Statistics and Sensor Data Fusion

3. Pattern Recognition and Bayes Optimal Classifier

Pattern Recognition and Bayes Optimal Classifier

Statistical Pattern Recognition:

Statistical pattern recognition is concerned with the **classification of objects** on the basis of **quantitative features**.

The term pattern indicates that the classification is facilitated by **characteristic structures** of the objects under consideration:

> "One of the most interesting aspects of the world is that it can be considered to be made up of **patterns**. A pattern is essentially an **arrangement**. It is characterized by the **order of the elements** of which it is made rather than by the intrinsic nature of these elements."
>
> *Norbert Wiener*

Pattern Recognition and Bayes Optimal Classifier

Mathematical Description of Patterns:

- Objects or patterns to be classified are described by features which constitute the different dimensions of the feature space.
- Within the framework of statistical pattern recognition, features are interpreted as either **discrete** or **continuous random variables** which are combined in feature vectors.
- The observed feature values of an object to be classified are interpreted as **realization** of the feature vector.
- Based on this realization of the feature vector, the object is classified, i.e. a **decision** is made about the underlying class.

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Feature Extraction and Feature Selection:

- The term feature extraction refers to the **definition** and **computation** of suitable features, which is problem-specific and depends on the objects to be classified:
  - Signals: Spectral analysis, Fourier transform, ...
  - Images: Segmentation, shape features, 2-dim. Fourier transform, ...
- The term feature selection refers to the choice of a **maximally informative subset** of features with respect to a specific classification task.

  In the case of feature selection, the set of all features to select from has to be determined already.

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Example of a Classification Problem:

In a fish cannery, the distinction between salmon and sea bass should be done automatically on the basis of **grayscale images**.

Therefore, the following **processing steps** are necessary:

1. Camera shot (original image)
2. Preprocessing of the image (noise filtering, calculation of grayscale values, segmentation, ...)
3. Feature extraction and selection (e.g. lightness and width)
4. Classification

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Possible Original Images:

https://www.indiamart.com/proddetail/atlantic-salmon-fish-21055714788.html
https://www.aoseafood.co.uk/buy/sea-bass/

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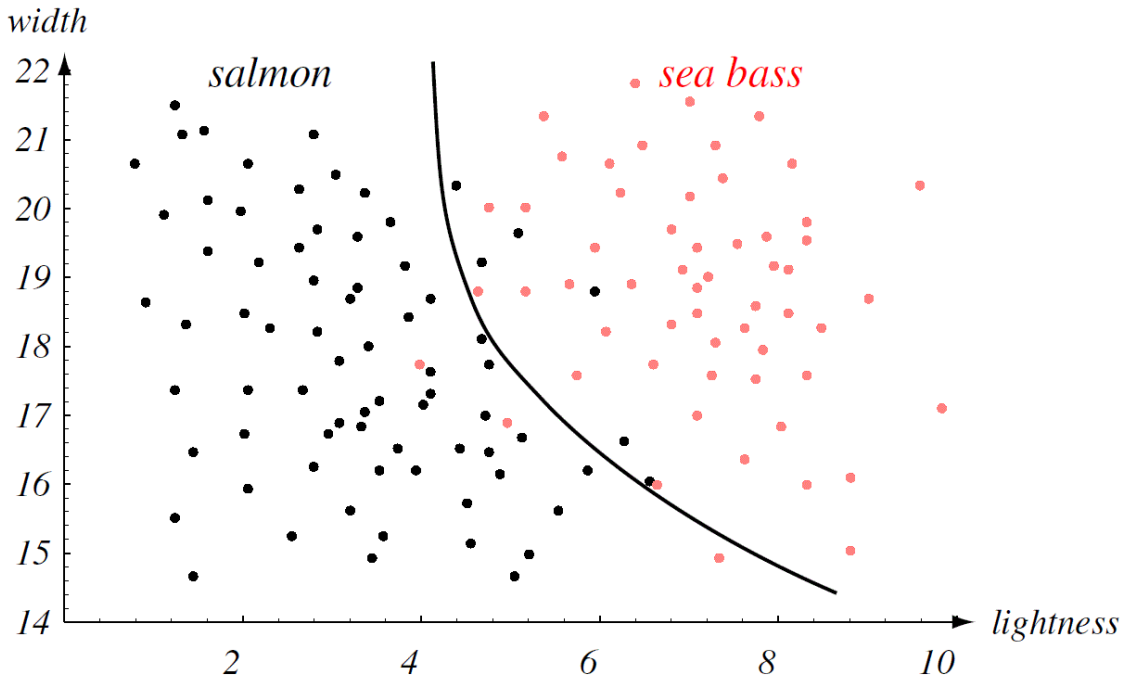
Possible Classification:



Based on the computed values of the two features **lightness** and **width**, a binary valued classifier might reach a decision on the type of the fish according to the plotted black line. Observe that the classifier will not be error-free in general.

