# Logarithms

Logarithms are important tools for variety of calculations dealing with science and technology, navigation, astronomy, space technology, research and analytical data. The concept of logarithm was introduced by John Napier (1550-1617), a Scottish mathematician. The knowledge of laws of indices or exponents is essential for the study of logarithms. Logarithm refers to the index of power in an exponential representation. Let us have a look at the following illustrations:

$$(i)\ 3^1 = 3 \qquad (ii)\ 3^2 = 9 \qquad (iii)\ 3^3 = 27 \qquad (iv)\ 3^4 = 81$$

In each exponentiation, we come across three numbers out of which two occur on the left-hand side and the third one occurs on the right-hand side. In each case, the bottom number on the LHS, is 3 which is referred to as the base; followed by the varying top numbers known as the indices, exponents or powers. The RHS is the value of the exponential form in the LHS. In any such illustration, 3 numbers are involved out of which if any two numbers are known, we can determine the third number. Quite often, given the base and value, we may need to find the exponent For example, asking the query as to what power raised to 3 yields 81, the answer is 4. Here, the power 4 is referred to as the 'logarithm of 81 to the base 3'. Symbolically, we write $\log_3 81 = 4$ which is said to be the logarithmic form of the exponential form shown in (iv) above. Thus, follows:

$$3^4 = 81 \iff \log_3 81 = 4.$$

Similarly, we can give the exponential and logarithmic forms of the above illustrations as follows:

$$3^1 = 3 \iff \log_3 3 = 1$$

$$3^2 = 9 \iff \log_3 9 = 2$$

$$3^3 = 27 \iff \log_3 27 = 3.$$

Thus, we can define logarithm as follows:

**The logarithm of a number to a given base is the index of the power to which the base must be raised to be equal to the number.**

In general, **if $a^x = y$, then $x$ is called the logarithm of $y$ to base $a$, briefly, we write $log_a y = x$. It is to be noted that the base $a$ is always taken positive except 1.** A positive number raised to any real power is always positive. Consequently, the number $y$ remains positive.

## Laws of Logarithms

The following laws of logarithms are quite useful:

$(1)\ \boldsymbol{log_a(mn) = log_a m + log_a n}$ $\qquad (2)\ \boldsymbol{log_a\left(\frac{m}{n}\right) = log_a m - log_a n}$

$(3)\ \boldsymbol{log_a m^n = n\, log_a m}$ $\qquad (4)\ \boldsymbol{log_a 1 = 0}$ $\qquad (5)\ \boldsymbol{log_a a = 1}$

$(6)\ \boldsymbol{log_a b = \frac{1}{log_b a}\ \ Or\ \ log_a b \times log_b a = 1}$

$(7)\ \boldsymbol{log_b a = \frac{log_c a}{log_c b}\quad OR\quad log_b a = log_c a \times log_b c}$ $\qquad (8)\ \boldsymbol{p = q^{log_q p}}$

Proof: (1) $Let\ log_a m = x \implies a^x = m \quad \& \quad let\ \ log_a n = y \implies a^y = n$

$Clearly,\quad a^x a^y = mn \implies a^{x+y} = mn \implies log_a(mn) = x + y = \log_a m + \log_a n.$

This proves formula (1).

This law can be extended to more than to factors:

$$\boldsymbol{log_a(mnp.....)} = \boldsymbol{\log_a m + \log_a n + \log_a p} + \ldots\ldots$$

(2) $Let\ log_a m = x \implies a^x = m \quad \& \quad let\ \ log_a n = y \quad \implies a^y = n$

$Then, we\ get\ \ \frac{a^x}{a^y} = \frac{m}{n} \implies a^{x-y} = \frac{m}{n} \implies log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n.$

This proves formula (2).

(3) $Let\ log_a m = x \implies a^x = m \implies (a^x)^n = m^n \implies a^{nx} = m^n \implies log_a m^n = nx = n\,log_a m$

$$\implies log_a m^n = n\,log_a m.$$

This proves formula (3).

(4) We have $a^0 = 1 \implies log_a 1 = 0.$

We can say that **log of 1 to any base is always zero**.

(5) We have $a^1 = a \implies log_a a = 1.$

We can say that **log of any number to the same base is always 1**.

(6) Let $log_a b = x \implies a^x = b.$

Taking log both sides to base b, we get $\log_b a^x = \log_b b \implies x\,log_b a = 1$

$$\implies \quad x = \frac{1}{log_b a} \quad \implies \quad log_a b = \frac{1}{log_b a}.$$

This proves formula (6).

