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Indian Statistical Institute
M.Tech. in Computer Science (2024-25)
Mid Semester Examination 2024

Paper: **Linear Algebra**
Time : **2 Hrs.**

Date: **14/09/2024**
Full Marks : **30**

Note: Symbols used have their usual meaning. *The figures in the margin indicate full marks.*
Answer any FIVE from the following questions. $5 \times 6 = 30$

1. Let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Q}$ be the fields of complex, real and rational numbers respectively. Determine whether the followings are vector spaces.

   (a) $\mathbb{C}$ over $\mathbb{R}$.

   (b) $\mathbb{C}$ over $\mathbb{Q}$.

   Find the dimension and a basis for each that is a vector space.

   [2 + 4]

2. (a) Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

   where all four blocks are $n \times n$ matrices. If A_{11} and A_{22} are nonsingular, show that A must also be nonsingular and that A^{-1} must be of the form

$$\begin{bmatrix} A_{11}^{-1} & X \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

   Also, determine X.

   (b) Let V be a vector space over a field $\mathbf{F}$ and $\mathbf{u}, \mathbf{v} \in V$. Prove or disprove the followings:

   i. If $\mathbf{w} \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{u} + \mathbf{w}$ and $\mathbf{v} + \mathbf{w}$ are linearly dependent.

   ii. If $\mathbf{w} \in V$ and $\mathbf{w} \notin \mathrm{Span}(\mathbf{u}, \mathbf{v})$ then $\mathbf{u} + \mathbf{w}$ and $\mathbf{v} + \mathbf{w}$ are linearly independent.

   [3 + 3]

3. (a) If A is a square matrix such that the linear equation system $Ax = 0$ has nonzero solutions, is it possible that $A'\mathbf{x} = \mathbf{b}$ has a unique solution for some column vector $\mathbf{b}$? Justify your answer.

   (b) Let W_1, W_2 and W_3 be the subspaces of a vector space V over $\mathbf{F}$. Does $W_1 \cap (W_2 + W_3) = (W_1 \cap W_2) + (W_1 \cap W_3)$ necessarily true? Justify your answer.

   (c) Prove or disprove: If v_1, v_2, v_3, v_4 is a basis of a vector space V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

   [2 + 2 + 2]

P.T.O.

4. (a) Find A^{100}, where $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

   (b) Prove or disprove: if the eigenvalues of A are $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$, then A is similar to the diagonal matrix $\operatorname{diag}(\lambda_2, \lambda_3, \ldots, \lambda_n, \lambda_1)$.

[4 + 2]

5. (a) Let A and B be n-square matrices. Show that the characteristic polynomial of the following matrix M is an even function; that is,

$$\text{if } \ |\lambda I - M| = 0, \quad \text{then } \ |-\lambda I - M| = 0,$$

   where

$$M = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}.$$

   (b) Let T be a linear transformation on a finite dimensional vector space V. Show that $\dim(\operatorname{Im}(T^2)) = \dim(\operatorname{Im}(T))$ if and only if

$$V = \operatorname{Im}(T) \oplus \operatorname{Ker}(T).$$

[3 + 3]

6. (a) Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is such that $\deg(Dp) = \deg(p) - 1$ for every nonconstant polynomial $p \in \mathcal{P}(\mathbb{R})$. Prove that D is surjective.

   (b) Let V be a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ and $RST = I$. Show that S is invertible and that $S^{-1} = TR$.

[3 + 3]

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Indian Statistical Institute

M.Tech. in Computer Science (2024-25)

End Semester Examination 2024

Paper: **Linear Algebra**

Time : **3 Hrs.**

Date: **21/11/2024**

Full Marks : **50**

Note: Symbols used have their usual meaning. *The figures in the margin indicate full marks.*
Answer any FIVE from the following questions. $5 \times 10 = 50$

1. (a) Let $T \in \mathcal{L}(V, W)$ and dim $V = n$. If $\{\alpha_1, \alpha_2, \dots, \alpha_s, \alpha_{s+1}, \dots, \alpha_n\}$ is a basis for V such that $\{\alpha_1, \alpha_2, \dots, \alpha_s\}$ is a basis for Ker(T), show that

   i. $\{T\alpha_{s+1}, \dots, T\alpha_n\}$ is a basis for Im(T).

   ii. $V = \text{Ker}(T) \oplus \text{Span}\{\alpha_{s+1}, \dots, \alpha_n\}$.

   iii. Is Ker(T)+Im(T) necessarily a direct sum when $V = W$? Justify your answer.