R. Duda, P. Hart, D. Stork, Pattern Classification, 2nd edition, Wiley, 2001

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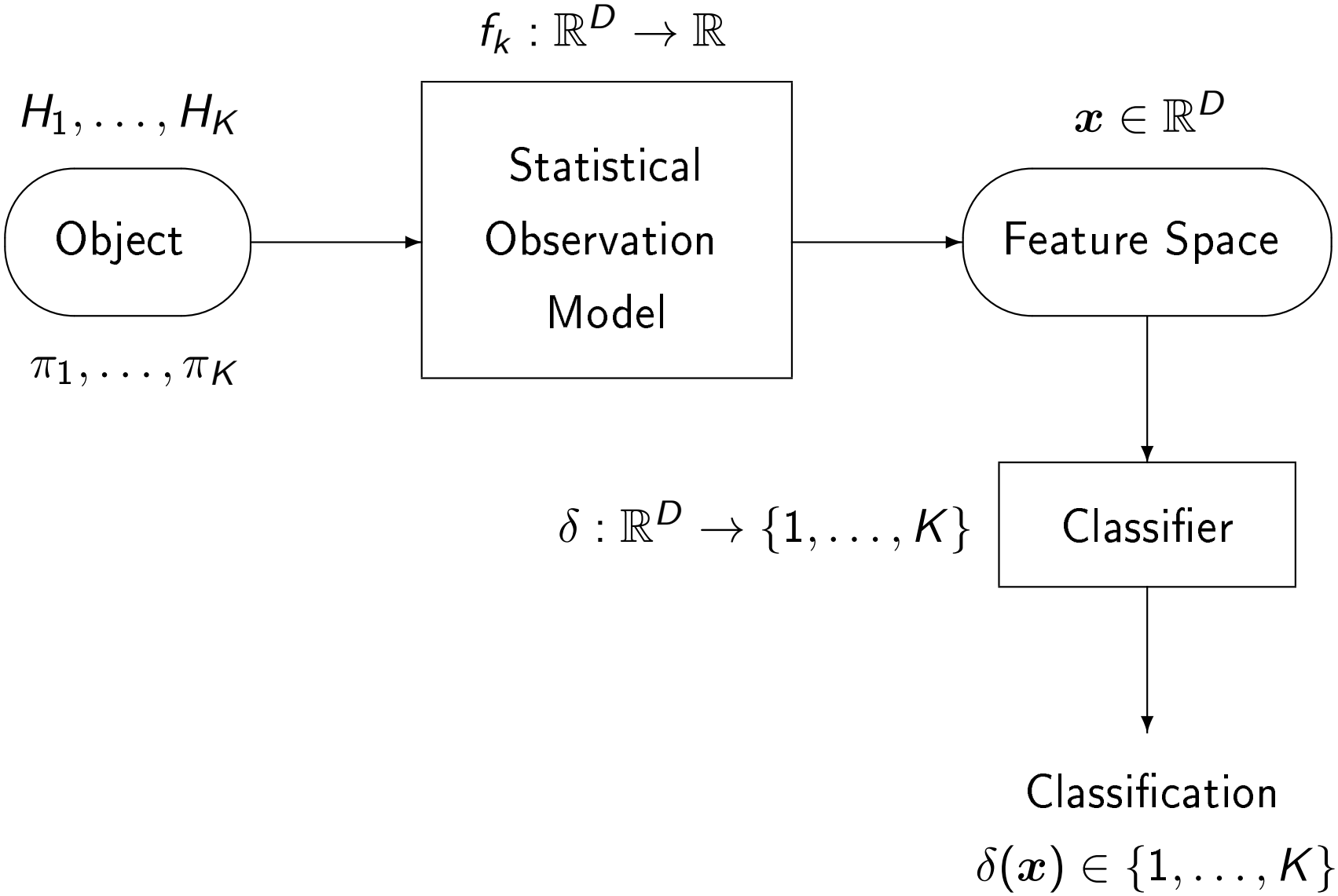
Statistical Model of Classification:



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Statistical Model of Classification:

1. An object belongs to one out of K classes represented by the **hypotheses** $H_1, \ldots, H_K$, which are characterized by the prior probabilities $\pi_1, \ldots, \pi_K$.

