Functions and Limits

Function: A relation involving two variables is called a function. A function is usually denoted by the letter f.

Example: Let us consider the mathematical relation $y = x^2 - 2x + 3$ which is a function.

A function connects the variables x and y in such a way that the variable y is expressed in terms of x. Any expression in terms of x, is referred to as a function of x and is usually written as $f(x)$. Clearly, $f(x)$ equals y, so, we can write $y = f(x)$. Here, the variable x can be given values freely or independently but the other variable y depends on x. So, we call the variable x as an *independent variable* and the variable y as a *dependent variable.* In a given function $f(x)$, we often find the value of y corresponding to a particular value of x. The value of y itself is called the *value of the function $f(x)$ at that particular value of x*. The value of the function $f(x)$ at $x = a$, is $f(a)$. We also consider the sets of values of the dependent and independent variables. If A is the set of values of x and B be the set of values of y, we say that the function f is defined from set A to set B. Symbolically, we write $f: A \to B$. Here, the set A is called the ***domain*** and the set B the ***range*** of the function f. If A and B consist of the real numbers, we call the function f as a ***real function***. For a given real function, we always look for its domain and range.

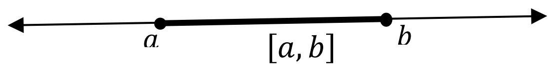
Intervals

An interval is a set of all real numbers lying between two given real numbers. The two given real numbers are called the end-points of the interval. We classify intervals on the basis of whether the end-points are included or not. If the end-points are included, the interval is called closed interval, else, it is called open interval. The end-points of an interval are written within the brackets [ ] for closed-interval and ( ) or] [for open-intervals separated by commas. We may also have a single end-point included. Thus, we have the following four types of intervals with the end-points a and b $(a < b)$:

(i) Closed interval: It includes both the end-points.

$$[a, b] = \{x | x \in R, \quad a \leq x \leq b\}$$

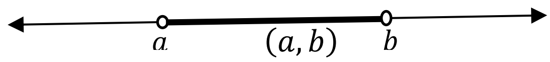
It can be shaded on the number line as follows:



(ii) Open interval: It excludes both the end-points.

$$(a, b) \ or \ ]a, b[\ = \{x | x \in R, \quad a < x < b\}$$

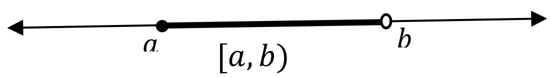
It can be shaded on the number line as follows:



(iii) Semi-closed or closed-open interval: It includes the left end-point and excludes the right end-point.

$$[a, b[\ or \ [a, b) = \{x | x \in R, \quad a \leq x < b\}$$

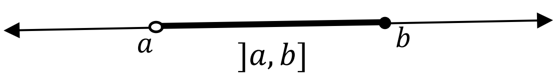
It can be shaded on the number line as follows:



(iv) Semi-open or open-closed interval: It excludes the left end-point and includes the right end-point.

$$(a, b] \ or \ ]a, b] = \{x | x \in R, \quad a < x \leq b\}$$

It can be shaded on the number line as follows:



The set of all real numbers can be written in the interval form as $R = (-\infty, \infty)$.

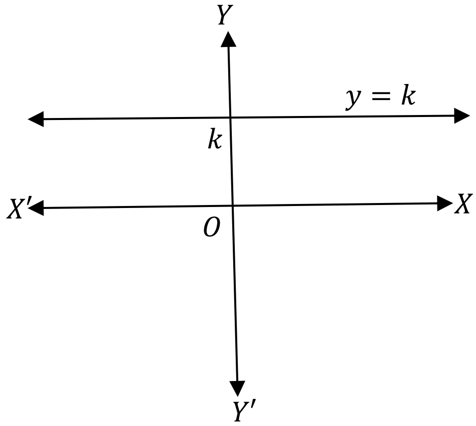
Types of real functions

1. Constant function: It is a function defined by a constant.

So, $y = f(x) = k$, k being a constant, is a constant function of x. Its domain is the set R of all real numbers and the range is the singleton set having k only.

Or, $\text{Domain} = R$ and $\text{Range} = \{k\}$.

