4 AC and three-phase AC

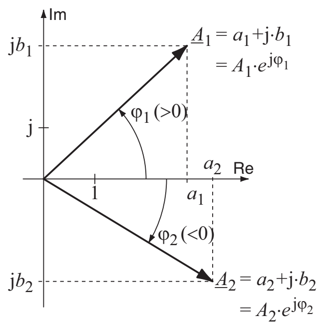
4.1 Complex AC calculation

In AC systems, time signals for currents and voltages are mostly sinusoidal with zero mean value. This type of sinusoidal oscillation is fully described by its amplitude, frequency and phase, so that the actual time function is an unnecessary redundancy.

Complex numbers offer a simpler and very compact form of representation and they can also be used for graphical calculations.

4.1.1 Complex Numbers

Representation forms Complex numbers are pairs of numbers, i.e. they cannot be plotted on a one-dimensional line like the integer, rational or real numbers, but can only be graphically represented on a two-dimensional plane. The complex plane is usually defined so that the x-axis (abscissa) represents the real axis and the y-axis (ordinate) represents the imaginary axis. A complex number can thus be described as a complex pointer by both its Cartesian coordinates (real and imaginary part) and its polar coordinates (amplitude and phase). Both notations can easily be converted into each other using the **Euler equation** shown in Figure 4.1.



$$e^{j\,\varphi} = \cos\varphi + j \cdot \sin\varphi \quad (j^2 = -1)$$

$$\underline{A} = A \cdot e^{j\,\varphi} = A \cdot (\cos\varphi + j \cdot \sin\varphi)$$

$$= A\cos\varphi + j\,A\sin\varphi = a + j\,b$$

$$A = \sqrt{a^2 + b^2}; \quad \tan\varphi = \frac{b}{a} \quad \varphi = \operatorname{atan}2(b, a)$$

Notation: $a = \mathrm{Re}(\underline{A}),\ b = \mathrm{Im}(\underline{A}),\ \varphi = \angle\,\underline{A}$

Figure 4.1: Complex plane

Basic arithmetic operations The following calculation rules apply to addition, subtraction, multiplication and division:

$$\underline{A}_1 = a_1 + \mathrm{j}\, b_1 = A_1 \mathrm{e}^{\mathrm{j}\,\varphi_1}; \quad \underline{A}_2 = a_2 + \mathrm{j}\, b_2 = A_2 \mathrm{e}^{\mathrm{j}\,\varphi_2}$$

$$\underline{A}_1 + \underline{A}_2 = a_1 + a_2 + \mathrm{j}(b_1 + b_2)$$

$$\underline{A}_1 - \underline{A}_2 = a_1 - a_2 + \mathrm{j}(b_1 - b_2)$$

$$\underline{A}_1 \cdot \underline{A}_2 = (A_1 \cdot A_2)\mathrm{e}^{\mathrm{j}(\varphi_1 + \varphi_2)} = a_1 a_2 - b_1 b_2 + \mathrm{j}(a_1 b_2 + a_2 b_1)$$

$$\frac{\underline{A}_1}{\underline{A}_2} = \frac{A_1}{A_2}\mathrm{e}^{\mathrm{j}(\varphi_1 - \varphi_2)} = \frac{a_1 a_2 + b_1 b_2 + \mathrm{j}(a_2 b_1 - a_1 b_2)}{a_2^2 + b_2^2}$$

4.1.2 RMS phasors

In AC engineering (especially in electrical power engineering applications) it is common to rotate the **complex plane $90°$ counterclockwise** so that the real axis is up, and the imaginary axis points to the left.

When a complex pointer is rotated around the origin by the angular velocity ω, its projection on the real axis at any time represents a sinusoidal oscillation. A „snapshot“ of the complex **phasor** for $\omega t = 0$ completely represents the time signal (see Figure 4.2).

Complex pointers and phasors with peak values have been used here till now. However, RMS instead of peak values are used usually for electric drives because it is easier to perform power calculations.

