# INDIAN STATISTICAL INSTITUTE

Mid Semestral Examination

M. Tech (CS), 2024-2025 (Semester – I)

*Probability and Stochastic Processes*

Date: 17.09.2024 Maximum Marks: 80 Duration: 3.0 Hours

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**Note:** Answer as much as you can but the maximum you can score in each of the groups is 40 for a total of 80. Please mention the questions attempted on the back of the cover page of your answer script.

$E[X]$ and var$[X]$ denote the expectation and variance of the random variable $X$, respectively.

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## Group A

(QA1) (i) If there are $m$ persons and $n$ possible birthdays, find out an expression for the probability that $m$ persons have distinct birthdays.

(ii) Now, consider the persons sequentially, one after another.

(a) Find out an expression, in terms of $i$ and $n$, for the probability that the first $i$ persons fail to have distinct birthdays.

(b) Find out a lower bound on $i$ so that the probability of all birthdays being distinct is at least $1/2$.

[4+(4+2)=10]

(QA2) Let $A_1, A_2, \ldots$ be a decreasing sequence of events, so that $A_1 \supseteq A_2 \supseteq \ldots$. Now define a limiting event $A$ as

$$A = \lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

Then, show that

$$\Pr(A) = \Pr\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \Pr(A_n)$$

[10]

(QA3) Consider $m+1$ boxes with the $i$-th box containing $i$ red balls and $m-i$ white balls, where $i = 0, \ldots, m$. We choose a box at random, where all boxes are equally likely to be chosen. Then, we choose a ball at random from that box, $n$ successive times with replacement (i.e., the ball drawn is replaced each time, and a new ball is selected independently). Suppose, a white ball is chosen each of the $n$ times. What is the probability that if we draw a ball one more time, it will be white? Estimate this probability for large $m$. [10]

(QA4) (i) Let $X$ be a random variable with PMF $p_X$, and let $g(X)$ be a function of $X$. Then, show that the expected value of the random variable $g(X)$ is given by

$$E[g(X)] = \sum_x g(x)p_X(x)$$

(ii) Let $X$ be a random variable with $\mathrm{var}[X] = 0$. What will be the PMF of $X$? Justify your result.

[6+4=10]

(QA5) (i) Let $X$ be a random variable that assumes only nonnegative values. Show that for all $a > 0$, $\Pr(X \geq a) \leq \frac{E[X^m]}{a^m}$, where $m$ is a positive integer $\geq 2$.

(ii) Show that for a random variable $X$ and any $t > 1$,

$$\Pr\left(|X - E[X]| \geq t \cdot \sqrt{\mathrm{var}[X]}\right) \leq \frac{1}{t^2}.$$

Prove all results that you need. [5+5=10]

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## Group B

(QB1) There are $n$ urns of which the $i$-th urn contains $i-1$ red balls and $n-i$ blue balls. You pick an urn at random and remove two balls at random without replacement. Find the probability that

(i) the second ball is blue;

(ii) the second ball is blue given the first ball is blue.

[3+7=10]

(QB2) Let $X_1, \ldots, X_n$ be independent random variables such that $E[X_i] \neq 0$. Show that

$$\frac{\mathrm{var}\left(\prod_{i=1}^{n} X_i\right)}{\prod_{i=1}^{n} E[X_i]^2} = \prod_{i=1}^{n}\left(\frac{\mathrm{var}(X_i)}{E[X_i]^2} + 1\right) - 1$$

[10]

(QB3) There are $n$ letters marked for $n$ envelopes. The letters are mixed up and put randomly inside the envelopes. A *match* occurs if a letter goes into the envelope it is marked for. What is the probability of exactly $k$ matches? [10]

(QB4) Let $X$ and $Y$ be independent Poisson variables with parameters $\lambda_1$ and $\lambda_2$, respectively. Prove that:

(i) $X + Y$ is Poisson with parameter $\lambda_1 + \lambda_2$;

(ii) the conditional distribution of $X$, given $X + Y = n$, is binomial; and find its parameters.

[4+6=10]

(QB5) (i) Consider $n$ independent tosses of a coin with probability of a head equal to $p$. Let $X$ and $Y$ be the number of heads and of tails, respectively. Compute the correlation coefficient of $X$ and $Y$.

(ii) Show that $\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{Cov}(X_i, Y_j)$.