(7) Let $log_b a = x \implies b^x = a.$

Taking log both sides at base c, we get $log_c b^x = log_c a \implies x\,log_c b = log_c a$

$$\implies \quad x = \frac{log_c a}{log_c b} \quad \implies \quad log_b a = \frac{log_c a}{log_c b}.$$

This proves formula (7).

(8) Let $log_q p = x \implies q^x = p \implies p = q^x = q^{log_q p}.$

This proves formula (8).

## The factorial notation and the number e

If $n \in N, we\ define\ 'factorial\ n'\ or\ 'n\ factorial'\ to\ be\ denoted\ by\ \ n!\ \ or$ $\underline{|n}$ and defined as the product of first n natural numbers. Examples:

$1! = 1, \quad 2! = 2 \times 1 = 2, \quad 3! = 3 \times 2 \times 1 = 6,$

$4! = 4 \times 3 \times 2 \times 1 = 24, \ldots\ldots\ldots$

$n! = n.(n-1).(n-2).(n-3)\ldots\ldots 3.2.1.$

Having learnt the factorial notation, we now introduce a very important number to be denoted by ‘e’ and defined as the sum of the infinite series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots\ldots\ldots to\ \infty.$$

Hence, we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots\ldots\ldots to\ \infty.$$

The number e is of great importance in higher mathematics. Its value lies between 2 and 3 and has the decimal form non-terminating and non-repeating. That is why e is an irrational number. Like the number $\pi$, its value was also given by many mathematicians to several decimal places. Professor J. C. Adams gave its value to more than 260 places of decimals. We give below the value of e correct up to 9 places of decimals:

$$e = 2.718281828.$$

So far as the origin of the number e is concerned, the names of two mathematicians are worth mentioning -Leonhard Euler and John Napier. It is because of this reason that the number e is named after them as Euler's number and Napier's number.

## Common logarithm and natural logarithm

In scientific computations, the numbers 10 and e are widely used as the base of logarithms. **The logarithms with base 10 are called common logarithms and those with base e are called natural logarithms or Napierian logarithms.**

The examples of common logarithms are $\log_{10} 2$, $\log_{10} 45$, $\log_{10} 134$, $etc.$
Sometimes, the base of a common logarithm is missing. For example, $\log 12 = \log_{10} 12$.

We talk about the characteristic and the mantissa of a common logarithm in decimal form. The integral part of a common logarithm is called the **characteristic** while the decimal part the **mantissa**. It is to be noted that the mantissa is always taken positive. For example, $\log_{10} 12 = 1.0792$ has the characteristic 1 and the mantissa $.0792$.

The examples of natural logarithms are $\log_e 2$, $\log_e 45$, $\log_e 134$, $etc.$

The natural logarithm of $x$ is $log_e x$ which is also written as $ln\, x$. So, $ln\, 2 = log_e 2$, $ln\, 42 = log_e 42$ and so on.

### SOLVED EXAMPLES

**Example 1**: *Write in logarithmic forms*:

$$(i)\ 10^3 = 1000 \qquad (ii)\ 7^{-2} = \frac{1}{49} \qquad (iii)\ e^y = x \qquad (iv)\ \left(\frac{2}{3}\right)^{-1} = \frac{3}{2}.$$

**Solution**: $(i)\ 10^3 = 1000 \implies log_{10} 1000 = 3 \quad (ii)\ 7^{-2} = \frac{1}{49} \implies log_7\left(\frac{1}{49}\right) = -2$

$$(iii)\ e^y = x \implies log_e x = y \quad (iv)\ \left(\frac{2}{3}\right)^{-1} = \frac{3}{2} \implies log_{\left(\frac{2}{3}\right)}\left(\frac{3}{2}\right) = -1$$

**Example 2**: *Write in exponential forms*:

$(i)\ log_2 32 = 5 \qquad (ii)\ log_{10}\left(\frac{1}{100}\right) = -2 \qquad (iii)\ log_5 1 = 0 \qquad (iv)\ log_t s = p$.

**Solution**: $(i)\ log_2 32 = 5 \implies 2^5 = 32 \quad (ii)\ log_{10}\left(\frac{1}{100}\right) = -2 \implies 10^{-2} = \frac{1}{100}$

$$(iii)\ log_5 1 = 0 \implies 5^0 = 1 \qquad (iv)\ log_t s = p \implies t^p = s$$

**Example 3**: Find the *logarithm* of $(i)$ 125 to the base 5 $(ii)$ 5832 to the base $3\sqrt{2}$.

**Solution**: $(i)\ Let\ \log_5 125 = x \implies 5^x = 125 \implies 5^x = 5^3 \implies x = 3$

Hence, $\log_5 125 = 3$.