2. The true class label cannot be observed directly, but instead a **feature vector** $\boldsymbol{x} \in \mathbb{R}^D$ is measured, which is interpreted as a realization of the random vector $\boldsymbol{X}$.

3. For an object from class k, the random vector $\boldsymbol{X}$ is distributed according to the class-conditional pdf $f_k : \mathbb{R}^D \to \mathbb{R}$.

4. A **classifier** $\delta : \mathbb{R}^D \to \{1, \ldots, K\}$, $\boldsymbol{x} \mapsto \delta(\boldsymbol{x})$, estimates the true class on the basis of the observed feature vector $\boldsymbol{x} \in \mathbb{R}^D$.

5. The goal is to construct an **optimal classifier** δ^* which minimizes the probability of error.

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Classifier and Decision Regions:

A classifier

$$\delta : \mathbb{R}^D \to \{1, \ldots, K\}, \quad \boldsymbol{x} \mapsto \delta(\boldsymbol{x})$$

corresponds to a **partition** of the feature space $\mathbb{R}^D$ into mutually exclusive subsets or decision regions $\mathcal{R}_k \subseteq \mathbb{R}^D$, where

$$\bigcup_{k=1}^{K} \mathcal{R}_k = \mathbb{R}^D, \quad \mathcal{R}_i \cap \mathcal{R}_j = \varnothing \quad \text{for } i \neq j$$

and

$$\delta(\boldsymbol{x}) = \begin{cases} 1, \text{ if } \boldsymbol{x} \in \mathcal{R}_1 \\ 2, \text{ if } \boldsymbol{x} \in \mathcal{R}_2 \\ \vdots \\ K, \text{ if } \boldsymbol{x} \in \mathcal{R}_K \end{cases}$$

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Classifier and Decision Regions:

A classifier $\delta : \mathbb{R}^D \to \{1, \ldots, K\}$ corresponds to a partition of the feature space $\mathbb{R}^D$ into decision regions $\mathcal{R}_k \subseteq \mathbb{R}^D$, where

$$\mathcal{R}_k = \{\boldsymbol{x} \in \mathbb{R}^D \mid \delta(\boldsymbol{x}) = k\}, \quad k = 1, \ldots, K$$

Error-free classification is only possible, if the sets

$$\mathcal{S}_k = \{\boldsymbol{x} \in \mathbb{R}^D \mid f_k(\boldsymbol{x}) > 0\}, \quad k = 1, \ldots, K$$

are mutually exclusive, that means if the class-conditional densities $f_k(\boldsymbol{x})$ **do not overlap**.

In this case, an error-free classifier is obtained via $\mathcal{R}_k = \mathcal{S}_k$, i.e.

$$\delta(\boldsymbol{x}) = k \iff \boldsymbol{x} \in \mathcal{S}_k$$

In general however, a classifier δ is not error-free.

