INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination:(2025-2026)

M.TECH (CS) I YEAR

Subject Name: Quantum Computation

Maximum Marks: 30 Duration: 2 hours Date: 11.09.2025

Answer any three of the following four questions

1. a) Consider the following linear operator A acting on a two dimensional Hilbert space.;

$$A = a_o I + a_x \sigma_x + a_y \sigma_y + a_z \sigma_z$$

where σ's are usual Pauli matrices and a_0, a_x, a_y, a_z are real numbers. Under what condition A is a density matrix.

b) Show that for every the density operator D with $D^2 = D$, there exists a unit vector $|\psi\rangle$, such that the following relation holds.

$$D = |\psi\rangle\langle\psi|$$

c) Let the initial density matrix of a qubit is $\frac{1}{2}(I+\frac{1}{3}\sigma_x+\frac{1}{2}\sigma_y)$. If spin measurement is performed along z-axis, what is the probability for the spin up result?

$4+3+3$

2. a) Let there is a cloning machine that can clone the following orthogonal states $|0\rangle$ and $|1\rangle$. Show that the machine will not be able to clone the following state $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$.

b) Consider a Swap operator U_s which acts in the following way;

$$U_s|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$$

for all possible states $|\psi\rangle$, $|\phi\rangle$. Then show that U_s can not be expressed as

$$U_s = U_1 \otimes U_2$$

where U_1 and U_2 are acting on particle 1 and particle 2 respectively.

c) Describe the realization of the swap operation (gate) by using C-Not gates.

$4+3+3$

3. a) Let Alice and Bob share the following state;

$$|\psi>_{AB}= \frac{1}{\sqrt{5}}(|0\rangle_A \otimes |0\rangle_B + \frac{2}{\sqrt{5}}|1\rangle_A \otimes |1\rangle_B)$$

where $|0\rangle$ and $|1\rangle$ are eigen states of σ_z.
(i) Show that the state can not be written in product form.
ii) Find the density matrix of the subsystem on Bob's side.
(ii) Derive the probability of conclusive teleportation of an unknown state by using this state. How many bits will be required for conclusive teleportation?

$2+2+6$

3. i) Consider a two qubits pure state $|\psi\rangle_{12}$ and four unitary operators $\{U_i; i = 1,2,3,4\}$ acting on the first particle. What is the necessary condition so that the four states $\{U_i \otimes I|\psi\rangle_{12}, i = 1,2,3,4\}$ form an orthogonal set?
ii) Find a two qubits state $|\psi\rangle_{12}$ for which $\{U_i \otimes I|\psi\rangle_{12}, i = 1,2,3,4\}$ form an orthogonal set.
iii) Discuss how quantum super dense coding can be realized using the $|\psi\rangle_{12}$ in (ii).

$3+3+4$

INDIAN STATISTICAL INSTITUTE

Semestral Examination:(2025-2026)

M.TECH (CS) I YEAR

Subject Name: Quantum Computation

Maximum Marks: 60 Duration: 3 hours Date: 21.11.2025

Answer any five of the following questions

1. Let the density matrix of a qubit be $\frac{1}{2}(I - \frac{1}{2}\sigma_x + \frac{1}{2}\sigma_y)$.

a) Determine whether the above density matrix represents a pure or a mixed state.

b) If spin measurement is performed on this state along x-axis, what is the probability for the spin down result?

c) Let you be given a two-qubit state which is one of the four orthogonal Bell states $|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle$ and $|\psi^-\rangle$. How do you determine the Bell state provided you are allowed to perform two-qubit C_{Not}, Hadamard gate H and single qubit measurement in the computational basis?

[3 + 4 + 5]

2. Consider the following three qubits state shared between Alice, Bob and Charlie stationed at distant laboratories:

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}}[|000\rangle + |111\rangle],$$

where $|0\rangle$ and $|1\rangle$ form an orthogonal basis in two dimensional Hilbert space.

i) Show that the state can neither be written in the form $|\psi\rangle_A \otimes |\phi\rangle_B \otimes |\chi\rangle_C$ nor in the form $|\eta\rangle_{AB} \otimes |\tau\rangle_C$.

ii) Let Alice be given another qubit whose state is not known to her. She has to prepare the state in Bob's lab. Show how this task can be made successful if all three parties cooperate by local operation and classical communication.

iii) Alice, Bob and Charlie stationed in three different labs share a 3-qubit state as described above. If Alice shares another 2-qubit state

$$|\phi\rangle_{AD} = \frac{1}{\sqrt{3}}|00\rangle + \frac{\sqrt{2}}{\sqrt{3}}|11\rangle$$

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with Dick sitting in another lab, then show that this state can be transferred to Bob and ~~Charlie~~ if all of them cooperate using local operations and classical communications.

[3 + 4 + 5]

3. Consider the following general (normalized) state shared between Alice, Bob and Eve

$$|\psi\rangle_{ABE} = a|00\rangle_{AB}|\phi_{00}\rangle_E + b|01\rangle_{AB}|\phi_{01}\rangle_E + c|10\rangle_{AB}|\phi_{10}\rangle_E + d|11\rangle_{AB}|\phi_{11}\rangle_E$$

where each of Alice and Bob has one qubit and $|\phi_{ij}\rangle_E$'s refer to states of Eve's system which may be associated with higher dimensional Hilbert space. The statistics obtained by Alice and Bob satisfy the following conditions:
i) The results of local spin measurements by Alice and Bob along any directions, are completely random.
ii) Alice and Bob's results are fully correlated when they both measure spin along Z-axis or X-axis.
a) Then show that Alice and Bob actually share a two-qubit maximally entangled state which is completely uncorrelated with Eve's system.
b) Describe a protocol of sharing secret key by using entanglement and discuss how the above result can be applied to guarantee secure key between Alice and Bob.

[8 + 4]

4. a) Let a single qubit unitary gate U be realized in the following way;

$$U = A\sigma_x B\sigma_x C$$

where A, B and C are also single qubit gates with $ABC = I$. By using appropriate diagram, show how two-qubit gate $C - U$ can be implemented by single qubit gates and C_{NOT} gates.
b) Let A be a Hermitian operator with $A^2 = I$. Find a arrangement that implements the measurement of A.
c) The function $f : \{0,1\}^n \to \{0,1\}$ is such that;
$f(x) = 1$, for $x = \omega$ and $f(x) = 0$ for $x \neq \omega$
i) In the classical world, how many queries are required to find ω?
ii) Show how a quantum algorithm can provide a quadratic speed up for this search problem.

[3 + 3 + (1 + 5)]

5. Consider a function $f : \{0, 1\}^n \to \{0, 1\}^n$. The function has a period given by n-bit string a: that is
$f(x) = f(y)$ iff $y = x \oplus a$.
a) Discuss how hard it is to find the period a in the classical world.
(b) Discuss the quantum algorithm by which the period can be found in polynomial time.

[2+10]

6. a) Discuss how phase error of a qubit in computational basis

$$|0\rangle \to |0\rangle \quad and \quad |1\rangle \to -|1\rangle$$

can be corrected by using the multi-qubit quantum code.

b) Consider the following two nine-qubit orthogonal states:

$$|0\rangle_L = \frac{1}{2\sqrt{2}}[|000\rangle + |111\rangle][|000\rangle + |111\rangle][|000\rangle + |111\rangle]$$

$$|1\rangle_L = \frac{1}{2\sqrt{2}}[|000\rangle - |111\rangle][|000\rangle - |111\rangle][|000\rangle - |111\rangle]$$

Show that by using these states as quantum codes, any error on a single qubit can be corrected.

[4 + 8]