$(ii)\ Let\ \log_{3\sqrt{2}} 5832 = x \implies \left(3\sqrt{2}\right)^x = 5832 = 8 \times 729 = 2^3 \times 3^6 = \left(\sqrt{2}\right)^6 \times 3^6 = \left(3\sqrt{2}\right)^6$

$\implies \left(3\sqrt{2}\right)^x = \left(3\sqrt{2}\right)^6 \implies x = 6$ Hence, $\log_{3\sqrt{2}} 5832 = 6$.

**Example 4**: *Express in terms of* $\log 2$ *and* $\log 3$: $(i)\ log\, 12 \quad (ii)\ log\, 96 \quad (iii)\ log\, 144$.
**Solution**: $(i)\ \log 12 = \log(2^2 \times 3) = \log 2^2 + \log 3 \qquad [Using\ \log(mn) = \log m + \log n]$.

$= 2\log 2 + \log 3 \qquad [Using \quad \log m^n = n \log m]$

$(ii)$ $\log 96 = \log(2^5 \times 3) = \log 2^5 + \log 3 \qquad [Using \quad \log(mn) = \log m + \log n]$ .

$= 5\log 2 + \log 3 \qquad [Using \quad \log m^n = n\log m]$

$(iii)$ $\log 144 = \log(2^4 \times 3^2) = \log 2^2 + \log 3^2 \qquad [Using \quad \log(mn) = \log m + \log n]$ .

$= 2\log 2 + 2\log 3 \qquad [Using \quad \log m^n = n\log m]$

**Example 5**: *Show that* $7\, log\dfrac{10}{9} - 2\, log\dfrac{25}{24} + 3\, log\dfrac{81}{80} = log\, 2$ .

**Solution**: Given that $LHS = 7\, log\dfrac{10}{9} - 2\, log\dfrac{25}{24} + 3\, log\dfrac{81}{80}$

$$= 7(\log 10 - \log 9) - 2(\log 25 - \log 24) + 3(\log 81 - \log 80)$$

$$\left[\text{Using} \quad \log\left(\frac{\text{m}}{\text{n}}\right) = \log \text{m} - \log \text{n}\right]$$

$$= 7[\log(2\times 5) - \log 3^2] - 2[\log 5^2 - \log(2^3 \times 3)] + 3[\log 3^4 - \log(2^4 \times 5)]$$

$$= 7[(\log 2 + \log 5) - 2\log 3] - 2[2\log 5 - (\log 2^3 + \log 3)] + 3[4\log 3 - (\log 2^4 + \log 5)]$$

$$[Using \quad \log(mn) = \log m + \log n \ \ \& \ \ \log m^n = n\log m\,]$$

$$= 7[(\log 2 + \log 5) - 2\log 3] - 2[2\log 5 - (3\log 2 + \log 3)] + 3[4\log 3 - (4\log 2 + \log 5)]$$

$$[Using \quad \log(mn) = \log m + \log n \ \ \& \ \ \log m^n = n\log m\,]$$

$$= 7\log 2 + 7\log 5 - 14\log 3 - 4\log 5 + 6\log 2 + 2\log 3 + 12\log 3 - 12\log 2 - 3\log 5$$

$$[Using \quad \log(mn) = \log m + \log n \ \ \& \ \ \log m^n = n\log m\,]$$

$= \log 2 = RHS$ .

**Example 6**: If $log_4 2^{x+1} = 3$, find the value of $x$ .

**Solution**: Given $log_4 2^{x+1} = 3 \implies 4^3 = 2^{x+1} \implies (2^2)^3 = 2^{x+1} \implies 2^6 = 2^{x+1}$

$$\implies x + 1 = 6 \implies x = 5 .$$

**Example 7**: Evaluate $\log_5\left(1+\frac{1}{5}\right) + \log_5\left(1+\frac{1}{6}\right) + \log_5\left(1+\frac{1}{7}\right) + \ldots\ldots\ldots + \log_5\left(1+\frac{1}{624}\right)$.

**Solution**: We have $\log_5\left(1+\frac{1}{5}\right) + \log_5\left(1+\frac{1}{6}\right) + \log_5\left(1+\frac{1}{7}\right) + \ldots\ldots\ldots + \log_5\left(1+\frac{1}{624}\right)$

$$= \log_5\frac{6}{5} + \log_5\frac{7}{6} + \log_5\frac{8}{7} + \ldots\ldots\ldots + \log_5\frac{625}{624} = \log_5\left(\frac{\cancel{6}}{5} \times \frac{\cancel{7}}{\cancel{6}} \times \frac{\cancel{8}}{\cancel{7}} \times \ldots\ldots\ldots \times \frac{625}{\cancel{624}}\right)$$

$$= \log_5\frac{625}{5} = \log_5 125 = \log_5 5^3 = 3\log_5 5 = 3 \times 1 = 3 .$$

**Example 6**: Evaluate $(i)\ 3^{4log_3 5}$ $\quad (ii)\ ln\, e^4$ $\quad (iii)\ e^{2\, ln\, 7}$ $\quad (iv)\ ln\, 3 + ln\dfrac{e}{3}$.

