$ |

^{SP} bezeichnet Funktionen, die bei Matlab zur ‘Signal Processing Toolbox’ gehören
^{CS} bezeichnet Funktionen, die bei Matlab zur ‘Control System Toolbox’ gehören

Discrete Time Fourier Transform DTFT & Miscellaneous

Definition

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k}$$

$$x[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) \cdot e^{j\Omega k} d\Omega$$

Eigenschaften der zeitdiskreten Fourier-Transformation

| | Zeitbereich | Frequenzbereich |
|-----------------------|--|--|
| Linearität | $c_1 x_1[k] + c_2 x_2[k]$ | $c_1 X_1(\Omega) + c_2 X_2(\Omega)$ |
| Faltung | $x[k] * y[k]$ | $X(\Omega) \cdot Y(\Omega)$ |
| Multiplikation | $x[k] \cdot y[k]$ | $\frac{1}{2\pi} X(\Omega) \circledast Y(\Omega)$ |
| Verschiebung | $x[k - k_0]$ | $X(\Omega) \cdot e^{-j\Omega k_0}$ |
| Modulation | $e^{j\Omega_0 k} \cdot x[k]$ | $X(\Omega - \Omega_0)$ |
| lineare Gewichtung | $k \cdot x[k]$ | $j \frac{d}{d\Omega} X(\Omega)$ |
| Zeitinversion | $x[-k]$ | $X(-\Omega)$ |
| konj. komplex | $x^*[k]$ | $X^*(-\Omega)$ |
| Realteil | $x_R[k]$ | $X_g(j\omega)$ |
| Imaginärteil | $j x_I[k]$ | $X_u(j\omega)$ |
| Parsevalsches Theorem | $\sum_{k=-\infty}^{\infty} x[k] \cdot y^*[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) \cdot Y^*(\Omega) d\Omega$ | |

$$\text{Frequency Response : } G(j\Omega) = G(z) \Big|_{z=e^{j\Omega}} \quad \Omega = \omega T = \omega \cdot \frac{2\pi}{\omega_A}$$

(only if $G(z)$ is stable!)

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$

$$\begin{aligned} x_g(t) &= \frac{1}{2} [x(t) + x(-t)] \\ x_u(t) &= \frac{1}{2} [x(t) - x(-t)] \end{aligned}$$

$$\text{DFT: } X[n] = \sum_{k=0}^{N-1} x[k] \cdot e^{-j(2\pi \frac{n}{N} k)}$$

Korrespondenzen der zeitdiskreten Fourier-Transformation

| Nr. | $x[k]$ | $X(\Omega)$ |
|-----|---|--|
| 1 | $\delta[k]$ | 1 |
| 2 | 1 | $2\pi \delta(\Omega)$ |
| 3 | $\text{III}_N[k]$ | $\frac{2\pi}{N} \text{III}_{\frac{2\pi}{N}}(\Omega)$ |
| 4 | $\varepsilon[k]$ | $\pi \delta(\Omega) + \frac{1}{1 - e^{-j\Omega}}$ |
| 5 | $\text{sgn}[k]$ | $\frac{1}{j \cdot \tan\left(\frac{\Omega}{2}\right)}$ |
| 6 | $\text{rect}_N[k]$ | $e^{-j\frac{(N-1)\Omega}{2}} \cdot \frac{\sin\left(\frac{N\Omega}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}$ |
| 7 | $e^{j2\pi T f_0 k} = e^{j\Omega_0 k}$ | $2\pi \delta(\Omega - \Omega_0)$ |
| 8 | $\cos(2\pi T f_0 k) = \cos(\Omega_0 k)$ | $\pi [\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)]$ |
| 9 | $\sin(2\pi T f_0 k) = \sin(\Omega_0 k)$ | $\pi j [\delta(\Omega + \Omega_0) - \delta(\Omega - \Omega_0)]$ |

$$\text{III}_N[k] = \sum_{n=-\infty}^{\infty} \delta[k - n N]$$

$X(\Omega)$ is periodically in $2\pi!$
Table shows only the interval $-\pi < \Omega < \pi$

All tables in this appendix are practically 1:1 taken from:
 T. Frey, M. Bossert, "Signal and systems theory", Publisher Vieweg + Teubner
 2nd edition, 2008

Z-Transform (single-sided)

