Mechanical System

Actuators - IRO6

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Why is the mechanical system relevant?

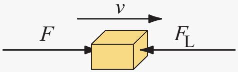
Robotics performs mechanical tasks

Mechanical system defines the requirements of the actuator

How are the requirements determined?

And if transmission elements are placed between the load and the actuator?

Translation



$$\sum F = 0: \qquad F - F_\mathrm{L} - m \cdot a = 0$$

Figure Equation of motion for translational movement

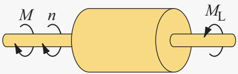
Relevant variables:

s	Displacement	in	$[m]$
$v = \dfrac{\mathrm{d}s}{\mathrm{d}t}$	Speed	in	$\left[\dfrac{m}{s}\right]$
$a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\mathrm{d}^2 s}{\mathrm{d}t^2}$	Acceleration	in	$\left[\dfrac{m}{s^2}\right]$
$j = \dfrac{\mathrm{d}a}{\mathrm{d}t} = \dfrac{\mathrm{d}^2 v}{\mathrm{d}t^2} = \dfrac{\mathrm{d}^3 s}{\mathrm{d}t^3}$	Jerk	in	$\left[\dfrac{m}{s^3}\right]$

Jerk j: important quantity for mechanical stress

$\Rightarrow$ „smooth acceleration“ avoids „shocks“ in the process

Rotation



$$\sum M = 0: \qquad M - M_L - J_{mech} \cdot \frac{d\Omega_m}{dt} = 0$$

Figure Equation of motion for rotational movement

Movement variables analogous to linear movement:

φ	angle	in	$[rad]$
$\Omega_m = \frac{d\varphi}{dt}$	angular velocity	in	$\left[\frac{rad}{s}\right]$
$\alpha_m = \frac{d\Omega_m}{dt} = \frac{d^2\varphi}{dt^2}$	angular acceleration	in	$\left[\frac{rad}{s^2}\right]$
$\sigma_m = \frac{d\alpha_m}{dt} = \frac{d^2\Omega_m}{dt^2} = \frac{d^3\varphi}{dt^3}$	angular jerk	in	$\left[\frac{rad}{s^3}\right]$

Mass m in linear movement corresponds to **moment of inertia** J_{mech}

Moment of inertia

Moment of inertia depends on the **geometry of the rotating body** and the **axis of rotation**:

⇒ For a point mass at a distance of r from the axis of rotation:

$$J_{\mathrm{mech}} = m \cdot r^2$$

⇒ For a solid cylinder with diameter d_{e} and length l:

$$J_{\mathrm{mech}} \left(= \int_V r^2 \rho_{\mathrm{mech}}(r)\,\mathrm{d}V \right) = \frac{\pi}{2}\,\rho_{\mathrm{mech}}\,l \left(\frac{d_{\mathrm{e}}}{2}\right)^4 = \frac{\pi}{32}\,\rho_{\mathrm{mech}}\,l\,d_{\mathrm{e}}^4 = \frac{1}{8} m\,d_{\mathrm{e}}^2$$

⇒ For a hollow cylinder with outside diameter d_{e}, inside diameter d_{i} and length l:

$$J_{\mathrm{mech}} = \frac{\pi}{32}\,\rho_{\mathrm{mech}}\,l \left(d_{\mathrm{e}}^4 - d_{\mathrm{i}}^4\right) = \frac{1}{8} m \left(d_{\mathrm{e}}^2 + d_{\mathrm{i}}^2\right)$$

Why $\left(d_{\mathrm{e}}^2 + d_{\mathrm{i}}^2\right)$? (Help: $a^2 - b^2 = (a+b)(a-b)$)

Relative values

Helpfull for dynamic processes (control technology!):

- Relate torques to nominal torque
- Relate (angular) velocities to nominal (angular) velocities

$$M - M_\mathrm{L} - J_\mathrm{mech}\,\frac{\mathrm{d}\Omega_\mathrm{m}}{\mathrm{d}t} = 0 \quad |:M_\mathrm{N}$$

$$m - m_\mathrm{L} - \frac{J_\mathrm{mech}}{M_\mathrm{N}}\,\frac{\mathrm{d}\Omega_\mathrm{m}}{\mathrm{d}t}\cdot\frac{\Omega_\mathrm{m,N}}{\Omega_\mathrm{m,N}} = 0$$

$$m - m_\mathrm{L} - T_\mathrm{J}\cdot\frac{\mathrm{d}\omega_\mathrm{m}}{\mathrm{d}t} = 0$$

$$\text{mit } \omega_\mathrm{m} = \frac{\Omega_\mathrm{m}}{\Omega_\mathrm{m,N}} = \frac{n}{n_\mathrm{N}}\,\text{bzw. } T_\mathrm{J} = 2\pi\cdot\frac{J_\mathrm{mech}}{M_\mathrm{N}}n_\mathrm{N} \tag{1.1}$$

