Hedging PNL Values for a Derivatives portfolio

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1 Introduction

A simple portfolio with long one option and short n stocks is explored in the current exercise. As the price of a underlying (stock) goes up the option price would increase in the portfolio and the short position in stock would become more negative. A delta neutral strategy applied on the portfolio would imply selling some stock (i.e. increasing short position). Similarly, if the stock price decreases, the call option price would decrease and delta strategy would imply buying some stock (reducing short position).

1.1 Hedging the PNL

Considering the PNL of the hedging portfolio, the initial position is $P_0 - nS_0$ and position after a price changes is $P_t - nS_t$ .When the price increases i.e. $P_t > P_0$, we increase our short position by x shares and our position becomes :$P_t - (n+x)S_t$.

Therefore, change in position is:

$$P_t - P_0 - n(S_t - S_0) - xS_t = 0 \Rightarrow \Delta P - n\Delta S - xS_t = 0 \Rightarrow x = \frac{\Delta P - n\Delta S}{S_t} \tag{1}$$

The purpose is not to keep the portfolio locally-delta-hedged (which would be equivalent to adjusting the portfolio according to the new delta of the option (i.e. $n + x = \frac{\partial P}{\partial S}$). The goal instead is to minimize the PNL volatility i.e. keep the change in PNL closest to zero as possible for small changes in P. A successful hedge would thus choose x and n to minimize the above PNL change. For small changes, the first-order approximation can work ($\delta_0 = \Delta P/\Delta S$). More particularly, if we choose delta to be constant for a sufficiently small interval dt, this change becomes:

$$(\delta_0 - n)\Delta S - xS_t$$

Setting above to zero gives us $x = \frac{(\delta_0 - n)\Delta S}{S_t}$. In this case of linear approximation, nothing needs to be done if $n = \delta_0$.

One may further use the Taylor polynomial for better approximations. Using second-order approximations(i.e.γ), for example, one obtains:

$\Delta P = P_t - P_0 = \delta_0 \Delta S + \gamma_0 (\Delta S)^2/2$

x can thus be approximated as :

$$x = \frac{(\delta_0 - n)\Delta S + \gamma_0(\Delta S)^2/2}{S_t} \tag{2}$$

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In a real-world world scenario, deltas and gammas can hardly be assumed as constant. One often relies on the Black-Scholes model to arrive at delta and gamma values for above analysis. Relieving the assumptions of constant volatility further (e.g. by using an Implied Volatility Function) could provide better approximations. What also matters in the real-world is the rebalancing frequency - if there is big jump in stock price (high ΔS) relative to the rebalancing interval, then the equation (1) doesn't provide a reasonable approximation.

It is a bit like a tailor who has been paid in advance for a suit to be delivered next year for the customer's son who grows exponentially in his teen years. Without knowledge of the height-profile of a typical teen, the tailor can only make guesses based on past measurements. If the customer's son grows really tall in a single year, the tailor would be quite inaccurate about the size of the suit. The higher is the volatility of the teen's height, after all, the more frequently would the suit need to be measured. One year can be too long for the "rebalancing" of the suit-size.

Of course, such a problem would not exist if the tailor knew how much the teen is going to grow in the next year. In our simulations, we could estimate x by using$P_t - P_0$directly i.e. by assuming that we know the price of the derivative as the stock-prices move:

$$x = \frac{(P_t - P_0) - n \cdot (\Delta S)}{S_t} \tag{3}$$

The assumption that we have apriori knowledge price of options and stocks at the rebalancing points is indeed not a realistic one. A real-world delta-strategy would aim to solve the equation by predicting the close-to market values of $\int dS$ and $\int dP$ while a simple delta-strategy only looks at short-term variations dP and dS.

$$x = \frac{\int dP - n * \int dS}{S_t}$$

2 Transaction Costs

Transactions in the real world are always at a cost. This is typically a fee based on the volume of the trade - which affects the PNL as well. Proceeding with the argument similar to above, let our initial position be $P_0 - nS_0$. Once the price changes our position becomes $P_t - nS_t$ and the delta-strategy would suggest shorting stocks when the price increases i.e. $P_t > P_0$ i.e. increase the short position by$x$ shares so that the position is - $P_t - (n + x)S_t - |x| \cdot c$. The modulus operator accounts for the fact that whether we buy or sell shares, we always lose a certain value proportional to the volume in the PNL.

