Metric Geometry
Midterm Assignment

Due: October 14, 2013 at 2:00 p.m.

Attempt all questions. Each part of each question is worth 10 points. You may consult notes and references as well as the instructor, but **do not collaborate or discuss with anyone else**.

1. Let X be a set and $d : X \times X \to \mathbb{R}$ be a function satisfying

   (a) For x, y in X, $d(x, y) \geq 0$.

   (b) For x in X, $d(x, x) = 0$.

   (c) For x, y in X, $d(x, y) = d(y, x)$.

   (d) For x, y, z in X, $d(x, y) + d(y, z) \geq d(x, z)$.

   Show that the relation $\sim$ on X given by $x \sim y$ if and only if $d(x, y) = 0$ is an equivalence relation. Further, show that we have a well-defined distance on $\bar{X} = X/\sim$ given by

   $$\bar{d}([x], [y]) = d(x, y)$$

   and that $(\bar{X}, \bar{d})$ is a metric space.

Henceforth, let $U \subset \mathbb{R}^2$ be an open set with $(0, 0) \in U$ and let $\langle \cdot, \cdot \rangle_p = g_p(\cdot, \cdot)$ be a Riemannian metric on U. Recall that this means that g_p gives an inner product on $T_pU = \mathbb{R}^2$ at each $p \in U$, varying smoothly in p.

Let ∇ be the corresponding connection and let $\frac{D}{Ds}$ denote the corresponding covariant derivative as usual. You may assume the following lemmas.

Lemma 1. *Suppose $f : U \to \mathbb{R}^n$ is a smooth function with $f(0, 0) = 0$. Then there are smooth functions $\varphi : U \to \mathbb{R}^n$ and $\psi : U \to \mathbb{R}^n$ so that*

$$f(x, y) = x\varphi(x, y) + y\psi(x, y)$$

for all $(x, y) \in U$.

Lemma 2. *Suppose $f : U \to \mathbb{R}^n$ is a smooth function with $f(x, 0) = 0$ for all $(x, 0) \in U$. Then there is a smooth function $\psi : U \to \mathbb{R}^n$ so that*

$$f(x, y) = y\psi(x, y)$$

for $(x, y) \in U$.

2. Show that if X is a vector field on U with $X(0, 0) = 0$ and $Y : U \to \mathbb{R}^2$ is a smooth vector field on U, then

$$(\nabla_X Y)(0, 0) = 0.$$

3. Deduce that if X_1 and X_2 are vector fields on U with $X_1(0, 0) = X_2(0, 0)$ and Y is a smooth vector field on U, then

$$(\nabla_{X_1} Y)(0, 0) = (\nabla_{X_2} Y)(0, 0).$$

4. Let $X = \hat{\imath}$ be the vector field on U given by the unit vector $\hat{\imath}$ in the positive x-direction and let Y be a vector field on U such that $Y(x, 0) = 0$ for $(x, 0) \in U$. Show that

$$(\nabla_X Y)(x, 0) = 0$$

for $(x, 0) \in U$.

5. Let $\alpha : (-1, 1) \to U$ be a smooth curve with $\alpha(0) = (0, 0)$ and $\dot{\alpha}(0) \neq 0$.

   (a) Show that there is an $\epsilon > 0$, an open set $W \subset U$ and a diffeomorphism $\Phi : W \to V$ with $V \subset \mathbb{R}^2$ open so that for all $s \in (-\epsilon, \epsilon)$, $\alpha(s) \in W$ and $\Phi(\alpha(s)) = (s, 0)$.

   (b) Deduce that if $Y : (-1, 1) \to \mathbb{R}^2$ is a smooth vector field along α, then there is a smooth vector field Z on W so that for all $s \in (-\epsilon, \epsilon)$, $Y(s) = Z(\alpha(s))$.

   (c) If Z is a vector field on W so that for all $s \in (-\epsilon, \epsilon)$, $Z(\alpha(s)) = 0$, show that there is a smooth function $\lambda : W \to \mathbb{R}$ with $\lambda(\alpha(s)) = 0$ for all $s \in (-\epsilon, \epsilon)$, and a vector field Z' on W so that

$$Z(p) = \lambda(p)Z'(p) \; \forall p \in W.$$

   (d) Deduce that if Z is a vector field on W so that for all $s \in (-\epsilon, \epsilon)$, $Z(\alpha(s)) = 0$, then

$$(\nabla_{\dot{\alpha}(0)} Z)(0, 0) = 0.$$

(e) Deduce that if $Y : (-1, 1) \to \mathbb{R}^2$ is a smooth vector field along α and Z_1 and Z_2 are vector fields on W so that for $s \in (-\epsilon, \epsilon)$, $Y(s) = Z_1(\alpha(s)) = Z_2(\alpha(s))$, then

$$(\nabla_{\dot{\alpha}(0)} Z_1)(0, 0) = (\nabla_{\dot{\alpha}(0)} Z_2)(0, 0).$$

Remark. It follows that the covariant derivative $\frac{D}{Ds}Y(0)$ is well-defined if $\dot{\alpha}(0) \neq 0$. If $\dot{\alpha}(0) = 0$, by definition $\frac{D}{Ds}Y(0) = 0$.