9. 05.06.2023

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Bilinear Transform and Frequency Warping

(z-Transform)

Laplace-Transform

R: $U(s) = R \cdot I(s)$

L: $U(s) = s \cdot L \cdot I(s)$

C: $U(s) = \frac{1}{sC} \cdot I(s)$

Steps of the Bilinear Transform:

1) Description of the circuit via Laplace-Transform
2) Substitute $s \approx G \frac{z-1}{z+1}$ with $\underbrace{G = \frac{2\pi f_0}{\tan(\pi f_0 T)}}_{\text{Frequency Warping}}$

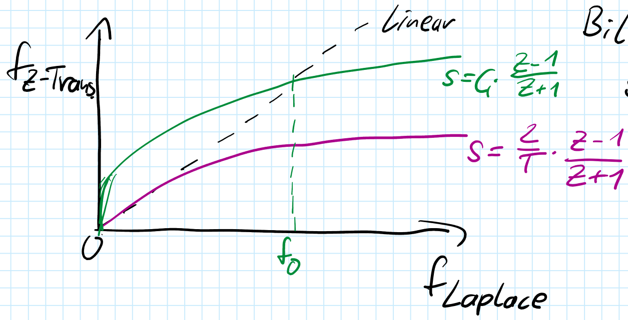
T: sampling time

f_0: frequency at which the approximation is precise:

low-/high-pass filter: cut-off frequency

bandpass : center frequency

=) allows a time-dicrete realisation of a system.

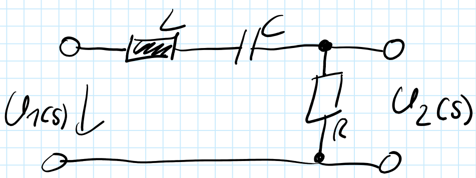


Bilinear Transform: $s = \frac{2}{T} \cdot \frac{z-1}{z+1}$

C capit. C

G small

Example: Series resonant circuit



Transfer-Function:

Transfer - Function:

$$H(s) = \frac{U_2(s)}{U_1(s)} = \frac{R}{R + sL + \frac{1}{sC}}$$

resonance frequency $f_0 = ?$

Definition of f_0 : $H(s)$ is completely real valued

$$sL + \frac{1}{s \cdot C} = 0 \Rightarrow sL = -\frac{1}{sC} \Rightarrow s^2 LC = -1$$

$$\Rightarrow s^2 = \frac{-1}{LC}$$

$$\boxed{\begin{aligned} s &= \sigma + j\omega = \sigma + j2\pi f_0 \\ \sigma &\overset{!}{=} 0 \end{aligned}}$$

$$\Rightarrow (j2\pi f_0)^2 = \frac{-1}{LC}$$

$$(2\pi f_0)^2 = \frac{1}{LC} \Rightarrow \boxed{f_0 = \frac{1}{2\pi\sqrt{LC}}}$$

Determine a time-discrete realisation of the filter by using the bilinear transform.

$$T = \frac{1}{1000}\,s \quad R = 100\,\Omega \quad L = 100\,mH \quad C = 100\,\mu F$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = 50{,}3\,Hz$$

$$c = \frac{2\pi f_0}{\tan(\pi f_0 T)} = 1983\,Hz$$

$$H(s) = \frac{R}{R + sL + \frac{1}{sC}} \cdot \frac{sC}{sC} = \frac{sRC}{sRC + s^2LC + 1}$$

Bilinear Transform: $s \approx c \cdot \frac{z-1}{z+1}$

$$H(z) = \frac{c \cdot \frac{z-1}{z+1} \cdot RC}{c \cdot \frac{z-1}{z+1} \cdot RC + c^2 \frac{(z-1)^2}{(z+1)^2} \cdot LC + 1} \cdot \frac{(z+1)^2}{(z+1)^2}$$

$$= \frac{c\,(z-1)(z+1)\,RC}{c\,(z-1)(z+1)\,RC + c^2 (z-1)^2 \cdot LC + (z+1)^2}$$

$$= \frac{c\,RC\,(z^2-1)}{c\,RC\,(z^2-1) + c^2 LC\,(z^2-2z+1) + (z^2+2z+1)}$$

$$= \frac{G\,RC\,(z^2-1)}{G\,RC\,(z^2-1)+G^2LC\,(z^2-2z+1)+(z^2+2z+1)}$$

$$= \frac{b_1 z^2 + b_2 z + b_3}{a_1 z^2 + a_2 z + a_3}$$

$$b_1 = G\,RC = 1983\,Hz \cdot 100\,\Omega \cdot 100\,\mu F = 19{,}83$$

Unit-check: $\frac{1}{s} \cdot \frac{V}{A} \cdot \frac{As}{V} = 1$

$$b_2 = 0$$

$$b_3 = -G\,RC = -19{,}83$$

$$a_1 = G\,RC + G^2LC + 1 = 60{,}15$$

$$a_2 = -2G^2LC + 2 = -76{,}65$$

$$a_3 = -G\,RC + G^2LC + 1 = 20{,}49$$

$$\Rightarrow H(z) = \frac{19{,}83 \cdot z^2 + 0 \cdot z - 19{,}83}{60{,}15 \cdot z^2 - 76{,}56\,z + 20{,}49}$$

e.g. Look at Pole-Null-Diagramm

Difference Equation and Block Diagramm?