[5+5=10]

# INDIAN STATISTICAL INSTITUTE

**End Semestral Examination**

M. Tech (CS), 2024-2025 (Semester – I)

*Probability and Stochastic Processes*

Date: 18.11.2024 | Maximum Marks: 100 | Duration: 3.5 Hours

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**Note:** This is a 3-page question paper with a 1-page addendum of standard normal table.

Answer as much as you can but the maximum you can score in each of the groups is 50 for a total of 100. Please mention the questions attempted on the back of the cover page of your answer script.

$E[X]$ and $\text{var}[X]$ denote the expectation and variance of the random variable $X$, respectively.

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## Group A

(QA1) (a) State and prove the union bound for a finite or infinite sequence of events $E_1, E_2, \ldots$.

(b) Show that if $E_1, E_2, \ldots, E_n$ are mutually independent, then so are $\overline{E_1}, \overline{E_2}, \ldots, \overline{E_n}$.

[(1+4)+5=10]

(QA2) (a) Let $E_i$, $i \geq 1$, be events such that $\Pr(E_i) = 1$ for all $i$. Show that $\Pr\left(\bigcap_{i=1}^{\infty} E_i\right) = 1$.

(b) Let $C_1, \ldots, C_n$ be disjoint events that form a partition of the state space. Let also $A$ and $B$ be events such that $\Pr(B \cap C_i) > 0$ for all $i$. Show that

$$\Pr(A \mid B) = \sum_{i=1}^{n} \Pr(C_i \mid B) \Pr(A \mid B \cap C_i).$$

[4+6=10]

(QA3) (i) Let $X$ be a random variable such that $E[X] = \mu$. Let $f$ be a convex function and there exists a value $c$ such that

$$f(x) = f(\mu) + f'(\mu)(x - \mu) + \frac{f''(c)(x-\mu)^2}{2}.$$

Show that $E[f(X)] \geq f(E[X])$.

(ii) Let $X$ be a non-negative continuous random variable. Show that

$$E[X] = \int_0^{\infty} \Pr(X > x)\ dx$$

[5+5=10]

(QA4) **(i)** Define moment generating function of a random variable $X$.

**(ii)** If $X_1, \ldots, X_n$ are independent standard normal random variables, then $\chi = \sum_{i=1}^{n} X_i^2$ is said to have the chi-squared distribution with $n$ degrees of freedom. Compute the moment generating function of a chi-squared random variable $\chi$ with $n$ degrees of freedom.

[2+8=10]

(QA5) (i) State the weak law and the strong law of large numbers.

(ii) Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables and assume that $E[X_1^4]$ is finite. Under the above assumption, prove the strong law of large numbers.

[(1+2)+7=10]

(QA6) (a) Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $E[X_i] = 1$. Define $Z_n = \prod_{i=1}^{n} X_i$. Then show that $\{Z_n, n \geq 1\}$ is a martingale.

(b) State and prove Wald's theorem on stopping times. You can assume the Martingale stopping theorem.

[3+(2+5)=10]

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## Group B

(QB1) Choose an integer uniformly at random from the range $[1, 10^6]$. Determine the probability that the number chosen is divisible by one or more of 4, 6 and 9. [10]

(QB2) Let $X_1, \ldots, X_n$ be independent random variables with $\Pr(X_i = 1) = \Pr(X_i = -1) = \frac{1}{2}$, for all $i = 1, \ldots, n$. Let $X = \sum_{i=1}^{n} X_i$. Then, for any $a > 0$, show that $\Pr(|X| \geq a) \leq 2e^{-a^2/2n}$. [10]

(QB3) Let $X$ and $Y$ have joint density function $f(x, y) = 2e^{-x-y}, 0 < x < y < \infty$.

**(i)** Are they independent?

**(ii)** Find their marginal density functions.

**(iii)** Find the covariance of $X$ and $Y$.

[4+4+2=10]

(QB4) A *tournament* on a set $V$ of $n$ players is an orientation $T = (V, E)$ of the edges of the complete graph on the set of vertices $V$. Thus for every two distinct elements $x$ and $y$ of $V$, either $(x, y)$ or $(y, x)$ belongs to $E$, but not both. A simple interpretation of *tournament* is in terms of games where each distinct pair $x, y$ of players, $x, y \in V$, play a single match; the outcome of the games are either win or loss. $(x, y)$ is in the *tournament* if and only if $x$ beats $y$.

$T$ has the property $S_k$ if for every set of $k$ players there is one who beats them all.