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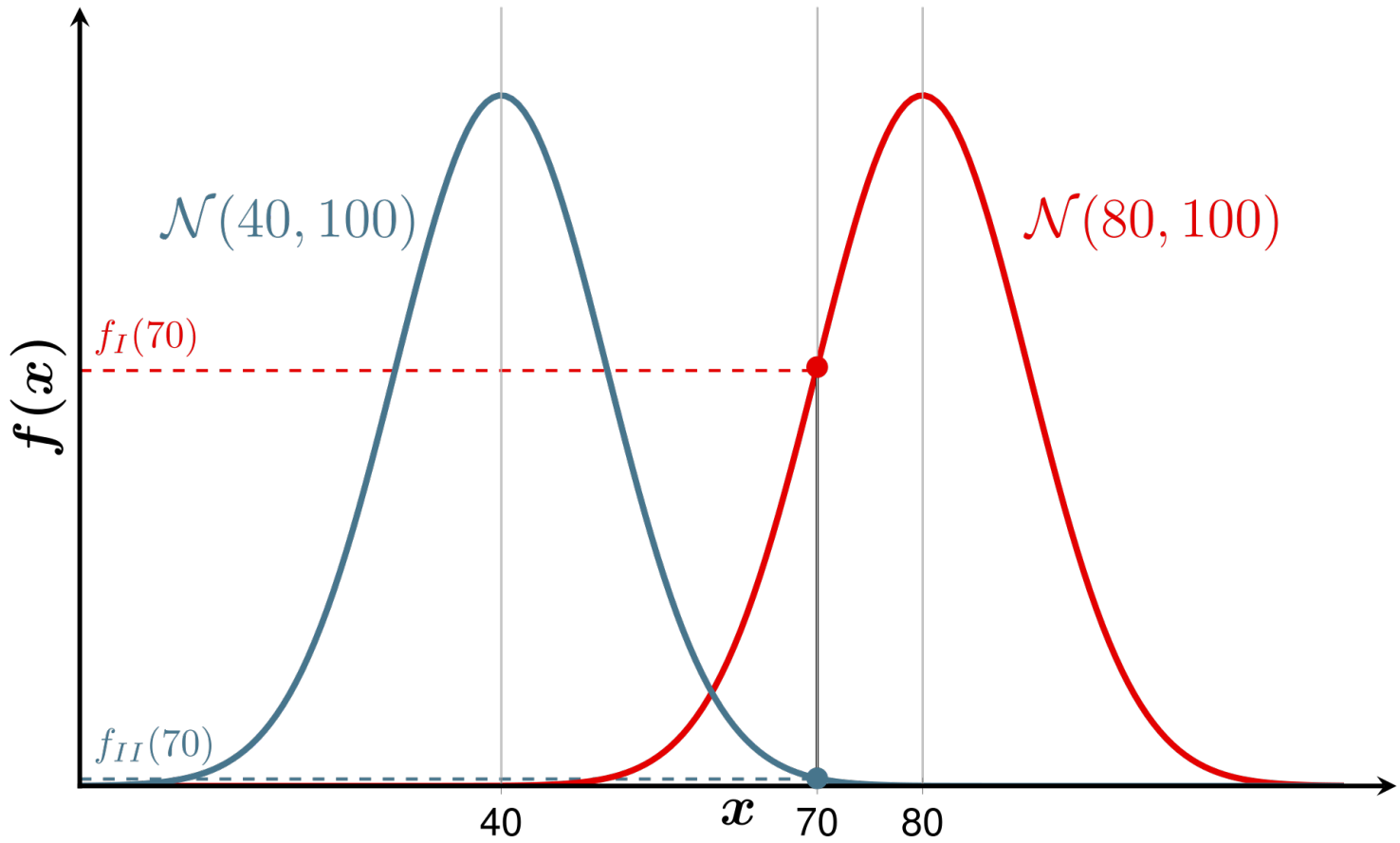
Example: Overlapping Densities



Von MartinThoma - Eigenes Werk, CC0, https://commons.wikimedia.org/w/index.php?curid=34672879

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Summary of the Different Probability Distributions:

1. Prior probabilities of the classes: $\pi_1, \ldots, \pi_K$
2. Class-conditional densities of the feature vectors: $f_k(\boldsymbol{x})$
3. Joint distribution over pairs: $f(\boldsymbol{x}, k) = \pi_k \cdot f_k(\boldsymbol{x})$

In the sequel, it is assumed that these probability distributions are **completely known** and can be used for the construction of the optimal classifier δ^*.

In Chapter 4, it will be shown how to estimate the parameters of the involved probability distributions based on a **data sample** or **training set**.

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Derived Probability Distributions:

From the joint distribution $f(\boldsymbol{x}, k)$, the marginal distribution of the feature vectors (independent of the class) is obtained as

$$f(\boldsymbol{x}) = \sum_{k=1}^{K} f(\boldsymbol{x}, k) = \sum_{k=1}^{K} \pi_k \cdot f_k(\boldsymbol{x})$$

Accordingly, the posterior distribution of the classes is given by the **conditional probabilities**

$$p(k\,|\,\boldsymbol{x}) = \frac{f(\boldsymbol{x}, k)}{f(\boldsymbol{x})} = \frac{\pi_k \cdot f_k(\boldsymbol{x})}{\sum_{j=1}^{K} \pi_j \cdot f_j(\boldsymbol{x})}$$

The posterior distribution is **normalized**, i.e.

