Delta Hedging for Options and Stocks Portfolio

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Introduction

If we assume a simple portfolio with long one option and short n stocks, then as the price of a stock goes up the option price would increase and the short position in stock would become more negative - a delta neutral strategy would imply selling some stock in this scenario(increasing short position). Similarly, if the stock price decreases, the call option price would decrease and delta strategy would imply buying some stock (reducing short position).

In this scenario, our initial position is $P_0 - nS_0$ while our position after a price changes becomes $P_t - nS_t$. With delta-strategy, when price increases i.e. $P_t > P_0$, we increase our short position by$x$ shares so that our position becomes - $P_t - (n + x)S_t$.

Therefore, change in position is:

$$P_t - P_0 - n(S_t - S_0) - xS_t = 0 \Rightarrow \Delta P - n\Delta S - xS_t = 0 \Rightarrow x = \frac{\Delta P - n\Delta S}{S_t} \quad (1)$$

Our goal is to keep this change closest to zero as possible for small changes in P. A successful delta-strategy would thus choose x and n accordingly. For small changes, indeed the first-order approximation could work ($\delta_0 = \Delta P/\Delta S$

). If we choose delta to be constant for a sufficiently small interval, then the change becomes:

$(\delta_0 - n)\Delta S - xS_t$.

Setting this to zero would give us $x = \frac{(\delta_0 - n)\Delta S}{S_t}$. Clearly, in the case of linear approximation, nothing needs to be done if $n = \delta_0$.

One may also use the Taylor polynomial for better approximations. Using second-order approximations(i.e.γ), for example, one obtains:

$\Delta P = P_t - P_0 = \delta_0 \Delta S + \gamma_0 (\Delta S)^2/2$

Using (1), x can be approximated as :

$$x = \frac{(\delta_0 - n)\Delta S + \gamma_0 (\Delta S)^2/2}{S_t} \tag{2}$$

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In a real-world world scenario, deltas and gammas can hardly be assumed as constant. One often relies on the Black-Scholes model to arrive at delta and gamma values for above analysis. Relieving the assumptions of constant volatility further (e.g. using IVF) could provide still better approximations. What also matters in the real-world is the rebalancing frequency - if there is big jump in stock price (high ΔS) relative to the rebalancing interval, then the equation (1) doesn't provide a reasonable approximation. In such a case, one is better off estimating x by using$P_t - P_0$directly i.e.

$$x = \frac{(P_t - P_0) - n \cdot (\Delta S)}{S_t} \tag{3}$$

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Of course, the assumption here is that we have apriori knowledge price of options and stocks at the rebalancing points. This is an unrealistic assumption but since our purpose here is only to demonstrate the case when ΔP can be estimated accurately, we calculatingΔP assuming that the model predicted options prices are real prices. A real-world delta-strategy would aim to solve the equation by predicting the close-to market values of $\int dS$ and $\int dP$. A simple delta-strategy only looks at short-term variations dP and dS.

$$x = \frac{\int dP - n * \int dS}{S_t}$$

Note that in (2), $\int dP$ is set as $P_t - P_0$because of the assumption stated above.

Simulation

We start with $n = \delta_0$ i.e. the first step of static hedging. Further, we evolve the stock prices using the geometric Brownian process $dS/S = \mu\Delta t + \sigma\epsilon\sqrt{\Delta t}$ and treat them as observed market prices for the analysis. With this assumption in place, the Black-Scholes deltas are market-deltas and hence we expect a suggested delta-neutral strategy to perfectly hedge the total exposure. In other words, if we start with a portfolio of an option and $\delta_0$stocks, then over a period of time T, while a delta-neutral strategy is active, we expect to observe that our PNL is not too different from its original value despite the prices having moved significantly above or below. This is indeed the true intention of a delta-neutral strategy - the performance of which is evaluated by looking at the deviation from original PNL. This performance does seem to get worse with increasing variance of stocks.