Replication Algorithm for Variance Swap Pricing

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1 Summary

The document describes the details of pricing a swap based on a replication-portfolio comprising of a log-contract and a stock or future position. The replication determines the price based on the volatilities across the skew while maintaining a constant dollar gamma for the portfolio. The main inputs to the model are

1. Stock prices or Futures Prices
2. Call options with strikes above and below a certain threshold close to the initial stock price S_0
3. Put options with strikes above and below a certain threshold close to the initial futures price F_0

The stochastic volatility models are currently out of scope for the current document.

2 Black Scholes Model

The basic idea for pricing is to construct a portfolio of European options weighted as the inverse strike squared gives a constant exposure to volatility. Notice that at each hedging interval, the PNL is given as follows for gamma Γ_i, the return R_i of the underlying, the variance $\sigma^2$and the time-step size δt

$$\frac{1}{2}\sum_i \Gamma_i S_i^2 (R_i^2 - \sigma^2 \delta t)$$

Notice also that $\theta = -\frac{1}{2}\Gamma S^2 \sigma^2$. A portfolio proportional to $\frac{1}{S_i}$ would evidently keep a constant second-derivative i.e. gamma Γ ($\propto \frac{1}{S_i^2}$) . More generally, we can say that the payoff $alnS + bS + c$ would have a constant Γ. A variance swap can thus be replicated with a log contract with options which are then delta-hedged. The following expression summarise how a combination of puts and calls approximates the payoff $alnS + bS + c$ (with price of underlying$S_T$at time T).

$$\int_0^{S_0} \frac{(K - S_T)^+}{K^2} dK + \int_{S_0}^{\infty} \frac{(S_T - K)^+}{K^2} dK = \frac{S_T}{S_0} - ln(\frac{S_T}{S_0}) \tag{1}$$

The replication can also be derived in terms of Stock prices using Black-Scholes assumptions (see Section 2.2).

2.1 Futures-based replication

We now discuss how the replication summarised with Equation 1 can be performed using futures prices F_t (at time t) and F_0. As discussed in [1, 2], the average variance over time $\frac{1}{T}\int_0^T \sigma_t dt$ can be written as

$$\frac{1}{T}\int_0^T \sigma_t dt = \frac{2}{T}(log(\frac{F_0}{F_t}) + \frac{F_t}{F_0} - 1) - \frac{2}{T}\int_0^T (\frac{1}{F_0} - \frac{1}{F_t})dF_t \tag{2}$$

The first-term can be replicated by taking a static position in a contract that pays $\frac{2}{T}(log(\frac{F_0}{F_t}) + \frac{F_t}{F_0} - 1)$ and the second-term can be replicated by a continuously rebalanced futures position. As detailed in [2], this can be performed using calls and puts with a position $\frac{1}{K^2}$ using the following expression

$$log(\frac{F_0}{F_T}) + \frac{F_T}{F_0} - 1) = \int_0^{F_0} \frac{(K - F_T)^+}{K^2} dK + \int_{F_0}^{\infty} \frac{(F_T - K)^+}{K^2} dK$$

Plugging this in Equation 2, we have

$$\frac{1}{T}\int_0^T \sigma_t dt = \frac{2}{T}(\int_0^{F_0} \frac{(K - F_T)^+}{K^2} dK + \int_{F_0}^{\infty} \frac{(F_T - K)^+}{K^2} dK) - \frac{2}{T}\int_0^T (\frac{1}{F_0} - \frac{1}{F_t})dF_t$$

The Black-Scholes assumptions allow us to replicate a variance swap by holding $\frac{1}{F_t} - \frac{1}{F_0}$ unites of futures at t and a set of calls and puts spread across futures price F_0. The forward cost of this portfolio is can be written as follows

$$K_{VAR}^2 = \frac{2e^{rT}}{T}(\int_0^{F_0} \frac{P_0(K)}{K^2} dK + \int_{F_0}^{\infty} \frac{C_0(K)}{K^2} dK)$$

Assuming a set of equally spaced strikes $K_0, ..., K_r, K_{r+1}, ..., K_n$ where $K_0, K_1, .., K_r$ are strikes for puts and $K_{r+1}, K_{r+2}, ..., K_n$ are strikes for calls. Since the options are equally-spaced we have $\Delta_K = K_{i+1} - K_i$ for all $i \in [1, n-1]$. The replication portfolio that approximates $\int_0^{F_0} \frac{P_0(K)}{K^2} dK + \int_{F_0}^{\infty} \frac{C_0(K)}{K^2} dK$ is

$$\sum_{i=0}^{r} \frac{\Delta_K P(K_i)}{K_i^2} + \sum_{i=r+1}^{n} \frac{\Delta_K C(K_i)}{K_i^2}$$

For a given variance notional exposure N_{var}, one must hold $\frac{2\Delta_K N_{var}}{TK^2}$ of each of option.

