STRAIGHT LINES

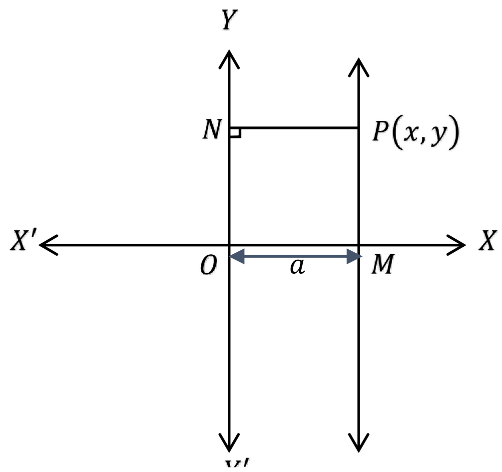
A straight line is treated as a locus of a point moving under certain condition. In this unit, we shall discuss the methods of finding equations of various types of straight lines. Firstly, we consider **the** lines which are parallel to the co-ordinate axes.

Equations of lines parallel to the axes

(i) Equation of a line parallel to the y-axis: Let us consider a line parallel to the y-axis which is at distance a from it. Clearly, the line is the locus of a point which is always at distance a from the y-axis. So, in order to find the equation of the line, we take any point $P(x, y)$ on the line. As in the figure, $PN \perp y$-axis and given that $PN = a$

$\Rightarrow \quad \boxed{x = a}$

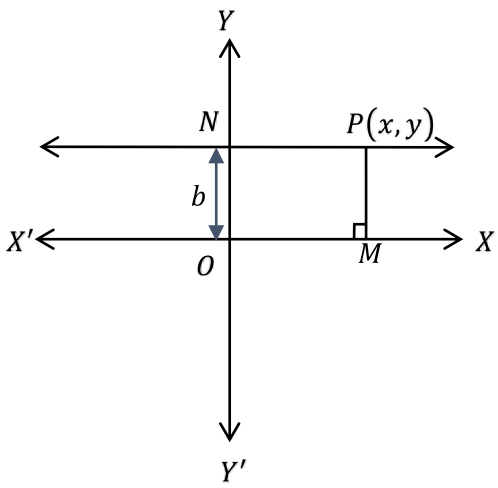
which is the required equation of the line. If $a > 0$, the line is on the right side of the y-axis and if $a < 0$, the line is on the left side of the y-axis. However, if $a = 0$, the line coincides with the y-axis. Hence, the equation of the y-axis, is $\boxed{x = 0}$.



(ii) Equation of a line parallel to the x-axis:
Let us consider a line parallel to the x-axis at distance b from it. Clearly, the line is the locus of a point which is always at distance b from the x-axis. So, in order to find the equation of the line, we take any point $P(x, y)$ on the line. As in the figure, $PM \perp x$-axis and given that $PM = b$

$\Rightarrow \quad \boxed{y = b}$

which is the required equation of the line. If $b > 0$, the line is above the x-axis and if $b < 0$, the line is below the x-axis. However, if $b = 0$, the line coincides with the x-axis. Hence, the equation of the x-axis, is $\boxed{y = 0}$.



Solved Examples

Example 1: Find the equation of the line parallel to the y-axis at distance -7 from it.

Solution: The equation of a line parallel to the y-axis is given by $x = a \implies x = -7 \implies x + 7 = 0$.

Example 2: Obtain the equation of the line parallel to the x-axis and passing through the point $(3, -2)$.

Solution: The equation of a line parallel to the x-axis is given by $y = b$. Since, the line passes through the point $(3, -2)$, we have $b = -2$ and hence, the equation of the line is

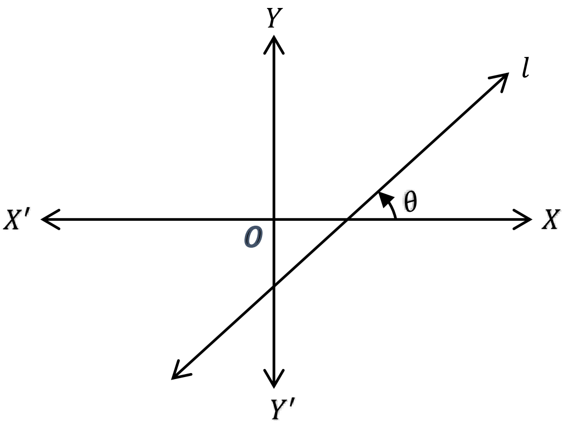
$$y = -2 \implies y + 2 = 0.$$

Example 3: Find the equations of the lines parallel to the axes and passing through the point $(3, 4)$.

Solution: The equation of the line parallel to the y-axis and passing through $(3, 4)$ is $x = 3$. The equation of the line parallel to the x-axis and passing through $(3, 4)$ is $y = 4$.

Inclination of a line: While finding the equation of a line, we often talk about a particular type of angle, referred to as the *inclination*, made by the line with the x-axis. It is defined below:

The inclination of a line is the angle made by the line with the positive direction of the x-axis in *anticlockwise direction*.



In the figure, the inclination of the line l is θ. It can be noted that the inclination of the x-axis is 0^0 and that of the y-axis is 90^0.

The slope or gradient of a line: The slope of a line is an important feature of the line which Is defined as follows:

The slope or gradient of a line is the tangent of the angle made by the line with the positive direction of the x-axis in anticlockwise direction.

OR

The slope of a line is the tangent of its inclination.

The slope of a line is usually denoted by the letter m. In the above figure, the slope of the line l, is $m = \tan\theta$. Clearly, the slope of the x-axis is $\tan 0^0 = 0$ and that of the y-axis is $\tan 90^0$ which is not defined.

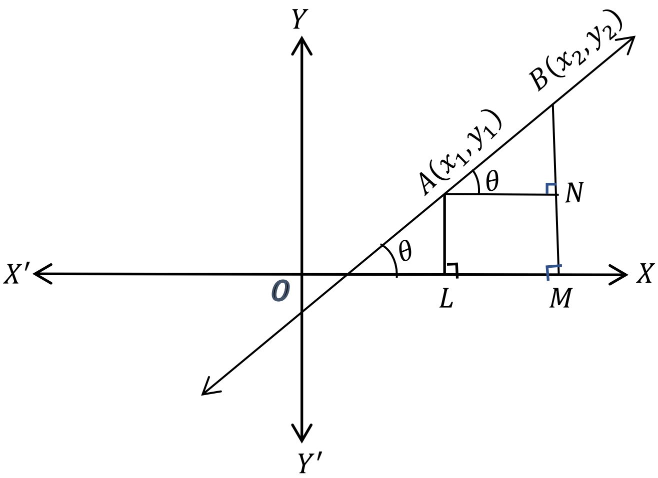
The slope of a line passing through two given points: Let a line pass through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let the inclination of the line be θ. . As in the figure, AL and BM are the perpendiculars drawn from the points A and B respectively on the x-axis. Again, $AN \perp BM$ is drawn. Clearly, $\angle BAN = \theta$.

Also, $AN = LM = OM - OL = x_2 - x_1$

$BN = BM - NM = BM - AL = y_2 - y_1$.

In right ΔANB, $\quad tan\,\theta = \dfrac{BN}{AN} = \dfrac{y_2 - y_1}{x_2 - x_1}$

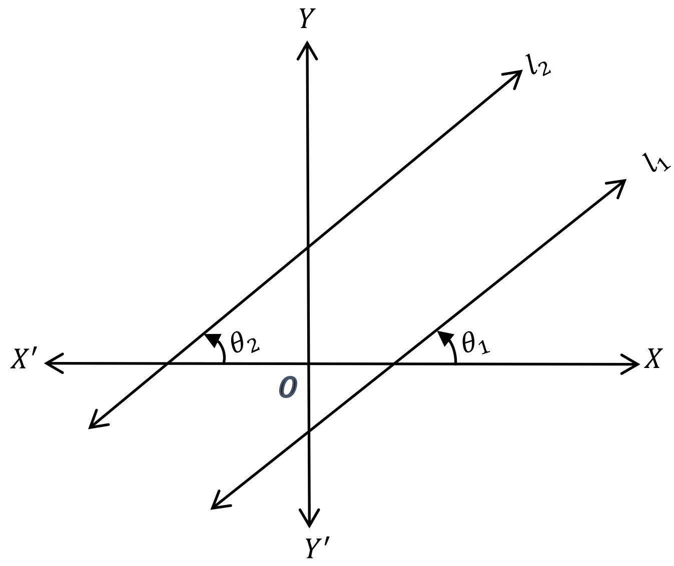
$$\Rightarrow \quad \boldsymbol{slope\ of\ the\ line\ \ m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}}$$



Relation between slopes of two parallel lines: Let l_1 and l_2 be two parallel lines with inclinations θ_1 and θ_2 and slopes m_1 and m_2 respectively. Then

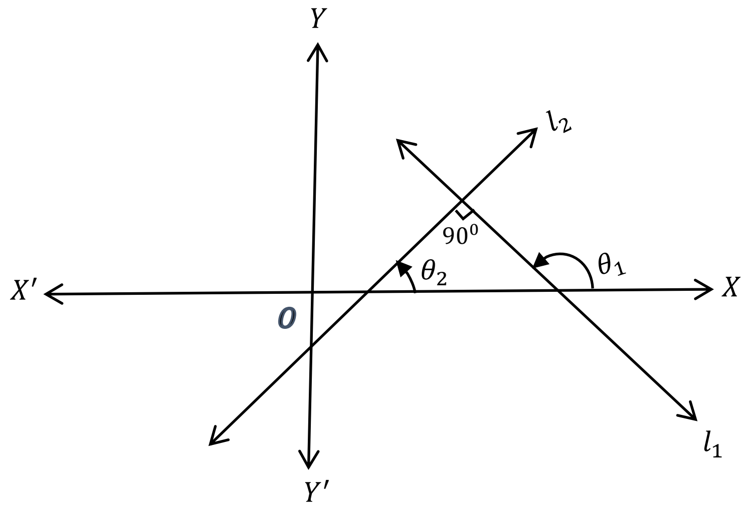
$m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$. Since, $l_1 \parallel l_2 \quad \Rightarrow \quad \theta_1 = \theta_2 \quad \Rightarrow \quad \tan\theta_1 = \tan\theta_2$

$\Rightarrow \quad \boldsymbol{m_1 = m_2}$.



Thus, we conclude that the *slopes of two parallel lines are equal*.

Relation between slopes of two perpendicular lines: Let $l_1 \perp l_2$ be two lines with inclinations θ_1 and θ_2 and slopes m_1 and m_2 respectively. Then, as in the figure, we have



$m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$. Since, $l_1 \perp l_2 \quad \Rightarrow \quad \theta_1 = 90^0 + \theta_2$

$\Rightarrow \quad \tan\theta_1 = \tan(90^0 + \theta_2) \qquad \Rightarrow \quad \tan\theta_1 = -\cot\theta_2 \qquad \Rightarrow \quad \tan\theta_1 = -\frac{1}{\tan\theta_2}$

$\Rightarrow \quad \boldsymbol{m_1 = -\frac{1}{m_2}} \qquad \Rightarrow \quad \boldsymbol{m_1\, m_2 = -1}$.

Thus, we conclude that the *product of the slopes of two perpendicular lines is equal* to -1.

Solved Examples

Example 1: Find the slope of a line inclined at an angle of (i) 60^0 $\quad (ii)$ 135^0 with the positive direction of the x-axis.

Solution: The slope of a line with inclination θ is given by $m = \tan\theta$. Hence,

(i) Slope $m = \tan 60^0 = \sqrt{3}$ $\qquad (ii)$ Slope $m = \tan 135^0 = \tan(180^0 - 45^0) = -\tan 45^0 = -1$.

Example 2: Obtain the slope of the line passing through the points $(3, -2)$ and $(-4, -1)$. Also write the slope of a line perpendicular to it.

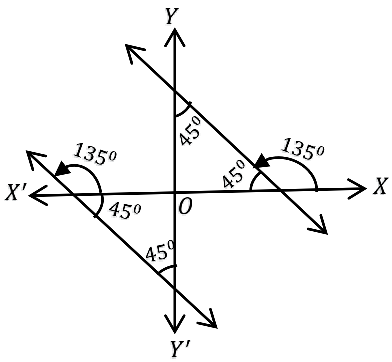
Solution: The slope of the line passing through two points, is given by the formula

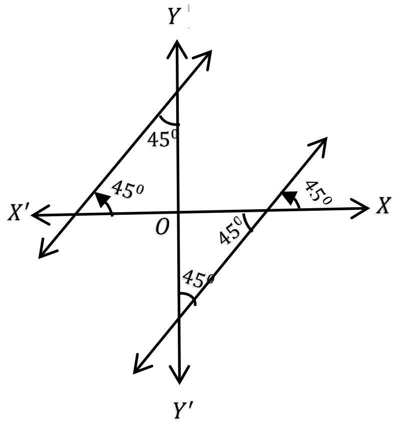
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-(-2)}{-4-3} = -\frac{1}{7}.$$

Further, since the product of the slopes two perpendicular lines is -1; so, the slope of a line perpendicular to the above line, is 7.

Example 3: Obtain the slope of a line equally inclined with the co-ordinate axes.