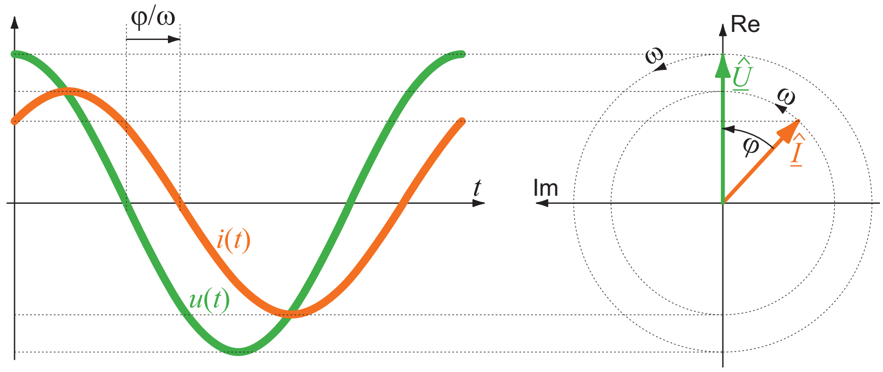


Figure 4.2: Time signal and phasor representation

Below is the RMS value of a cosine function with period T:

$$U = \sqrt{\frac{1}{T}\int_0^T u^2(t)\,\mathrm{d}t} \qquad U: \text{ RMS value}$$

$$= \sqrt{\frac{1}{T}\hat{U}^2\int_0^T \cos^2\omega t\,\mathrm{d}t} = \sqrt{\frac{1}{T}\hat{U}^2\left[\frac{\cos\omega t\cdot\sin\omega t}{2\omega} + \frac{t}{2}\right]_0^T} = \sqrt{\frac{1}{T}\hat{U}^2\frac{T}{2}} = \frac{\hat{U}}{\sqrt{2}}$$

In the case of sinusoidal periodic functions (and only then!!) the RMS value is therefore:

$$U = \frac{\hat{U}}{\sqrt{2}} \qquad \text{or} \qquad I = \frac{\hat{I}}{\sqrt{2}}$$

RMS values and DC quantities are usually denoted with capital letters and time-dependent quantities with lower case letters. Complex quantities are marked with an underscore and peak values with a „hat“. Sinusoidal quantities can therefore be described with complex RMS phasors.

According to the standard, the angle φ is counted from the current vector to the voltage vector. The angles are counted mathematically positive, i.e. counterclockwise is $\varphi > 0$ (the current is lagging), clockwise is $\varphi < 0$ (the current is leading).

$$u(t) = \sqrt{2}\,U\cos(\omega t) = \mathrm{Re}\left(\sqrt{2}\,U\mathrm{e}^{\mathrm{j}\,\omega t}\right) = \mathrm{Re}\left(\sqrt{2}\,\underline{U}\mathrm{e}^{\mathrm{j}\,\omega t}\right) \quad \text{with} \quad \underline{U} = U\mathrm{e}^{\mathrm{j}\,0}$$

$$i(t) = \sqrt{2}\,I\cos(\omega t - \varphi) = \mathrm{Re}\left(\sqrt{2}\,I\mathrm{e}^{\mathrm{j}\,\omega t}\mathrm{e}^{-\mathrm{j}\,\varphi}\right) = \mathrm{Re}\left(\sqrt{2}\,\underline{I}\mathrm{e}^{\mathrm{j}\,\omega t}\right) \quad \text{with} \quad \underline{I} = I\mathrm{e}^{-\mathrm{j}\,\varphi}$$

4.1.3 Impedances

The following applies to the voltage drop across inductances in general:

$$u_\mathrm{L} = L\frac{\mathrm{d}i_\mathrm{L}}{\mathrm{d}t} \qquad i_\mathrm{L} = \mathrm{Re}\left(\sqrt{2}\,\underline{I}_\mathrm{L}\mathrm{e}^{\mathrm{j}\,\omega t}\right)$$

$$\frac{\mathrm{d}i_\mathrm{L}}{\mathrm{d}t} = \mathrm{Re}\left(\sqrt{2}\,\underline{I}_\mathrm{L}\frac{\mathrm{d}\mathrm{e}^{\mathrm{j}\,\omega t}}{\mathrm{d}t}\right) = \mathrm{Re}\left(\mathrm{j}\,\omega\cdot\sqrt{2}\,\underline{I}_\mathrm{L}\mathrm{e}^{\mathrm{j}\,\omega t}\right)$$

$$\underline{U}_\mathrm{L} = \mathrm{j}\,\omega L\cdot\underline{I}_\mathrm{L} = \mathrm{j}\,X_\mathrm{L}\cdot\underline{I}_\mathrm{L} \qquad \left(\frac{\mathrm{d}}{\mathrm{d}t}\,\hat{=}\,\mathrm{j}\,\omega\right)$$