$$\sum_{k=1}^{K} p(k\,|\,\boldsymbol{x}) = 1$$

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Bayes Optimal Classifier:

The classifier δ^* which minimizes the **probability of error** is called Bayes optimal classifier and can be obtained as

$$\delta^*(\boldsymbol{x}) = \underset{k=1,\ldots,K}{\operatorname{argmax}}\ p(k\,|\,\boldsymbol{x}) = \underset{k=1,\ldots,K}{\operatorname{argmax}}\ \frac{\pi_k \cdot f_k(\boldsymbol{x})}{\sum_{j=1}^{K} \pi_j \cdot f_j(\boldsymbol{x})}$$

The Bayes optimal classifier maximizes the **posterior probability** of the class k given the realization of the feature vector $\boldsymbol{x} \in \mathbb{R}^D$.

However, for its implementation we need to know explicitly both

- the prior probabilities $\pi_1, \ldots, \pi_K$
- the class-conditional densities $f_1(\boldsymbol{x}), \ldots, f_K(\boldsymbol{x})$

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Classifiers and Discriminant Functions:

In many cases, a classifier $\delta = \delta(\boldsymbol{x})$ is expressed in terms of a so-called discriminant function $g = g(\boldsymbol{x}, k)$:

$$\delta(\boldsymbol{x}) = \underset{k=1,\ldots,K}{\text{argmax}}\ g(\boldsymbol{x}, k)$$

For example, the **Bayes optimal classifier** can be expressed in terms of a discriminant function which is obtained via suitable transformations of the **posterior probability** $p(k|\boldsymbol{x})$:

1. $g(\boldsymbol{x}, k) = f(\boldsymbol{x}) \cdot p(k|\boldsymbol{x}) = f(\boldsymbol{x}, k) = \pi_k \cdot f_k(\boldsymbol{x})$
2. $g(\boldsymbol{x}, k) = \log(f(\boldsymbol{x}, k)) = \log(\pi_k) + \log(f_k(\boldsymbol{x}))$
3. $g(\boldsymbol{x}, k) = \log(p(k|\boldsymbol{x})) = \log(\pi_k \cdot f_k(\boldsymbol{x})) - \log\left(\sum_{j=1}^{K} \pi_j \cdot f_j(\boldsymbol{x})\right)$

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Example: Multivariate Normal Distributions $\mathcal{N}_D(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$

The classes should be characterized by the **prior probabilities** $\pi_1, \ldots, \pi_K$, the feature vectors should be characterized by the **class-conditional densities**

$$f_k(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{D}{2}} |\boldsymbol{\Sigma}_k|^{\frac{1}{2}}} \exp\Big(-\tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu}_k)^T \boldsymbol{\Sigma}_k^{-1} (\boldsymbol{x} - \boldsymbol{\mu}_k) \Big)$$

where $\boldsymbol{\mu}_k \in \mathbb{R}^D$ and $\boldsymbol{\Sigma}_k \in \mathbb{R}^{D \times D}$ symmetric and positive definite for $k = 1, \ldots, K$.

For the sequel, it is assumed that the prior probabilities $\pi_1, \ldots, \pi_K$, the mean vectors $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_K \in \mathbb{R}^D$ and the covariance matrices $\boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_K \in \mathbb{R}^{D \times D}$ are completely known.

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Calculation of the Discriminant Function:

For the discriminant function $g(\boldsymbol{x}, k)$, we choose

$$g(\boldsymbol{x}, k) = \log(\pi_k \cdot f_k(\boldsymbol{x})) = \log(\pi_k) + \log(f_k(\boldsymbol{x}))$$

$$= \log(\pi_k) - \tfrac{1}{2}\log((2\pi)^D|\boldsymbol{\Sigma}_k|) - \tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_k)^T\boldsymbol{\Sigma}_k^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_k)$$

$$= \log(\pi_k) - \tfrac{1}{2}\log((2\pi)^D|\boldsymbol{\Sigma}_k|) - \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{\Sigma}_k^{-1}\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{\Sigma}_k^{-1}\boldsymbol{\mu}_k$$

$$+\tfrac{1}{2}\boldsymbol{\mu}_k^T\boldsymbol{\Sigma}_k^{-1}\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{\mu}_k^T\boldsymbol{\Sigma}_k^{-1}\boldsymbol{\mu}_k$$

In the general case of **different covariance matrices** $\boldsymbol{\Sigma}_k$, the discriminant function $g(\boldsymbol{x}, k)$ is quadratic in $\boldsymbol{x}$.