Solution: If a line is equally inclined with the co-ordinate axes, it makes equal angles with both the axes as shown in the following figures:





The line may pass through anyone of the four quadrants giving rise to its inclination either 135^0 (for the 1st and 3rd quadrants) or 45^0 (for the 2nd and 4th quadrants) and hence, the required slope is

$$(i)\ \ m = \tan 135^0 = \tan(180^0 - 45^0) = -\tan 45^0 = -1 \quad \text{or} \quad (ii)\ \ \tan 45^0 = 1\,.$$

$$\Rightarrow \qquad m = \pm 1\,.$$

Assignment No. 1

1. Find the equation of the line parallel to the x-axis at the distance -5 from it. (Answer: $y = -5$)
2. Obtain the equation of the line parallel to the y-axis and passing through the point $(1, -4)$. (Answer: $x = 1$)
3. Find the equation of the line parallel to the axes and passing through the point $(6, -2)$. (Answer: $x = 6, \quad y = -2$)
4. Find the slope of a line inclined at an angle of $(i)\ 30^0 \quad (ii)\ 120^0$ with the positive direction of the x-axis. {Answer: $(i) 1/\sqrt{3}, \quad (ii) -\sqrt{3}$ }
5. Obtain the slope of the line passing through the points $(1, -1)$ and $(-3,\ 5)$. Also write the slope of a line perpendicular to it. $\left(Answer:\ -\frac{3}{2},\ \frac{2}{3}\right)$
6. For which value of x the line passing through the points $(3, -x)$ and $(2x,\ 4)$ has its slope 6 ? (Answer:2)
7. Obtain the slope of a line equally inclined with the co-ordinate axes. (Answer: ± 1)

Equations of lines in various standard forms

Equation of a line in slope-point form

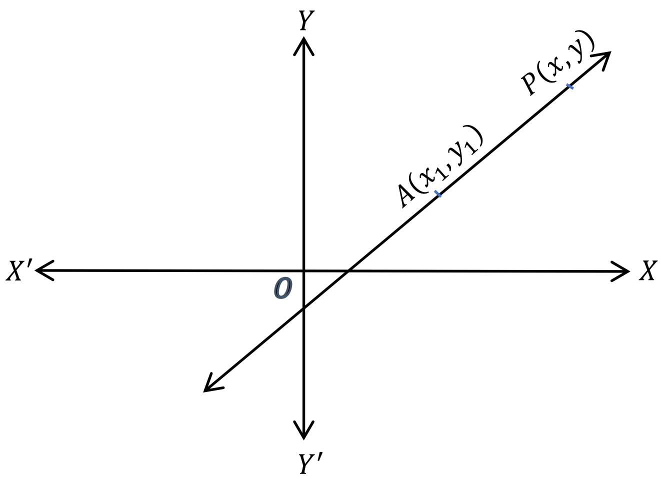
To find the equation of a line with a given slope and passing through a given point.

Let the slope of the line be m and the given point be $A(x_1, y_1)$ through which the line passes. Considering the line as a locus and adopting the procedure of finding its equation, we choose any point $P(x, y)$ on the line. Then the line passes through two points (x_1, y_1) and (x, y) and hence, its slope is $\frac{y-y_1}{x-x_1}$.

$$\text{But,} \qquad \text{given that slope} = m \qquad \Rightarrow \qquad \frac{y - y_1}{x - x_1} = m$$

$$\Rightarrow \qquad \boxed{\boldsymbol{y - y_1 = m(x - x_1)}}$$

Which is the equation of the line in the slope-point form.

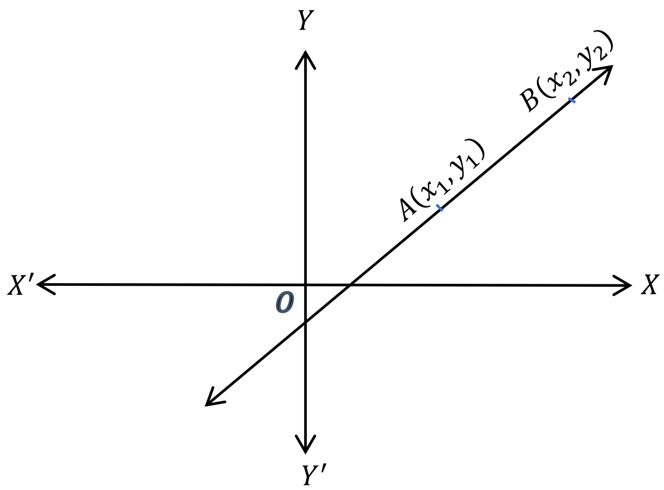


If the line passes through the origin, then $x_1 = 0$ and $y_1 = 0$ and hence, the equation of the line passing through the origin is given by $\boxed{\boldsymbol{y = mx}}$.

Equation of a line in two-point form

To find the equation of a line passing through two given points.

Let a line pass through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$. Clearly, the slope of the line is $\frac{y_2-y_1}{x_2-x_1}$.



Since, the line passes through the point (x_1, y_1), hence, its equation using the slope-point form, is given by $y - y_1 = m(x - x_1)$

$$\Rightarrow \quad \boxed{\boldsymbol{y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)}}$$

which is the equation of the line in two-point form.

Solved Examples

Example 1: Find the equation of the line inclined at an angle of 120^0 with the positive direction of the x-axis and passing through the origin.

Solution: The slope of the line is $m = \tan\theta = \tan 120^0 = \tan(180^0 - 60^0) = -\tan 60^0 = -\sqrt{3}$. Since, the line passes through the origin, its equation, is given by $y = mx \implies y = -\sqrt{3}x$.

Example 2: Find the equation of the line passing through the point $(3, -1)$ and making an angle of 150^0 with the positive direction of the x-axis.

Solution: The slope of the line is $m = \tan\theta = \tan 150^0 = \tan(180^0 - 30^0) = -\tan 30^0 = -\frac{1}{\sqrt{3}}$. Since, the line passes through the point $(3, -1)$, hence, its equation using the slope-point form, is given by $y - y_1 = m(x - x_1) \implies y - (-1) = -\frac{1}{\sqrt{3}}(x - 3) \implies \sqrt{3}(y + 1) = x + 3$
$\implies x - \sqrt{3}y + 3 - \sqrt{3} = 0$.

Example 3: Obtain the equation of the line passing through the points $(3, -2)$ and $(-4, -1)$.

Solution: The line passes through two points. Hence, its equation using two-point form, is given by the formula

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) \implies y - (-2) = \left[\frac{-1 - (-2)}{-4 - 3}\right](x - 3)$$

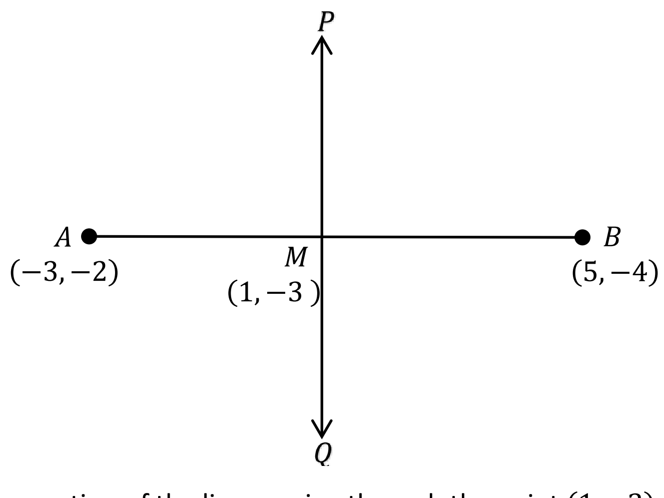
$$\implies y + 2 = -\frac{1}{7}(x - 3) \implies 7y + 14 = -x + 3 \implies x + 7y + 11 = 0.$$

Example 4: Obtain the equation of the perpendicular bisector of the line-segment joining the points $(-3, -2)$ and $(5, -4)$.

Solution: As in the figure, PQ is the perpendicular bisector of the line-segment joining the points $A(-3, -2)$ and $B(5, -4)$. The perpendicular bisector PQ passes through the mid-point M of the line-segment AB and is perpendicular to AB. The co-ordinates of the mid-point M are

$$\left(\frac{-3 + 5}{2}, \frac{-2 - 4}{2}\right) \ i.e. (1, -3) \quad and \quad \text{the slope of } AB = \frac{-4 + 2}{5 + 3} = -\frac{2}{8} = -\frac{1}{4}.$$

$\implies$ The slope of the perpendicular bisector PQ is 4 .



Thus, we need to find the equation of the line passing through the point $(1, -3)$ and having the slope 4. Hence, its equation using the slope-point form, is given by the formula
$y - y_1 = m(x - x_1) \implies y - (-3) = 4(x - 1) \implies y + 3 = 4x - 4$
$\implies 4x - y - 7 = 0$ which is the required equation of the perpendicular-bisector.

ASSIGNMENT NO. 2

1. Find the equation of the line inclined at an angle of 135^0 with the positive direction of the x-axis and passing through the origin. (Answer: $x + y = 0$)

2. Find the equation of the line passing through the point $(2, -3)$ and making an angle of 120^0 with the positive direction of the x-axis. (Answer: $\sqrt{3}x + y + 3 - 2\sqrt{3} = 0$)

3. Obtain the equation of the line passing through the mid-point of the line-segment joining $(2, -1)$ and $(-4, 5)$ and making an angle of 150^0 with the positive direction of the x-axis.
(Answer: $x + \sqrt{3}y + 1 - 2\sqrt{3} = 0$)

4. Obtain the equation of the perpendicular bisector of the line-segment joining the points $(-1, -2)$ and $(3, -4)$. (Answer: $2x - y = 5$)

5. Find the equation of the line passing through the points $(1, -2)$ and $(-4, -5)$.
(Answer: $3x - 5y = 13$)

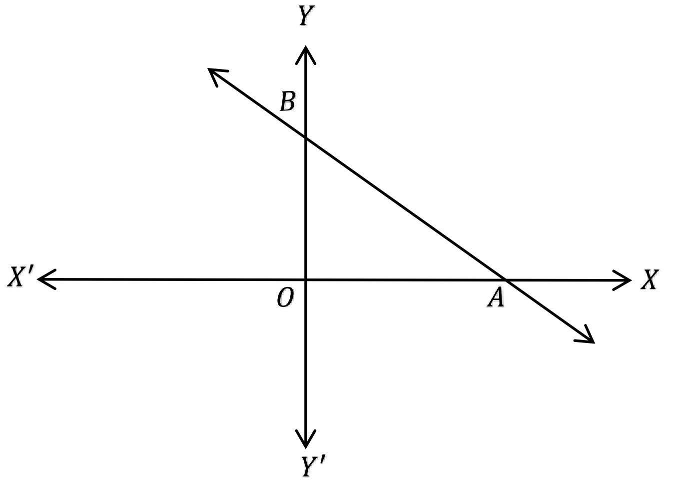
6. Find the equations of the sides of the triangle whose vertices are $(1, -1), (-2, 4)$ and $(3, -5)$.
(Answer: $5x + 3y = 2$, $9x + 5y = 2$ & $2x + y = 1$)

7. Find the equation of the line passing through the point $(1, -3)$ and parallel to the line whose slope is -5. (Answer: $5x + y = 2$)

8. Find the equation of the line passing through the points $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$.
$\left(Answer: x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}\right)$

Intercepts of a line on the axes

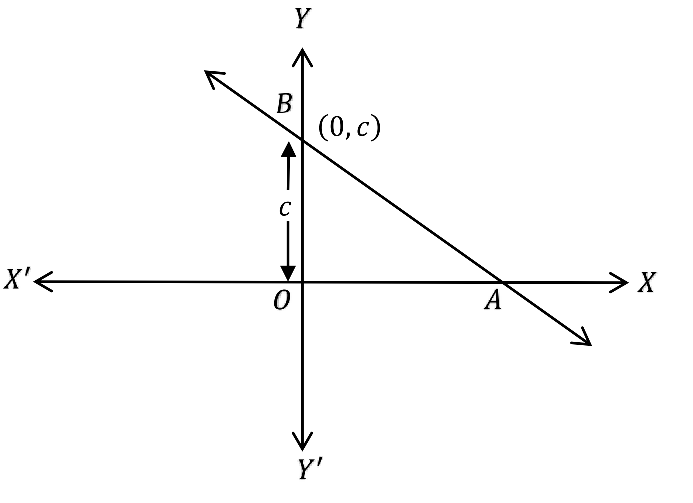
If a line intersects the x-axis and the y-axis at the points A and B respectively, then OA and OB are called the intercepts of the line on the axes. An intercept may be positive or negative according as whether it lies in the positive or negative direction of the axis.



In the above figure, OA is the x-intercept and OB the y-intercept of the line AB.