Its graph can be given as follows:



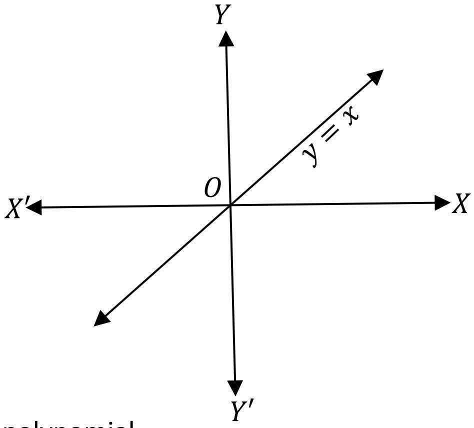
2. Identity function: It is the function having both the variables equal.

So, $y = f(x) = x$ is the identity function of x.

Obviously, this function is defined for all real numbers. Hence,

$\text{Domain} = \text{R}$ and $\text{Range} = \text{R}$.

Its graph can be given as follows:



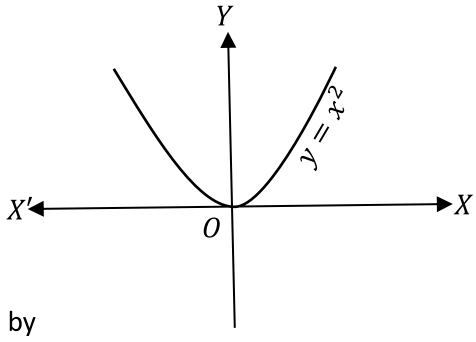
3. Polynomial function: It is a function defined by a polynomial.

So, $y = f(x)$, where $f(x)$ is a polynomial in x. Its domain is the set R of all real numbers and the range depends on $f(x)$.

Example: Let $y = f(x) = x^2$

Then its domain is R and the range is the set of all non-negative real numbers or the interval $[0, \infty)$

Its graph can be given as follows:

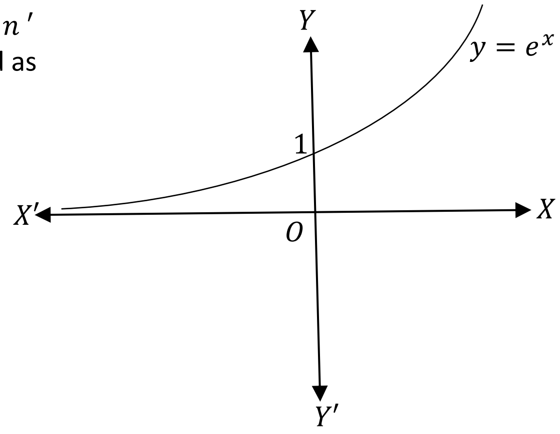


4. Exponential function: It is a function defined by

$y = f(x) = e^x$; where $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \ldots \ldots . . to\ \infty$ and

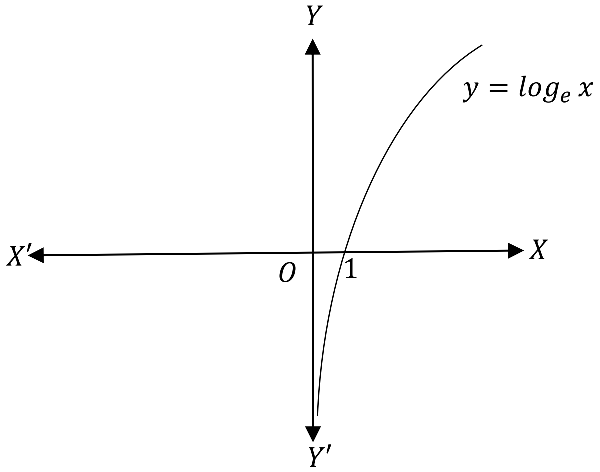
$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \ldots \ldots . to\ \infty$. The number e is very popular in calculus. It is an irrational number whose approximate value is $e = 2.718281828 \ldots$.The number e is also used as the base of logarithms which are called natural logarithms or Naperian logarithms.

Factorial notation: If $n \in N, we\ define\ 'factorial\ n'$ $or\ 'n\ factorial'\ to\ be\ denoted\ by\ n!$ and defined as the product of first n natural numbers. Examples:
$1! = 1,\quad 2! = 2 \times 1 = 2,\quad 3! = 3 \times 2 \times 1 = 6,$
$4! = 4 \times 3 \times 2 \times 1 = 24, \ldots\ldots\ldots$
$n! = n.(n-1).(n-3)\ldots\ldots3.2.1.$

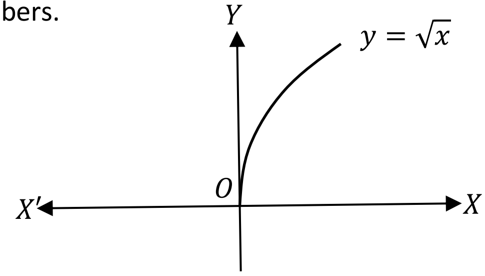


The domain of this function is the set of all real numbers and the range is the set of all positive real numbers.
Or, $\text{Domain} = \text{R}$ and $\text{Range} = (0, \infty)$.