and analogously for capacities:

$$i_\mathrm{C} = C\frac{\mathrm{d}u_\mathrm{C}}{\mathrm{d}t} \qquad u_\mathrm{C} = \mathrm{Re}\left(\sqrt{2}\,\underline{U}_\mathrm{C}\mathrm{e}^{\mathrm{j}\,\omega t}\right)$$

$$\frac{\mathrm{d}u_\mathrm{C}}{\mathrm{d}t} = \mathrm{Re}\left(\sqrt{2}\,\underline{U}_\mathrm{C}\frac{\mathrm{d}\mathrm{e}^{\mathrm{j}\,\omega t}}{\mathrm{d}t}\right) = \mathrm{Re}\left(\mathrm{j}\,\omega\cdot\sqrt{2}\,\underline{U}_\mathrm{C}\mathrm{e}^{\mathrm{j}\,\omega t}\right)$$

$$\underline{I}_\mathrm{C} = \mathrm{j}\,\omega C\cdot\underline{U}_\mathrm{C} = \frac{1}{\mathrm{j}\,X_\mathrm{C}}\cdot\underline{U}_\mathrm{C} \qquad \mathrm{j}\,X_\mathrm{C} = \frac{1}{\mathrm{j}\,\omega C} = -\mathrm{j}\,\frac{1}{\omega C}$$

A complex resistance is then represented as follows:

$$\underline{Z} = R + \mathrm{j}\,X = Z\cdot\mathrm{e}^{\mathrm{j}\,\varphi}; \quad Z = \sqrt{R^2 + X^2}; \quad \tan\varphi = \frac{X}{R} \tag{4.1}$$

The real part R is called **resistance**, the imaginary part X **reactance**, the respective reciprocal values $G = 1/R$ **conductance** or $Y = 1/X$ **Admittance**.

4.1.4 Power

If the voltage is in the real axis of the complex plane, the active and reactive power flow can be determined only by the position of the current vector. A distinction is