Equation of a line in slope-intercept form

To find the equation of a line when its slope and the y-intercept are given.
Let a line with slope m, cut off the intercept c on the y-axis. Clearly, the line passes through the point $(0, c)$. Hence, its equation using the slope-point form, is given by $y - y_1 = m(x - x_1)$



$$\Rightarrow \quad y - c = m(x - 0)$$

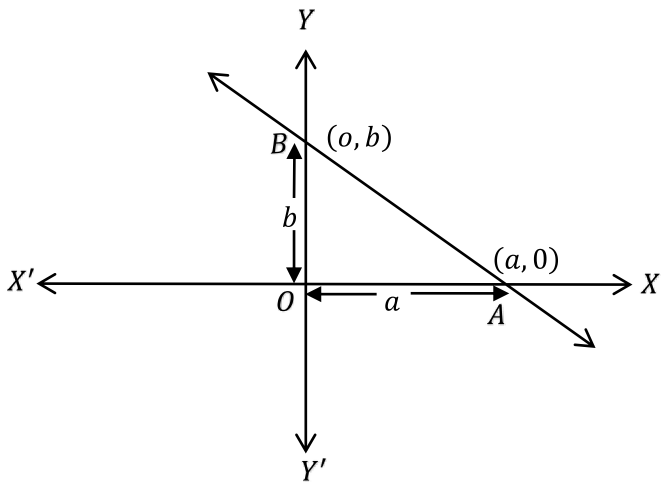
$$\Rightarrow \quad \boxed{y = mx + c}$$

which is the equation of the line in slope-intercept form. If the line passes through the origin, then $c = 0$ and hence, its equation becomes $\boldsymbol{y = mx}$ which is already found in the slope-point form.

Equation of a line in intercept form

To find the equation of a line when its intercepts on both the axes are given.
Let a line cut off the intercepts $OA = a$ and $OB = b$ on the x-axis and the y-axis respectively. Clearly, the line passes through the point $(a, 0)$ and $(0, b)$, Hence, its equation using the two-point form, is given by

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$



$$\Rightarrow \quad y - 0 = \left(\frac{b - 0}{0 - a}\right)(x - a) \qquad \Rightarrow \qquad ay = -bx + ab \qquad \Rightarrow \qquad bx + ay = ab$$

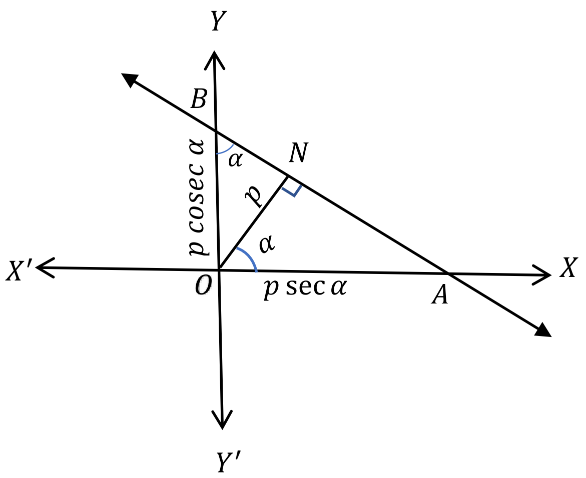
Dividing both sides by ab, we get

$$\boxed{\frac{\boldsymbol{x}}{\boldsymbol{a}} + \frac{\boldsymbol{y}}{\boldsymbol{b}} = \mathbf{1}}$$

which is the equation of the line in the intercept form.

Equation of a line in normal form or perpendicular form

To find the equation of a line when two features about the perpendicular drawn from the origin to the line, are given. As in the figure, ON is the normal to the line. We are given two informations about the normal which include its length $ON = p$ and its inclination $\angle AON = \alpha$. Now, there can be several ways to find the equation of the line. Obviously, OA and OB are the intercepts made by the line on the axes.



In right Δ^s ONA and ONB, $we\ have$ $\frac{OA}{ON} = sec\ \alpha$ $\quad and \quad$ $\frac{OB}{ON} = cosec\ \alpha$

$\Rightarrow \quad \frac{OA}{p} = \sec\alpha$ $\quad$ and $\quad$ $\frac{OB}{p} = \operatorname{cosec}\alpha$ $\quad \Rightarrow \quad OA = p\ sec\ \alpha \quad and \quad OB = p\ cosec\ \alpha$

Hence, the equation of the line using the intercept form, is $\frac{x}{a} + \frac{y}{b} = 1 \quad \Rightarrow \quad \frac{x}{OA} + \frac{y}{OB} = 1$

$$\Rightarrow \quad \frac{x}{p\ sec\ \alpha} + \frac{y}{p\ cosec\ \alpha} = 1 \qquad \Rightarrow \qquad \frac{x\cos\alpha}{p} + \frac{y\sin\alpha}{p} = 1$$

$$\Rightarrow \qquad \boxed{\boldsymbol{x\cos\alpha + y\sin\alpha = p}} \quad \underline{\qquad\qquad} \quad (1)$$

which is the equation of the line in the normal form.

Remark: In the normal form of the equation (1), the length p of the normal is always taken positive.

Solved Examples

Example 1: Obtain the equation of the line whose normal from the origin measures 5 units and the inclination of the normal is 120^0.
Solution: We are given that the length of the normal $p = 5$ and the inclination of the normal, is $\alpha = 120^o$.
Hence, the equation of the line using the normal form, is given by

$$x\cos\alpha + y\sin\alpha = p \quad \Rightarrow \quad x\cos 120^o + y\sin 120^0 = 5$$
$$\Rightarrow \quad x\cos(180^o - 60^o) + y\sin(180^o - 60^o) = 5$$
$$\Rightarrow \ -x\cos 60^o + y\sin 60^o = 5$$
$$\Rightarrow \ -x.\frac{1}{2} + y.\frac{\sqrt{3}}{2} = 5$$
$$\Rightarrow \quad x - \sqrt{3}y + 10 = 0.$$

Example 2: Obtain the equation of the line having the length of the normal from the origin 8 units with the inclination of the normal 135^0.
Solution: We are given that the length of the normal $p = 8$ and the inclination of the normal, is $\alpha = 135^o$.
Hence, the equation of the line using the normal form, is given by

$$x\cos\alpha + y\sin\alpha = p \quad \Rightarrow \quad x\cos 135^o + y\sin 135^0 = 8$$
$$\Rightarrow \quad x\cos(180^o - 45^o) + y\sin(180^o - 45^o) = 8$$
$$\Rightarrow \ -x\cos 45^o + y\sin 45^o = 8$$
$$\Rightarrow \ -x.\frac{1}{\sqrt{2}} + y.\frac{1}{\sqrt{2}} = 8$$
$$\Rightarrow \quad x - y + 8\sqrt{2} = 0.$$

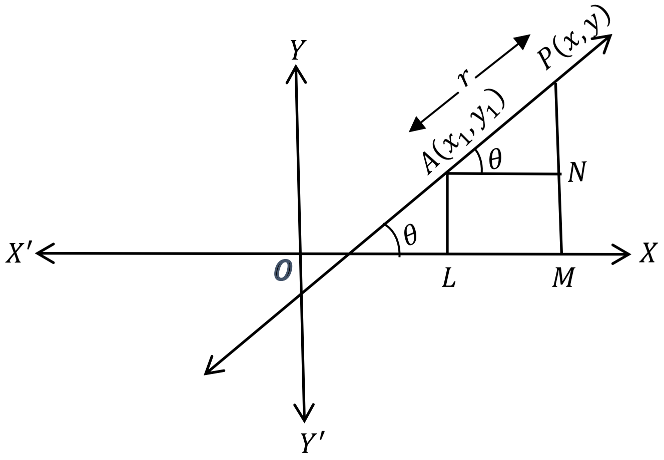
Equation of a line in parametric or symmetric form

We find the equation of a line when its inclination is given and a point through which the line passes, is also given. Let the given point be $A(x_1, y_1)$ and the inclination of the line be θ. Let $P(x, y)$ be point on the line at distance r from the point A. As in the figure, AL and PM are perpendiculars drawn from A and P respectively on the x-axis. Again, $AN \perp PM$ is drawn. Clearly, $\angle PAN = \theta$.

Also, $AN = LM = OM - OL = x - x_1$ $\quad$ & $\quad$ $PN = PM - NM = PM - AL = y - y_1$

$$In\ right\ \Delta ANP, \quad \frac{AN}{AP} = cos\ \theta \qquad \& \qquad \frac{PN}{AP} = sin\ \theta$$

$$\Rightarrow \quad \frac{x - x_1}{r} = \cos\theta \qquad \& \qquad \frac{y - y_1}{r} = \sin\theta$$



$$\Rightarrow\ x - x_1 = r\cos\theta \qquad \& \qquad y - y_1 = r\sin\theta \qquad ----(1)$$

$$\Rightarrow\ \frac{x-x_1}{\cos\theta} = r \qquad \& \qquad \frac{y-y_1}{\sin\theta} = r$$

$$\Rightarrow\ \boxed{\frac{\boldsymbol{x-x_1}}{\boldsymbol{\cos\theta}} = \frac{\boldsymbol{y-y_1}}{\boldsymbol{\sin\theta}} = \boldsymbol{r}} \qquad -----(2)$$

From (1), we have $x = x_1 + r\cos\theta$ & $y = y_1 + r\sin\theta$. These relations determine the co-ordinates of the point at distance r from the point (x_1, y_1). If r is allowed to take any value, then $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ represent the co-ordinates of any point on the line.

Thus, the co-ordinates of any point on the line can be given as $(x_1 + r\cos\theta, y_1 + r\sin\theta)$, where r is an arbitrary constant known as a parameter. Hence, equation (2), is known as the parametric or symmetric form of the equation of the line.

Solved Examples

Example 1: Find the equation of the line which is inclined at an angle of 120^0 with the positive direction of the x-axis and cuts off an intercept of -5 units on the y-axis.

Solution: The slope of the line is $m = \tan\theta = \tan 120^0 = \tan(180^0 - 60^0) = -\tan 60^0 = -\sqrt{3}$.
The y-intercept of the line, is $c = -5$.
Hence, the equation of the line using the slope-intercept form, is given by

$$y = mx + c \Rightarrow y = -\sqrt{3}x - 5 \quad \Rightarrow \quad \sqrt{3}x + y + 5 = 0$$

Example 2: Find the equation of the line which cuts off the intercepts -4 and 3 respectively on the axes.

Solution: The x-intercept $a = -4$ and the y-intecept $b = 3$.
Hence, the equation of the using the intercept form, is given by

$$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \quad \frac{x}{-4} + \frac{y}{3} = 1 \quad \Rightarrow \quad 3x - 4y + 12 = 0.$$

Example 3: Obtain the equation of the line passing through the point $(3, -2)$ and making equal intercepts on the axes.

Solution: The equation of the line using intercept form, is given by the formula

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \Rightarrow \quad \frac{x}{a} + \frac{y}{a} = 1 \quad \Rightarrow \quad x + y = a \quad ----(1)$$

(Taking $a = b$ as both the intercepts are equal)

Since, the line (1) passes through the point $(3, -2)$, we have $3 - 2 = a \quad \Rightarrow \quad a = 1$.

Hence, the required equation of the line, is $x + y = 1$.

Example 4: Obtain the equation of the line passing through the point $(9, -8)$ and the sum of whose intercepts on the axes, is 7.

Solution: The equation of the line using intercept form, is given by the formula

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad ----(1)$$

Since, the line (1) passes through the point $(9, -8)$, we have

$$\frac{9}{a} + \frac{-8}{b} = 1 \quad \Rightarrow \quad 9b - 8a = ab \ ------ (2)$$

Also, given that $\qquad a + b = 7 ------(3)$

From (2) and (3), we have $9(7-a) - 8a = a(7-a) \quad \Rightarrow \quad a^2 - 24a + 63 = 0$

$$\Rightarrow \quad (a-21)(a-3) = 0 \quad \Rightarrow \quad a = 21 \ or \ 3.$$

If $a = 21$, then $b = 7 - 21 = -14 \ and$ hence, from (1) the line is

$$\frac{x}{21} + \frac{y}{-14} = 1 \quad or \quad 2x - 3y = 42 \qquad ----(4)$$

If $a = 3$, then $b = 7 - 3 = 4 \ and$ hence, from (1) the line is

$$\frac{x}{3} + \frac{y}{4} = 1 \quad or \quad 4x + 3y = 12 \qquad ----(5)$$

Hence, (4) and (5) are the required equations of the line.

