1 Simulating Bounces in a Tunnel

1.1 Random Number Generator

For exponetially distributed η, the CDF is $F_\eta(x) = P(\eta \leq x) = 1 - e^{-\lambda x} = 1 - e^{-\frac{x}{\bar{\eta}}}$. Using inverse-transform, $f(x) = -\bar{\eta} \cdot ln(1-x)$ for $x \in U[0,1]$ gives exponentially distributed η with mean and stdev$\bar{\eta}$. The uniform random numbers are generated in the program using a Marsaglia's MWC generator.

1.2 Computing Average Exit Time

The horizontal velocity $vcos\alpha$faces no decay so the time of exit must be exactly $t_{exit} = \frac{L}{vcos\alpha}$. This only depends on α and a one-dimensional integration (sampling) suffices the evaluation of $E(t_{exit})$.

1.3 Computing Average Number of Bounces

A simulation of bounces is performed until the time the ball exits the tunnel. Time is incremented in units of bounce-times (i.e. time spent until ball covers the distance h with constant velocity). The total time elapsed until bounce i would thus be $\frac{h}{vsin\alpha}(1+(1+\eta_1)+(1+\eta_1)(1+\eta_2)+...)$. Here $\eta_j$is the j^{th}draw of the exponentially distributed η and the vertical velocity is multiplied by $1-\frac{\eta_j}{1+\eta_j}$at every bounce. η_j is modeled as an exponentially distributed random-variable with mean $\bar{\eta}$. The program thus calculates the time at every bounce until it exceeds $t_{exit} = \frac{L}{vcos\alpha}$ to calculate $E(N)$ across η_i-space for every sampled$\alpha \in [0, \frac{\pi}{4}]$.

1.4 Computing Variance of results

Running average of $t_{exit}, t_{exit}^2, N$and $N^2$are maintained to calculated $\sigma_{t_{exit}}$and σ_N. No result-vectors are stored for optimization purposes.

1.5 Theoretical Estimates and Simulation Results

1.5.1 Exit Time

Since exit-time depends only on distribution ofα, it is simply the expectation of t_{exit}over α.

$E(t_{exit}) = \int_0^{\pi/4} \frac{L}{cos\alpha} f(\alpha) d\alpha$where $f(\alpha) = \frac{4}{\pi}$. Thus, $E(t_{exit}) = \int_0^{\pi/4} \frac{4L}{\pi cos\alpha} d\alpha \sim \mathbf{112.22}$ (used for verification).

1.5.2 Number of Bounces

The number of bounces N is the lowest i such that $\frac{h}{sin\alpha}(1+(1+\eta_1)+(1+\eta_1)(1+\eta_2)+...+(1+\eta_1)(1+\eta_2)...(1+\eta_i) > \frac{L}{cos\alpha}$. In other words, $N = inf\{i \in \mathbb{N} : \frac{h}{sin\alpha}(1+(1+\eta_1)+(1+\eta_1)(1+\eta_2)+...+(1+\eta_1)(1+\eta_2)...(1+\eta_i) > \frac{L}{cos\alpha}\}$

If we let $S_i = 1 + (1+\eta_1) + (1+\eta_1)(1+\eta_2) + ... + (1+\eta_1)(1+\eta_2)...(1+\eta_i)$, then

$$N = inf\{i \in \mathbb{N} : S_i \geq \frac{Ltan\alpha}{h}\}$$

Alternately, we can formulate N as a state-model where$n$ denotes the number of bounces. The expectation would be $\Sigma_0 i\phi_i$where,

$\phi_0 = P(N=0) = P(exit|n=0) \cdot P(n=0) = P(1 \geq \frac{Ltan\alpha}{h})$

$\phi_1 = P(N=1) = P(exit|n=1)P(n=1)$=$P(exit|n=1) \cdot P(n=1|n=0) = P(exit|n=1)(1-\phi_0)$

$\Rightarrow \phi_1 = P(S_2 \geq \frac{Ltan\alpha}{h}|S_1 \leq \frac{Ltan\alpha}{h})(1-\phi_0)$

$\phi_2 = P(N=2) = P(exit|n=2)P(n=2) = P(exit|n=2)P(n=2|n=1)P(n=1|n=0)$= $P(S_3 \geq \frac{Ltan\alpha}{h}) \cdot P(S_2 \leq \frac{Ltan\alpha}{h}|S_1 \leq \frac{Ltan\alpha}{h}) \cdot (1-\phi_0)$=$P(S_3 \geq \frac{Ltan\alpha}{h}|S_1, S_2 \leq \frac{Ltan\alpha}{h}) \cdot (1 - \frac{\phi_1}{1-\phi_0})(1-\phi_0)$

$\Rightarrow \phi_2 = P(S_3 \geq \frac{Ltan\alpha}{h}|S_1, S_2 \leq \frac{Ltan\alpha}{h})(1-\phi_0-\phi_1)$.

This is not a Markov-chain since $S_i$and ϕ_i depend on past values. More intuitively, as there are multiple ways to reach a certain number of bounces n, the probability of whether the next bounce would occur in the tunnel or not depends on previous states (not just the current number of bounces).

However, since we are sure that the vertical velocity only decreases in magnitude, we can be assured that the number of bounces is always bounded by the case when $\eta_i = 0 \forall i \in \mathbb{N}$.

S_i is not a martingale ($E(S_{i+1} - S_i|\eta_1, \eta_2, \eta_3..) = (1+\eta_1)(1+\eta_2)...(1+\eta_i) \cdot E(1+\eta_{i+1}|\eta_1, \eta_2, \eta_3..) = 1+\bar{\eta} \geq 0$) - the average N is bounded by the case when $\eta_i = 0 \forall i \in \mathbb{N}$ where $E(N)$ is approximated by $\int_0^{\frac{\pi}{4}} \frac{4}{\pi} floor(\frac{Ltan\alpha}{h}) d\alpha \sim$ 8.8(verified with results). $E(N)$ with exponentially distributed $\eta_i(\bar{\eta} = 0.15)$is found to be $\mathbf{5.1}$.