5. Logarithmic function: It is a function defined by natural logarithm of independent variable**.**
So, $y = f(x) = log_e x$, is a logarithmic function. The natural logarithm of $x, i.e.\ log_e x$ is also written as $ln\,x$. The logarithms are defined for positive real numbers. Hence, its domain is the set of all positive real numbers and the range is the set R of all real numbers.
Or, $\text{Domain} = (0, \infty)$ and $\text{Range} = \text{R}$.
Its graph is shown in the figure.
To give concept of logarithm, we have
$3^2 = 9,\quad then\ the\ exponent\ 2\ is\ called\ the\ logarihm\ of\ 9\ to\ base\ 3\ to\ be\ denoted\ by\ \log_3 9 = 2.$
$If\ a^m = b, then\ the\ exponent\ m\ is\ called\ the\ logarihm\ of\ b\ to\ base\ a\ to\ be\ denoted\ by$
$log_a b = m.$



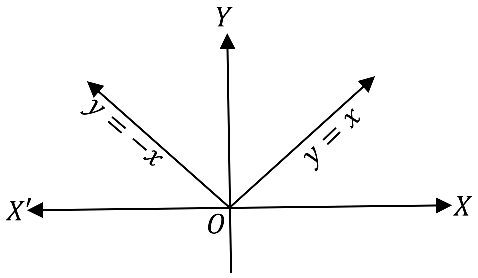
6. Square root function: It is a function defined by the square root of independent variable.
So, $y = f(x) = \sqrt{x}$ is the square root function of x. The square root is defined only for non-negative real numbers. Hence, its domain is the set of all non-negative real numbers and the range is also the set of all non-negative real numbers.
$\text{Domain} = [0, \infty)$ and $\text{Range} = [0, \infty)$.
Its graph can be given as follows.



7. Modulus function: It is defined by the modulus or absolute value of independent variable.

So, $y = f(x) = |x| = \begin{cases} -x, & if\ x < 0 \\ x, & if\ x \geq 0 \end{cases}$. Clearly, domain $=$ R and Range $= [0, \infty)$.

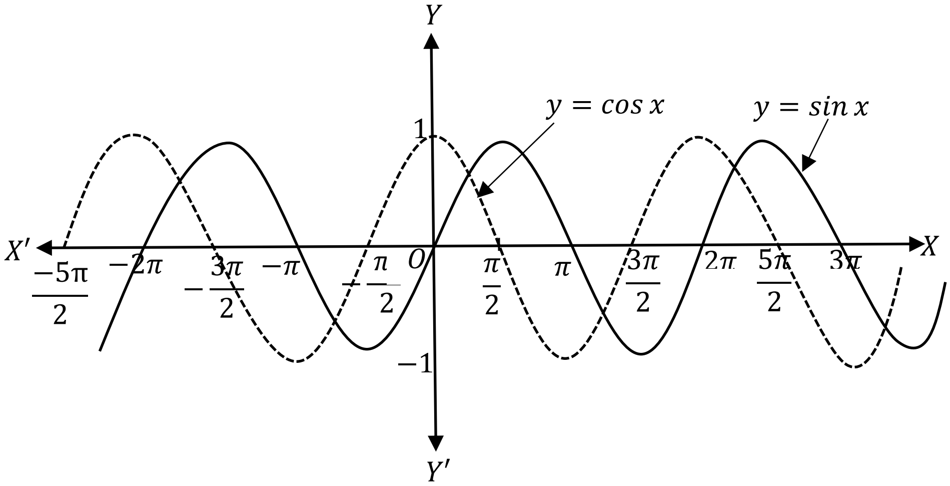
Its graph is shown in the figure.