Ramp-up time T_J: Acceleration time (without load) with nominal torque from standstill to the nominal speed

Example 1-1: Operating point drive/load

Given is a system consisting of an electric motor and a mechanical load, e.g. a robot arm. The speed-torque equation for the motor is $M = M_0 \cdot (1 - 0.1\frac{n}{n_{max}})$ with $M_0 = 2\,Nm$ and $n_{max} = 3000\,\mathrm{min}^{-1}$. The equation for the load torque acting on the motor shaft is $M_L = M_{0,L} \cdot (1 + 3\frac{n}{n_{max}})$ with $M_{0,L} = 0.75\,Nm$.

a. What is the stationary operating point of the system?
b. Thanks to an inverter, M_0 for the motor can be varied as a function of the input voltage. The acceleration during startup should be as constant as possible. Draw qualitatively the operating points of the motor in the M-n characteristic.

Mechanical transmission elements in the balance equation

- Direct coupling drive - load

⇒ Balance equation ⇒ Driving forces from the movement task

- With mechanical transmission elements
  - Conversion between rotation and translation
  - Speed adjustment

⇒ Moving masses with 2 or more speeds → Balance equation?

Law of conservation of energy

Mass of the load as additional (transformed) moment of inertia on the motor shaft
OR
Motor inertia as additional (transformed) mass on the load

- Case 1: Gear transmission
- Case 2: Converter from rotation to translation by wheels, rollers or drums (vehicles, cranes, elevators or conveyor belts)
- Case 3: Spindle

Transformed moment of inertia, case 1: „Gear drive“

Gearbox with the ratio i:

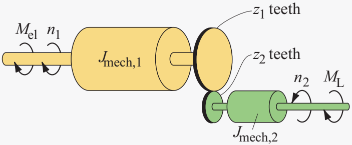


Figure Basic kinematics of a gear transmission

for example for a spur gear drive gear with z_1, driven gear with z_2 teeth:

$$i_{12} = \frac{z_1}{z_2} \quad \Rightarrow \quad n_1 \cdot z_1 = n_2 \cdot z_2 \quad \Rightarrow \quad n_2 = \frac{z_1}{z_2} \cdot n_1 = i_{12} \cdot n_1 \tag{1.2}$$

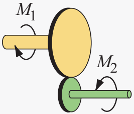
Kinetic energy stored in the moment of inertia $J_{\mathrm{mech},2}$:

$$W_{\mathrm{kin}} = \frac{1}{2} J_{\mathrm{mech},2}\, \Omega^2_{\mathrm{m},2} = \frac{1}{2} J_{\mathrm{mech},2} \cdot (i_{12} \cdot \Omega_{\mathrm{m},1})^2 = \frac{1}{2} \left(i^2_{12} \cdot J_{\mathrm{mech},2} \right) \cdot \Omega^2_{\mathrm{m},1}$$

$$= \frac{1}{2} \cdot J'_{\mathrm{mech},2} \cdot \Omega^2_{\mathrm{m},1} \qquad \text{mit } J'_{\mathrm{mech},2} = i^2_{12} \cdot J_{\mathrm{mech},2} \tag{1.3}$$

Case 1: Equation of Motion

The same tangential forces on both gears:



$$\frac{M_1}{z_1} = \frac{M_2}{z_2} \quad \Rightarrow \quad M_1 = i_{12} \cdot M_2$$

Figure Equilibrium of forces in a gear transmission

Equation of motion for shaft 1:

$$\Sigma M = 0: \quad M_{\mathrm{el}} - i_{12} \cdot M_{\mathrm{L}} - \left(J_{\mathrm{mech},1} + J'_{\mathrm{mech},2}\right) \cdot \frac{\mathrm{d}\Omega_{\mathrm{m},1}}{\mathrm{d}t} = 0 \qquad \left(i_{12} = \frac{z_1}{z_2}\right) \qquad (1.4)$$