Assignment No. 3

1. Find the equation of the line which is inclined at an angle of 45^0 with the positive direction of the x-axis and cuts off an intercept of -4 units on the y-axis. (Answer: $x - y = 4$)

2. Find the equation of the line which cuts off the intercepts -4 and 6 respectively on the axes. (Answer: $3x - 2y + 12 = 0$)

3. Obtain the equation of the line passing through the point $(2, \ 3)$ and making equal intercepts on the axes. (Answer: $x + y = 5$)

4. Obtain the equation of the line passing through the point $(3, \ 4)$ and the sum of whose intercepts on the axes, is 14. (Answer: $x + y = 7, \quad 4x + 3y = 24$)

5. Obtain the equation of the line whose normal from the origin measures 7 units and the inclination of the normal is 225^0. (Answer: $x + y + 7\sqrt{2} = 0$)

Equation of a line in general form

While discussing the equations of straight lines in standard forms, we have noticed that an equation of a line is always linear in x and y. So, the general form of the equation of a line can be taken as $ax + by + c = 0$, where a, b and c are constants. Intuitively, each point on a line satisfies a linear equation. If two linear equations represent the same line, their corresponding constants need to be proportional. Let the equations

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ represent the same line, then

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

A straight line has certain features which are unique. Clearly, the slope m, intercepts a and b on the axes, the length p of the normal and its inclination α, are the unique features of a line. Given the equation of a line, we can determine these unique features by reducing the equation to the standard forms involving such features as discussed below:

Reduction of general equation of a line to the slope-intercept form

$$(\boldsymbol{y = mx + c} \ \textbf{form})$$

Let the general equation of a line be $\qquad Ax + By + C = 0$ ------------------- (1)

$$\Rightarrow \quad By = -Ax - C$$

Dividing both sides by B (if $B \neq 0$), we get $\quad y = -\frac{A}{B}x - \frac{C}{B}$

which is in the form $\quad y = mx + c$, where

$$\boxed{\boldsymbol{Slope\ m = -\frac{A}{B} = -\frac{coefficient\ of\ x}{coefficient\ of\ y}}} \quad and$$

$$\boxed{\boldsymbol{y - intercept \quad c = -\frac{C}{B} = -\frac{constant\ term}{coefficient\ of\ y}}}$$

Reduction of general equation of a line to the intercept form $\left(\frac{x}{a} + \frac{y}{b} = 1 \text{ form}\right)$

Let the general equation of a line be $\quad Ax + By + C = 0$ ------------------- (1)

$$\Longrightarrow \quad Ax + By = -C$$

Dividing both sides by $-C$ (if $C \neq 0$), we get $\quad \frac{A}{-C}x + \frac{B}{-C}y = 1$

$$\Longrightarrow \quad \frac{x}{\left(\frac{-C}{A}\right)} + \frac{y}{\left(\frac{-C}{B}\right)} = 1$$

$$which\ is\ in\ the\ intercept\ form \quad \frac{x}{a} + \frac{y}{b} = 1$$

where,

$$\boldsymbol{x\ intercept \quad a = -\frac{C}{A} = -\frac{constant\ term}{coefficient\ of x}} \quad and$$

$$\boldsymbol{y\ intercept \quad b = -\frac{C}{B} = -\frac{constant\ term}{coefficient\ of\ y}}$$

Reduction of general equation of a line to the normal form $(x\cos\alpha + y\sin\alpha = p)$

Let the general equation of a line be $\quad Ax + By + C = 0$ ------------------- (1)

$$\Longrightarrow \quad Ax + By = -C \text{ --------------------- (2)}$$

Dividing both sides by $\sqrt{A^2 + B^2}$, we get

$$\frac{A}{\sqrt{A^2 + B^2}}x + \frac{B}{\sqrt{A^2 + B^2}}y = \frac{-C}{\sqrt{A^2 + B^2}}$$

which is in the normal form $\quad x\cos\alpha + y\sin\alpha = p$, if $C < 0$, where

$$\cos\alpha = \frac{A}{\sqrt{A^2 + B^2}}, \qquad \sin\alpha = \frac{B}{\sqrt{A^2 + B^2}} \ \ \&\ \ length\ of\ normal\ p = \frac{-C}{\sqrt{A^2 + B^2}}, if\ C < 0.$$

However, if $C > 0$, then from (2), we have $-Ax - By = C$

Dividing both sides by $\sqrt{A^2 + B^2}$, we get

$$-\frac{A}{\sqrt{A^2+B^2}}x+\frac{-B}{\sqrt{A^2+B^2}}y=\frac{C}{\sqrt{A^2+B^2}}$$

which is in the normal form $x\cos\alpha + y\sin\alpha = p$, if $C>0$, where

$$\Rightarrow \cos\alpha = \frac{-A}{\sqrt{A^2+B^2}}, \quad \sin\alpha = \frac{-B}{\sqrt{A^2+B^2}} \quad \& \quad length\ of\ normal\ p = \frac{C}{\sqrt{A^2+B^2}}, if\ C>0.$$

Thus, we have,

$$\boldsymbol{The\ length\ of\ the\ normal}\ \boxed{\boldsymbol{p} = \left|\frac{C}{\sqrt{A^2+B^2}}\right| \quad \& \quad \tan\boldsymbol{\alpha} = \frac{\boldsymbol{B}}{\boldsymbol{A}} \quad \boldsymbol{or} \quad \boldsymbol{\alpha} = \tan^{-1}\left(\frac{\boldsymbol{B}}{\boldsymbol{A}}\right)}.$$

Solved Examples

Example 1: Find the slope and the y-intercept of the line represented by the equation $3x-4y=12$.

Solution: The equation of line $3x-4y=12 \Rightarrow 3x-4y-12=0$

$$Slope\ m = -\frac{A}{B} = -\frac{coefficient\ of\ x}{coefficient\ of\ y} = -\frac{3}{-4} = \frac{3}{4}$$

$$y\ intercept\quad c = -\frac{C}{B} = -\frac{constant\ term}{coefficient\ of\ y} = -\frac{-12}{-4} = -3.$$

Example 2: Convert the equation $3x-4y=12$ into the slope-intercept form $y=mx+c$.

Solution: The given equation is $3x-4y=12 \Rightarrow -4y=-3x+12$

$$\Rightarrow \quad y=\frac{3}{4}x-3$$

which is in the form $y=mx+c$, where, $m=\frac{3}{4}$ and $c=-3$.

Example 3: Find the intercepts of the line $3x-4y=12$ on the co-ordinate axes.

Solution: The equation of line, is $3x-4y=12 \Rightarrow 3x-4y-12=0$

By formula, the x-intercept$=-\frac{C}{A}=-\frac{-12}{3}=4$ and the y-intercept$=-\frac{C}{B}=-\frac{-12}{-4}=-3$

Example 4: Convert the equation $2x-5y+12=0$ into the normal form of the equation of the line.

Solution: The given equation is $2x-5y+12=0 \Rightarrow 2x-5y=-12$

$$\Rightarrow -2x+5y=12$$

Dividing both sides by $\sqrt{(-2)^2+5^2}=\sqrt{29}$, we get

$$-\frac{2}{\sqrt{29}}x+\frac{5}{\sqrt{29}}y=\frac{12}{\sqrt{29}}$$

which is in the normal form $x\cos\alpha + y\sin\alpha = p$.

Assignment No. 4

1. Find the slope and the y-intercept of the line $2x-3y=10$. (Answer: $2/3, -10/3$)
2. Find the intercepts made by the line $3x-4y=12$ on the axes. (Answer: -3 and 4)
3. Convert the equation $2x-5y-18=0$ into the slope-intercept form $y=mx+c$.
   $\left(Answer: y=\frac{2}{5}x-\frac{18}{5}\right)$
4. Convert the equation $3x-4y-10=0$ into the intercept form $\frac{x}{a}+\frac{y}{b}=1$.
   $\left(Answer: \frac{x}{10/3}+\frac{y}{-5/2}=1\right)$
5. Convert the equation $3x-4y+12=0$ into the normal form of the equation of the line.
   $\left(Answer: -\frac{3}{5}x+\frac{4}{5}y=\frac{12}{5}\right)$
6. If the area of the triangle formed by the line $x+y=a$ with the co-ordinate axes is 50 sq. units, find the values of a. (Answer: $a=\pm 10$)
7. If the area of the triangle formed by the line $a^2x+4y=12$ with the co-ordinate axes is 2 square

units, find the value of a. (Answer: $a = \pm 3$)

8. Write the sum of the intercepts of the line $3x - 4y = 12$ on the co-ordinate axes.

(Answer: 1)

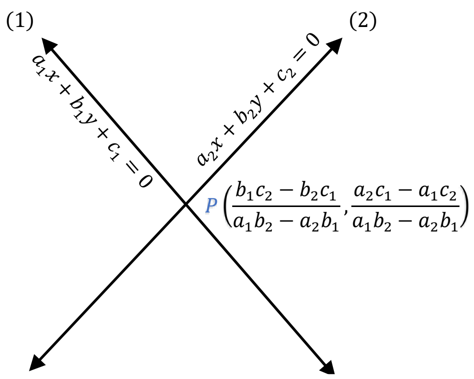
The point of intersection of two lines

If two lines are non-parallel, they intersect each other at a point which is known as their point of intersection. This point of intersection is the point common to both the lines. Hence, it satisfies the equations of both the lines. In other words, the point of intersection is obtained by solving the equations of both the lines simultaneously. Let two lines be

$a_1x + b_1y + c_1 = 0$ -------------- (1)

and $a_2x + b_2y + c_2 = 0$ -------------- (2)

Solving equations (1) and (2) simultaneously by cross-multiplication method, we get



Intersecting lines

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Longrightarrow \quad x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \quad \& \quad y = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\ ; \quad provided \quad a_1b_2 - a_2b_1 \neq 0.$$

$$Hence, the\ point\ of\ intersection\ of\ lines\ (1)\ and\ (2), is\ \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right).$$

Solved Examples

Example 1: Find the point of intersection of the lines $4x - 3y = 10$ and $2x + 5y + 6 = 0$.

Solution: The lines are $4x - 3y - 10 = 0$ ---------- (1)

$2x + 5y + 6 = 0$ ---------- (2)

Solving the equations (1) and (2) simultaneously by cross-multiplication method, we get

$$\frac{x}{(-3).6 - 5.(-10)} = \frac{y}{2.(-10) - 4.6} = \frac{1}{4.5 - 2.(-3)}$$

$$\Longrightarrow \quad \frac{x}{32} = \frac{y}{-44} = \frac{1}{26} \qquad \Longrightarrow \ x = \frac{32}{26} = \frac{16}{13}\ \ \&\ y = -\frac{44}{26} = -\frac{22}{13}$$

$$Hence, the\ point\ of\ intersection\ of\ lines, is\ \left(\frac{16}{13}, -\frac{22}{13}\right)$$

Example 2: Find the area of the triangle formed by the lines $3x - 4y + 4a = 0$, $2x - 3y + 4a = 0$ and $5x - y + a = 0$.

Solution: The lines are $3x - 4y + 4a = 0$ ---------- (1)

$2x - 3y + 4a = 0$ ---------- (2)

& $5x - y + a = 0$ ----------- (3)

Solving the equations (1) and (2), we get

$$\frac{x}{-16a+12a}=\frac{y}{8a-12a}=\frac{1}{-9+8} \quad \Longrightarrow \quad x=4a, \quad y=4a.$$

So, one vertex of the triangle, is $(4a,4a)$.

Solving the equations (2) and (3), we get

$$\frac{x}{-3a+4a}=\frac{y}{20a-2a}=\frac{1}{-2+15} \quad \Longrightarrow \quad x=\frac{a}{13}, \quad y=\frac{18a}{13}$$

So, the second vertex of the triangle, is $(a/13, 18a/13)$.

Solving the equations (1) and (3), we get

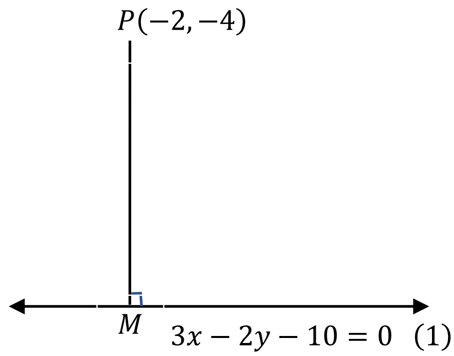
$$\frac{x}{-4a+4a}=\frac{y}{20a-3a}=\frac{1}{-3+20} \quad \Longrightarrow \quad x=0, \quad y=a.$$

So, the third vertex of the triangle, is $(0,a)$.

$$Hence, the\ area\ of\ the\ triangle=\frac{1}{2}\left|4a\left(\frac{18a}{13}-a\right)+\frac{a}{13}(a-4a+0)\right|=\frac{17}{26}a^2\ sq.units.$$

Example 3: Find the co-ordinates of the foot of perpendicular drawn from the point $(-2,-4)$ on the line $3x-2y=10$.

Solution: The given line is $3x-2y-10=0$ -------------- (1)



As in the figure, the point M is the foot of the perpendicular from $P(-2,-4)$ on the line (1).

The slope of the line (1), is 3/2. So, the slope of PM, is -2/3.

Clearly, the equation of PM, is

$$y+4=-\frac{2}{3}(x+2) \quad \Longrightarrow \quad 2x+3y+16=0 \quad --- \quad (2)$$

Solving (1) and (2), we get

$$\frac{x}{-32+30}=\frac{y}{-20-48}=\frac{1}{9+4} \quad \Longrightarrow \quad x=-\frac{2}{3}, \quad y=-\frac{68}{13}.$$

So, the foot of the perpendicular, is $\left(-\frac{2}{3},-\frac{68}{13}\right)$.

Example 4: Find the orthocentre of the triangle whose sides are $7x+y=10$, $x-2y+5=0$ and $x+y+2=0$.

Solution: The sides of the triangle are $7x+y-10=0$ ---------- (1)

$x-2y+5=0$ ---------- (2)

& $x+y+2=0$ ----------- (3)

The orthocentre of a triangle is the point at which the three altitudes of the triangle meet. We find the equations of any two altitudes, then their point of intersection is the orthocentre.

Solving the equations (1) and (2), we get

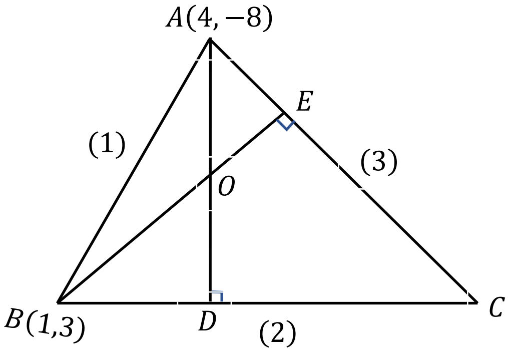
$$\frac{x}{5-20}=\frac{y}{-10-35}=\frac{1}{-14-1} \quad \Longrightarrow \quad x=1, \quad y=3$$

So, the vertex B of the triangle, is $(1,3)$.