Definition

$$X(z) = \sum_{k=0}^{\infty} x[k] z^{-k}$$

$$x[k] = \frac{1}{2\pi j} \oint_{\mathcal{C}} X(z) z^{k-1} dz \quad \mathcal{C}: \text{pos. orientierte Kurve in } \mathcal{K}$$

Eigenschaften und Rechenregeln

| | Zeitbereich | Bildbereich |
|-------------------------|---|--|
| Linearität | $c_1 x_1[k] + c_2 x_2[k]$ | $c_1 X_1(z) + c_2 X_2(z)$ |
| Faltung | $x[k] * y[k]$ | $X(z) \cdot Y(z)$ |
| Dämpfung | $a^k \cdot x[k]$ | $X\left(\frac{z}{a}\right)$ |
| lineare Gewichtung | $k \cdot x[k]$ | $-z \cdot \frac{d}{dz} X(z)$ |
| konj. komplexes Signal | $x^*[k]$ | $X^*(z^*)$ |
| Zeitinversion | $x[-k]$ | $X\left(\frac{1}{z}\right)$ |
| diskrete Ableitung | $x[k] - x[k-1]$ | $X(z) \cdot \frac{z-1}{z}$ |
| diskrete Integration | $\sum_{i=0}^k x[i]$ | $X(z) \cdot \frac{z}{z-1}$ |
| periodische Fortsetzung | $\sum_{i=0}^{\infty} x[k-iN_p]$ | $X(z) \cdot \frac{1}{1-z^{-N_p}}$ |
| Upsampling | $x\left[\frac{k}{N}\right]$ | $X(z^N)$ |
| Verschiebung links | $x[k+k_0], k_0 > 0$ | $z^{k_0} X(z) - \sum_{i=0}^{k_0-1} x[i] z^{k_0-i}$ |
| Verschiebung rechts | $x[k-k_0], k_0 > 0$ | $z^{-k_0} X(z)$ |
| Anfangswertsatz | $x[0] = \lim_{z \rightarrow \infty} X(z), \text{ falls Grenzwert existiert}$ | |
| Endwertsatz | $\lim_{k \rightarrow \infty} x[k] = \lim_{z \rightarrow 1} (z-1) X(z), \text{ falls } X(z) \text{ nur Pole mit } z < 1 \text{ oder bei } z=1$ | |

Korrespondenzen der z-Transformation

| Nr. | $x[k]$ | $X(z)$ |
|-----|---|--|
| 1 | $\delta[k]$ | 1 |
| 2 | $\delta[k-k_0]$ | z^{-k_0} |
| 3 | $\varepsilon[k]$ | $\frac{z}{z-1}$ |
| 4 | $k \cdot \varepsilon[k]$ | $\frac{z}{(z-1)^2}$ |
| 5 | $a^k \cdot \varepsilon[k]$ | $\frac{z}{z-a}$ |
| 6 | $\binom{k}{m} a^{k-m} \cdot \varepsilon[k]$ | $\frac{z}{(z-a)^{m+1}}$ |
| 7 | $\sin(\Omega_0 k) \cdot \varepsilon[k]$ | $\frac{z \cdot \sin(\Omega_0)}{z^2 - 2z \cdot \cos(\Omega_0) + 1}$ |
| 8 | $\cos(\Omega_0 k) \cdot \varepsilon[k]$ | $\frac{z \cdot [z - \cos(\Omega_0)]}{z^2 - 2z \cdot \cos(\Omega_0) + 1}$ |
| 9 | $a^k \cdot \varepsilon[-k-1]$ | $-\frac{z}{z-a}$ |
| 10 | $a^{ k }, a < 1$ | $\frac{z \cdot (a - \frac{1}{a})}{(z-a)(z - \frac{1}{a})}$ |
| 11 | $\frac{1}{k!} \cdot \varepsilon[k]$ | $e^{\frac{1}{z}}$ |

$$z = e^{sT}$$

$$\text{Bilinear Transf.: } s = \frac{2}{T} \cdot \frac{z-1}{z+1} \quad \text{or} \quad z = \frac{1+s\frac{T}{2}}{1-s\frac{T}{2}}$$

$$\text{Step-Inv. Transf.: } G(z) = \frac{z-1}{z} \cdot Z \left\{ L^{-1} \left\{ \frac{G(s)}{s} \right\}_{t=kT} \right\}$$