Case 1: Equation of motion for side 2

It depends on which side you want to refer to:

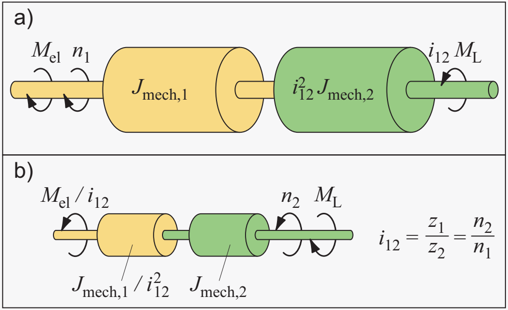


Figure kinematic replacement models

Equation of motion related to side 2:

$$\Sigma M = 0 : \quad M_{el}/i_{12} - M_L - \left(J_{mech,1}/i_{12}^2 + J_{mech,2}\right) \cdot \frac{d\Omega_{m,2}}{dt} = 0$$

Transformed moment of inertia, case 2: „Drums“

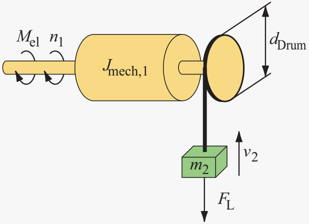


Figure Basic kinematics of an elevator

$$v_2 = \Omega_{m,1}\frac{d_{Drum}}{2} = i_{12}\cdot\Omega_{m,1}$$

$$\Rightarrow \quad \Omega_{m,1} = \frac{v_2}{i_{12}} = \frac{v_2}{d_{Drum}/2}$$

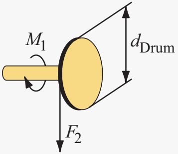


Figure Equilibrium of forces in an elevator mechanism

$$M_1 = F_2\cdot\frac{d_{Drum}}{2} = i_{12}\cdot F_2$$

Case 2: Equation of motion for side 1 or 2

$$
\begin{aligned}
W_{\mathrm{kin}} &= \frac{1}{2} m_2\, v_2^2 = \frac{1}{2} m_2 \left(i_{12} \cdot \Omega_{\mathrm{m},1}\right)^2 \\
&= \frac{1}{2} \left(i_{12}^2 \cdot m_2\right) \cdot \Omega_{\mathrm{m},1}^2 \\
&= \frac{1}{2} \cdot J'_{\mathrm{mech},2} \cdot \Omega_{\mathrm{m},1}^2 \qquad \text{mit } J'_{\mathrm{mech},2} = i_{12}^2 \cdot m_2 \qquad (1.5)
\end{aligned}
$$

Equation of motion for side 1:

$$
\Sigma M = 0: \quad M_{\mathrm{el}} - i_{12} \cdot F_2 - \left(J_{\mathrm{mech},1} + J'_{\mathrm{mech},2}\right) \cdot \frac{\mathrm{d}\Omega_{\mathrm{m},1}}{\mathrm{d}t} = 0 \qquad \left(i_{12} = \frac{d_{\mathrm{Drum}}}{2}\right) \qquad (1.6)
$$

or for the moving mass m_2:

$$
\Sigma F = 0: \quad M_{\mathrm{el}}/i_{12} - F_2 - \left(J_{\mathrm{mech},1}/i_{12}^2 + m_2\right) \cdot \frac{\mathrm{d}v_2}{\mathrm{d}t} = 0 \qquad \left(i_{12} = \frac{d_{\mathrm{Drum}}}{2}\right) \qquad (1.7)
$$

Transformed moment of inertia, case 3: „Spindle“

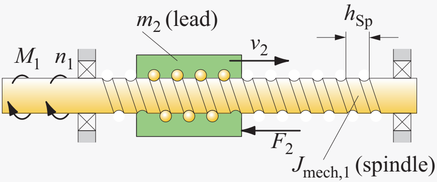


Figure Linear axis with ball roller spindle

The following applies:

Work: $$M_1 \cdot 2\pi = F_2 \cdot h_{Sp} \quad \Rightarrow \quad \frac{F_2}{M_1} = \frac{2\pi}{h_{Sp}}$$

Power: $$M_1 \cdot \Omega_1 = F_2 \cdot v_2 \quad \Rightarrow \quad \frac{v_2}{\Omega_1} = \frac{M_1}{F_2} = \frac{h_{Sp}}{2\pi} = i_{12} \qquad (1.8)$$