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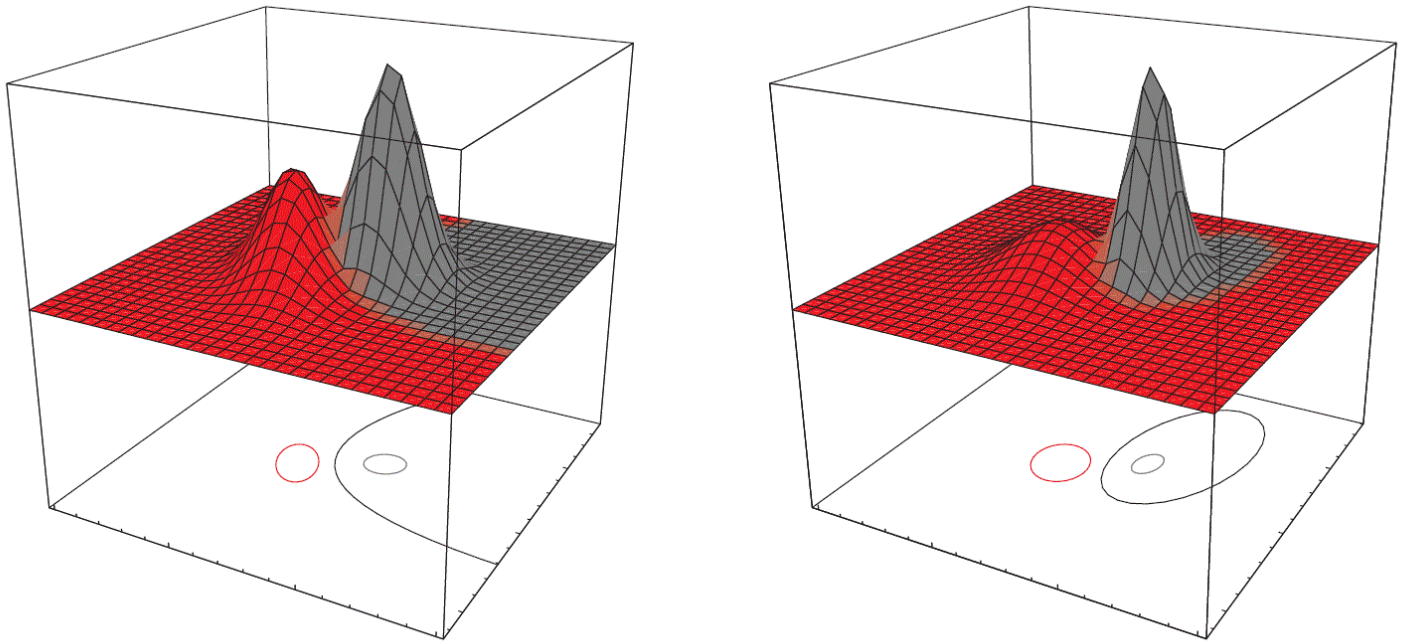
Example: Two Bivariate Normal Distributions with $\boldsymbol{\Sigma}_1 \neq \boldsymbol{\Sigma}_2$



The boundary between the decision regions $\mathcal{R}_1$ and $\mathcal{R}_2$ of the two classes is given by a quadric (parabola, ellipse).

R. Duda, P. Hart, D. Stork, Pattern Classification, 2nd edition, Wiley, 2001

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Bayes Optimal Classifier for Identical Covariance Matrices:

In the case of **identical covariance matrices**, i.e. $\boldsymbol{\Sigma}_k = \boldsymbol{\Sigma}$ for $k = 1, \ldots, K$, the discriminant function corresponding to the Bayes optimal classifier δ^* is given by

$$g(\boldsymbol{x}, k) = \log(\pi_k) - \tfrac{1}{2}\log((2\pi)^D|\boldsymbol{\Sigma}|) - \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x}$$

$$+\tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_k + \tfrac{1}{2}\boldsymbol{\mu}_k^T\boldsymbol{\Sigma}^{-1}\boldsymbol{x} - \tfrac{1}{2}\boldsymbol{\mu}_k^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_k$$

Since constant terms can be omitted, we eventually obtain

$$g(\boldsymbol{x}, k) = \log(\pi_k) + \boldsymbol{x}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_k - \tfrac{1}{2}\boldsymbol{\mu}_k^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_k$$

In the special case of **identical covariance matrices** $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_k$, the discriminant function $g(\boldsymbol{x}, k)$ is linear in $\boldsymbol{x}$.