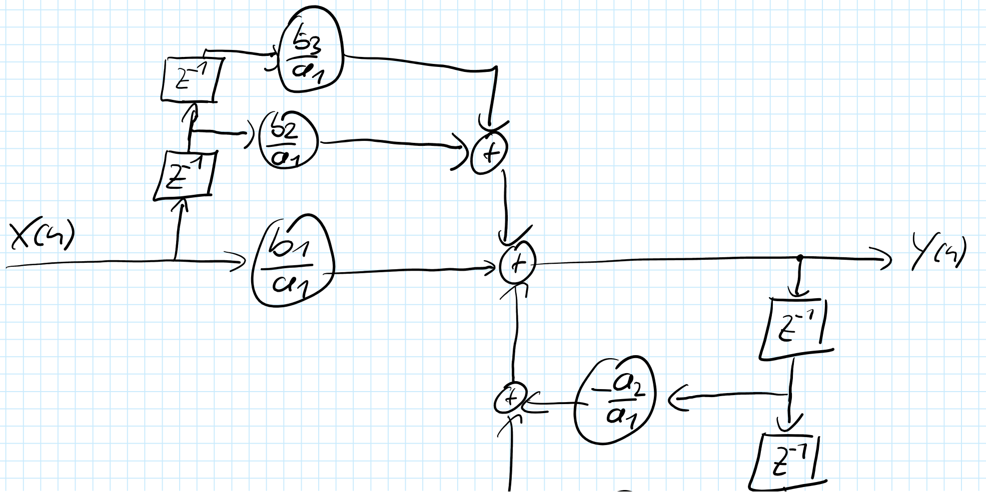
$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_1 z^2 + b_2 z + b_3}{a_1 z^2 + a_2 z + a_3} \cdot \frac{z^{-2}}{z^{-2}} = \frac{b_1 + b_2 z^{-1} + b_3 z^{-2}}{a_1 + a_2 z^{-1} + a_3 z^{-2}}$$

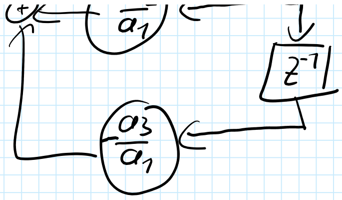
$$Y(z) \cdot (a_1 + a_2 z^{-1} + a_3 z^{-2}) = X(z)\,(b_1 + b_2 z^{-1} + b_3 z^{-2})$$

Time discrete form

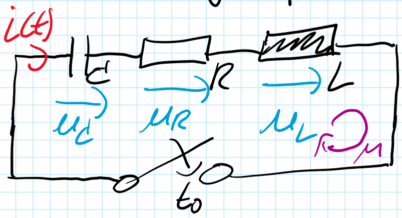
$$a_1 \cdot y(n) + a_2 \cdot y(n-1) + a_3 \cdot y(n-2) = b_1 \cdot x(n) + b_2 \cdot x(n-1) + b_3 \cdot x(n-2)$$

$$y(n) = \frac{b_1}{a_1} x(n) + \frac{b_2}{a_1} x(n-1) + \frac{b_3}{a_1} x(n-2) - \frac{a_2}{a_1} y(n-1) - \frac{a_3}{a_1} y(n-2)$$





Example: discharge process



$$u_C(t=0) = 12V$$

$$u_C(t) = \frac{1}{C} \cdot Q(t)$$

$$u_R(t) = R \cdot i(t) = R \cdot \frac{dQ(t)}{dt} = R \cdot C \frac{du_C(t)}{dt}$$

$$u_L(t) = L \cdot \frac{di(t)}{dt} = LC \frac{d^2 u_c(t)}{dt}$$

M: $u_C(t) + u_R(t) + u_L(t) = 0$

$$u_C(t) + RC \frac{du_C(t)}{dt} + LC \frac{d^2 u_C(t)}{dt^2} = 0$$

$$\circ\!\!-\!\!\bullet$$

$$U(s) + RC \cdot s \cdot U(s) + LC \cdot s^2 \cdot U(s) = 0$$

Apply Bilinear Transform: $s \approx c \frac{z-1}{z+1}$

$$U(z) + RC\, c \frac{z-1}{z+1} U(z) + LC\, c^2 \frac{(z-1)^2}{(z+1)^2} U(z) = 0 \quad | \cdot (z+1)^2$$

$$\vdots$$

$$U(z)\left[(1 + cRC + cLC) + z^{-1}(2 - 2c^2LC) + z^{-2}(1 - cRC + c^2LC)\right] = 0$$

discrete form

$$u(n) = \frac{-1}{1 + cRC + c^2LC}\left[(2 - 2c^2LC) \cdot u(n-1) + (1 - cRC + c^2LC) \cdot u(n-2)\right]$$

$\Rightarrow$ Python / Matlab ... $u(0) = 12V$

$u(-1) = 12V$