Transformed moment of inertia, case 3: „Spindle“

Consider again the kinetic energy stored in m_2:
m_2 acts as if $J_{\mathrm{mech},1}$ was enlarged by J'_{m}:

$$\frac{1}{2} m_2 v_2^2 = \frac{1}{2} J'_{\mathrm{m}} \Omega_1^2$$

$$\Rightarrow \qquad J'_{\mathrm{m}} = m_2 \left(\frac{v_2}{\Omega_1} \right)^2 = m_2 \cdot i_{12}^2 = m_2 \left(\frac{h_{\mathrm{Sp}}}{2\pi} \right)^2 \tag{1.9}$$

...or $J_{\mathrm{mech},1}$ acts as if m_2 was enlarged by m'_1:

$$\frac{1}{2} m'_1 v_2^2 = \frac{1}{2} J_{\mathrm{mech},1} \Omega_1^2$$

$$\Rightarrow \qquad m'_1 = J_{\mathrm{mech},1} \left(\frac{\Omega_1}{v_2} \right)^2 = J_{\mathrm{mech},1} \cdot i_{12}^{-2} = J_{\mathrm{mech},1} \left(\frac{2\pi}{h_{\mathrm{Sp}}} \right)^2 \tag{1.10}$$

Note: This consideration (refer everything to the side with m_2) is often done in machine tool construction!

Optimal gear ratio

The time for a positioning process depends from the gear ratio:

- gear ratio too high: speed too low on the output side
- gear ratio too small: effective moment of inertia too large

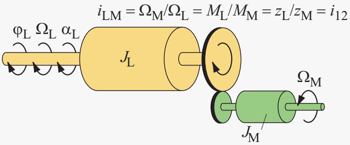


Figure Drive with gearbox

with no load or friction: $$\alpha_L = \frac{i \cdot M_M}{J_L + i^2 J_M} = \frac{M_M}{i \cdot J_M + J_L/i} \qquad (1.11)$$

Speed may be limited when accelerating!

Optimal gear ratio without speed limitation

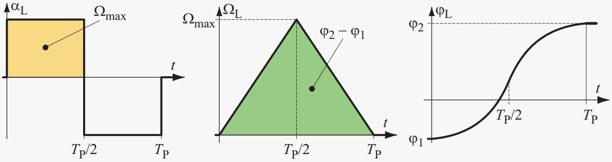


Figure Time-optimal positioning process without limitation

- Speed: two straight sections
- Displacement: two pieces of parabola

$$\alpha_L = \frac{d\Omega_L}{dt} = \frac{\Omega_{max}}{T_P/2} \quad \Rightarrow \quad \Omega_{max} = \alpha_L \cdot \frac{T_P}{2}$$

$$\Delta\varphi_L = \varphi_2 - \varphi_1 = \frac{T_P \cdot \Omega_{max}}{2} = \frac{\alpha_L}{4} \cdot T_P^2$$

$$\Rightarrow \quad T_P = \sqrt{\frac{4\Delta\varphi_L}{\alpha_L}} = \sqrt{\frac{4\Delta\varphi_L}{M_M} \cdot \left(i \cdot J_M + \frac{J_L}{i}\right)} \tag{1.12}$$

Optimal gear ratio without speed limitation

Find optimum (minimum):

$$\frac{\partial T_\mathrm{P}}{\partial i} = \frac{\partial}{\partial i}\sqrt{\frac{4\Delta\varphi_\mathrm{L}}{M_\mathrm{M}}\cdot\left(i\cdot J_\mathrm{M}+\frac{J_\mathrm{L}}{i}\right)} = \sqrt{\frac{4\Delta\varphi_\mathrm{L}}{M_\mathrm{M}}}\cdot\frac{\partial}{\partial i}\sqrt{i\cdot J_\mathrm{M}+\frac{J_\mathrm{L}}{i}}$$

$$= \sqrt{\frac{4\Delta\varphi_\mathrm{L}}{M_\mathrm{M}}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{i\cdot J_\mathrm{M}+\frac{J_\mathrm{L}}{i}}}\cdot\left(J_\mathrm{M}-\frac{J_\mathrm{L}}{i^2}\right)\overset{!}{=}0$$

$$\Rightarrow \quad J_\mathrm{M}-\frac{J_\mathrm{L}}{i_\mathrm{opt}^2}=0 \quad \Rightarrow \quad i_\mathrm{opt}=\sqrt{\frac{J_\mathrm{L}}{J_\mathrm{M}}} \tag{1.13}$$

