MTH 393A
Additive Combinatorics and Incidence Geometry: The Kakeya Problem

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About the Project

This project, done under the supervision of Prof. Nitin Saxena, Dept. of CSE and Prof. Shobha Madan, Dept. of MTH is an exploration of the Kakeya Problem in Incidence Geometry by using methods from the field of Additive Combinatorics. The presentations for this project were also attended by Mr. Vijay Keswani, an undergraduate in the Dept. of CSE and Prof. Rajat Mittal, Dept. of CSE.

We consider the Kakeya Problem both in the reals ($\mathbb{R}^n$) and in finite fields ($\mathbb{F}^n$). Both our report and this presentation following [1] quite closely.

Notation

For us, $f \ll g$, $g \gg f$ and $f = \mathcal{O}(g)$ all mean the same thing, viz.

$$\exists C > 0, f \leq Cg$$

The implicit constant may sometimes depend on some quantities, in which case those quantities will be in the subscript, such as $f \ll_\epsilon g$ to mean $f \leq C_\epsilon g$.
We also use the non-standard notation $f \sim g$ for when $f \ll g$ and $f \gg g$ simultaneously.

Notation: Additive Combinatorics

We will be using Additive Combinatorics throughout this presentation. Thus, we take this opportunity to present some notation. If $(G, +)$ is an additively presented commutative group, and $A, B \subset G$ are arbitrary subsets. Then the sumset $A + B$ is defined as follows:

$$A + B = \{a + b : a \in A, b \in B\}$$

This brings us to the most basic inequality we have, which is

$$|A + B| \leq |A||B|$$

The Kakeya Problem: Essential Statment

Put in words, the Kakeya Problem requires the notion of a Kakeya set, which is essentially the following: given a geometry (in particular, we only work with $\mathbb{R}^n$ and $\mathbb{F}^n$ with a well-defined notion of "lines" and a well-defined notion of "direction(s)" of a line, a Kakeya set is a set which has a "line" in every "direction". With this definition, and a given suitable notion of "size" of a set, the Kakeya problem then becomes the problem of how "big" a Kakeya set must be. That is, the problem seeks to lower bound the "size" of a Kakeya set.

In the real setting, the notion of "line" is taken to be a unit line segment, and the notion of direction is taken to be the colloquial direction (that is, a unit vector). The definition is, thus, as follows:

Definition (Kakeya sets in $\mathbb{R}^n$)

A *Kakeya set* $K \subset \mathbb{R}^n$ is a compact set such that for every $x \in S^{n-1}$, there exists a $y \in K$ such that

$$l = \{y + tx : t \in [0, 1]\} \subset K$$

To state a cogent Kakeya problem, we now need a suitable notion for size. It should be quite obvious that cardinality is a useless measure of size in this case - all Kakeya sets are uncountable.

A standard denotation of size in analysis is measure: more specifically the Lesbesge measure or the Jordan-Riemann measure $\mu(A)$ of a subset of $\mathbb{R}^n$ which is colloquially known as distance (for $n = 1$), area (for $n = 2$), volume (for $n = 3$) and hypervolume (for general n).

However, it was shown by Besicovitch in 1928 [2] that there exists a Kakeya set in $\mathbb{R}^n$ which has Lesbesgue measure equal to zero - the Lesbesgue measure is thus a useless way to lower bound the size of a Kakeya set. This is the reason why a real Kakeya set is also called a Besicovitch set.

Dimension

The suitable notion of "size" we will use will be a notion of dimension - in particular the fractal dimension known as the upper Minkowski dimension.

Definition (Upper Minkowski Dimension)

For any bounded set K, suppose $B_\epsilon(K)$ is the minimal number of balls of radius ϵ using which K can be completely covered (this number exists since K is completely contained in a compact set). The dimension of K, or more properly, the *Upper Minkowski Dimension* of K is given by

$$\dim K = \limsup_{\epsilon \to 0} \frac{B_\epsilon(K)}{\log(1/\epsilon)}$$

Loosely speaking, if $\dim K \leq d$, then something like $\sim (1/\epsilon)^d$ balls of radius ϵ will be needed to cover K.