Solving the equations (1) and (3), we get

$$\frac{x}{2+10}=\frac{y}{-10-14}=\frac{1}{1+2} \quad \Longrightarrow \quad x=4 \quad y=-8.$$

So, the vertex A of the triangle, is $(4,-8)$.



The slope of the side $BC = \frac{1}{2}$ $\Longrightarrow$ The slope of the altitude $AD = -2$

$\Longrightarrow$ The equation of the altitude AD, is $y+8=-2(x-4)$

$\Longrightarrow$ $2x+y=0$ ------------- (4)

The slope of the side $AC = -1$ $\Longrightarrow$ The slope of the altitude $BE = 1$

$\Longrightarrow$ The equation of the altitude AD, is $y-3=1.(x-1)$

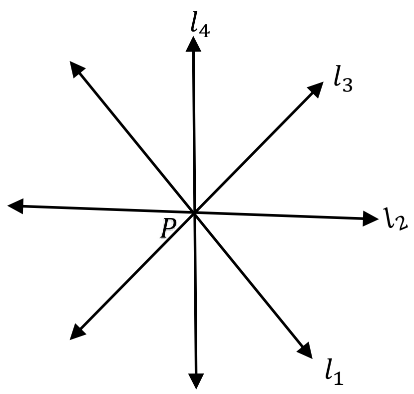
$\Longrightarrow$ $x-y+2=0$ ------------- (5)

Solving the equations (4) and (5), we get $x=-\frac{2}{3},\ y=\frac{4}{3}$.

$$\text{So, the orthocentre is } \left(-\frac{2}{3},\frac{4}{3}\right).$$

Concurrent Lines

Three or more lines are said to be concurrent if they pass through the same point. The point through which the lines pass, is called the point of concurrence.



Concurrent lines

In the figure, lines $l_1, l_2,\ l_3$ and l_4 are concurrent with the point P as their point of concurrence.

Concurrency of three lines

In order to prove the concurrency of given lines, we can find the point of intersection of any two of them and verify whether the remaining lines also pass through the same point. If so, the given lines are concurrent.

Let us consider three lines

$$a_1x + b_1y + c_1 = 0 \quad \text{-------------- (1)}$$
$$a_2x + b_2y + c_2 = 0 \quad \text{-------------- (2)}$$
and $$a_3x + b_3y + c_3 = 0 \quad \text{-------------- (3)}$$

$$\text{Clearly, the point of intersection of lines (1) and (2), is } \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right).$$

If the above lines are concurrent, then the line (3) also passes through this point and we have

$$a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

$$\Rightarrow \quad \boldsymbol{a_3(b_1c_2 - b_2c_1) + b_3(a_2c_1 - a_1c_2) + c_3(a_1b_2 - a_2b_1) = 0} \quad \text{------------- (4)}$$

which is the condition of concurrency of the three lines (1), (2) and (3).

This condition may also be given in the determinant form as follows:

$$\begin{vmatrix} \boldsymbol{a_1} & \boldsymbol{b_1} & \boldsymbol{c_1} \\ \boldsymbol{a_2} & \boldsymbol{b_2} & \boldsymbol{c_2} \\ \boldsymbol{a_3} & \boldsymbol{b_3} & \boldsymbol{c_3} \end{vmatrix} = \boldsymbol{0} \quad \textbf{------------- (5)}$$

There is yet another way to establish concurrency of three lines. If three constants p, q and r can be found so that

$p(a_1x + b_1y + c_1) + q(a_2x + b_2y + c_2) + r(a_3x + b_3y + c_3) = 0$ identically, then the three lines are concurrent.

For in this case, we have

$$a_3x + b_3y + c_3 = -\frac{p}{r}(a_1x + b_1y + c_1) - \frac{q}{r}(a_2x + b_2y + c_2) \quad \_\_\_\_\_\_(6)$$

The co-ordinates of the point of intersection of the first two of the lines make the right-hand side of (6) zero. Hence, the same co-ordinates make the left-hand side also zero. Thus, the point of intersection of the first two lines satisfy the equation of the third line and hence, the three lines are concur rent.

Remark: If the right-hand sides of the equations of three lines are zero and the sum of their left-hand sides yields zero, then the three lines are concurrent.

Hence, if $P = 0$, $Q = 0$ and $R = 0$ be three lines, then the lines are concurrent if $P + Q + R = 0$.

Solved Examples

Example 1: Examine whether the lines $2x - 3y + 5 = 0$, $3x + 4y = 7$ and $9x - 5y + 8 = 0$ are concurrent or not. If yes, find the point of concurrence.

Solution: The lines are $2x - 3y + 5 = 0$ ---------- (1)

$3x + 4y - 7 = 0$ ---------- (2)

& $9x - 5y + 8 = 0$ ----------- (3)

Solving the equations (1) and (2), we get

$$\frac{x}{21 - 20} = \frac{y}{15 + 14} = \frac{1}{8 + 9} \quad \Rightarrow \quad x = \frac{1}{17}, \quad y = \frac{29}{17}.$$

$$So, the\ point\ of\ intersection\ of\ the\ lines\ (1)\ and\ (2), is\ \left(\frac{1}{17}, \frac{29}{17}\right).$$

We now verify if the line (3) passes through this point. From (3), we have

$$9, \frac{1}{17} - 5.\frac{29}{17} + 8 = \frac{9 - 145}{17} + 8 = -\frac{136}{17} + 8 = -8 + 8 = 0\,.$$

which is true. So, the above lines pass through the same point. Hence, they are concurrent.

$$The\ point\ of\ concurrence, is\ \left(\frac{1}{17}, \frac{29}{17}\right).$$

Example 2: For which value of λ, the lines $y = x + 1$, $y = 2(x + 1)$ and $y = \lambda x + 3$ are concurrent ?

Solution: The lines are $x - y + 1 = 0$ ---------- (1)

$2x - y + 2 = 0$ ---------- (2)

& $\lambda x - y + 3 = 0$ ----------- (3)

For concurrency of the above lines, we must have

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 2 \\ \lambda & -1 & 3 \end{vmatrix} = 0 \qquad \Rightarrow -3+2++6-2\lambda+\lambda-2=0 \qquad \Rightarrow \qquad \lambda = 3.$$

Example 3: If $l+m+n=0$, show that the lines $lx+my+n=0$, $mx+ny+l=0$ and $nx+ly+m=0$ are concurrent.

Solution: The lines are $P=0$ ---------- (1), where, $P \equiv lx+my+n$

$Q=0$ ---------- (2), where, $Q \equiv mx+ny+l$

& $R=0$ ----------- (3), where, $R \equiv nx+ly+m$

Now, $P+Q+R=(lx+my+n)+(mx+ny+l)+(nx+ly+m)$

$= (l+m+n)x+(m+n+l)y+(n+l+m) = 0.x+0.y+0 = 0.$

which is identically zero, hence, the given lines are concurrent.

Assignment No. 5

1. Find the point of intersection of the lines $2x-5y=10$ and $x+3y+4=0$.

$Answer: \left(\frac{10}{11}, -\frac{18}{11}\right)$

2. Find the area of the triangle formed by the lines $y=c$, $y=mx$ and $x=0$.

$\left(Answer: \frac{c^2}{2m} sq.units\right)$

3. Show the area of the triangle whose sides have the equations $y=m_1x+c_1$, $y=m_2x+c_2$ $and\ \ x=0,\ is\ \ \frac{1}{2}\left|\frac{(c_1-c_2)^2}{m_1-m_2}\right|\ sq.units.$

4. Find the orthocentre of the triangle whose sides are $7x+y=10$, $x-2y+5=0$ and $x+y+2=0$.

$Answer: \left(-\frac{2}{3}, \frac{4}{3}\right)$

5. Find the orthocentre of the triangle whose vertices are $(0,0)$, $(2,-1)$ and $(-1,3)$.

$Answer: (-4,-3)$

6. Find the co-ordinates of the foot of perpendicular drawn from the point $(1,-4)$ on the line $2x-5y=7$.

$Answer: \left(-\frac{1}{29}, -\frac{41}{29}\right)$

Assignment No. 6

1. Examine whether the lines $x-4y=9$, $3x+2y+1=0$ and $5x-y=7$ are concurrent or not. If yes, find the point of concurrence.

$Answer: yes, (1,-2)$

2. For which value of λ, the lines $y=x+1$, $y=2(x+1)$ and $y=\lambda x+3$ are concurrent?

(Answer: 3)

3. If the lines $a_1x+b_1y+1=0$, $a_2x+b_2y+1=0$ and $a_3x+b_3y+1=0$ are concurrent, show that the points (a_1, b_1), (a_2, b_2) and (a_3, b_3) are collinear.

4. Show that the following lines are concurrent:

$(b-c)x+(c-a)y+(a-b)=0,$

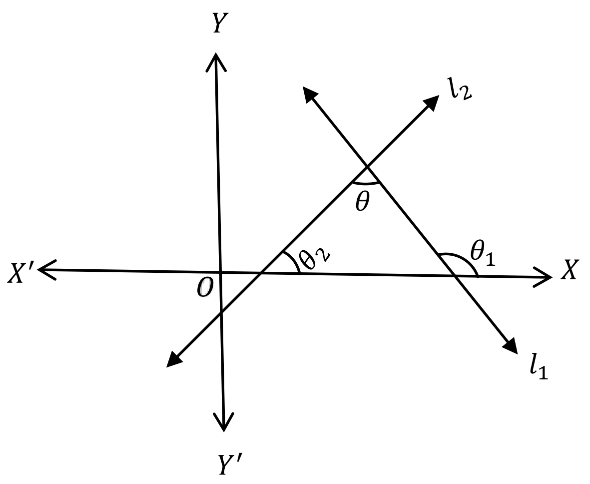
$(c-a)x+(a-b)y+(b-c)=0$

and $(a-b)x+(b-c)y+(c-a)=0.$

5.The co-ordinates of the vertices of a triangle are $(2, 3)$, $(-1, 5)$ and $(3,-4)$. Show that the medians of the triangle are concurrent.

Angle between two lines

Let two lines l_1 and l_2 with slopes m_1 and m_2 intersect each other at an angle θ.



As in the figure, θ_1 and θ_2 are the inclinations of the lines l_1 and l_2. Then $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$. Also, $\theta + \theta_2 = \theta_1 \Rightarrow \theta = \theta_1 - \theta_2$

$$\Rightarrow \tan\theta = \tan(\theta_1 - \theta_2) \Rightarrow \tan\theta = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} \Rightarrow \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

If θ is one angle between the lines l_1 and l_2, then the other angle between them can be taken as $(180^o - \theta)$. If θ is an acute angle, then $(180^o - \theta)$ is an obtuse angle. Moreover, $\tan(180^o - \theta) = -\tan\theta$ which is negative. Thus, we have, $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$

For the acute angle, $\tan\theta > 0$ and hence,

$$\boldsymbol{\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \quad or \quad \theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|}.$$

However, if the equations of two lines be in the general form as

$a_1 x + b_1 y + c_1 = 0$ -------------- (1)

and $a_2 x + b_2 y + c_2 = 0$ -------------- (2), then the angle θ between them is given by

$$tan\,\theta = \left|\frac{-\frac{a_1}{b_1} - \left(-\frac{a_2}{b_2}\right) =}{1 + \left(-\frac{a_1}{b_1}\right).\left(-\frac{a_2}{b_2}\right)}\right| = \left|\frac{\frac{a_1 b_2 - a_2 b_1}{b_1 b_2}}{\frac{a_1 a_2 + b_2 b_2}{b_1 b_2}}\right| = \left|\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_2 b_2}\right|$$

$$\Rightarrow \boldsymbol{\tan\theta = \left|\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_2 b_2}\right| \quad or \quad \theta = \tan^{-1}\left|\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_2 b_2}\right|}$$

Solved Examples

Example 1: Find the angle between the lines $3x - 5y = 7$ and $4x + 3y = 9$.

Solution: The lines are $3x - 5y - 7 = 0$ --------------- (1) and $4x + 3y - 9 = 0$ ----------- (2)

The slope of the line (1), is $m_1 = -\frac{3}{-5} = \frac{3}{5}$ and that of the line (2), is $m_2 = -\frac{4}{3}$.

If θ be the angle between the above lines, then by formula

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{\frac{3}{5} - \left(-\frac{4}{3}\right)}{1 + \frac{3}{5}.\left(-\frac{4}{3}\right)}\right| = \left|\frac{\frac{29}{15}}{\frac{1}{5}}\right| = \frac{29}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{29}{3}\right).$$

Example 2: Find the angle between the lines $y = (2 - \sqrt{3})x + 4$ and $(2 + \sqrt{3})x - y = 11$.

Solution: The given lines are $(2 - \sqrt{3})x - y + 4 = 0$ --------------- (1)

and $(2 + \sqrt{3})x - y - 11 = 0$ -------------- (2)

The slope of the line (1), is $m_1 = (2 - \sqrt{3})$ and that of the line (2), is $m_2 = (2 + \sqrt{3})$.