8. Trigonometric function: It is defined by a trigonometric ratio.

The following are trigonometric functions with domain and range:

S. No.	Trigonometric function	Domain	Range
(i)	$y = f(x) = \sin x$	R	$[-1, 1]$
(ii)	$y = f(x) = \cos x$	R	$[-1, 1]$
(iii)	$y = f(x) = \tan x$	$R - \left\{(2n+1)\frac{\pi}{2} \mid n \in I\right\}$	R
(iv)	$y = f(x) = \cot x$	$R - \{n\pi \mid n \in I\}$	R
(v)	$y = f(x) = \sec x$	$R - \left\{(2n+1)\frac{\pi}{2} \mid n \in I\right\}$	$R - (-1, 1)$ or $(-\infty, -1] \cup [1, \infty)$
(vi)	$y = f(x) = cosec\, x$	$R - \{n\pi \mid n \in I\}$	$R - (-1, 1)$ or $(-\infty, -1] \cup [1, \infty)$



8. Rational function: A function in the form $\frac{p(x)}{q(x)}$; where, $p(x)$ and $q(x)$ are polynomials in x and $q(x) \neq 0$, is called a rational function. The domain and range of a rational function depends on the polynomials $p(x)$ and $q(x)$.

Example: Let $y = f(x) = \dfrac{x^2 - 4}{x - 2}$ *be a rational function which is defined for all* real numbers other than 2. We can also write

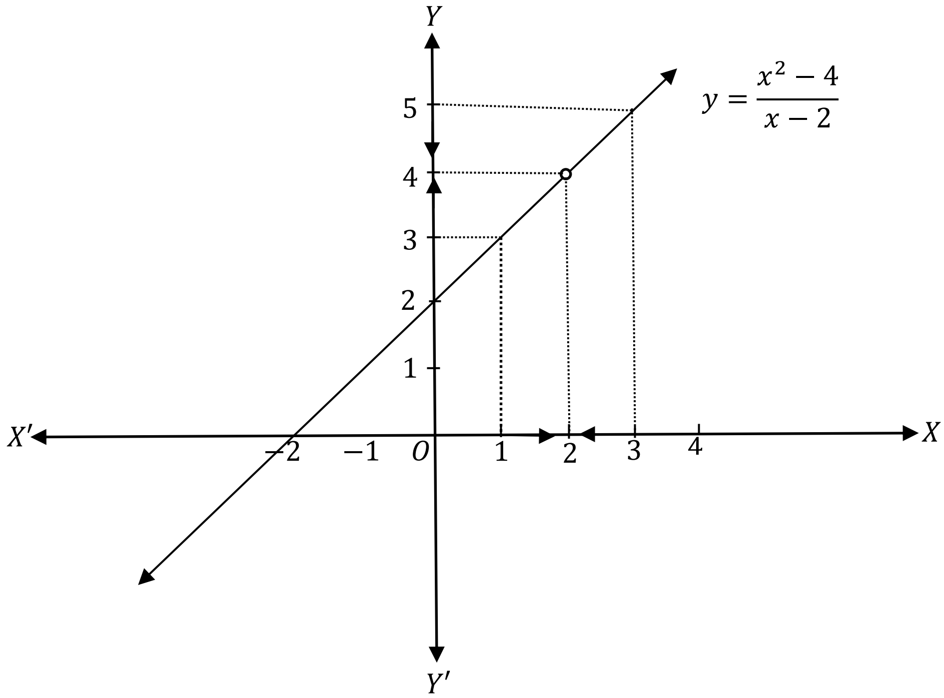
$$y = f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x - 2} = x + 2;\ \ \forall\ x \neq 2.$$

Clearly, the domain of the function is the set of all real numbers except 2 and the range is the set of all real numbers except 4. Thus, domain$= R - \{2\}$ *or* $(-\infty, 2) \cup (2, \infty)$ & range$= R - \{4\}$ *or* $(-\infty, 4) \cup (4, \infty)$.

The function $f(x)$ is not defined at $x = 2$. If, however, we try to find the value of $f(x)$ at $x = 2$, we get

$$f(2) = \frac{2^2 - 4}{2 - 2} = \frac{0}{0} \text{ which is absurd or meaningless.}$$

Thus, y or $f(x)$ is not available at $x = 2$. In such a case, we examine $f(x)$ in the neighbourhood of $x = 2$. The graph of the function is shown below:



At $x = 2, y$ is not defined. We can take the values of x close to 2 on its left-hand side or right-hand side, moving slowly and slowly towards 2. Simultaneously, at each value of x, we also find the value of y.

