1 Hedging Portfolio in a Black-Scholes economy

1.1 Self-financing portfolio

The well-known Black-Scholes equation is obtained with a self-financing tracking portfolio of a bond and a stock - a combination that replicates the payoff of an option. Since value of $V = \alpha B + \Delta S$ cannot change except by changing weights α, Δ - the adjustments of dV at every increment dS leads us to the Black-Scholes equation.

$dV = \alpha dB + \Delta dS = \alpha rBdt + \Delta(\mu Sdt + \sigma SdW) \Rightarrow$

$$dV = (\alpha rB + \mu\Delta S)dt + \sigma\Delta SdW \tag{1}$$

The Ito's lemma provides a differential equation for dV:
$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}(\mu Sdt + \sigma SdW) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt$

$$dV = (\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2)dt + \sigma S\frac{\partial V}{\partial S}dW \tag{2}$$

Equating (1) and (2): $\alpha = \frac{V_t + V_{SS}\sigma^2 S^2}{rB}$ and hence $rV = r\alpha B + r\Delta S$ i.e.

$$rV = \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + rS\frac{\partial V}{\partial S} \tag{3}$$

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1.2 Hedge Portfolio in the BS-economy

Black-Scholes equation also implies the risk-free growth of the hedge portfolio. Consider the hedge-portfolio as $\Pi = V - \Delta S$. This would in turn imply:$d\Pi = dV - \Delta S$.

Using (2), we obtain:$d\Pi = dV - \Delta dS = (\frac{\partial V}{\partial t} + \mu S(\frac{\partial V}{\partial S} - \Delta) + \frac{\partial^2 V}{\partial S^2}\sigma^2 S^2)dt + (\sigma S\frac{\partial V}{\partial S} - \Delta)dW$. Setting local risk-lessness results in $\Delta = \frac{\partial V}{\partial S}$. Thus, $d\Pi = \Pi rdt \Rightarrow (V - \frac{\partial V}{\partial S}S)r = \frac{\partial V}{\partial t} + \frac{\partial^2 V}{\partial S^2}\sigma^2 S^2$.

Following no-arbitrage arguments, the hedge-portfolio must grow at the risk-free rate : $d\Pi_t = \Pi_t rdt$. What it means for the current exercise is that a delta-neutral strategy (which sets $\Delta = \frac{\partial V}{\partial S}$) would result in growth of hedging-portfolio as$\Pi_t = \Pi_0 e^{-rT}$. This can be verified with simulations.

1.3 Minimizing Volatility

The best way to minimize the volatility of $d\Pi$ in the B-S economy is keeping he coefficient of dW as zero. This is exactly what a delta-neutral hedging accomplishes. What it cannot guarantee, however, is minimizing the volatility of Π_t.

Using directly formulas, the delta-neutral strategy would have the following payoff: $\Pi_t = P_t - N(d_1)S_t = Ke^{-rT}N(d_2)$. The volatility of Π_t is therefore ought to be the same as that of $N(d_2)$ (this can also be experimentally verified). To minimize the volatility of Π_t, let's assume that we deviate from the delta-neutral hedging in a way that we maintain $\Delta_t + \epsilon_t = N(d_1) + \epsilon_t$stocks instead of $\Delta_t = \frac{\partial P}{\partial S}$ stocks. This would lead to the hedging portfolio value move as $\overline{\Pi_t} = P_t - N(d_1)S_t - \epsilon_t S_t = \Pi_t - \epsilon_t S_t = Ke^{-rT}N(d_2) - \epsilon_t S_t$.

Modifying the delta-hedging methods and evaluating performance of such a strategy (that doesn't track delta aggressively) is commented upon as part of this exercise. The effect of this deviation on the volatility of $\Pi_t$is computed through simulations as well.