If θ be the angle between the above lines, then by formula

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{(2 - \sqrt{3}) - (2 + \sqrt{3})}{1 + (2 - \sqrt{3}).(2 + \sqrt{3})}\right| = \left|\frac{-2\sqrt{3}}{2}\right| = \sqrt{3} = \tan 60^0 \Rightarrow \theta = 60^o.$$

Example 3: Find the angle between the lines $(m^2 - mn)y = (mn + n^2)x + n^3$ and $(mn + m^2)y = (mn - n^2)x + m^3$.

Solution: The given lines are $(mn + n^2)x - (m^2 - mn)y + n^3 = 0$ --------------- (1)

and $(mn - n^2)x - (mn + m^2)y + m^3 = 0$ --------------- (2)

If θ be the angle between the above lines, then by formula

$$\tan\theta = \left|\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_2 b_2}\right| = \left|\frac{(mn + n^2).\{-(mn + m^2)\} - (mn - n^2).\{-(m^2 - mn)\}}{(mn + n^2).(mn - n^2) + \{-(m^2 - mn)\}.\{-(mn + m^2)\}}\right|$$

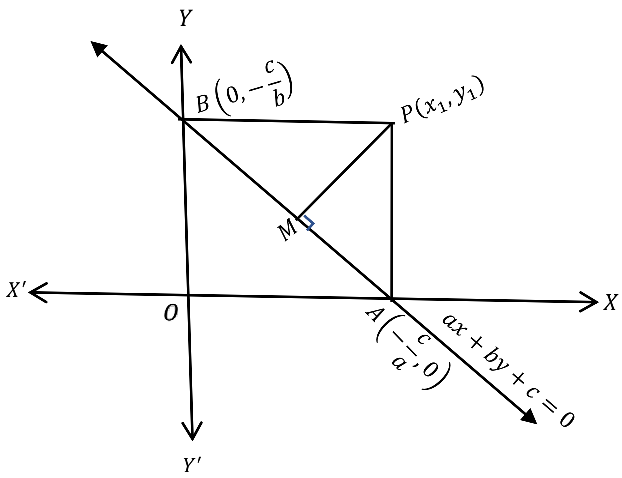
$$\Rightarrow \tan\theta = \left|\frac{-m^2n^2 - m^3n - mn^3 - m^2n^2 + m^3n - m^2n^2 - m^2n^2 + mn^3}{m^2n^2 - n^4 + m^3n + m^4 - m^2n^2 - m^3n}\right|$$

$$\Rightarrow \tan\theta = \left|\frac{-4m^2n^2}{m^4 - n^4}\right| \Rightarrow \theta = \tan^{-1}\left|\frac{4m^2n^2}{m^4 - n^4}\right|$$

which is the required angle between the two lines.

Distance of a line from a point

Let the given line be $ax + by + c = 0$ --------------- (1) and the given point be $P(x_1, y_1)$.



As in the figure, $PM \perp AB$ is drawn. Also, PA and PB are joined. Then the length PM of the perpendicular is the distance of the given line AB from the given point $P(x_1, y_1)$. Clearly, the points of intersection of the line AB with the axes, are $A\left(-\frac{c}{a}, 0\right)$ and $B\left(0, -\frac{c}{b}\right)$. In the

$$Area\ of\ \Delta PAB = \frac{1}{2}.AB.PM = \frac{1}{2}\sqrt{\left(0+\frac{c}{a}\right)^2 + \left(-\frac{c}{b}-0\right)^2}\ .PM = \frac{1}{2}\sqrt{\frac{c^2}{a^2}+\frac{c^2}{b^2}}.PM$$

$$= \frac{c}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}.PM = \frac{c}{2}\sqrt{\frac{b^2+a^2}{a^2b^2}}.PM = \frac{c}{2ab}\sqrt{a^2+b^2}.PM \text{ \_\_\_\_\_\_\_\_\_(2)}$$

$$Again, area\ of\ \Delta PAB = \frac{1}{2}\left|x_1\left(0+\frac{c}{b}\right) - \frac{c}{a}\left(-\frac{c}{b}-y_1\right) + 0(y_1-0)\right| = \frac{1}{2}\left|\frac{cx_1}{b}+\frac{c^2}{ab}+\frac{cy_1}{a}\right|$$

$$= \frac{c}{2}\left|\frac{x_1}{b}+\frac{c}{ab}+\frac{y_1}{a}\right| = \frac{c}{2}\left|\frac{ax_1+c+by_1}{ab}\right| = \frac{c}{2ab}|ax_1+by_1+c| \text{ \_\_\_\_\_\_\_\_ (3)}$$

From (2) and (3), we have

$$\frac{c}{2ab}\sqrt{a^2+b^2}.PM = \frac{c}{2ab}|ax_1+by_1+c|$$

$$\Rightarrow \quad PM = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$

$$\Rightarrow \quad \boldsymbol{Distance} = \left|\frac{\boldsymbol{ax_1+by_1+c}}{\sqrt{\boldsymbol{a^2+b^2}}}\right|$$

This formula is known as perpendicular distance formula.

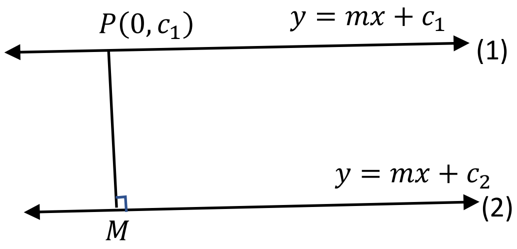
If the given point is the origin, then $x_1 = 0$ and $y_1 = 0$. **Hence, the distance of the line from the origin, is given by $\left|\frac{c}{\sqrt{a^2+b^2}}\right|$ which is the length of the normal to the line drawn from the origin.**

Distance between two parallel lines

Let two parallel lines be $\quad y = mx + c_1$ ----------------- (1)

and $\quad y = mx + c_2$ ------------------ (2)

As the lines (1) and (2) are parallel to each other, they have the same slope m and their y-intercepts are c_1 and c_2 respectively. In order to find the distance between the above lines, we can choose any point on one line and then find the distance of the other line from it.



On the line (1), if $x = 0$, then $y = c_1 \implies (0, c_1)$ is a point on the line (1). So, as in the figure, the point $P(0, c_1)$ lies on the line (1). From P, we find the distance of the line (2) as the perpendicular distance PM. Thus, we have

$$PM = \left|\frac{m.0 - c_1 + c_2}{\sqrt{m^2 + (-1)^2}}\right| = \left|\frac{c_1 - c_2}{\sqrt{1 + m^2}}\right|.$$

$$\implies \textbf{Distance between two parallel lines} = \left|\frac{\boldsymbol{c_1 - c_2}}{\sqrt{\boldsymbol{1 + m^2}}}\right| \_\_\_\_\_\_\_\_\_(\mathbf{3})$$

where c_1 *and* c_2 are the y-intercepts of the lines and m their slope.
However, if the equations of two parallel lines be in general form as
$ax + by + c_1 = 0$ -------------- (4)
and $ax + by + c_2 = 0$ -------------- (5), then the distance between them is given by

$$\left|\frac{\text{difference of y intercepts}}{\sqrt{1 + \text{slope}^2}}\right| = \left|\frac{\frac{c_1}{b} - \frac{c_2}{b}}{\sqrt{1 + \frac{a^2}{b^2}}}\right| = \left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$$

$$\textbf{Then, the distance between two parallel lines} = \left|\frac{\boldsymbol{c_1 - c_2}}{\sqrt{\boldsymbol{a^2 + b^2}}}\right| \_\_\_\_\_\_\_\_\_(\mathbf{6})$$

where, c_1 *and* c_2 are simply the constant terms of the lines (4) and (5).

Remark: Students are advised to note the contrast between the formulae (3) and (6) regarding the meanings of symbols c_1 *and* c_2.

Solved Examples

Example 1: Find the distance of the line $3x = 15 - 4y$ from the point $(4, -7)$. Also, find the distance of the line from the origin.

Solution: The given line, is $3x + 4y - 15 = 0$ and the given point, is $(4, -7)$.

$$\implies \text{By formula, distance} = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right| = \left|\frac{3.4 + 4.(-7) - 15}{\sqrt{3^2 + 4^2}}\right| = \left|\frac{-31}{5}\right| = \frac{31}{5}.$$

$$\text{Also, distance from the origin} = \left|\frac{c}{\sqrt{a^2 + b^2}}\right| = \left|\frac{-15}{\sqrt{3^2 + 4^2}}\right| = \left|\frac{-15}{5}\right| = 3.$$

Example 2: Find the distance of the line $x \cos\beta + y \sin\beta = a$ from the point $(a\cos\alpha, a\sin\alpha)$.

Solution: The given line, is $x\cos\beta + y\sin\beta - a = 0$ and the given point, is $(a\cos\alpha, a\sin\alpha)$.

$$\text{By formula, distance} = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right| = \left|\frac{a\cos\alpha.\cos\beta + a\sin\alpha.\sin\beta - a}{\sqrt{cos^2\beta + sin^2\beta}}\right|$$

$$= |a\{\cos(\alpha - \beta) - 1\}| = |a\{1 - \cos(\alpha - \beta)\}| = 2a\sin^2\left(\frac{\alpha - \beta}{2}\right).$$

***Example 3**: If p be the length of perpendicular from the origin to a line whose intercepts on the axes are a and b, show that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

Solution: Let a and b be the x-intercept and y-intercept respectively of the line. Then the equation of line is

$$\frac{x}{a}+\frac{y}{b}=1 \quad \Rightarrow \frac{x}{a}+\frac{y}{b}-1=0 \;------\; (1)$$

$$Given\ that\ \ p=\left|\frac{-1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2}}\right| \quad \Rightarrow \ p^2=\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}} \ \ (Squaring\ both\ sides)$$

$$\Rightarrow \ \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2} \quad (Taking\ reciprocals\ both\ sides)$$

***Example 4**:If p and p' be the lengths of perpendiculars from the origin upon the straight lines $x\sec\theta+y\operatorname{cosec}\theta=a$ and $x\cos\theta-y\sin\theta=a\cos 2\theta$ respectively, prove that $4p^2+{p'}^2=a^2$.

Solution: The given lines are $x\sec\theta+y\operatorname{cosec}\theta-a=0$ ----------------- (1)

& $x\cos\theta-y\sin\theta-a\cos 2\theta=0$ --------------- (2)

Given that p and p' are the lengths of perpendiculars from the origin on the lines (1) and (2) respectively, we have

$$p=\left|\frac{-a}{\sqrt{\sec^2\theta+\operatorname{cosec}^2\theta}}\right| \quad \Rightarrow \ p=\left|\frac{a}{\sqrt{\frac{1}{\cos^2\theta}+\frac{1}{\sin^2\theta}}}\right|=\left|\frac{a}{\sqrt{\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta\cos^2\theta}}}\right|=|a\sin\theta\cos\theta|$$

$$\Rightarrow \quad p^2=a^2\sin^2\theta\cos^2\theta \quad \text{----------------- (3)}$$

$$\& \quad p'=\left|\frac{-a\cos 2\theta}{\sqrt{(\cos\theta)^2+(-\sin\theta)^2}}\right| \quad \Rightarrow \ p'=\left|\frac{a\cos 2\theta}{\sqrt{\cos^2\theta+\sin^2\theta}}\right|=|a\cos 2\theta|$$

$$\Rightarrow \quad {p'}^2=a^2\cos^2 2\theta \quad \text{------------------- (4)}$$

Now, $4p^2+{p'}^2=4a^2\sin^2\theta\cos^2\theta+a^2\cos^2 2\theta=a^2(4\sin^2\theta\cos^2\theta+\cos^2 2\theta)$

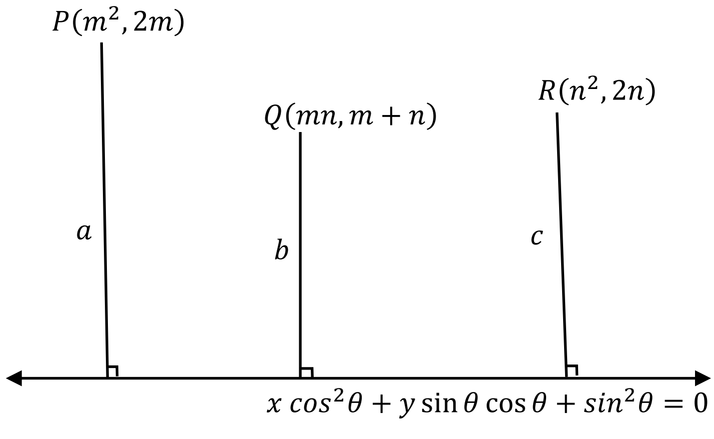
$=a^2[(2\sin\theta\cos\theta)^2+\cos^2 2\theta]=a^2(\sin^2 2\theta+\cos^2 2\theta)$

$=a^2.$

***Example 5:** Prove that the lengths of perpendiculars from the points $P(m^2,\ 2m)$, $Q(mn,\ m+n)$ and $R(n^2,2n)$ to the line $x\,cos^2\theta+y\sin\theta\cos\theta+sin^2\theta=0$ are in G.P. .