On the LHS of 2 or for $x < 2$, we have:

$$\begin{aligned}
x &= 1 &&\Rightarrow\quad y = x + 2 = 1 + 2 = 3\\
x &= 1.1 &&\Rightarrow\quad y = x + 2 = 1.1 + 2 = 3.1\\
x &= 1.2 &&\Rightarrow\quad y = x + 2 = 1.2 + 2 = 3.2\\
x &= 1.3 &&\Rightarrow\quad y = x + 2 = 1.3 + 2 = 3.3\\
x &= 1.4 &&\Rightarrow\quad y = x + 2 = 1.4 + 2 = 3.4\\
x &= 1.5 &&\Rightarrow\quad y = x + 2 = 1.5 + 2 = 3.5\\
x &= 1.6 &&\Rightarrow\quad y = x + 2 = 1.6 + 2 = 3.6\\
x &= 1.7 &&\Rightarrow\quad y = x + 2 = 1.7 + 2 = 3.7\\
x &= 1.8 &&\Rightarrow\quad y = x + 2 = 1.8 + 2 = 3.8\\
x &= 1.9 &&\Rightarrow\quad y = x + 2 = 1.9 + 2 = 3.9\\
x &= 1.91 &&\Rightarrow\quad y = x + 2 = 1.91 + 2 = 3.91\\
x &= 1.92 &&\Rightarrow\quad y = x + 2 = 1.92 + 2 = 3.92\\
x &= 1.96 &&\Rightarrow\quad y = x + 2 = 1.96 + 2 = 3.96\\
x &= 1.99 &&\Rightarrow\quad y = x + 2 = 1.99 + 2 = 3.99\\
x &= 1.991 &&\Rightarrow\quad y = x + 2 = 1.991 + 2 = 3.991\\
x &= 1.99995 &&\Rightarrow\quad y = x + 2 = 1.99995 + 2 = 3.99995\\
x &= 1.9999876 &&\Rightarrow\quad y = x + 2 = 1.9999876 + 2 = 3.9999876
\end{aligned}$$

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and so on so forth. This process will never stop. The above data, depicts that x takes values in such a way that it comes closer and closer to 2 from the left-hand side and, at the same time, y comes closer and closer to 4. In other words, as x approaches 2 from the left-hand side, the variable y approaches 4. Symbolically, we write $x \longrightarrow 2^-$ $\Rightarrow$ $y \longrightarrow 4$. Here, the arrow-mark is read as '*approaches or tends to'.*

On the RHS of 2 or for $x > 2$, we have:

$$x = 3 \qquad \Rightarrow \qquad y = x + 2 = 3 + 2 = 5$$
$$x = 2.9 \qquad \Rightarrow \qquad y = x + 2 = 2.9 + 2 = 4.9$$
$$x = 2.8 \qquad \Rightarrow \qquad y = x + 2 = 2.8 + 2 = 4.8$$
$$x = 2.7 \qquad \Rightarrow \qquad y = x + 2 = 2.7 + 2 = 4.7$$
$$x = 2.6 \qquad \Rightarrow \qquad y = x + 2 = 2.6 + 2 = 4.6$$
$$x = 2.5 \qquad \Rightarrow \qquad y = x + 2 = 2.5 + 2 = 4.5$$
$$x = 2.4 \qquad \Rightarrow \qquad y = x + 2 = 2.4 + 2 = 4.4$$
$$x = 2.3 \qquad \Rightarrow \qquad y = x + 2 = 2.3 + 2 = 4.3$$
$$x = 2.2 \qquad \Rightarrow \qquad y = x + 2 = 2.2 + 2 = 4.2$$
$$x = 2.1 \qquad \Rightarrow \qquad y = x + 2 = 2.1 + 2 = 4.1$$
$$x = 2.09 \qquad \Rightarrow \qquad y = x + 2 = 2.09 + 2 = 4.09$$
$$x = 2.07 \qquad \Rightarrow \qquad y = x + 2 = 2.07 + 2 = 4.07$$
$$x = 2.02 \qquad \Rightarrow \qquad y = x + 2 = 2.02 + 2 = 4.02$$
$$x = 2.0001 \qquad \Rightarrow \qquad y = x + 2 = 2.0001 + 2 = 4.0001$$
$$x = 2.00003 \qquad \Rightarrow \qquad y = x + 2 = 2.00003 + 2 = 4.00003$$
$$x = 2.00000001 \qquad \Rightarrow \qquad y = 2.00000001 + 2 = 4.00000001$$
$$x = 2.00000000002 \qquad \Rightarrow y = 2.00000000002 + 2 = 4.00000000002$$

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and so on so forth. This process will never stop. The above data, depicts that x takes values in such a way that it comes closer and closer to 2 from the right-hand side and, at the same time, y comes closer and closer to 4. In other words, as x approaches 2 from the right-hand side, the variable y approaches 4. Symbolically, we write $x \longrightarrow 2^+$ $\Rightarrow$ $y \longrightarrow 4$.