Properties of Dimension

We would like to see if this definition of dimension agrees with our intuitive notion of dimension.

Property (Monotonicity)

Suppose $L \subset K \subset \mathbb{R}^n$ *such that* K *and* L *are both bounded. Then,*

$$\dim L \leq \dim K$$

Property (Upper Bound)

Suppose $K \subset \mathbb{R}^n$ *is a bounded set. Then,*

$$\dim K \leq n$$

Properties of Dimension

Property (Dimension of sets of non-zero measure)

Let $K \subset \mathbb{R}^n$ be a bounded subset having positive Lesbesgue measure $\mu(K) > 0$. Then,

$$\dim K = n$$

Property (Affine Dimension)

Let A be an affine subspace of $\mathbb{R}^n$ having affine dimension k. Then, for any ball B having a non-trivial intersection with A we have that

$$\dim A \cap B = k$$

This last property essentially shows that the "affine" concept of dimension gives the same value of dimension as the upper Minkowski dimension.

Properties of Dimension

Property (Tensoring)

Let K be a bounded subset of $\mathbb{R}^n$. It then follows that K^t is a bounded subset of $\mathbb{R}^{nt}$ under some suitable canonical maps. Thus, we have

$$\dim K^t = t \dim K$$

This is a fairly useful property here.

We can now state the Kakeya conjecture in the real setting.

Conjecture (Kakeya problem for the reals)

Let $K \subset \mathbb{R}^n$ be a Kakeya set, then we have

$$\dim K = n$$

where $\dim K$ is the upper Minkowski dimension (hereafter referred as dimension) of K.

What is known?

The Kakeya conjecture is completely settled in the case $n = 2$, that is, in the Euclidean plane.

For general values of n, several lower bounds (each progressively better) are known. In particular, it is known that if $K \subset \mathbb{R}^n$ is a Kakeya set, then $\dim K$ is at least $(1/2)n$. This can also be improved to $(4/7)n$.

All of this was considered in the project, and is proved in the report. Please see it for the details.

Kakeya(β)

Conjecture (Kakeya(β))

We say that Kakeya(β) holds over $\mathbb{R}^n$, if for any Kakeya set $K \subset \mathbb{R}^n$, we have the following bound

$$\dim K \geq \frac{n}{\beta}$$

Thus, the full Kakeya conjecture amounts to showing that Kakeya(1) holds over $\mathbb{R}^n$ for all n.

We will deduce Kakeya(2) by applying the "tensoring" trick to the following bound:

Theorem ($(n-1)/2$ bound)

If $K \subset \mathbb{R}^n$ is a Kakeya set, then

$$\dim K \geq \frac{n-1}{2}$$

To go from this theorem to Kakeya(2) we use what is known as the "tensoring" trick. Note that for a Kakeya set $K \subset \mathbb{R}^n$, $K^t \subset \mathbb{R}^{nt}$ is also a Kakeya set. Hence, by applying the above to that, we see that

$$t \dim K = \dim K^t \geq \frac{nt - 1}{2}$$

Dividing throughout by t and taking the limit $t \to \infty$, we see that

$$\dim K \geq \frac{n}{2}$$

giving Kakeya(2).

An Additive Combinatorial Problem

Conjecture (SD(R, β))

For any positive real number β, and some $R \subset \mathbb{N}$, we say SD(R, β) holds over an abelian group G, if the following holds:
For any subsets $A, B \subset G$ such that $|A|, |B| \leq N$, and for any $\Gamma \subset A \times B$, suppose that for all $r \in R$, we have that

$$|\{a + rb : (a, b) \in \Gamma\}| \leq N$$

Then, we have that

$$|\{a - b : (a, b) \in \Gamma\}| \leq N^{\beta}$$

Theorem (Reduction)

Suppose SD(R, β) *holds over* $\mathbb{R}^n$ *with* $R = \{1, 2, \cdots, r\}$, *then* Kakeya(β) *holds over* $\mathbb{R}^n$.