Fourier Transform

Definition

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} d\omega$$

Eigenschaften der Fourier-Transformation

| | Zeitbereich | Frequenzbereich |
|-----------------------|---|--|
| Linearität | $c_1 x_1(t) + c_2 x_2(t)$ | $c_1 X_1(j\omega) + c_2 X_2(j\omega)$ |
| Faltung | $x(t) * y(t)$ | $X(j\omega) \cdot Y(j\omega)$ |
| Multiplikation | $x(t) \cdot y(t)$ | $\frac{1}{2\pi} X(j\omega) * Y(j\omega)$ |
| Verschiebung | $x(t - t_0)$ | $X(j\omega) \cdot e^{-j\omega t_0}$ |
| Modulation | $e^{j\omega_0 t} \cdot x(t)$ | $X(j[\omega - \omega_0])$ |
| lineare Gewichtung | $t \cdot x(t)$ | $-\frac{d}{d(j\omega)} X(j\omega)$ |
| Differentiation | $\frac{d}{dt} x(t)$ | $j\omega \cdot X(j\omega)$ |
| Integration | $\int_{-\infty}^t x(\tau) d\tau$ | $\frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega)$ |
| Skalierung | $x(at)$ | $\frac{1}{ a } \cdot X\left(\frac{j\omega}{a}\right)$ |
| Zeitinversion | $x(-t)$ | $X(-j\omega)$ |
| konj. komplex | $x^*(t)$ | $X^*(-j\omega)$ |
| Realteil | $x_R(t)$ | $X_{R^*}(j\omega)$ |
| Imaginärteil | $j x_I(t)$ | $X_{I^*}(j\omega)$ |
| Dualität | $X(t) [X(jt)]$ | $2\pi x(-\omega)$ |
| Parsevalsches Theorem | $\int_{-\infty}^{\infty} x(t) \cdot y^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \cdot Y^*(j\omega) d\omega$ | |

Korrespondenzen der Fourier-Transformation

| Nr. | $x(t)$ | $X(j\omega)$ |
|-----|---|---|
| 1 | $\delta(t)$ | 1 |
| 2 | 1 | $2\pi \delta(\omega)$ |
| 3 | $\text{III}_T(t)$ | $\frac{2\pi}{ T } \text{III}_{\frac{2\pi}{T}}(\omega)$ |
| 4 | $\varepsilon(t)$ | $\pi \delta(\omega) + \frac{1}{j\omega}$ |
| 5 | $\text{sgn}(t)$ | $\frac{2}{j\omega}$ |
| 6 | $\frac{1}{\pi t}$ | $-j \text{sgn}(\omega)$ |
| 7 | $\text{rect}\left(\frac{t}{T}\right)$ | $ T \cdot \text{si}\left(\frac{T}{2}\omega\right)$ |
| 8 | $\text{si}\left(\pi \frac{t}{T}\right)$ | $ T \cdot \text{rect}\left(\frac{T}{2\pi}\omega\right)$ |
| 9 | $\Lambda\left(\frac{t}{T}\right)$ | $ T \cdot \text{si}^2\left(\frac{T}{2}\omega\right)$ |
| 10 | $\text{si}^2\left(\pi \frac{t}{T}\right)$ | $ T \cdot \Lambda\left(\frac{T}{2\pi}\omega\right)$ |
| 11 | $e^{j\omega_0 t}$ | $2\pi \delta(\omega - \omega_0)$ |
| 12 | $\cos(\omega_0 t)$ | $\pi [\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$ |
| 13 | $\sin(\omega_0 t)$ | $\pi j [\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$ |
| 14 | $e^{-a^2 t^2}$ | $\frac{\sqrt{\pi}}{a} e^{-\frac{\omega^2}{4a^2}}$ |
| 15 | $e^{-\frac{ t }{T}}$ | $\frac{2T}{1+(T\omega)^2}$ |

Frequency Response: $G(j\omega) = G(s)|_{s=j\omega}$

$\text{si}(t) = \frac{\sin(t)}{t}$ $\text{III}_T(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$

(only if $G(s)$ is stable!)