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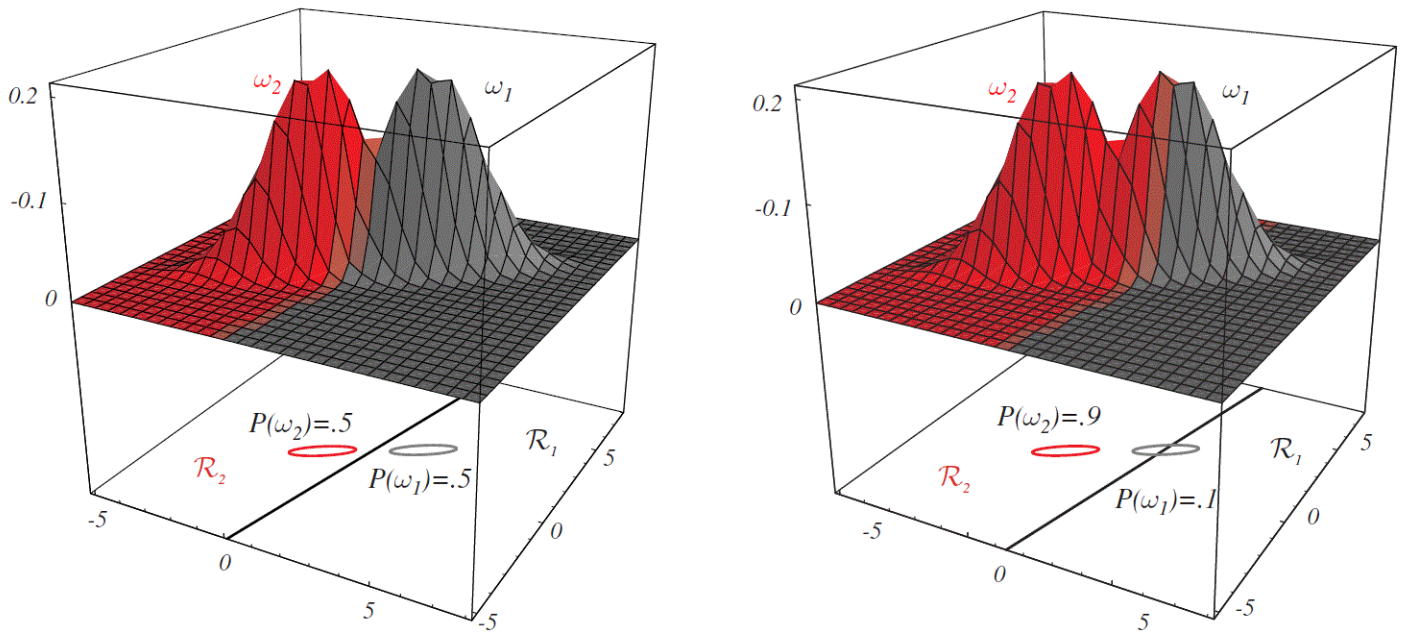
Example: Two Bivariate Normal Distributions with $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$



The boundary between the decision regions $\mathcal{R}_1$ and $\mathcal{R}_2$ is given by a line. For $\pi_1 = \pi_2$, the line passes through the point $(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)/2$, otherwise it is shifted in the direction of the less probable class.

R. Duda, P. Hart, D. Stork, Pattern Classification, 2nd edition, Wiley, 2001

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Exercise:

Consider the following binary classification problem, where the class-conditional densities are given by

$$f_k(\boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}|\boldsymbol{\Sigma}_k|^{1/2}} \cdot \exp\left(-\tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_k)^T\boldsymbol{\Sigma}_k^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_k)\right), \quad k = 1, 2$$

with

$$\boldsymbol{\mu}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad \boldsymbol{\mu}_2 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 9 & 4 \\ 2 & 4 & 9 \end{pmatrix}$$

The observed feature vector should be $\boldsymbol{x} = (1, 2, 2)^T$. Determine the corresponding posterior probabilities $p(k|\boldsymbol{x})$ for the two cases

(i) $\pi_1 = \pi_2 = 0.5$ (ii) $\pi_1 = 0.4,\ \pi_2 = 0.6$