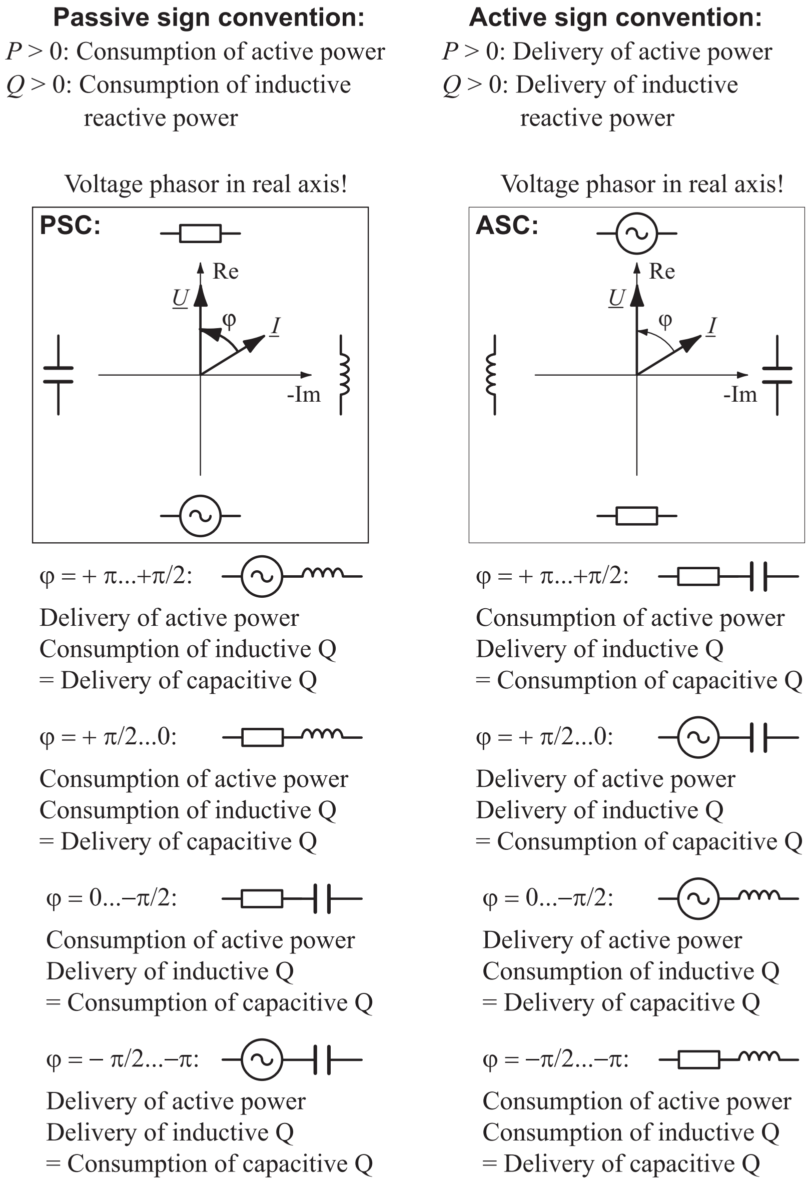


Figure 4.3: Direction of active and reactive power flow for passive i.e. consumer sign convention (PSC) and active i.e. generator sign convention (ASC)

needed for passive i.e. consumer sign convention (PSC) or active i.e. generator sign convention (ASC) as can be seen in Figure 4.3

The complex apparent power can be determined from the product of the complex voltage vector and the conjugate complex current vector:

$$
\begin{aligned}
&\text{apparent power} && \underline{S} && = \underline{U} \cdot \underline{I}^* = P + \mathrm{j}\,Q \quad \left(\underline{I} = I \cdot \mathrm{e}^{-\mathrm{j}\,\varphi} \Leftrightarrow \underline{I}^* = I \cdot \mathrm{e}^{+\mathrm{j}\,\varphi}\right) \\
&\text{active power} && P && = U\,I\,\cos\varphi \\
&\text{reactive power} && Q && = \sqrt{S^2 - P^2} = \sqrt{(U\,I)^2 - (U\,I\,\cos\varphi)^2} = U\,I\,\sin\varphi
\end{aligned}
$$

The resulting complex power from the calculation with RMS voltage and current phasors represents the time average i.e. active and reactive power are also averaged over time.

Although the reactive power is purely an arithmetic variable, the active power can also be clearly displayed over time:

$$p(t) = u(t) \cdot i(t) = \sqrt{2}\,U\cos(\omega t) \cdot \sqrt{2}\,I\cos(\omega t - \varphi)$$
$$= 2\,U\,I \cdot \cos(\omega t) \cdot \cos(\omega t - \varphi) = 2\,U\,I\,\frac{1}{2}\,(\cos\varphi + \cos(2\omega t - \varphi))$$

$$p(t) = U\,I\,\cos\varphi + U\,I\,\cos(2\omega t - \varphi) \qquad = P + S\,\cos(2\omega t - \varphi)$$

The active power P is the mean value of the instantaneous power $p(t)$, whereas the instantaneous power itself oscillates with the amplitude S and twice the mains frequency around the mean value P. The time curves for a phase shift of $\varphi = 30°$ (current lags behind the voltage) are shown as an example.

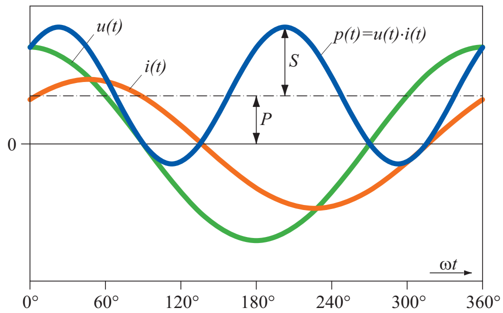


Figure 4.4: Instantaneous power $p(t)$ for $\varphi = 30°$

4.2 Three-phase AC

It is not possible with a single-phase AC line to transmit power that is constant over time. The effective power P that can be used on average is superimposed on a power component oscillating at twice the supply frequency with the amplitude S, so that the instantaneous power $p(t)$ oscillates between $P - S$ and $P + S$.

$$u_1(t) = \sqrt{2}\,U\cos(\omega t)$$
$$u_2(t) = \sqrt{2}\,U\cos(\omega t - \frac{2\pi}{3})$$
$$u_3(t) = \sqrt{2}\,U\cos(\omega t + \frac{2\pi}{3})$$
$$i_1(t) = \sqrt{2}\,I\cos(\omega t - \varphi)$$
$$i_2(t) = \sqrt{2}\,I\cos(\omega t - \frac{2\pi}{3} - \varphi)$$
$$i_3(t) = \sqrt{2}\,I\cos(\omega t + \frac{2\pi}{3} - \varphi)$$