The change in the PNL assuming transaction costs, therefore, becomes:

$$P_t - P_0 - n(S_t - S_0) - xS_t - |x| \cdot c = 0 \Rightarrow \Delta P - n\Delta S - xS_t - |x| \cdot c = 0$$

If $x > 0$ (i.e. we're shorting more shares),

$$x = \frac{\Delta P - n\Delta S}{S_t + c} = \frac{(\delta_0 - n)\Delta S}{S_t + c} \tag{4}$$

whereas if $x < 0$ (i.e. we're buying-shares or reducing our short-position):

$$x = \frac{\Delta P - n\Delta S}{S_t - c} = \frac{(\delta_0 - n)\Delta S}{S_t - c} \tag{5}$$

Here the approximation$\delta_0 = \frac{\Delta P}{\Delta S}$ was used as before[1].

3 Liquidity concerns

Another complication of the real world comes because of changes in price of the underlier with increase in trading volumes. If we assume an Amivest ratio of $\alpha = \frac{\partial S}{\partial V}$, then selling x shares at $S_t$with Amivest ratio α would reduce the price of the underlying to $S_t - dS = S_t - \alpha x$. The net PNL after selling would therefore change from $P_0 - nS_0$to $P_t - (n+x)(S_t - \alpha x)$. The change (which needs to be close to zero) would therefore become: $P_t - (nS_t - n\alpha x + S_t x - \alpha x^2) - P_0 + nS_0 = \Delta P - n\Delta S + n\alpha x - S_t x + \alpha x^2$. Using $\delta_0 = \frac{\partial P}{\partial S}$,we can solve for x in the following quadratic equation:

$$\alpha x^2 + (n\alpha - S_t)x + (\delta_0 - n)\Delta S = 0 \tag{6}$$

This leads to the solution $x = \frac{(S_t - n\alpha) \pm \sqrt{(S_t - n\alpha)^2 - 4\alpha(\delta_0 - n)\Delta S}}{2\alpha}$

4 A generalized model

In a model with both liquidity and transaction costs, the PNL change would become:
$\Delta PNL = P_t - (nS_t - n\alpha x + S_t x - \alpha x^2) - P_0 + nS_0 - |x|c$
Setting the above to zero leads to:

$$\Delta P - n\Delta S + n\alpha x - S_t x - |x|c + \alpha x^2 = 0 \tag{7}$$

If $x > 0$, we have $\alpha x^2 + (n\alpha - S_t - c)x + (\delta_0 - n)\Delta S = 0$ and
$x = \frac{(S_t - n\alpha - c) \pm \sqrt{(S_t - n\alpha - c)^2 - 4\alpha(\delta_0 - n)\Delta S}}{2\alpha}$
If $x < 0$, we have $\alpha x^2 + (n\alpha - S_t + c)x + (\delta_0 - n)\Delta S = 0$ and
$x = \frac{(S_t - n\alpha + c) \pm \sqrt{(S_t - n\alpha + c)^2 - 4\alpha(\delta_0 - n)\Delta S}}{2\alpha}$

5 Simulation

We start with $n = \delta_0$ i.e. the first step of static hedging. Further, we evolve the stock prices using the geometric Brownian process $dS/S = \mu\Delta t + \sigma\epsilon\sqrt{\Delta t}$ and treat them as observed market prices for the analysis. With this assumption in place, the Black-Scholes deltas are

[1] Notice that the fluctuations in the x are higher than the no-transaction costs case and thus, the net PNL would be far more sensitive to the hedge-frequency (i.e. the time when x shares are added/reduced).

market-deltas and hence we expect a suggested delta-neutral strategy to perfectly hedge the total exposure. In other words, if we start with a portfolio of an option and $\delta_0$stocks, then over a period of time T, while a delta-neutral strategy is active, we expect to observe that our PNL is not too different from its original value despite the prices having moved significantly above or below. This is indeed the true intention of a delta-neutral strategy - the performance of which is evaluated by looking at the deviation from original PNL. This performance does seem to get worse with increasing variance of stocks.