Solution: The given line is $x\,cos^2\theta+y\sin\theta\cos\theta+sin^2\theta=0$ ------------------- (1)

Let the lengths of perpendiculars from the points P,Q and R be a,b and c respectively, we need to prove that a,b and c are in G.P., i.e., $b^2=ac$.



$$Clearly, \quad a=\left|\frac{m^2cos^2\theta+2m\sin\theta\cos\theta+sin^2\theta}{\sqrt{(cos^2\theta)^2+(\sin\theta\cos\theta)^2}}\right|=\left|\frac{(m\cos\theta+\sin\theta)^2}{\sqrt{\cos^4\theta+sin^2\theta\,cos^2\theta}}\right|$$

$$\Rightarrow \quad a = \left|\frac{(m\cos\theta + \sin\theta)^2}{\sqrt{\cos^2\theta(cos^2\theta + sin^2\theta)}}\right| \qquad \Rightarrow \quad a = \left|\frac{(m\cos\theta + \sin\theta)^2}{\cos\theta}\right| \text{\_\_\_\_\_\_\_}(2)$$

$$Also, \quad b = \left|\frac{mn\, cos^2\theta + (m+n)\sin\theta\cos\theta + sin^2\theta}{\sqrt{(cos^2\theta)^2 + (\sin\theta\cos\theta)^2}}\right|$$

$$\Rightarrow \quad b = \left|\frac{(m\cos\theta + \sin\theta)(n\cos\theta + \sin\theta)}{\cos\theta}\right| \text{\_\_\_\_\_\_\_\_}(3)$$

$$and, \quad c = \left|\frac{n^2 cos^2\theta + 2n\sin\theta\cos\theta + sin^2\theta}{\sqrt{(cos^2\theta)^2 + (\sin\theta\cos\theta)^2}}\right| \quad \Rightarrow \quad a = \left|\frac{(n\cos\theta + \sin\theta)^2}{\cos\theta}\right| \text{\_\_\_\_\_\_\_}(4)$$

From, (2), (3) and (4), we get $b^2 = ac$.
This proves that a, b and c are in G.P..

***Example 6**: Prove that the product of lengths of perpendiculars on the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ from the points $\left(\pm\sqrt{a^2 - b^2},\ 0\right)$, is b^2. (BTE, Delhi, 2014, 2016)

Solution: The given line is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta - 1 = 0$ ------------------- (1)

The length of the perpendicular from the point $\left(\sqrt{a^2 - b^2},\ 0\right)$ on the line (1), is

$$\left|\frac{\frac{\sqrt{a^2 - b^2}}{a}\cos\theta - 1}{\sqrt{\left(\frac{\cos\theta}{a}\right)^2 + \left(\frac{\sin\theta}{b}\right)^2}}\right| = p\ (say)$$

$$\Rightarrow \left|\frac{\frac{\sqrt{a^2 - b^2}}{a}\cos\theta - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right| = p \ ----(2)$$

The length of the perpendicular from the point $\left(-\sqrt{a^2 - b^2},\ 0\right)$ on the line (1), is

$$\left|\frac{\frac{-\sqrt{a^2 - b^2}}{a}\cos\theta - 1}{\sqrt{\left(\frac{\cos\theta}{a}\right)^2 + \left(\frac{\sin\theta}{b}\right)^2}}\right| = q\ (say)$$

$$\Rightarrow \left|\frac{\frac{\sqrt{a^2 - b^2}}{a}\cos\theta + 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right| = q \ ----(3)$$

Multiplying (2) and (3), we get

$$pq = \left|\frac{\frac{\sqrt{a^2 - b^2}}{a}\cos\theta - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right| \left|\frac{\frac{\sqrt{a^2 - b^2}}{a}\cos\theta + 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right| = \left|\frac{\frac{a^2 - b^2}{a^2}\cos^2\theta - 1}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}\right|$$

$$\Rightarrow \quad pq = \left|\frac{\frac{a^2\cos^2\theta - b^2\cos^2\theta - a^2}{a^2}}{\frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2}}\right|$$

$$\Rightarrow \quad pq = \left|\left(\frac{a^2 - a^2\cos^2\theta + b^2\cos^2\theta}{a^2}\right).\left(\frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}\right)\right|$$

$$\Rightarrow \quad pq = \left|\frac{b^2\{a^2(1 - \cos^2\theta) + b^2\cos^2\theta\}}{b^2\cos^2\theta + a^2\sin^2\theta}\right| = b^2\left|\frac{a^2\sin^2\theta + b^2\cos^2\theta}{b^2\cos^2\theta + a^2\sin^2\theta}\right| = b^2.$$

Example 7: Find the distance between the parallel lines $3x = 15 - 4y$ and $6x + 8y + 5 = 0$.

Solution: The parallel lines are

$$3x + 4y - 15 = 0 \text{ ------------- } (1)$$
$$\& \qquad 6x + 8y + 5 = 0 \text{ ------------- } (2)$$

Clearly, the slope of both the line is $m = (-3/4)$, their y-intercepts are $c_1 = 15/4$ and $c_2 = -5/8$. By formula,

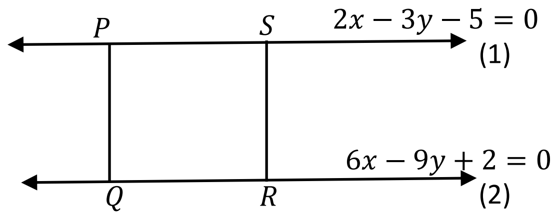
$$\text{The distance between two parallel lines} = \left|\frac{c_1 - c_2}{\sqrt{1+m^2}}\right| = \left|\frac{\frac{15}{4} - \left(-\frac{5}{8}\right)}{\sqrt{1+\left(\frac{-3}{4}\right)^2}}\right| = \left|\frac{\frac{35}{8}}{\frac{5}{4}}\right| = \frac{7}{2}.$$

***Example 8**: Find the area of the square whose one pair of opposite sides lie on the parallel lines $2x = 5 + 3y$ and $6x = 9y - 2 = 0$.

Solution: The parallel lines are

$$2x - 3y - 5 = 0 \text{ ------------ } (1)$$
$$\& \qquad 6x - 9y + 2 = 0 \text{ ------------ } (2)$$

In the figure, $PQRS$ is a square whose one pair of opposite sides PS and QR lie on the parallel lines.



The slope of parallel lines is $m = (2/3)$, their y-intercepts are $c_1 = -5/3$ and $c_2 = 2/9$. By formula,

$$\text{PQ} = \text{The distance between two parallel lines} = \left|\frac{c_1 - c_2}{\sqrt{1+m^2}}\right| = \left|\frac{-\frac{5}{3} - \frac{2}{9}}{\sqrt{1+\left(\frac{2}{3}\right)^2}}\right| = \left|\frac{\frac{17}{9}}{\frac{\sqrt{13}}{3}}\right| = \frac{17}{3\sqrt{13}}.$$

$$\text{As in the figure, the area of the square } PQRS = PQ^2 = \left(\frac{17}{3\sqrt{13}}\right)^2 = \frac{289}{117} \text{ sq. units.}$$

Assignment No. 7

1. Find the distance of the line $3x = 5 + 4y$ from the origin. Also, find the distance of the line from the point $(-2, -4)$. $(Answer: 1,\ 1)$

2. Find the length of the perpendicular drawn from the point $(2, -3)$ to the line $6x - y - 14 = 0$. $\left(Answer: 1/\sqrt{37}\right)$

***3.** If p be the length of perpendicular from the origin to a line whose intercepts on the axes are a and b, show that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

***4.** If p and p' be the lengths of perpendiculars from the origin upon the straight lines $x \sec\theta + y \operatorname{cosec}\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$ respectively, prove that $4p^2 + {p'}^2 = a^2$.

***5.** Prove that the lengths of perpendiculars from the points $P(m^2,\ 2m)$, $Q(mn,\ m+n)$ and $R(n^2, 2n)$ to the line $x\,cos^2\theta + y\sin\theta\cos\theta + sin^2\theta = 0$ are in G.P. .

***6.** Prove that the product of lengths of perpendiculars on the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ from the points $\left(\pm\sqrt{a^2 - b^2},\ 0\right)$, is b^2.

7. Find the distance between the parallel lines $x = 10 - 5y$ and $3x + 15y + 7 = 0$. $\left(Answer: 37/3\sqrt{26}\right)$

8. Find the area of the square whose one pair of opposite sides lie on the parallel lines $x = 4 + 3y$ and $2x = 6y - 7 = 0$. $\left(Answer:\ \frac{45}{8}\ sq.units\right)$

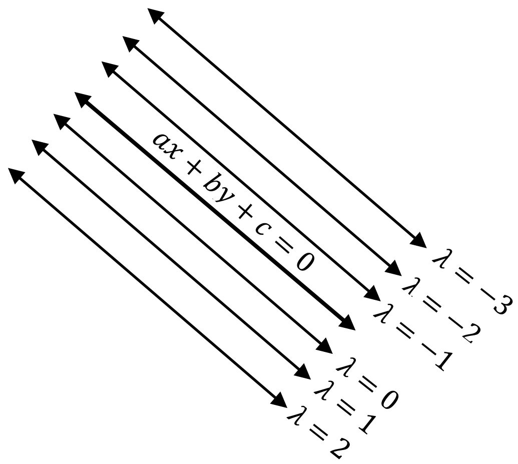
9. Find the angle between the lines $x - 4y = 6$ and $6x - y = 13$. $\left(Answer:\ \tan^{-1}\left(\frac{23}{10}\right)\right)$

10. Find the angle between the lines $3x + y + 5 = 0$ and $x + 2y + 8 = 0$. $(Answer: \ 45^o)$

Family of lines

By 'family of lines', we mean the set or collection of all lines having a common feature. Such lines can be represented by a common equation involving an arbitrary constant, often called as a parameter. Each member-line of the family corresponds to a fixed value of the parameter. Below, we discuss three types of family of lines.

1. **Family of lines parallel to a given line:** Let the given line be $ax + by + c = 0$ ----------- (1)



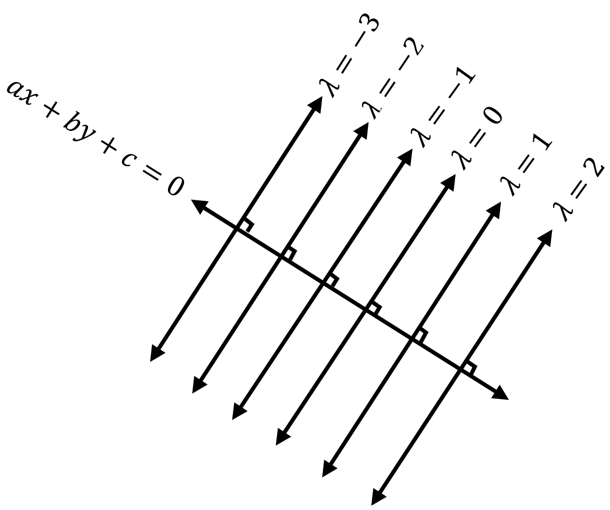
The slope of the line (1), is $(-a/b)$, and any line parallel to it, will also have the same slope, i.e., the same coefficients of x and y. **So, the equation of a line parallel to the line (1), can be taken as $\boldsymbol{ax + by + \lambda = 0}$ --------------------- (2),** where λ is an arbitrary constant, called parameter. For every value of the parameter λ, the equation obtained from (2), represents a line which is parallel to the given line. If we go on giving different values to λ, we shall be getting different lines each of which is parallel to the line (1). In the figure, some lines are shown for particular values of λ. This way, the equation (2), represents the family of lines parallel to the given line. In order to find a particular member-line of the family, an additional condition is required which determines the corresponding value of λ.

Thus, we get the following procedure of getting the equation of a line parallel to a given line:

Working Rule: 1. The equation of the given line is noted down.

2. Its constant term is replaced by the parameter λ.
3. The value of λ, is found using the additional condition.
4. The parameter λ is replaced by its value in the above equation to get the required equation.

2. **Family of lines perpendicular to a given line:** Let the given line be $ax + by + c = 0$ ----- (1)



Since, the slope of the line (1), is $\left(-\frac{a}{b}\right)$, *so*, the slope of a line perpendicular to *it*, is $\frac{b}{a}$

and this is possible when the coefficients of x and y of the given equation are interchanged and the sign of one of them is also changed.

Hence, the equation of any line perpendicular to the above line, is

$\boldsymbol{bx - ay + \lambda = 0}$ ------------ ***(2),*** where λ is a parameter.

Clearly, the equation (2) represents the family of lines perpendicular to the given line.

Thus, we get the following procedure of getting the equation of a line perpendicular to a given line:
Working Rule: 1. The equation of the given line is noted down.

2. The coefficients of x and y are interchanged by changing the sign of one of them.
3. The constant term of the given equation is replaced by the parameter λ which is found using the additional condition.
4. The parameter λ is replaced by its value in the above equation so as to get the required equation of the line.