In the report, we show that $\mathrm{SD}(1, 2, 7/4)$ holds over all abelian groups that do not embed $\mathbb{Z}/2\mathbb{Z}$, thereby establishing that Kakeya$(7/4)$ holds over $\mathbb{R}^n$.

Kakeya Sets in $\mathbb{F}^n$

In the finite field setting, there is no longer any useful notion of length. Hence, the notion of "line" is taken to be an entire line, and the notion of direction is taken to be the any non-zero vector in $\mathbb{F}^n$:

Definition (Kakeya sets in $\mathbb{F}^n$)

A *Kakeya set* $K \subset \mathbb{F}^n$ is a set such that for every $x \in \mathbb{F}^n - \{0\}$, there exists a $y \in K$ such that

$$l = \{y + tx : t \in \mathbb{F}\} \subset K$$

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Suitable notion of “size”

One clear difference between the real setting and the finite field setting is the fact that unlike reals, we have only one really meaningful notion of size of sets in the finite field setting - cardinality. Since all sets are finite, it makes most sense to work completely with cardinality.

Now, as we will see later, there is a sort of similarity to considering the finite field case with the size of the field q going to infinity, to the real case with balls of radius ϵ covering the Kakeya set as $\epsilon \to 0$. As a matter of fact, using the relation $q \sim 1/\epsilon$ gives a nice heuristic method for translating expected results back and forth between the finite field and real setting.

Dvir's theorem for finite fields

We can now state the Kakeya problem for the finite field setting, solved completely by Zeev Dvir (the author of the survey) in [3].

Theorem (Kakeya problem in Finite Fields)

For any positive integer n, for all finite fields $\mathbb{F}$ of cardinality q we have that for any Kakeya set $K \subset \mathbb{F}^n$, we have

$$|K| \gg_n |\mathbb{F}|^n = q^n$$

where the implicit constant is independent of the underlying field and only depends on the dimension of the ambient space over the field.

In other words, for any positive integer n, there exists an absolute constant C_n such that for any Kakeya set K in any finite field $\mathbb{F}$ we have that

$$|K| \geq C_n |\mathbb{F}|^n = C_n q^n$$

Projective Space

Definition (Projective Space over $\mathbb{F}$)

Let $\mathbb{F}^{n+1}$ be the $n+1$ dimensional linear space over $\mathbb{F}$. We define the equivalence relation $\sim_P$ for $x, y \in \mathbb{F}^{n+1} - \{0\}$ as follows: $x \sim_P y$ if and only if there exists a non-zero $\lambda \in \mathbb{F}^*$ such that $x = \lambda y$. We call the resulting quotient space under this relation as the *projective space of dimension n over* $\mathbb{F}$, denoted as $\mathbb{PF}^n$.

We will call the process of taking the equivalence relations *projectivizing*. Furthermore, all linear maps from $\mathbb{F}^{n+1}$ that remain well-defined after projectivization shall be known as projective maps from $\mathbb{PF}^n$.

Projective Space

Points in $\mathbb{PF}^n$ shall be denoted by the $n+1$ homogenous coordinates (which are unique up to multiplication by a non-zero scalar) $x = (x_0 : x_1 \cdots : x_n)$.

Now note that the n-dimensional affine space $\mathbb{F}^n$ can be embedded into $\mathbb{PF}^n$ by mapping the point $(x_1, \cdots, x_n) \in \mathbb{F}^n$ to $(1 : x_1 : \cdots : x_n) \in \mathbb{PF}^n$, and this map will respect the structure (whereby projective maps will reduce to affine maps for the embedded affine space). Once this embedding has been fixed, the points in $\mathbb{PF}^n$ having $x_0 = 0$, that is, points of the form $(0 : x_1 : \cdots : x_n)$ are known as the *points at infinity*. The set of all these points is then known as the hyperplane at infinity, analogous to real projective case.

Projective Space

Now consider any line l in $\mathbb{F}^n$ say

$$a_0 + a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

It is easy to see that after projectivizing, $x_0 = 1$, so by homogenizing the line will now become

$$a_0x_0 + a_1x_1 + \cdots + a_nx_n = 0$$

Projective Space

we will first need a suitable notion of "polynomial" over a projective space. In particular, we need the notion of "zero" to be well-defined for these polynomials.

