Worksheet 6

Exercise 1

Bivariate random vector $X = (X_1, X_2)^T$ with the joint pdf

$$f_X(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)} & , \; x_1 \geq 0, \; x_2 \geq 0 \\ 0 & , \; \text{otherwise} \end{cases}$$

(a) $f_X(x_1, x_2) \geq 0$ for all $x_1, x_2 \in \mathbb{R}$ ✓

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_X(x_1, x_2)\, dx_2\, dx_1 = \int_0^{\infty}\int_0^{\infty} \overbrace{e^{-x_1} \cdot e^{-x_2}}^{e^{-(x_1+x_2)}}\, dx_2\, dx_1 =$$

$$= \int_0^{\infty} \left[e^{-x_1} \cdot (-e^{-x_2}) \right]_0^{\infty} dx_1 = \int_0^{\infty} e^{-x_1}\, dx_1 = \left[-e^{-x_1} \right]_0^{\infty} = 1 \; ✓$$

(b) $$P(X \in [\tfrac{1}{2}, 1] \times [0, \tfrac{1}{2}]) = \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} e^{-x_1} \cdot e^{-x_2}\, dx_2\, dx_1 =$$

$$= \int_{\frac{1}{2}}^{1} \left[e^{-x_1} \cdot (-e^{-x_2}) \right]_0^{\frac{1}{2}} dx_1 = \int_{\frac{1}{2}}^{1} e^{-x_1} \cdot \left(-e^{-\frac{1}{2}} + 1 \right) dx_1 =$$

$$= \left(1 - e^{-\frac{1}{2}}\right) \cdot \left[-e^{-x_1} \right]_{\frac{1}{2}}^{1} = \left(1 - e^{-\frac{1}{2}}\right) \cdot \left(-e^{-1} + e^{-\frac{1}{2}}\right) =$$

$$= -e^{-1} + e^{-\frac{1}{2}} + e^{-\frac{3}{2}} - e^{-1} = e^{-\frac{1}{2}} + e^{-\frac{3}{2}} - 2e^{-1} \approx \underline{\underline{0.094}}$$

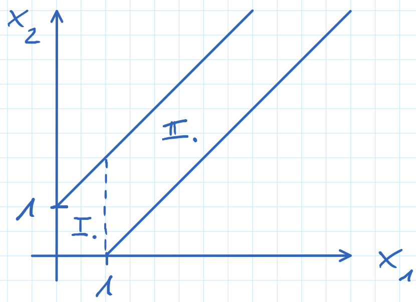
(c) $$P(X_2 > X_1) = \iint_{x_2 > x_1} f_X(x_1, x_2)\, dx_2\, dx_1 =$$

$$= \int_{-\infty}^{\infty}\int_{x_1}^{\infty} f_X(x_1, x_2)\, dx_2\, dx_1 = \int_0^{\infty}\int_{x_1}^{\infty} e^{-x_1} \cdot e^{-x_2}\, dx_2\, dx_1 =$$

$$= \int^{\infty} \left[e^{x_1} \cdot (-e^{-x_2}) \right]^{\infty} dx_1 = \int^{\infty} e^{-2x_1}\, dx_1 = \left[-\tfrac{1}{2} e^{-2x_1} \right]^{\infty} = \tfrac{1}{2}$$

$$= \int_0^\infty \left[e^{x_1} \cdot (-e^{-x_2}) \right]_{x_1}^\infty dx_1 = \int_0^\infty e^{-2x_1}\, dx_1 = \left[-\frac{1}{2} e^{-2x_1} \right]_0^\infty = \underline{\underline{\frac{1}{2}}}$$