Positioning time:

$$T_\mathrm{P,opt} = \sqrt{\frac{4\Delta\varphi_\mathrm{L}}{M_\mathrm{M}}\cdot\left(\sqrt{\frac{J_\mathrm{L}}{J_\mathrm{M}}}\cdot J_\mathrm{M}+\sqrt{\frac{J_\mathrm{M}}{J_\mathrm{L}}}\cdot J_\mathrm{L}\right)} = \sqrt{\frac{4\Delta\varphi_\mathrm{L}}{M_\mathrm{M}}\cdot\left(\sqrt{J_\mathrm{L}\cdot J_\mathrm{M}}+\sqrt{J_\mathrm{M}\cdot J_\mathrm{L}}\right)}$$

$$= \sqrt{\frac{8\Delta\varphi_\mathrm{L}}{M_\mathrm{M}}\cdot\sqrt{J_\mathrm{L}\cdot J_\mathrm{M}}}$$

Optimal gear ratio: sensitivity analysis

$$\frac{T_\mathrm{P}}{T_\mathrm{P,opt}} = \frac{\sqrt{\dfrac{4\Delta\varphi_\mathrm{L}}{M_\mathrm{M}} \cdot \left(i \cdot J_\mathrm{M} + \dfrac{J_\mathrm{L}}{i}\right)}}{\sqrt{\dfrac{8\Delta\varphi_\mathrm{L}}{M_\mathrm{M}} \cdot \sqrt{J_\mathrm{L} \cdot J_\mathrm{M}}}} \ldots = \sqrt{\frac{1}{2} \cdot \left(\frac{i}{i_\mathrm{opt}} + \frac{i_\mathrm{opt}}{i}\right)} \qquad (1.14)$$

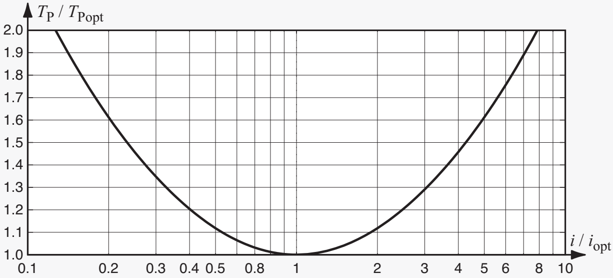


Figure Dependence of the adjustment time on the gear ratio

Optimal gear ratio with limited load speed

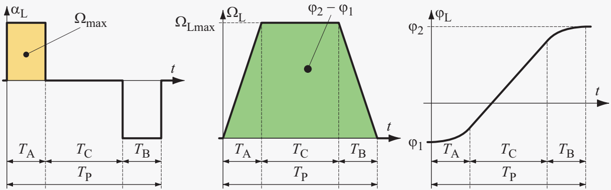


Figure Time-optimized positioning process with limited load speed

$$\alpha_L = \frac{d\Omega_L}{dt} = \frac{\Omega_{Lmax}}{T_A} = \frac{\Omega_{Lmax}}{T_B} = \frac{M_M}{i \cdot J_M + J_L/i}$$

$$\Rightarrow \quad T_A = T_B = \frac{\Omega_{Lmax}}{M_M} \cdot \left(i \cdot J_M + \frac{J_L}{i} \right)$$

Optimal gear ratio with limited load speed

Area under $\Omega_\mathrm{L}(t)$ is equal to the displacement angle covered $\Delta\varphi = \varphi_2 - \varphi_1$:

$$\Delta\varphi = \varphi_2 - \varphi_1 = \left(\frac{T_\mathrm{A} + T_\mathrm{B}}{2} + T_\mathrm{C}\right) \cdot \Omega_\mathrm{Lmax} \quad \Rightarrow \quad T_\mathrm{C} = \frac{\Delta\varphi}{\Omega_\mathrm{Lmax}} - T_\mathrm{A}$$

$$T_\mathrm{P} = T_\mathrm{A} + T_\mathrm{C} + T_\mathrm{B} = 2T_\mathrm{A} + T_\mathrm{C} = 2T_\mathrm{A} + \frac{\Delta\varphi}{\Omega_\mathrm{Lmax}} - T_\mathrm{A}$$

$$= \frac{\Delta\varphi}{\Omega_\mathrm{Lmax}} + \frac{\Omega_\mathrm{Lmax}}{M_\mathrm{M}} \cdot \left(i \cdot J_\mathrm{M} + \frac{J_\mathrm{L}}{i}\right)$$