   [2 + 2 + 2]

   (b) A function $f \in C(\mathbb{R})$ is even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$, and f is odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Let W_1 and W_2 be the collections of even and odd continuous functions on $\mathbb{R}$, respectively. Show that W_1 and W_2 are subspaces of $C(\mathbb{R})$. Show further that $C(\mathbb{R}) = W_1 \oplus W_2$.

   [4]

2. (a) Let $A \in \mathcal{L}(V)$ be an idempotent linear transformation on an n-dimensional vector space V, that is, $A^2 = A$. Prove or disprove the followings:

   i. $I - A$ is idempotent.

   ii. Ker $A = \{x - Ax | x \in V\} = \text{Im}(I - A)$

   iii. If $V = U \oplus W$, then there exists a unique linear transformation B such that $B^2 = B$, $\text{Im}(B) = U$, $\text{Ker}(B) = W$.

   [1 + 2 + 3]

   (b) Let A and B be linear transformations on an n-dimensional vector space V over the complex field $\mathbf{C}$ satisfying $AB = BA$. Show that

   i. A and B have at least one common eigenvector (not necessarily belonging to the same eigenvalue).

   ii. The matrix representations of A and B are both upper-triangular under some basis.

   [2 + 2]

3. (a) Let L be a linear transformation on a vector space V, dim $V = n$. If $L^{n-1}x \neq 0$, but $L^n x = 0$, for some $x \in V$, show that

P.T.O.

$$x, Lx, \dots, L^{n-1}x$$

are linearly independent, and thus form a basis of V. What are the eigenvalues of L? Find the matrix representation of L under the basis. [5]

(b) Let A be an $n \times n$ matrix with singular value decomposition $U\Sigma V^T$ and let $B = \begin{pmatrix} 0 & A^T \\ A & O \end{pmatrix}$.

Show that if $\mathbf{x}_i = \begin{pmatrix} \mathbf{v}_i \\ \mathbf{u}_i \end{pmatrix}$, $\mathbf{y}_i = \begin{pmatrix} -\mathbf{v}_i \\ \mathbf{u}_i \end{pmatrix}$, $i = 1, 2, \dots, n$, then the $\mathbf{x}_i$'s and $\mathbf{y}_i$'s are eigenvectors of B. How do the eigenvalues of B relate to the singular values of A? [5]

4. (a) Let A be an $n \times n$ nonsingular matrix having distinct eigenvalues. If B is a matrix satisfying $AB = BA^{-1}$, show that B^2 is diagonalizable. [3]

(b) Let A be an $n \times n$ matrix with an eigenvalue λ of multiplicity n. Show that A is diagonalizable if and only if $A = \lambda I$. [3]

(c) Let A be a $n \times n$ matrix with Schur decomposition $UTU^\dagger$. Show that if the diagonal entries of T are all distinct, then there is an upper triangular matrix R such that ~~$X = UR$~~A diagonalizes A.
X=UR [4]

5. (a) Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $\{\beta_1, \beta_2, \dots, \beta_n\}$ be the two sets of vectors of an inner product space V of dimension n. Does there always exist a linear transformation that maps each α_i to β_i? Show that if

$$\langle \alpha_i | \alpha_j \rangle = \langle \beta_i | \beta_j \rangle \quad i, j = 1, 2, \dots, n,$$

then there exists an orthogonal linear transformation T (i.e., $\langle u|v \rangle = \langle Tu|Tv \rangle$) such that

$$T\alpha_i = \beta_i \quad i = 1, 2, \dots, n.$$

[4]

(b) Let X be a $n \times m$ matrix and $XX^\dagger = I$. Does X necessarily unitary? If not, find the condition when it is unitary. [3]

(c) Let H be a Hermitian matrix, and let $U = e^{iH} = \sum_k \frac{(iH)^k}{k!}$. Prove that U is unitary.

[3]

6. (a) Let A be an $n \times n$ matrix and let $\|.\|_M$ be a matrix norm that is compatible with some vector norm on $\mathbb{R}^n$. Show that if λ is an eigenvalue of A, then $|\lambda| \leq \|A\|_M$. [3]

(b) Let A be an $n \times n$ matrix with distinct real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Let λ be a scalar that is not an eigenvalue of A and let $B = (A - \lambda I)^{-1}$. Show that

i. the scalars $\mu_j = 1/(\lambda_j - \lambda)$, $j = 1, 2, \dots, n$ are the eigenvalues of B.

ii. if $\mathbf{x}_j$ is an eigenvector of B belonging to μ_j, then $\mathbf{x}_j$ is an eigenvector of A belonging to λ_j.

iii. if the power method is applied to B, then the sequence of vectors will converge to an eigenvector of A belonging to the eigenvalue that is closest to λ.

[2 + 2 + 3]