(d) $$P(|X_1 - X_2| < 1) = \iint_{|x_1 - x_2| < 1} f_X(x_1, x_2)\, dx_2\, dx_1 =$$



$$= \int_{-\infty}^{\infty} \int_{x_1 - 1}^{x_1 + 1} f_X(x_1, x_2)\, dx_2\, dx_1 =$$

$$= \int_0^1 \int_0^{x_1+1} e^{-x_1} \cdot e^{-x_2}\, dx_2\, dx_1 +$$

$$+ \int_1^\infty \int_{x_1-1}^{x_1+1} e^{-x_1} \cdot e^{-x_2}\, dx_2\, dx_1 =$$

$$= \int_0^1 \left[e^{-x_1} \cdot (-e^{-x_2}) \right]_0^{x_1+1} dx_1 + \int_1^\infty \left[e^{-x_1} \cdot (-e^{-x_2}) \right]_{x_1-1}^{x_1+1} dx_1 =$$

$$= \int_0^1 e^{-x_1} \cdot (1 - e^{-x_1 - 1})\, dx_1 + \int_1^\infty e^{-x_1} \cdot (-e^{-x_1 - 1} + e^{-x_1 + 1})\, dx_1 =$$

$$= \int_0^1 e^{-x_1}\, dx_1 - \int_0^\infty e^{-2x_1 - 1}\, dx_1 + \int_1^\infty e^{-2x_1 + 1}\, dx_1 =$$

$$= \left[-e^{-x_1} \right]_0^1 - e^{-1} \cdot \left[-\frac{1}{2} e^{-2x_1} \right]_0^\infty + e \cdot \left[-\frac{1}{2} e^{-2x_1} \right]_1^\infty =$$

$$= 1 - e^{-1} - \frac{1}{2} e^{-1} + \frac{1}{2} e^{-1} = 1 - e^{-1} \approx \underline{\underline{0.632}}$$

Exercise 2

Normally distributed bivariate random vector $X = (X_1, X_2)^T$ with the joint pdf

$$f_X(x_1, x_2) = c \cdot \exp\left(-\frac{2}{7} x_1^2 + \frac{2}{7} x_1 + \frac{6}{7} x_2 + \frac{2}{7} x_1 x_2 - \frac{4}{7} x_2^2 - \frac{4}{7} \right) \overset{!}{=}$$

$$\overset{!}{=} \frac{1}{\dots} \cdot \exp\left(-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right)$$

$$\overset{\nabla}{\underset{-}{\circ}} \frac{1}{2\pi |\Sigma|^{\frac{1}{2}}} \cdot \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

symmetry of Σ

$$\Rightarrow |\Sigma| = \sigma_{11} \cdot \sigma_{22} - \sigma_{12} \cdot \sigma_{21}, \quad \sigma_{12} = \sigma_{21}$$

$$\Sigma^{-1} = \frac{1}{|\Sigma|} \cdot \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{pmatrix}$$

(a)

Consider the product

$$(x-\mu)^T \Sigma^{-1} (x-\mu) =$$

$$= (x_1-\mu_1, x_2-\mu_2) \cdot \frac{1}{|\Sigma|} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{pmatrix} \cdot \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix} =$$

$$= \frac{1}{|\Sigma|} \Big(\sigma_{22} x_1^2 - \mu_1 \sigma_{22} x_1 - \sigma_{21} x_1 x_2 + \mu_2 \sigma_{21} x_1 - \mu_1 \sigma_{22} x_1 + \mu_1^2 \sigma_{22} +$$

$$+ \mu_1 \sigma_{21} x_2 - \mu_1 \mu_2 \sigma_{21} + \sigma_{11} x_2^2 - \mu_2 \sigma_{11} x_2 - \sigma_{12} x_1 x_2 +$$

$$+ \mu_1 \sigma_{12} x_2 - \mu_2 \sigma_{11} x_2 + \mu_2^2 \sigma_{11} + \mu_2 \sigma_{12} x_1 - \mu_1 \mu_2 \sigma_{12} \Big) =$$

$$= \frac{1}{|\Sigma|} \Big(\sigma_{22} x_1^2 + (-\mu_1 \sigma_{22} + \mu_2 \sigma_{21} - \mu_1 \sigma_{22} + \mu_2 \sigma_{12}) \cdot x_1 +$$

$$+ (\mu_1 \sigma_{21} - \mu_2 \sigma_{11} + \mu_1 \sigma_{12} - \mu_2 \sigma_{11}) \cdot x_2 + (-\sigma_{21} - \sigma_{12}) \cdot x_1 x_2 +$$

$$+ \sigma_{11} x_2^2 + \mu_1^2 \sigma_{22} - \mu_1 \mu_2 \sigma_{21} + \mu_2^2 \sigma_{11} - \mu_1 \mu_2 \sigma_{12}$$