Search minimum:

$$\frac{\partial T_\mathrm{P}}{\partial i} = \frac{\partial}{\partial i}\left[\frac{\Delta\varphi}{\Omega_\mathrm{Lmax}} + \frac{\Omega_\mathrm{Lmax}}{M_\mathrm{M}} \cdot \left(i \cdot J_\mathrm{M} + \frac{J_\mathrm{L}}{i}\right)\right] \ldots \stackrel{!}{=} 0$$

$$\Rightarrow \qquad i_\mathrm{opt} = \sqrt{\frac{J_\mathrm{L}}{J_\mathrm{M}}}$$

⇒ The limitation of the load speed plays no role in selecting the optimal gear ratio.

Optimal gear ratio with limited motor speed

Maximum output speed Ω_{Lmax} now dependent on gear ratio:

$$T_{\text{P}} = \frac{\Delta\varphi}{\Omega_{\text{Lmax}}} + \frac{\Omega_{\text{Lmax}}}{M_{\text{M}}} \cdot \left(i \cdot J_{\text{M}} + \frac{J_{\text{L}}}{i} \right) \qquad \text{here:} \quad \Omega_{\text{Lmax}} = \frac{\Omega_{\text{Mmax}}}{i}$$

$$= \frac{i \cdot \Delta\varphi}{\Omega_{\text{Mmax}}} + \frac{\Omega_{\text{Mmax}}}{M_{\text{M}}} \cdot \left(J_{\text{M}} + \frac{J_{\text{L}}}{i^2} \right)$$

Determination of the optimal gear ratio:

$$\frac{\partial T_{\text{P}}}{\partial i} = \frac{\Delta\varphi}{\Omega_{\text{Mmax}}} + \frac{\Omega_{\text{Mmax}}}{M_{\text{M}}} \cdot \left(0 - 2\frac{J_{\text{L}}}{i^3} \right) \overset{!}{=} 0$$

$$\Rightarrow \quad i_{\text{opt}} = \sqrt[3]{\frac{2\,\Omega_{\text{Mmax}}^2 \cdot J_{\text{L}}}{\Delta\varphi \cdot M_{\text{M}}}} \tag{1.15}$$

⇒ Consideration almost purely academic

(Energy) Optimal Movement

Example subway and S-Bahn:

- average distance between stops approximately 1000 m with travel time 70 s to 80 s
- energy-optimal: roll (sailing) for as long as possible

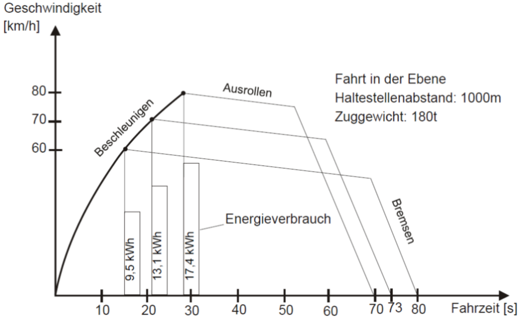


Figure Speed-time diagram of an S-Bahn, source: Hamburger Hochbahn
Geschwindigkeit = Speed
Fahrzeit = Travel time
Beschleunigen = Accelerating
Ausrollen = Rolling
Bremsen = Braking
Fahrt in der Ebene = Plain terrain
Haltestellenabstand = Distance between stations
Zuggewicht = Train weight

Example 1-2: Drives for a passenger elevator

A passenger elevator with a maximum payload of 650 kg (max. total mass of the elevator car 1900 kg, mass of the counterweight 1565 kg) should cover the distance from the basement (2nd basement floor) to the 20th floor in 30 s. Each floor has a height of 3.5 m

a. Estimate the necessary drive power and driving speed!
b. What values do you estimate for the acceleration, the acceleration time and the distance covered in the acceleration phase?
c. By what factor would the torque be greater during the acceleration phase than during the time at constant speed?
d. What would be the gear ratio with a cable drum diameter of 320 mm and a motor speed of 1500 $\min^{-1}$?
e. Is this ratio time-optimal? How much can the travel time be shortened? Remember that it is very unlikely that an elevator can travel 22 floors without stopping! (The motor moment of inertia can be assumed to be 0.04 $kg\,m^2$)