INDIAN STATISTICAL INSTITUTE

Mid Semester Examination

M.Tech CS, 2024 - 2025 (Semester – I)

Discrete Mathematics

Date: 12 September 2024 Maximum Marks: 60 Duration: 2 hours

General comment. Answer as much as you can, but the maximum you can score from both Group-A and Group-B is 30.

Notations and definitions. $\mathbb{N}$ denotes the set of natural numbers. Given any set X, 2^X denotes the collection of all subsets of X.

A permutation $\pi : [n] \to [n]$ is a derangement if for all $i \in [n]$ we have $\pi(i) \neq i$.

Group-A

(AQ1) Let P be a partially ordered finite set. The minimum number m of disjoint chains which together contain all elements of P is equal to the maximum number M of elements in an antichain of P. [10]

(AQ2) Show that the points $p_0, \ldots, p_n$ in $\mathbb{R}^n$ are ~~linearly~~ finely independent if and only if the vectors $v_1, \ldots, v_n$, where $v_i = p_i - p_0$ for all $i \in [n]$, are linearly independent. [10]

(AQ3) Count the number of derangements on the set $[n]$. [10]

(AQ4) A graph G is 2-connected if and only if there exists, for any two vertices of G, a cycle in G containing these two vertices. [10]

Group-B

(BQ1) Show that every non-star tree T is isomorphic to a subgraph of the complement graph $\overline{T}$. [10]

(BQ2) (a) A set C of propositions is called logically closed if it satisfies both the following conditions:

(i) C contains all tautologies (propositions with truth tables containing only T in the output column)

(ii) For any propositions p and q, if both p and $(p \implies q)$ are in C, then so is q.

If C is a logically closed set, then prove that if the propositions p and $\neg p$ are in C, then C contains every proposition.

(b) Write the contrapositive statement of the following:

If we have good crops then either it rained well or the fertilization worked properly.

[5+5=10]

(BQ3) Let $\{A_1, \ldots, A_m\}$ be a collection of m distinct subsets of $[n]$ satisfying the following properties:

- For any two distinct subsets A_i and A_i, we have $A_i \not\subseteq A_j$.
- For all A_i and A_j, $A_i \cap A_j \neq \emptyset$.
- For all A_i and A_j, $A_i \cup A_j \neq [n]$

Show that $m \leq \binom{n}{\lfloor n/2 \rfloor - 1}$. [10]

(BQ4) (a) Let k be some fixed positive integer and $P(n)$ be a mathematical statement that satisfies the following properties:

(i) All of $P(0), P(1), \cdots, P(k-1)$ are true;

(ii) $P(n) \implies P(n+k)$, for any $n \geq 0$.

Then $P(n)$ is true for every $n \in \mathbb{N}$.

(b) Show that

$$\binom{n}{k} \leq (en/k)^k .$$

[5+5=10]

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INDIAN STATISTICAL INSTITUTE

Semester Examination

M.Tech CS, 2024 - 2025 (Semester – I)

Discrete Mathematics

Date: 25 November 2024 | Maximum Marks: 100 | Duration: 3 hours

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General comment. Answer as much as you can, but the maximum you can score from both Group-A and Group-B is 60 and 40 respectively.

Notations and definitions. $\mathbb{N}$ denotes the set of natural numbers. Given any set X, 2^X denotes the collection of all subsets of X.

Group-A

(AQ1) Let $A_1, \ldots, A_n, B_1, \ldots, B_n$ be finite subsets of $\mathbb{N}$ such that for all i and j in $[n]$, we have

$$A_i \cap B_i = \emptyset, \ \forall i \in [n] \text{ and}$$
$$A_i \cap B_j \neq \emptyset, \text{ for all } i \neq j \in [n].$$

Show that

$$\sum_{i \in [n]} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

[15]

(AQ2) Consider the graph $G = ([n], E)$.

(a) Formulate, with explanation, an Integer Linear Program that computes the smallest vertex cover for the graph G.

(b) Formulate, with explanation, a Linear Program (LP) that computes the smallest fractional vertex cover for the graph G, and using this LP design a 2-factor approximation for the vertex cover problem.

[7 + 8 = 15]

(AQ3) Consider a planar graph $G = (V, E)$ with a planar drawing having f faces, $|V| = n \geq 3$ and $|E| = m$.

(a) Show that $n - m + f = 2$.

(b) If G is a bipartite planar graph then $m \leq 2n - 4$.

[$9 + 6 = 15$]

(AQ4) (a) Using recursion-tree solve the following recurrence

$$T(n) = 3T(n/4) + \Theta(n^2).$$

(b) Show that, for all $n \in \mathbb{N}$, we have

$$\ln n \leq \sum_{i=1}^{n} \frac{1}{i} \leq \ln n + 1.$$

[$9 + 6 = 15$]

(AQ5) If G is a simple graph with $n \geq 3$ vertices such that for every pair of non-adjacent vertices u, v of G we have $d(u) + d(v) \geq n$, then G contains a *Hamiltonian cycle*. [15]

Group-B

(BQ1) Let $R(r, k)$ denote the number of partitions of the integer r into k parts.

(a) Show that $R(r, k) = R(r-1, k-1) + R(r-k, k)$.

(b) Show that $\sum_{k=1}^{r} R(n-r, k) = R(n, r)$.

[$6+4 = 10$]

(BQ2) Show that König's Theorem and Dilworth's Theorem are equivalent. [10]

(BQ3) Let $P_1, \ldots, P_N$ be non-empty finite subsets of $\mathbb{R}^n$, and

$$z \in \operatorname{conv}\left(\bigcup_{i \in [N]} P_i\right).$$

Show that there exists a subset $I \subseteq [N]$ with $|I| = n + 1$, such that

$$z \in \operatorname{conv}\left(\bigcup_{i \in I} P_i\right).$$

[10]

(BQ4) For all $n \geq 5$, any planar drawing of K_n will have at least $\frac{1}{5}\binom{n}{4}$ many crossings. [10]

(BQ5) Let $(\Omega, \mathbb{P})$ be a finite probability space. Suppose that n independent events $A_1, A_2, \ldots, A_n \subseteq \Omega$ exist such that $0 < \mathbb{P}[A_i] < 1$ for each $i \in [n]$. Show that $|\Omega| \geq 2^n$. [10]