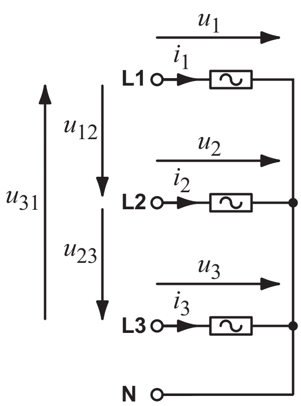


Figure 4.5: Phase and line voltages in three-phase systems

With a three-phase system, however, it is possible to transmit power that is constant over time. Here, three AC voltage sources of the same frequency, each phase-shifted by $360°/3 = 120°$, are combined as shown in Figure 4.2.

Two types of voltage can be distinguished: phase voltages $u_1,\ u_2,\ u_3$ and line voltages $u_{12},\ u_{23},\ u_{31}$. In addition to the indices 1, 2 and 3, the indices a, b and c and U, V, W are also commonly used. A neutral conductor N can be connected to the star point but is not always available. For this reason, line voltages and currents (RMS values) are always specified on rating plates, although phase voltages and currents are usually used for calculations!

In Figure 4.2 the phasor diagram for phase and line voltages is shown.

$$
\begin{aligned}
\underline{U}_{12} &= \underline{U}_1 - \underline{U}_2 \\
\underline{U}_{23} &= \underline{U}_2 - \underline{U}_3 \\
\underline{U}_{31} &= \underline{U}_3 - \underline{U}_1 \\
\underline{U}_{12} &= \underline{U}_{23} = \underline{U}_{31} \\
U_{12} &= \sqrt{3}\, U_1 \\
U_1 &= U_\mathrm{N}/\sqrt{3} \\
U_{12} &= U_\mathrm{N}
\end{aligned}
$$

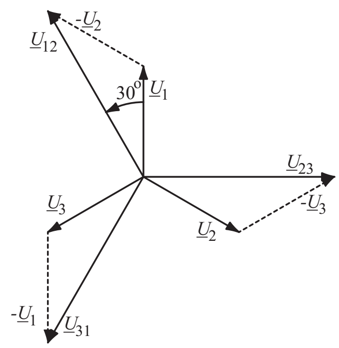


Figure 4.6: Phasor diagram three-phase system

The three line voltages can be easily constructed from the three phase voltages. The vectors of the three line voltages are greater than the phase voltages by a factor $\sqrt{3}$ and form a voltage system that precedes the phase voltage system by $30°$.

In order to consider the instantaneous power transmitted in the entire three-phase system, it is assumed that the network is loaded with the same impedances in all three phases (symmetrical load). This means that all phase currents have the same effective value and are also phase-shifted by $120°$. Now the total instantaneous power is considered:

$$
\begin{aligned}
p(t) &= u_1(t)\cdot i_1(t) + u_2(t)\cdot i_2(t) + u_3(t)\cdot i_3(t) \\
p(t) &= 2\,U\,I\ [\cos(\omega t)\cdot\cos(\omega t-\varphi) + \\
&\qquad + \cos\left(\omega t - \frac{2\,\pi}{3}\right)\cdot\cos\left(\omega t - \frac{2\,\pi}{3} - \varphi\right) + \\
&\qquad + \cos\left(\omega t + \frac{2\,\pi}{3}\right)\cdot\cos\left(\omega t + \frac{2\,\pi}{3} - \varphi\right)\Big]
\end{aligned}
$$

In the individual strands, there are again double-frequency components, which, however, compensate each other in total:

$$\begin{aligned} p(t) = U\,I\, \Big[&\cos\varphi + \cos(2\omega t - \varphi) + \\ &+ \cos\varphi + \cos\left(2\omega t + \frac{2\,\pi}{3} - \varphi\right) \\ &+ \cos\varphi + \cos\left(2\omega t - \frac{2\,\pi}{3} - \varphi\right)\Big] \end{aligned}$$

$$p(t) = 3\,U\,I\,\cos\varphi = P; \qquad S = 3\,U\,I; \qquad Q = 3\,U\,I\,\sin\varphi; \qquad S^2 = P^2 + Q^2$$

The total instantaneous power is therefore constant, although the instantaneous power transmitted in the individual phases oscillates as in a single-phase system. The consequence of this is that the transmission of electrical power in energy technology is always carried out with the help of three-phase current, as this can significantly reduce the use of materials. In the case of symmetrical conditions, not all three phases need to be considered. Single-phase equivalent circuit diagram only models the part of the three-phase system between the terminal 1 and the neutral conductor N.

