Delta Hedging for Options and Stocks Portfolio

July 21, 2014

1 Introduction

If we assume a simple portfolio with long one option and short n stocks, then as the price of a stock goes up the option price would increase and the short position in stock would become more negative - a delta neutral strategy would imply selling some stock in this scenario(increasing short position). Similarly, if the stock price decreases, the call option price would decrease and delta strategy would imply buying some stock (reducing short position).

1.1 Hedging Portfolio

In this scenario, our initial position is $P_0 - nS_0$ while our position after a price changes becomes $P_t - nS_t$. With delta-strategy, when price increases i.e. $P_t > P_0$, we increase our short position by$x$ shares so that our position becomes - $P_t - (n+x)S_t$.

Therefore, change in position is:

$$P_t - P_0 - n(S_t - S_0) - xS_t = 0 \Rightarrow \Delta P - n\Delta S - xS_t = 0 \Rightarrow x = \frac{\Delta P - n\Delta S}{S_t} \quad (1)$$

Our goal is to keep this change closest to zero as possible for small changes in P. A successful delta-strategy would thus choose x and n accordingly. For small changes, indeed the first-order approximation could work ($\delta_0 = \Delta P/\Delta S$). If we choose delta to be constant for a sufficiently small interval, then the change becomes:

$(\delta_0 - n)\Delta S - xS_t$.

Setting this to zero would give us $x = \frac{(\delta_0 - n)\Delta S}{S_t}$. Clearly, in the case of linear approximation, nothing needs to be done if $n = \delta_0$.

One may also use the Taylor polynomial for better approximations. Using second-order approximations(i.e.γ), for example, one obtains:

$\Delta P = P_t - P_0 = \delta_0 \Delta S + \gamma_0 (\Delta S)^2/2$

Using (1), x can be approximated as :

$$x = \frac{(\delta_0 - n)\Delta S + \gamma_0 (\Delta S)^2/2}{S_t} \tag{2}$$

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In a real-world world scenario, deltas and gammas can hardly be assumed as constant. One often relies on the Black-Scholes model to arrive at delta and gamma values for above analysis. Relieving the assumptions of constant volatility further (e.g. using IVF) could provide still better approximations. What also matters in the real-world is the rebalancing frequency - if there is big jump in stock price (high ΔS) relative to the rebalancing interval, then the equation (1) doesn't provide a reasonable approximation. In such a case, one is better off estimating x by using$P_t - P_0$directly i.e.

$$x = \frac{(P_t - P_0) - n \cdot (\Delta S)}{S_t} \tag{3}$$

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Of course, the assumption here is that we have apriori knowledge price of options and stocks at the rebalancing points. This is an unrealistic assumption but since our purpose here is only to demonstrate the case when ΔP can be estimated accurately, we calculatingΔP assuming that the model predicted options prices are real prices. A real-world delta-strategy would aim to solve the equation by predicting the close-to market values of $\int dS$ and $\int dP$. A simple delta-strategy only looks at short-term variations dP and dS.

$$x = \frac{\int dP - n * \int dS}{S_t}$$

Note that in (2), $\int dP$ is set as $P_t - P_0$because of the assumption stated above.

1.2 Transaction Costs

Transactions in the real world are at a cost. This is a fee based on the volume of the trade - which would affect the PNL as well. Proceeding with the argument similar to above, let our initial position be $P_0 - nS_0$. One the price changes our position becomes $P_t - nS_t$ and the delta-strategy would suggest shorting stocks when the price increases i.e. $P_t > P_0$ i.e. increase the short position by$x$ shares so that the position is - $P_t - (n + x)S_t - |x| \cdot c$.

Therefore, change in position is:

$$P_t - P_0 - n(S_t - S_0) - xS_t - |x| \cdot c = 0 \Rightarrow \Delta P - n\Delta S - xS_t - |x| \cdot c = 0$$

If $x > 0$,

$$x = \frac{\Delta P - n\Delta S}{S_t - c} = \frac{(\delta_0 - n)\Delta S}{S_t + c} \tag{4}$$

whereas if $x < 0$ (buying-shares):

$$x = \frac{\Delta P - n\Delta S}{S_t - c} = \frac{(\delta_0 - n)\Delta S}{S_t - c} \tag{5}$$

Here the approximation$\delta_0 = \frac{\Delta P}{\Delta S}$ was used as before.

1.3 Liquidity concerns

Another complication of the real world comes because of changes in price with increase in trading volumes. If we assume an Amivest ratio of $A = \frac{\partial P}{\partial V}$. Notice that this means we cannot use $\delta_0 = \frac{\Delta P}{\Delta S}$ any more, rather the equation becomes:

$$dP = \frac{\partial P}{\partial S} dS + \frac{\partial P}{\partial V} dV = \delta_0 dS + A_0 dV \tag{6}$$

Increasing the short position by x shares number of traded shares, now results in the following equations:

$$P_t - (n+x)S_t - |x| \cdot c - (P_0 - nS_0) = 0 \Rightarrow P_t - P_0 - n(S_t - S_0) - xS_t - |x| \cdot c = 0$$

$$\Rightarrow \Delta P - n\Delta S = xS_t + |x| \cdot c$$

$\Rightarrow \delta_0 \Delta S + A_0 x - n\Delta S = xS_t + |x| \cdot c \Rightarrow (\delta_0 - n)\Delta S = x(S_t - A_0) + |x| \cdot c$
If $x > 0$,
$x = \frac{(\delta_0 - n)\Delta S}{S_t - A_0 + c}$
and if $x < 0$,
$x = \frac{(\delta_0 - n)\Delta S}{S_t - A_0 - c}$
Clearly, it is no more delta itself that determines the hedge direction.

Simulation

We start with $n = \delta_0$ i.e. the first step of static hedging. Further, we evolve the stock prices using the geometric Brownian process $dS/S = \mu\Delta t + \sigma\epsilon\sqrt{\Delta t}$ and treat them as observed market prices for the analysis. With this assumption in place, the Black-Scholes deltas are market-deltas and hence we expect a suggested delta-neutral strategy to perfectly hedge the total exposure. In other words, if we start with a portfolio of an option and $\delta_0$stocks, then over a period of time T, while a delta-neutral strategy is active, we expect to observe that our PNL is not too different from its original value despite the prices having moved significantly above or below. This is indeed the true intention of a delta-neutral strategy - the performance of which is evaluated by looking at the deviation from original PNL. This performance does seem to get worse with increasing variance of stocks.