Laplace Transform (single-sided)

Definition

$$X(s) = \int_0^\infty x(t) e^{-st} dt$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) \cdot e^{st} ds, \quad \sigma \in \mathcal{K}$$

Eigenschaften und Rechenregeln $x(t) = 0, t < 0$

| | Zeitbereich | Bildbereich |
|------------------------|---|--|
| Linearität | $c_1 x_1(t) + c_2 x_2(t)$ | $c_1 X_1(s) + c_2 X_2(s)$ |
| Faltung | $x(t) * y(t)$ | $X(s) \cdot Y(s)$ |
| Multiplikation | $x(t) \cdot y(t)$ | $X(s) * Y(s)$ |
| Verschiebung rechts | $x(t - t_0), t_0 > 0$ | $e^{-st_0} \cdot X(s)$ |
| Verschiebung links | $x(t + t_0), t_0 > 0$ | $e^{st_0} \cdot X(s) - \int_0^{t_0} x(t) \cdot e^{-s(t+t_0)} dt$ |
| Dämpfung | $e^{at} \cdot x(t)$ | $X(s - a)$ |
| Gewichtung | $t \cdot x(t)$ $t^2 \cdot x(t)$ | $-\frac{d}{ds} X(s)$ $\frac{d^2}{ds^2} X(s)$ |
| Differentiation | $\dot{x}(t)$ $\ddot{x}(t)$ | $s \cdot X(s) - x(0.)$ $s^2 \cdot X(s) - s \cdot x(0.) - \dot{x}(0.)$ |
| Integration | $\int_0^t x(\tau) d\tau$ | $\frac{1}{s} \cdot X(s)$ |
| Skalierung | $x(at)$ | $\frac{1}{ a } \cdot X\left(\frac{s}{a}\right)$ |
| konj. komplexes Signal | $x^*(t)$ | $X^*(s^*)$ |
| Anfangswertsatz | $x(0.) = \lim_{s \rightarrow \infty} s \cdot X(s),$ | falls $x(0.)$ existiert |
| Endwertsatz | $\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} s \cdot X(s),$ | falls $\lim_{t \rightarrow \infty} x(t)$ existiert |

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot y(t - \tau) d\tau$$

$$G_{PT_2}(s) = \frac{K \cdot \omega_0^2}{s^2 + 2D\omega_0 s + \omega_0^2} = \frac{K}{\frac{s^2}{\omega_0^2} + \frac{2Ds}{\omega_0} + 1}$$

Korrespondenzen der Laplace-Transformation

| Nr. | $X(s)$ | $x(t)$ |
|-----|---------------------------------|---|
| 1 | 1 | $\delta(t)$ |
| 2 | $\frac{1}{s}$ | $\varepsilon(t)$ |
| 3 | $\frac{1}{s^2}$ | $t \cdot \varepsilon(t)$ |
| 4 | $\frac{1}{s^3}$ | $\frac{t^2}{2} \cdot \varepsilon(t)$ |
| 5 | $\frac{1}{s + a}$ | $e^{-at} \cdot \varepsilon(t)$ |
| 6 | $\frac{1}{(s + a)^2}$ | $t e^{-at} \cdot \varepsilon(t)$ |
| 7 | $\frac{1}{s(s + a)}$ | $\frac{1}{a} (1 - e^{-at}) \cdot \varepsilon(t)$ |
| 8 | $\frac{1}{(s + a)(s + b)}$ | $\frac{1}{a - b} (e^{-bt} - e^{-at}) \cdot \varepsilon(t)$ |
| 9 | $\frac{1}{s(s + a)^2}$ | $\frac{1}{a^2} [1 - (1 + at)e^{-at}] \cdot \varepsilon(t)$ |
| 10 | $\frac{1}{s(s + a)(s + b)}$ | $\frac{1}{ab} [1 + \frac{1}{a-b} (be^{-at} - ae^{-bt})] \cdot \varepsilon(t)$ |
| 11 | $\frac{s}{(s + a)^2}$ | $(1 - at)e^{-at} \cdot \varepsilon(t)$ |
| 12 | $\frac{s}{(s + a)(s + b)}$ | $\frac{1}{a - b} (ae^{-at} - be^{-bt}) \cdot \varepsilon(t)$ |
| 13 | $\frac{s + a}{s(s + b)}$ | $\left(\frac{a}{b} + \frac{b-a}{b} e^{-bt}\right) \cdot \varepsilon(t)$ |
| 14 | $\frac{1}{(s + a)(s + b)^2}$ | $\frac{1}{(a-b)^2} (e^{-at} - e^{-bt} + (a-b)te^{-bt}) \cdot \varepsilon(t)$ |
| 15 | $\frac{\omega}{s^2 + \omega^2}$ | $\sin \omega t \cdot \varepsilon(t)$ |
| 16 | $\frac{s}{s^2 + \omega^2}$ | $\cos \omega t \cdot \varepsilon(t)$ |