Show that if $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$, then there is a tournament on $n$ vertices that has the property $S_k$. [10]

[**Hints:** Can you use probabilistic methods? You can consider a random tournament on $V$ by choosing either edge $(i,j)$ or $(j,i)$, where each of these two choices is equally likely. Consider a fixed subset $K$ of $V$, $|K| = k$ and let $\mathcal{E}_K$ denote the event that there is no vertex that beats all the members of $K$. What is $\Pr(\mathcal{E}_K)$?]

(QB5) (a) Let $X_i$, $i = 1, \dots, 10$ be independent random variables, each uniformly distributed over $[0,1]$. Calculate an approximation to $\Pr\left(\sum\limits_{i=1}^{10} X_i > 6\right)$.

(b) Let $A \sim B$ means $A$ can be approximated by $B$. Prove that, for $x \geq 0$, as $n \to \infty$:

$$\sum_{k:|k-n|\leq x\sqrt{n}} \frac{n^k}{k!} \sim e^n \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\, du$$

[5+5=10]

(QB6) A *queue* is a line where customers wait for service. Time is divided into steps of equal length. At each time step, exactly one of the following occurs:

- If the queue has fewer than $n$ customers, then with probability $p$, a new customer joins the queue.
- If the queue is not empty, then with probability $q$, the head of the line is served and leaves the queue.
- With the remaining probability, the queue is unchanged.

Let $X_t$ be the number of customers in the queue at time $t$.

(i) Show that $X_t$ qualifies to be a finite-state Markov chain.

(ii) Find the transition matrix corresponding to the above Markov chain.

(iii) Show that this Markov chain is irreducible and aperiodic.

(iv) Find the unique stationary distribution for the above Markov chain.

[2+2+2+4=10]

**STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.**

| Z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| -3.9 | .00005 | .00005 | .00004 | .00004 | .00004 | .00004 | .00004 | .00004 | .00003 | .00003 |
| -3.8 | .00007 | .00007 | .00007 | .00006 | .00006 | .00006 | .00006 | .00005 | .00005 | .00005 |
| -3.7 | .00011 | .00010 | .00010 | .00010 | .00009 | .00009 | .00008 | .00008 | .00008 | .00008 |
| -3.6 | .00016 | .00015 | .00015 | .00014 | .00014 | .00013 | .00013 | .00012 | .00012 | .00011 |
| -3.5 | .00023 | .00022 | .00022 | .00021 | .00020 | .00019 | .00019 | .00018 | .00017 | .00017 |
| -3.4 | .00034 | .00032 | .00031 | .00030 | .00029 | .00028 | .00027 | .00026 | .00025 | .00024 |
| -3.3 | .00048 | .00047 | .00045 | .00043 | .00042 | .00040 | .00039 | .00038 | .00036 | .00035 |
| -3.2 | .00069 | .00066 | .00064 | .00062 | .00060 | .00058 | .00056 | .00054 | .00052 | .00050 |
| -3.1 | .00097 | .00094 | .00090 | .00087 | .00084 | .00082 | .00079 | .00076 | .00074 | .00071 |
| -3.0 | .00135 | .00131 | .00126 | .00122 | .00118 | .00114 | .00111 | .00107 | .00104 | .00100 |
| -2.9 | .00187 | .00181 | .00175 | .00169 | .00164 | .00159 | .00154 | .00149 | .00144 | .00139 |
| -2.8 | .00256 | .00248 | .00240 | .00233 | .00226 | .00219 | .00212 | .00205 | .00199 | .00193 |
| -2.7 | .00347 | .00336 | .00326 | .00317 | .00307 | .00298 | .00289 | .00280 | .00272 | .00264 |
| -2.6 | .00466 | .00453 | .00440 | .00427 | .00415 | .00402 | .00391 | .00379 | .00368 | .00357 |
| -2.5 | .00621 | .00604 | .00587 | .00570 | .00554 | .00539 | .00523 | .00508 | .00494 | .00480 |
| -2.4 | .00820 | .00798 | .00776 | .00755 | .00734 | .00714 | .00695 | .00676 | .00657 | .00639 |
| -2.3 | .01072 | .01044 | .01017 | .00990 | .00964 | .00939 | .00914 | .00889 | .00866 | .00842 |
| -2.2 | .01390 | .01355 | .01321 | .01287 | .01255 | .01222 | .01191 | .01160 | .01130 | .01101 |
| -2.1 | .01786 | .01743 | .01700 | .01659 | .01618 | .01578 | .01539 | .01500 | .01463 | .01426 |
| -2.0 | .02275 | .02222 | .02169 | .02118 | .02068 | .02018 | .01970 | .01923 | .01876 | .01831 |
| -1.9 | .02872 | .02807 | .02743 | .02680 | .02619 | .02559 | .02500 | .02442 | .02385 | .02330 |
| -1.8 | .03593 | .03515 | .03438 | .03362 | .03288 | .03216 | .03144 | .03074 | .03005 | .02938 |
| -1.7 | .04457 | .04363 | .04272 | .04182 | .04093 | .04006 | .03920 | .03836 | .03754 | .03673 |
| -1.6 | .05480 | .05370 | .05262 | .05155 | .05050 | .04947 | .04846 | .04746 | .04648 | .04551 |
| -1.5 | .06681 | .06552 | .06426 | .06301 | .06178 | .06057 | .05938 | .05821 | .05705 | .05592 |
| -1.4 | .08076 | .07927 | .07780 | .07636 | .07493 | .07353 | .07215 | .07078 | .06944 | .06811 |
| -1.3 | .09680 | .09510 | .09342 | .09176 | .09012 | .08851 | .08691 | .08534 | .08379 | .08226 |
| -1.2 | .11507 | .11314 | .11123 | .10935 | .10749 | .10565 | .10383 | .10204 | .10027 | .09853 |
| -1.1 | .13567 | .13350 | .13136 | .12924 | .12714 | .12507 | .12302 | .12100 | .11900 | .11702 |
| -1.0 | .15866 | .15625 | .15386 | .15151 | .14917 | .14686 | .14457 | .14231 | .14007 | .13786 |
| -0.9 | .18406 | .18141 | .17879 | .17619 | .17361 | .17106 | .16853 | .16602 | .16354 | .16109 |
| -0.8 | .21186 | .20897 | .20611 | .20327 | .20045 | .19766 | .19489 | .19215 | .18943 | .18673 |
| -0.7 | .24196 | .23885 | .23576 | .23270 | .22965 | .22663 | .22363 | .22065 | .21770 | .21476 |
| -0.6 | .27425 | .27093 | .26763 | .26435 | .26109 | .25785 | .25463 | .25143 | .24825 | .24510 |
| -0.5 | .30854 | .30503 | .30153 | .29806 | .29460 | .29116 | .28774 | .28434 | .28096 | .27760 |
| -0.4 | .34458 | .34090 | .33724 | .33360 | .32997 | .32636 | .32276 | .31918 | .31561 | .31207 |
| -0.3 | .38209 | .37828 | .37448 | .37070 | .36693 | .36317 | .35942 | .35569 | .35197 | .34827 |
| -0.2 | .42074 | .41683 | .41294 | .40905 | .40517 | .40129 | .39743 | .39358 | .38974 | .38591 |
| -0.1 | .46017 | .45620 | .45224 | .44828 | .44433 | .44038 | .43644 | .43251 | .42858 | .42465 |
| -0.0 | .50000 | .49601 | .49202 | .48803 | .48405 | .48006 | .47608 | .47210 | .46812 | .46414 |

**STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.**

| Z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | .50000 | .50399 | .50798 | .51197 | .51595 | .51994 | .52392 | .52790 | .53188 | .53586 |
| **0.1** | .53983 | .54380 | .54776 | .55172 | .55567 | .55962 | .56356 | .56749 | .57142 | .57535 |
| **0.2** | .57926 | .58317 | .58706 | .59095 | .59483 | .59871 | .60257 | .60642 | .61026 | .61409 |
| **0.3** | .61791 | .62172 | .62552 | .62930 | .63307 | .63683 | .64058 | .64431 | .64803 | .65173 |
| **0.4** | .65542 | .65910 | .66276 | .66640 | .67003 | .67364 | .67724 | .68082 | .68439 | .68793 |
| **0.5** | .69146 | .69497 | .69847 | .70194 | .70540 | .70884 | .71226 | .71566 | .71904 | .72240 |
| **0.6** | .72575 | .72907 | .73237 | .73565 | .73891 | .74215 | .74537 | .74857 | .75175 | .75490 |
| **0.7** | .75804 | .76115 | .76424 | .76730 | .77035 | .77337 | .77637 | .77935 | .78230 | .78524 |
| **0.8** | .78814 | .79103 | .79389 | .79673 | .79955 | .80234 | .80511 | .80785 | .81057 | .81327 |
| **0.9** | .81594 | .81859 | .82121 | .82381 | .82639 | .82894 | .83147 | .83398 | .83646 | .83891 |
| **1.0** | .84134 | .84375 | .84614 | .84849 | .85083 | .85314 | .85543 | .85769 | .85993 | .86214 |
| **1.1** | .86433 | .86650 | .86864 | .87076 | .87286 | .87493 | .87698 | .87900 | .88100 | .88298 |
| **1.2** | .88493 | .88686 | .88877 | .89065 | .89251 | .89435 | .89617 | .89796 | .89973 | .90147 |
| **1.3** | .90320 | .90490 | .90658 | .90824 | .90988 | .91149 | .91309 | .91466 | .91621 | .91774 |
| **1.4** | .91924 | .92073 | .92220 | .92364 | .92507 | .92647 | .92785 | .92922 | .93056 | .93189 |
| **1.5** | .93319 | .93448 | .93574 | .93699 | .93822 | .93943 | .94062 | .94179 | .94295 | .94408 |
| **1.6** | .94520 | .94630 | .94738 | .94845 | .94950 | .95053 | .95154 | .95254 | .95352 | .95449 |
| **1.7** | .95543 | .95637 | .95728 | .95818 | .95907 | .95994 | .96080 | .96164 | .96246 | .96327 |
| **1.8** | .96407 | .96485 | .96562 | .96638 | .96712 | .96784 | .96856 | .96926 | .96995 | .97062 |
| **1.9** | .97128 | .97193 | .97257 | .97320 | .97381 | .97441 | .97500 | .97558 | .97615 | .97670 |
| **2.0** | .97725 | .97778 | .97831 | .97882 | .97932 | .97982 | .98030 | .98077 | .98124 | .98169 |
| **2.1** | .98214 | .98257 | .98300 | .98341 | .98382 | .98422 | .98461 | .98500 | .98537 | .98574 |
| **2.2** | .98610 | .98645 | .98679 | .98713 | .98745 | .98778 | .98809 | .98840 | .98870 | .98899 |
| **2.3** | .98928 | .98956 | .98983 | .99010 | .99036 | .99061 | .99086 | .99111 | .99134 | .99158 |
| **2.4** | .99180 | .99202 | .99224 | .99245 | .99266 | .99286 | .99305 | .99324 | .99343 | .99361 |
| **2.5** | .99379 | .99396 | .99413 | .99430 | .99446 | .99461 | .99477 | .99492 | .99506 | .99520 |
| **2.6** | .99534 | .99547 | .99560 | .99573 | .99585 | .99598 | .99609 | .99621 | .99632 | .99643 |
| **2.7** | .99653 | .99664 | .99674 | .99683 | .99693 | .99702 | .99711 | .99720 | .99728 | .99736 |
| **2.8** | .99744 | .99752 | .99760 | .99767 | .99774 | .99781 | .99788 | .99795 | .99801 | .99807 |
| **2.9** | .99813 | .99819 | .99825 | .99831 | .99836 | .99841 | .99846 | .99851 | .99856 | .99861 |
| **3.0** | .99865 | .99869 | .99874 | .99878 | .99882 | .99886 | .99889 | .99893 | .99896 | .99900 |
| **3.1** | .99903 | .99906 | .99910 | .99913 | .99916 | .99918 | .99921 | .99924 | .99926 | .99929 |
| **3.2** | .99931 | .99934 | .99936 | .99938 | .99940 | .99942 | .99944 | .99946 | .99948 | .99950 |
| **3.3** | .99952 | .99953 | .99955 | .99957 | .99958 | .99960 | .99961 | .99962 | .99964 | .99965 |
| **3.4** | .99966 | .99968 | .99969 | .99970 | .99971 | .99972 | .99973 | .99974 | .99975 | .99976 |
| **3.5** | .99977 | .99978 | .99978 | .99979 | .99980 | .99981 | .99981 | .99982 | .99983 | .99983 |
| **3.6** | .99984 | .99985 | .99985 | .99986 | .99986 | .99987 | .99987 | .99988 | .99988 | .99989 |
| **3.7** | .99989 | .99990 | .99990 | .99990 | .99991 | .99991 | .99992 | .99992 | .99992 | .99992 |
| **3.8** | .99993 | .99993 | .99993 | .99994 | .99994 | .99994 | .99994 | .99995 | .99995 | .99995 |
| **3.9** | .99995 | .99995 | .99996 | .99996 | .99996 | .99996 | .99996 | .99996 | .99997 | .99997 |