INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination (Second Semester) : 2023-2024

Course Name : M. MATH & M.TECH.(CS)

Subject name : LOGIC FOR COMPUTER SCIENCE

Date : March 3, 2025 Maximum Marks : 40 Duration : 2 hours

Answer all questions. Notations are as used in the class.

1. Let $X = \{p, |, \neg, \rightarrow,), (\}$. Let Y denote the set of all finite strings over X. Is Y countable? Justify your answer. [5]

2. Give an algorithm to check whether a given *CPL* formula φ is a tautology. Use your algorithm to check whether the following formulas are tautologies: [5] + [1.5] + [1.5]

   (a) $\neg\varphi \rightarrow (\varphi \rightarrow \psi)$.

   (b) $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi)$.

3. Let Δ be a consistent and complete set of *CPL* formulas. Show the following: [2] + [5]

   (a) $\delta \in \Delta$ iff $\Delta \vdash \delta$.

   (b) $\varphi \vee \psi \in \Delta$ iff $\varphi \in \Delta$ or, $\psi \in \Delta$

4. Let $\mathcal{A}$ be a non-empty set. A binary relation $<$ on $\mathcal{A}$ is said to be a pre-order if $<$ is reflexive, and transitive. The relation $<$ is dense if for any $x, y \in \mathcal{A}$, with $x < y$, there is some $z \in \mathcal{A}$ such that $x < z < y$. Find a suitable first order language and give axioms in it for a dense pre-order. In addition. give axioms for the pre-order to become a bounded pre-order. [4]

5. Let $\mathcal{R}$ be the structure $(\mathbb{R}; +_{\mathbb{R}}, \cdot_{\mathbb{R}})$, $\mathbb{R}$ being the set of real numbers, $+_{\mathbb{R}}$ and $\cdot_{\mathbb{R}}$ correspond to the usual addition and multiplication, respectively. Let $\mathcal{L}$ be a first order language with equality, two binary function symbols $+$ and $\cdot$, interpreted as $+_{\mathbb{R}}$ and $\cdot_{\mathbb{R}}$, respectively in the structure $\mathcal{R}$. A formula $\varphi(x)$ in $\mathcal{L}$ with exactly one free variable x is said to define a set S of real numbers if ($s \in S$ iff $\varphi(x)$ holds in $\mathcal{R}$ with x assigned to s).

   (a) Give a formula in $\mathcal{L}$ that defines the set $\{x \in \mathbb{R} : x \geq 0\}$ in the structure $\mathcal{R}$.

   (b) Give a formula in $\mathcal{L}$ that defines the set $\{0\}$ in the structure $\mathcal{R}$. [6]

6. Assume that a first order language has equality and a two place predicate symbol P. For each of the following conditions, find a sentence φ such that a structure $\mathcal{A}$ is a model of φ iff the following conditions are met. [5]

   (a) The domain of the structure $\mathcal{A}$ has exactly three members.

   (b) The interpretation of P in the structure $\mathcal{A}$, $P^{\mathcal{A}}$ is a function on the domain of $\mathcal{A}$.

7. Let $\mathbb{N}$ be the set of natural numbers. Let L be a first-order language with equality together with a predicate symbol $<$, constant symbols 0, 1, and function symbols $+$ and $\times$, which are interpreted in $\mathbb{N}$ in the usual way. Using compactness theorem, show that there is a model of Th($\mathbb{N}$), containing a number c, say, which is larger than any element of $\mathbb{N}$. [5]

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End Semester Examination (Even Semester) : 2024-2025

Course Name : M.MATH & M.TECH.(CS)
Subject name : LOGIC FOR COMPUTER SCIENCE
Date : May 2, 2025 Maximum Marks : 100 Duration : 3 hours

Answer all questions. Notations are as used in the class. 2 marks are reserved for neatness.

1. When asked for the ages of her three children, Ms. Delta says that:
   (a) Alice is her youngest child if Bill is not her youngest child.
   (b) Alice is not her youngest child if Carl is not her youngest child.

   Note that only one of the three children can be her youngest child, and no two can both be the youngest child. Who is Ms. Delta's youngest child? Use semantic consequence relation of CPL to justify your answer. [8]

2. Are the CPL formulas $(p \vee q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ logically equivalent? Justify your answer. [5]

3. Write the following CPL formulas in negation normal form. [3 + 5 = 8]
   (a) $\neg(p \vee (\neg q \wedge r))$
   (b) $\neg(\neg(p \rightarrow q) \rightarrow (p \rightarrow \neg q))$

4. Let L be a first-order language. Give an algorithm to check whether a variable-free L-formula φ is true in a given finite L-structure $S = (D, \mathcal{I})$. [10]

5. Let L be a finite first-order language whose vocabulary consists of constant symbols and relation symbols only. Let $\mathbb{A}$ and $\mathbb{B}$ be two finite elementarily equivalent L-structures. Are $\mathbb{A}$ and $\mathbb{B}$ isomorphic? Justify your answer. [8]

6. Let L be a sound and complete first-order logic. Show that L is compact. [6]

7. Consider the following collection of graphs:
   (a) the collection of graphs containing a complete subgraph of size $k \geq 1$
   (b) the collection of graphs containing a set of $k \geq 1$ nodes such that all nodes that are not in this set have a neighbour in the set.
   (c) the collection of graphs whose maximum out-degree is finite

   Check whether these collections of graphs are definable in the first-order language of graphs containing a binary relation symbol, only. Justify your answers. [2 + 3 + 5 = 10]

8. Mention whether the following statements are true or false, with justifications. [20]
   (a) There exists a non-compact logic.
   (b) A finite Kripke model cannot be bisimilar to an infinite Kripke model.
   (c) There is a valid formula in the modal logic for transitive models which is not derivable.
   (d) Down-closed sets in basic modal logic are always maximally consistent.
   (e) Partial-order frames can be characterized by a modal formula.

9. Consider a frame $\mathsf{F} = (W, R)$. Show that the following properties of R are not definable in basic modal logic. [10]
   (a) Converse seriality: for each world w, there exists a world v such that Rvw.
   (b) Acyclicity: for each world w, there is no finite path of non-zero length from w to itself.

10. (a) Let $\mathsf{M}_1 = (W_1, R_1, V_1)$ and $\mathsf{M}_2 = (W_2, R_2, V_2)$ be two Kripke models, and Z be a bisimulation between W_1 and W_2, with $(w_1, w_2) \in Z$. Show that for any basic modal logic formula φ, $\mathsf{M}_1, w_1 \models \Diamond^* \varphi$ if and only if $\mathsf{M}_2, w_2 \models \Diamond^* \varphi$. [7]
    (b) Give an example of a valid formula in MLTC containing both $\Diamond^*$ and $\Box^*$ operators. Justify your answer. [5]

$\neg((p \rightarrow q) \vee (p \rightarrow \neg q))$
$(p \wedge \neg q) \wedge (p \wedge q)$