Consider the original joint pdf according to

$$f_X(x_1, x_2) = c \cdot \exp\left(-\frac{1}{2}\left(\frac{4}{7} x_1^2 - \frac{4}{7} x_1 - \frac{12}{7} x_2 - \frac{4}{7} x_1 x_2 + \frac{8}{7} x_2^2 + \frac{8}{7}\right)\right)$$

and perform <u>comparison of coefficients:</u>

$$\sigma_{22} = \frac{4}{7} |\Sigma|, \quad \sigma_{11} = \frac{8}{7} |\Sigma|$$

$$\mu_2\sigma_{21} - 2\mu_1\sigma_{22} + \mu_2\sigma_{12} = -\frac{4}{7}|\Sigma|$$

$$\mu_1\sigma_{21} - 2\mu_2\sigma_{11} + \mu_1\sigma_{12} = -\frac{12}{7}|\Sigma|$$

$$-\sigma_{12} - \sigma_{21} = -\frac{4}{7}|\Sigma|$$

$$\mu_1^2\sigma_{22} - \mu_1\mu_2\sigma_{21} + \mu_2^2\sigma_{11} - \mu_1\mu_2\sigma_{12} = \frac{8}{7}|\Sigma|$$

$$\Rightarrow \sigma_{22} = \frac{1}{2}\sigma_{11}, \quad \sigma_{12} = \sigma_{21} = \frac{\sigma_{11}}{4}$$

$$\Rightarrow \Sigma = \begin{pmatrix} \sigma_{11} & \frac{\sigma_{11}}{4} \\ \frac{\sigma_{11}}{4} & \frac{\sigma_{11}}{2} \end{pmatrix}$$

<u>System of equations :</u>

$$\left.\begin{aligned} \mu_2\frac{\sigma_{11}}{4} - 2\mu_1\frac{\sigma_{11}}{2} + \mu_2\frac{\sigma_{11}}{4} &= -\frac{\sigma_{11}}{2} \\ \mu_1\frac{\sigma_{11}}{4} - 2\mu_2\sigma_{11} + \mu_1\frac{\sigma_{11}}{4} &= -\frac{3}{2}\sigma_{11} \end{aligned}\right\} \overset{\sigma_{11}>0}{\Longleftrightarrow} \boxed{\begin{aligned} 2\mu_1 - \mu_2 &= 1 \\ \frac{4}{3}\mu_2 - \frac{1}{3}\mu_1 &= 1 \end{aligned}}$$

$$\Rightarrow \underline{\underline{\mu = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}}$$

$$\sigma_{11} = \frac{8}{7}|\Sigma| = \frac{8}{7}\left(\frac{\sigma_{11}^2}{2} - \frac{\sigma_{11}^2}{16}\right) = \frac{1}{2}\sigma_{11}^2$$

$$\Rightarrow \sigma_{11} = 2$$

$$\Rightarrow \underline{\underline{\Sigma = \begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}}}, \quad |\Sigma| = \frac{7}{4}$$

$$\Rightarrow c = \frac{1}{2\pi|\Sigma|^{\frac{1}{2}}} = \underline{\underline{\frac{1}{\pi\sqrt{7}}}}$$

(b) Marginal densities :

$$f_{X_1}(x_1) = \frac{1}{\sqrt{4\pi}} e^{-\frac{(x_1-1)^2}{4}}, \qquad f_{X_2}(x_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_2-1)^2}{2}}$$

(c) X_1 and X_2 are <u>not</u> statistically independent,

(c) X_1 and X_2 are <u>not</u> statistically independent,

since $\sigma_{12} = \sigma_{21} \neq 0$.