2.2 Stocks-based replication

The Black-Scholes assumptions readily lead to a replciation algorithm that is now described (see [3]). Notice first that $d(log\ S_t) = (\mu - \frac{\sigma^2}{2})dt + \sigma dZ_t$ (for normal variable Z_t) leads to $\frac{1}{2}\sigma^2 dt = \frac{dS_t}{S_t} - d(log\ S_t)$. This in turn implies that

$$\frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}(\int_0^T \frac{dS_t}{S_t} - log(\frac{S_T}{S_0})) \tag{3}$$

Using $E(\int_0^T \frac{dS_t}{S_t}) = rT$ and a boundary S_* between calls and puts, we make use of the following identity (see Derman)

$$log(\frac{S_T}{S_0}) = log(\frac{S_*}{S_0}) + log(\frac{S_T}{S_*})$$

$$= log(\frac{S_*}{S_0}) - (\int_0^{S_*} \frac{(K-S_T)^+}{K^2} + \int_{S_*}^{\infty} \frac{(S_T-K)^+}{K^2} - \frac{S_T - S_*}{S_*})$$

The Equation 3 becomes

$$E(\int_0^T \sigma_t^2 dt) = \frac{2}{T}(rT - (\frac{S_0}{S_*}e^{rT} - 1) - log(\frac{S_*}{S_0}) + e^{rT}\int_0^{S*} \frac{1}{K^2}P(K)dK + e^{rT}\int_{S_*}^{\infty} \frac{1}{K^2}C(K)dK$$

Notice that $K_{Var} \equiv E(\int_0^T \sigma_t^2 dt)$ is called the strike of the variance swap..
We can rewrite K_{var} as

$$K_{var} = \frac{2}{T}(rT - (\frac{S_0}{S_*}e^{rT} - 1) - log(\frac{S_*}{S_0})) + e^{rT}\Pi_{CP}$$

The term Π_{CP} is the following value of the portfolio of puts and calls where the weights of the portfolio are calculated using a recursive algorithm described in Section 2.2.1

$$\Pi_{CP} = \sum_i w(K_{ip})P(S, K_{ip}) + \sum_i w(K_{ic})C(S, K_{ic}) \tag{4}$$

2.2.1 Algorithm to replicate Log-contract

With the portfolio in Equation 4, let's consider calls with strikes $K_{1c,}K_{2c}$ etc. such that $S_* < K_{1c} < K_{2c} < ...$ and puts K_{1p}, K_{2p}etc. such that $S_* > K_{1c} > K_{2c}...$ - where S^* is the boundary between calls and puts. We can set this to initial value of the stock i.e. $S_* = S_0$.

Let's consider the strike K_0 at the boundary (reference) stock-price S_*. The weight associated with the option struck at K_0 would be set to

$$w_c(K_0) = \frac{f(K_{1c}) - f(K_0)}{K_{1c} - K_0}$$

Here, $f(S_T)$ is defined as

$$f(S_T) = \frac{2}{T}(\frac{S_T - S_*}{S_*} - log(\frac{S_T}{S_*}))$$

The (units) weights for calls in the portfolio are given by

$$w_c(K_{n,c}) = \frac{f(K_{n+1,c}) - f(K_{n,c})}{K_{n,c} - K_{n+1,c}} - \sum_{i=0}^{n-1} w_c(K_{i,c})$$

Similarly, we can obtain the weights associated with puts by first setting

$$w_p(K_0) = \frac{f(K_{1p}) - f(K_0)}{K_{1p} - K_0}$$

and then holding other weights of puts determined by

$$w_p(K_{n,p}) = \frac{f(K_{n+1,p}) - f(K_{n,p})}{K_{n,p} - K_{n+1,p}} - \sum_{i=0}^{n-1} w_p(K_{i,p})$$

2.3 Greeks

The greeks relevant for variance swaps can also be calculated using the dynamics of Equation 1.

References

[1] P. Allen, S. Einchcomb, and N. Granger, "Variance swaps," *London: JP Morgan (November)*, 2006.

[2] P. Carr and D. Madan, "Towards a theory of volatility trading," *Volatility: New estimation techniques for pricing derivatives*, vol. 29, pp. 417–427, 1998.

[3] K. Demeterfi, E. Derman, M. Kamal, and J. Zou, "More than you ever wanted to know about volatility swaps," *Goldman Sachs quantitative strategies research notes*, vol. 41, pp. 1–56, 1999.