It is not very difficult to see that the notion of polynomials over $\mathbb{PF}^n$ would actually be set of all homogenous polynomials in $\mathbb{F}[x_0, x_1, \cdots, x_n]$. That is, all polynomials whose monomials have the same degree. Clearly the polynomial will remain well-defined as a function after projectivization, since homogeneity means

$$f(ax_0, ax_1, \cdots, ax_n) = a^d f(x_0, x_1, \cdots, x_n)$$

where $d = \deg f$, and a is a non-zero element of $\mathbb{F}$. Hence, $x \sim_P y$ implies that $f(x) \sim_P f(y)$.

Due to this, we will denote the set of homogenous polynomials over the $n + 1$ variables $(x_0, x_1, \cdots, x_n)$ as $\mathbb{PF}[x_0 : x_1 : \cdots : x_n]$.

In particular, note that the standard embedding of $\mathbb{F}^n$ into $\mathbb{PF}^n$ actually gives an embedding of $\mathbb{F}[x_1, \cdots, x_n]$ into $\mathbb{PF}[x_0 : x_1 : \cdots : x_n]$ as follows: for any polynomial $f \in \mathbb{F}[x_1, \cdots, x_n]$ of degree d, multiply every monomial in f with a power x_0^r such that the degree of the monomial becomes equal to d (and makes the result a homogenous polynomial in the $n+1$ variables, say f^h). In other words, we consider the map $f \mapsto f^h$ given by

$$f^h(x_0, x_1, \cdots, x_n) := x_0^d f(x_1/x_0, \cdots, x_n/x_0)$$

Projective Space

To see that this map is injective (and thus an embedding) note that substituting $x_0 = 1$ in f^h gives back f. Also note that this also demonstrates that the given embedding is consistent with the embedding of $\mathbb{F}^n$ into $\mathbb{PF}^n$.

Finally, note that setting $x_0 = 0$ gives the restriction of f^h to the hyperplane at infinity, and is in fact, equal to the homogenous part of highest degree of f.

Schwartz-Zippel

Lemma (Schwartz-Zippel Theorem,$\star$)

Let $f \in \mathbb{F}[x_1, \cdots, x_n]$ be a polynomial of degree d. Then there are at most dq^{n-1} points in $\mathbb{F}^n$ on which f vanishes.

The above theorem is easily proved by induction. In any case, the important information in this is that for every non-vanishing polynomial with degree less than the size of the field, there is at least one point in $\mathbb{F}^n$ on which the polynomial does not vanish. This fact is much easier to prove by induction than the Schwartz-Zippel Theorem.

We now begin the proof of the Kakeya problem in finite fields. Following [4], we prove the following two lemmas.

Lemma

Let $E \subset \mathbb{F}^n$ be a set such that $|E| < \binom{n+d}{d}$ for some $d \geq 0$. Then there exists a non-zero polynomial $f \in \mathbb{F}[x_1, \cdots, x_n]$ with degree at most d which vanishes on E.

Morally speaking, this lemma states that if any set is small enough, there is at least one low-degree polynomial which vanishes on it entirely. Thus, if there is a set on which no low-degree polynomial vanishes, it must follow that that set must be large.

In other words, we get the contrapositive which allows us to bound below the size of a set by showing that no polynomial of low degree vanishes on it. Explicitly, if no polynomial of degree less than d vanishes on a set E, then we must have

$$|E| \geq \binom{n+d}{d} = \binom{n+d}{n}$$

Lemma

If $K \subset \mathbb{F}^n$ is a Kakeya set, and f is a polynomial of degree at most $q-1$ which vanishes on K, then $f \equiv 0$.

Combining the lemmata above, we clearly get that for a Kakeya set K, we have

$$|K| \geq \binom{n+(q-1)}{n} \geq \frac{q^{n-1}(q-1)}{n!} \gg_n q^n$$

which solves the Kakeya problem in finite fields with an implicit constant of the order of $1/n!$.

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