**Solution**: We have $(i)\ 3^{4log_3 5} = 3^{log_3 5^4} = 5^4 = 625$ $\quad$ (Using formula $p = q^{log_q p}$) .

$(ii)$ $ln\, e^4 = \log_e e^4 = 4\log_e e = 4 \times 1 = 4$ $\quad$ (Using formula $log_a a = 1$).

$(iii)$ $e^{2\, ln\, 7} = e^{log_e 7^2} = 7^2 = 49$ $\quad$ (Using formula $p = q^{log_q p}$) .

$$(iv)\ \ ln\, 3 + ln \frac{e}{3} = \ln\left(3 \times \frac{e}{3}\right) = ln\, e = 1.$$

### ***ASSIGNMENT***

$Q.1. Write\ in\ logarithmic\ forms$:

$(i)\ \ 4^3 = 64 \qquad (ii)\ \ 5^{-2} = \frac{1}{25} \qquad (iii)\ \ \sqrt{4} = 2 \qquad (iv)\ \ \left(\frac{3}{4}\right)^{-3} = \frac{64}{27}$.

$Q.2. Write\ in$ exponential $forms$:

$(i)\ \ log_5 125 = 3 \qquad (ii)\ \ log_6\left(\frac{1}{36}\right) = -2 \qquad (iii)\ \ ln\, s = t \qquad (iv)\ \ log_{\frac{1}{3}} 27 = p$ .

Q. 3. Find the *logarithm* of $(i)$ 81 to the base 3 $\qquad (ii)$ 64 to the base $\sqrt{2}$ .

Q. 4. Express $\frac{1}{2}\log 9 - 3\log 4 + 3\log 2$ as the logarithm of a single number.

Q. 5. Prove that $(i)\ \ log\frac{11}{5} + log\frac{14}{3} - log\frac{22}{15} = log\, 7 \qquad (ii)\ \ log\frac{9}{14} + log\frac{35}{24} - log\frac{15}{16} = 0$ .

Q. 6. Show that $(i)\ \ 3\, log\, 4 + 2\, log\, 5 - \frac{1}{3}\, log\, 64 - \frac{1}{2} log\, 16 = 2$

$(ii)\ \ \frac{1}{2} log\, 9 + 2\, log\, 6 + \frac{1}{4}\, log\, 81 - log\, 12 = 3\log 3$ .

Q. 7. Evaluate $\log_5\left(1 + \frac{1}{2}\right) + \log_5\left(1 + \frac{1}{3}\right) + \log_5\left(1 + \frac{1}{4}\right) + \ldots\ldots\ldots + \log_5\left(1 + \frac{1}{127}\right)$.

Q. 8. Evaluate $(i)\ 7^{2 log_7 5} \qquad (ii)\ \ ln\, e^3 \qquad (iii)\ \ e^{2\, ln\, 4} \qquad (iv)\ 2\, ln\, 3 - ln\, \frac{9}{e}$.

$Q.9. Solve\ for\ x$:

$(i)\ \ 3^{2x+1} = 243 \qquad (ii)\ \ \frac{log\, 144}{log\, 12} = x \qquad (iii)\ \ log_x 4 + log_x 16 + log_x 64 = 12$ .

### ***ANSWERS***

Q. 1. $(i)\ \ log_4 64 = 3 \qquad (ii)\ \ log_5\left(\frac{1}{25}\right) = -2 \qquad (iii)\ \ log_4 2 = \frac{1}{2} \qquad (iv)\ \ log_{\frac{3}{4}} \frac{16}{9} = -2$ .

Q. 2. $(i)\ \ 5^3 = 125 \qquad (ii)\ \ 6^{-2} = \frac{1}{36} \qquad (iii)\ \ e^t = s \qquad (iv)\ \ \left(\frac{3}{4}\right)^{-2} = \frac{16}{9}$.

Q. 3 (i) 4 (ii) 12 Q. 4. $\log\frac{3}{8}$ Q. 6. 7 Q. 8. (i) 25 (ii) 3 (iii) 16 (iv) 1 .

Q. 9. (i) 2 (ii) 2 (iii) 2 .

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