3. **Family of lines passing through the point of intersection of two given lines:** Let two given lines be

$a_1x + b_1y + c_1 = 0$ -------------- (1)

and $a_2x + b_2y + c_2 = 0$ -------------- (2)

If $P(x_1, y_1)$ be the point of intersection of the lines (1) and (2), then the equations (1) and (2) will be satisfied by the coordinates of the point P and hence, we have

$a_1x_1 + b_1y_1 + c_1 = 0$ -------------- (3)

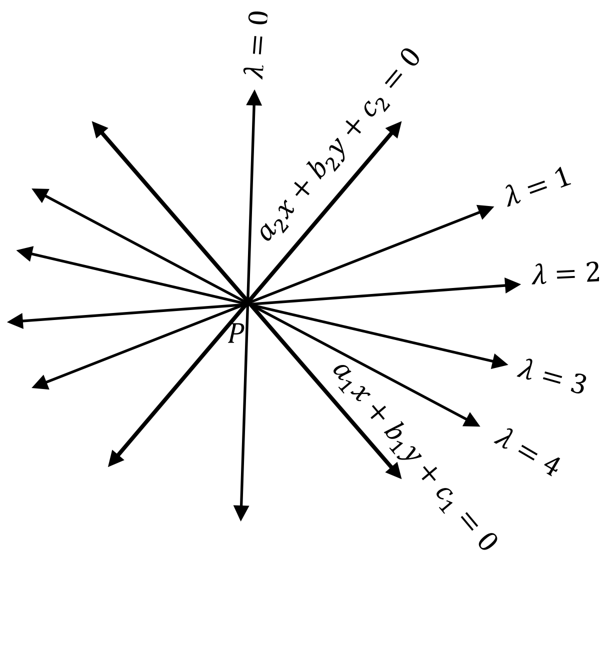
and $a_2x_1 + b_2y_1 + c_2 = 0$ -------------- (4)

Let us consider the equation

$(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$ ----------------- (5), where λ is an arbitrary constant.

The equation (5) being linear in x and y represents a straight line. It can be verified that the equation (5) represents such a line which always passes through the point of intersection of the given lines (1) and (2), irrespective of the value of λ; since because the equation (5) is satisfied by the coordinates of $P(x_1, y_1)$,

i.e., *using* (3) *and* (4), *we have* $(a_1x_1 + b_1y_1 + c_1) + \lambda(a_2x_1 + b_2y_1 + c_2) = 0 + \lambda.0 = 0.$
which is true.



Thus, **we conclude that any line passing through the point of intersection of the lines (1) and (2), is given by** $(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$ **or** $S_1 + \lambda S_2 = 0$, **where** $S_1 \equiv a_1x + b_1y + c_1$ *or LHS of* (1) & $S_2 \equiv a_2x + b_2y + c_2$ *or LHS of* (2).

If λ is a parameter, the above equation represents the family of lines passing through the point of intersection of the given lines.

Finally, we get the following procedure of getting the equation of a line passing through the point of intersection of two given lines:

Working Rule: 1. The equations of the given lines are noted down with all terms on their left hand sides so that the right-hand sides are zero.

2. The equation is written in the form $S_1 + \lambda S_2 = 0$, where S_1 and S_2 are the left hand sides of both the equations.
3. The parameter λ is found using the additional condition.
4. The parameter λ is replaced by its value in the above equation so as to get the required equation of the line.

Solved Examples

Example 1: Find the equation of the line parallel to the line $3x = 2y - 4$ and passing through the point $(2, 7)$.

Solution (first method): The given line is $3x - 2y + 4 = 0$ ------------- (1)

Any line parallel to the line (1), is given by $3x - 2y + \lambda = 0$ ---------- (2)

Since, the line (2) passes through the point $(2, 7)$, we have $3.2 - 2.7 + \lambda = 0 \quad \Rightarrow \quad \lambda = 8$

Hence, from (2), the required equation of the line, is $3x - 2y + 8 = 0$.

Second method: The slope of the line $3x - 2y + 4 = 0$, is 3/2. So, the slope of the line parallel to it, is 3/2.

Since, the line passes through the point $(2, 7)$, hence, its equation using the slope-point form, is given by $y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - 7 = \frac{3}{2}(x - 2) \quad \Rightarrow \quad 2y - 14 = 3x - 6$

$$\Rightarrow \quad 3x - 2y + 8 = 0.$$

Example 2: Find the equation of the line perpendicular to the line $4x - 3y = 8$ and passing through the point $(2, 3)$.

Solution (first method): The given line is $4x - 3y - 8 = 0$ ------------- (1)

Any line perpendicular to the line (1), is given by $3x + 4y + \lambda = 0$ ---------- (2)

Since, the line (2) passes through the point $(2, 3)$, we have $3.2 + 4.3 + \lambda = 0 \quad \Rightarrow \quad \lambda = -18$

Hence, from (2), the required equation of the line, is $3x + 4y - 18 = 0$.

Second method: The slope of the line $4x - 3y - 8 = 0$, is 4/3. So, the slope of the line perpendicular to it, is $-3/4$.

Since, the line passes through the point $(2, 3)$, hence, its equation using the slope-point form, is given by $y - y_1 = m(x - x_1) \quad \Rightarrow \quad y - 3 = -\frac{3}{4}(x - 2) \quad \Rightarrow \quad 4y - 12 = -3x + 6$

$$\Rightarrow \quad 3x + 4y - 18 = 0.$$

Example 3: Show that the equation of the line perpendicular to the line $x \sec\theta + y \operatorname{cosec}\theta = a$ and passing through the point $(a\cos^3\theta, a\sin^3\theta)$, is $x\cos\theta - y\sin\theta = a\cos 2\theta$.

Solution (first method):The given line is $x\sec\theta + y\operatorname{cosec}\theta - a = 0$ ------------- (1)

Any line perpendicular to the line (1), is given by $x\operatorname{cosec}\theta - y\sec\theta + \lambda = 0$ ---------- (2)

Since, the line (2) passes through the point $(a\cos^3\theta, a\sin^3\theta)$, we have

$a\cos^3\theta . \operatorname{cosec}\theta - a\sin^3\theta . \sec\theta + \lambda = 0$

$$\Rightarrow \quad \lambda = -\left(\frac{a\cos^3\theta}{\sin\theta} - \frac{a\sin^3\theta}{\cos\theta}\right) = -\left(\frac{a\cos^4\theta - a\sin^4\theta}{\sin\theta\cos\theta}\right)$$
$$= -a\left\{\frac{(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)}{\sin\theta\cos\theta}\right\} = -a\cos 2\theta\,.\sec\theta\,\mathrm{cosec}\,\theta$$
Hence, from (2), the required equation of the line, is
$x\,\mathrm{cosec}\,\theta - y\sec\theta - a\cos 2\theta\,.\sec\theta\,\mathrm{cosec}\,\theta = 0 \quad \Rightarrow \quad x\cos\theta - y\sin\theta = a\cos 2\theta.$
Second method): The slope of the line $x\sec\theta + y\,\mathrm{cosec}\,\theta - a = 0$, is $-\frac{\sec\theta}{\mathrm{cosec}\,\theta} = -\tan\theta$.
The slope of a line perpendicular to the above line, is $\cot\theta$.
Since, the line passes through the point $(a\cos^3\theta, a\sin^3\theta)$, so, its equation is
$y - a\sin^3\theta = \cot\theta\,(x - a\cos^3\theta) \quad \Rightarrow \quad x\,cot\,\theta - y = a\,cot\,\theta\cos^3\theta - a\,sin^3\,\theta.$
$$\Rightarrow \quad x\cos\theta - y\sin\theta = (a\cos^4\theta - a\sin^4\theta) = a(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$$
$$\Rightarrow \quad x\cos\theta - y\sin\theta = a\cos 2\theta.$$
Example 4: Find the equation of the line passing through the point of intersection of the lines $4x - y = 7$ and $5x - 2y + 3 = 0$ and having the slope -3.
Solution (first method): The given lines, are $4x - y - 7 = 0$ ------------- (1)
& $5x - 2y + 3 = 0$ ------------ (2)
Any line passing through the point of intersection of the lines (1) and (2), is given by
$S_1 + \lambda S_2 = 0 \quad \Rightarrow \quad (4x - y - 7) + \lambda(5x - 2y + 3) = 0$
$\Rightarrow$ $(4+5\lambda)x - (1 + 2\lambda)y - 7 + 3\lambda = 0$ ---------- (3)
Since, the slope of the line (3), is -3, we have
$$-\frac{(4 + 5\lambda)}{-(1 + 2\lambda)} = -3 \quad \Rightarrow \quad 4 + 5\lambda = -3(1 + 2\lambda) \quad \Rightarrow \quad 5\lambda + 6\lambda = -3 - 4 \quad \Rightarrow \quad \lambda = -\frac{7}{11}.$$
Substituting the value of λ in (3), we get
$$\left\{4 + 5\left(-\frac{7}{11}\right)\right\}x - \left\{1 + 2\left(-\frac{7}{11}\right)\right\}y - 7 + 3\left(-\frac{7}{11}\right) = 0.$$
$$\frac{9}{11}x + \frac{3}{11}y - \frac{98}{11} = 0$$
Hence, the required equation of the line, is $9x + 3y - 98 = 0$.
Second method: The given lines, are $4x - y - 7 = 0$ ------------- (1)
& $5x - 2y + 3 = 0$ ------------ (2)
The point of intersection of the lines (1) and (2), is given by
$$\frac{x}{-3 - 14} = \frac{y}{-35 - 12} = \frac{1}{-8 + 5} \quad \Rightarrow \quad x = \frac{17}{3},\ y = \frac{47}{3}.$$
The point of intersection of the lines (1) and (2), is $\left(\frac{17}{3}, \frac{47}{3}\right)$.
Hence, the equation of the line using the slope-point form, is
$$y - \frac{47}{3} = -3\left(x - \frac{17}{3}\right) \quad \Rightarrow \quad 3y - 47 = -9x + 51$$
$\Rightarrow 9x + 3y - 98 = 0$.
Example 5: Find the equations of the lines passing through the point of intersection of the lines $3x - y = 2$ and $x - 4y + 3 = 0$ and equally inclined to the axes.
Solution (first method): The given lines, are $3x - y - 2 = 0$ ------------- (1)
& $x - 4y + 3 = 0$ ------------ (2)
Any line passing through the point of intersection of the lines (1) and (2), is given by
$S_1 + \lambda S_2 = 0 \quad \Rightarrow \quad 3x - y - 2 + \lambda(x - 4y + 3) = 0$
$\Rightarrow$ $(3+\lambda)x - (1 + 4\lambda)y - 2 + 3\lambda = 0$ ---------- (3)
The slope of the line (3), is $-\frac{(3+\lambda)}{-(1+4\lambda)} = \frac{3+\lambda}{1+4\lambda}$
Since, the slope of the lines equally inclined to the axes, is ± 1, we have
$$\frac{3 + \lambda}{1 + 4\lambda} = \pm 1 \quad \Rightarrow \quad 3 + \lambda = \pm 1, (1 + 4\lambda) \quad \Rightarrow \quad 3 + \lambda = \pm(1 + 4\lambda).$$
Taking the positive and the negative signs of the right hand side separately, we get

$$3+\lambda=(1+4\lambda) \quad \& \quad 3+\lambda=-(1+4\lambda) \quad \Rightarrow \quad \lambda=\frac{2}{3}, \quad -\frac{4}{5}$$

Substituting the value of λ in (3), we get

$$\left(3+\frac{2}{3}\right)x-\left(1+4.\frac{2}{3}\right)y-2+3.\frac{2}{3}=0 \quad \& \quad \left(3-\frac{4}{5}\right)x-\left(1-4.\frac{4}{5}\right)y-2-3.\frac{4}{5}=0.$$

$$\Rightarrow \quad \frac{11}{3}x-\frac{11}{3}y=0 \quad \& \quad \frac{11}{5}x+\frac{11}{5}y-\frac{22}{5}=0.$$

Hence, the required equations of the lines, are $x=y$ & $x+y-2=0$.

Second method: The given lines, are $3x-y-2=0$ ------------- (1)

& $x-4y+3=0$ ------------ (2)

The point of intersection of the lines (1) and (2), is given by

$$\frac{x}{-3-8}=\frac{y}{-2-9}=\frac{1}{-12+1} \quad \Rightarrow \quad x=1, \ y=1.$$

The point of intersection of the lines (1) and (2), is $(1,1)$.

We know that the slope of the lines equally inclined to the axes, is ± 1.

Hence, the equation of the line using the slope-point form, is

$$y-1=1.(x-1) \quad \& \quad y-1=-(x-1) \quad \Rightarrow \quad x=y \quad \& \quad x+y-2=0.$$

Assignment No. 8

1. Find the equation of the line parallel to the line $3x=2y-5$ and passing through the point $(4,5)$. $(Answer: 3x-2y-2=0)$

2. Find the equation of the line parallel to the line $2x-3y-5=0$ and passing through the point $(4,5)$. $(Answer: 2x-3y+7=0)$

3. Find the equation of the line perpendicular to the line $4x-3y=8$ and passing through the point $(2,3)$. $(Answer: 3x+4y-18=0)$

4. Find the equation of the line perpendicular to the line $7x+8y=5$ and passing through the point $(-6,10)$. $(Answer: 8x-7y+118=0)$

5. Show that the equation of the line perpendicular to the line $x\sec\theta+y\,\mathrm{cosec}\,\theta=a$ and passing through the point $(a\cos^3\theta, a\sin^3\theta)$, is $x\cos\theta-y\sin\theta=a\cos 2\theta$.

6. Find the equation of the line passing through the point of intersection of the lines $x+2y-5=0$ and $3x+7y-17=0$ and parallel to the line $4x-3y+7=0$.
$(Answer: 4x-3y+2=0)$

7. Find the equation of the line passing through the point of intersection of the lines $x+2y+3=0$ and $3x+4y+7=0$ and making an angle of 135^o with the positive direction of the x-axis. $(Answer: x+y+2=0)$

8. Find the equations of the lines passing through the point of intersection of the lines $3x-y=2$ and $x-4y+3=0$ and equally inclined to the axes. $(Answer: x=y, \ x+y=2)$

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