4.3 Tasks

Example 4-1: Reactive power compensation

The inductive reactive power of a single-phase AC motor (nominal voltage $230\,\mathrm{V}$, active power $250\,\mathrm{W}$, power factor $\cos\varphi = 0,7$) should be compensated with a capacitor connected in parallel to the motor. Which capacitance is needed?

Example 4-2: Active and reactive power

The current and voltage curve (Figure 4.7) of a consumer on the AC network ($230\,\mathrm{V}/50\,\mathrm{Hz}$) was oscillographed. State the active and reactive power of the consumer.

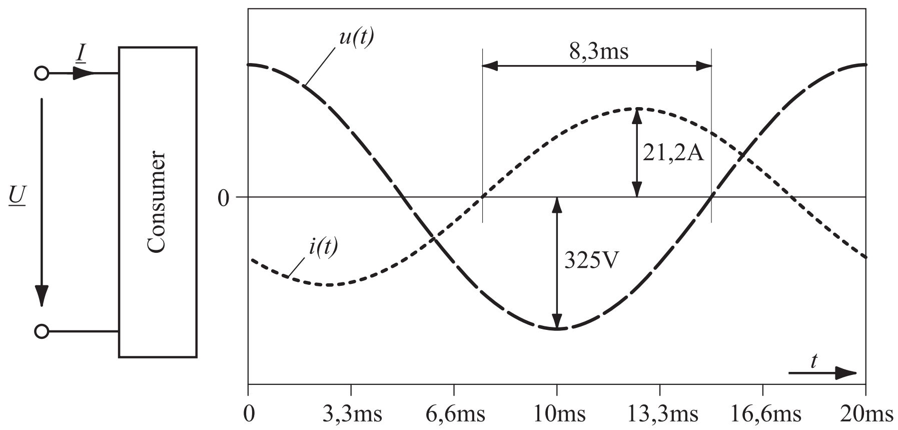
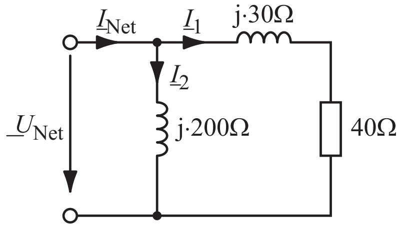


Figure 4.7: Current and voltage curves

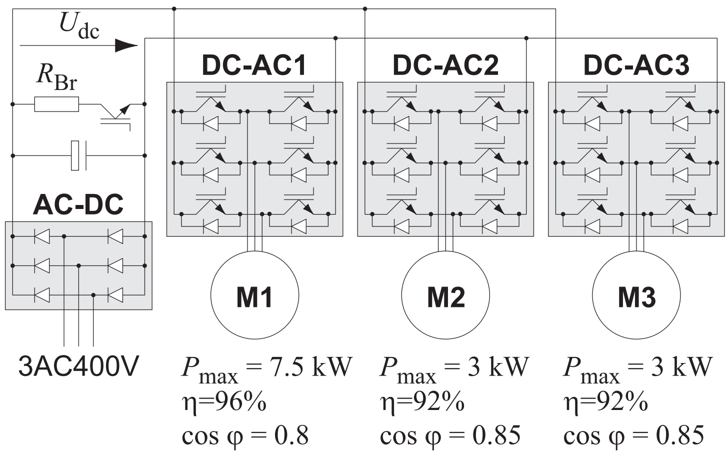
Example 4-3: Network with complex impedances

Given the following network ($\underline{U}_{\mathrm{Net}} = 230\,\mathrm{V}e^{\mathrm{j}\,0^\circ}$), state all the currents.
Hint: first determine $\underline{I}_1$ and $\underline{I}_2$ and finally $\underline{I}_{\mathrm{Net}}$.



Example 4-4: Drive System

A total of three drives are required for a gantry robot, each of which is fed via its own inverter from a common DC link:



The specified values can be used for both motor and generator operation. What apparent power should the three inverters (AC-DC1..3) be designed for? What apparent power should the rectifier (AC-DC) be designed for?