$From\ the\ above\ analysis, we\ notice\ that, though\ y\ is\ not\ defined\ at\ x = 2,\ but\ as$ $x\ approaches\ 2\ either\ from\ the\ left\ or\ from\ the\ right, y\ approaches\ 4$. We write $x \longrightarrow 2 \quad \Rightarrow \quad y \longrightarrow 4$.

Obviously, the number 4 is not the value of $y\ or\ f(x)$, rather, it is called the limit of $y\ or\ f(x)$ at $x = 2$. We are now in a position to define the limit not a function.

Limit of a function

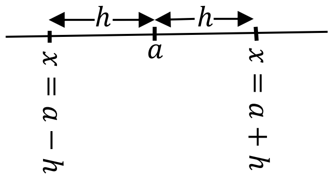
Let $y = f(x)$ be a function of x such that y tends to the number l as x tends to the number a. Then l is called the limit of y or $f(x)\ at\ x = a$. We write

$$\lim_{x \to a} y = \lim_{x \to a} f(x) = l.$$

One-sided limits: When the variable x approaches the number 'a' from a particular side of 'a' either from the left or from the right and side by side the variable y approaches a fixed number. In such a case, the fixed number is called *one-sided limit* which is precisely called the *left-hand limit (LHL)* or *right-hand limit (RHL),* according as x approaches a from its left or right. We can formulate LHL and RHL using another variable, say h, representing the gap between x and a. Obviously, *as x approaches a*, the '*gap between x and a' or h keeps on reducing*. When x is very close to a, then h is very small or very close to 0. Thus, if $x = a \pm h$ and $x \longrightarrow a$, then $h \longrightarrow 0$. If h is a small positive real number, then $x = a + h$ implies that x lies on the right-hand side of a and $x = a - h$ implies that x lies on the left-hand side of a. Thus, we have the following results:

Left-hand limit (LHL): The left-hand limit of $f(x)$ at $x = a$, is given by

$LHL = \lim\limits_{x \to a^-} f(x) = \lim\limits_{h \to 0} f(a-h)\,;\ h$ being a small positive real number.



Right-hand limit (RHL): The right-hand limit of $f(x)$ at $x = a$, is given by

$RHL = \lim\limits_{x \to a^+} f(x) = \lim\limits_{h \to 0} f(a+h)\,;\ h$ being a small positive real number.

Intuitively, **if $LHL = RHL$, then $\lim\limits_{x \to a} f(x)$ exists.** Moreover,

$\boldsymbol{\lim\limits_{x \to a} f(x) = \lim\limits_{h \to 0} f(a+h) = \lim\limits_{h \to 0} f(a-h)}$**;** h being a small real number, positive or negative.

However, if $LHL \neq RHL$, we say that $\lim\limits_{x \to a} f(x)$ does not exist.

Difference between the value and limit of a function

As already discussed, the value of a function $f(x)$ at a given point $x = a$, is $f(a)$. While, the limit of $f(x)$ at $x = a$, is the number to which $f(x)$ tends as x tends to a. There are instances when the value and limit both may or may not exist, only one of them exists. Even if both exist, they may not be equal. If at all the value and the limit are equal, $f(x)$ is referred to as a *continuous function*. **Thus, a function $f(x)$ is said to be a continuous function at $x = a$ if and only if $\lim\limits_{x \to a} f(x) = f(a)$.**

Indeterminate Forms

While finding the value of a function at a given point of x, we may come across the function taking the forms

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad \infty - \infty,\quad 0 \times \infty,\quad 0^0,\quad \infty^0 \ or\ 1^\infty .$$

Each one of the above forms, is absurd and meaningless, and is termed as indeterminate form. In such situation, we seek the limit of the function.

Methods of finding limits

Keeping in view, the above aspects, methods of finding limits are devised. So long as the function is defined at a given point of x, often the value of the function itself becomes the limit. Thus, **$\lim\limits_{x \to a} f(x) = f(a)$, if $f(a)$ exists and $f(x)$ is continuous. We may call this process of finding limit as the *method of direct substitution*.**

The following are some examples:

Solved Examples

Example 1: Evaluate $(i)\ \lim\limits_{x \to -1} (2x^2 - 3x + 4)$ $\quad (ii)\ \lim\limits_{x \to \pi/4} (3\sin^2 x - 2\tan x)$

Solution: $(i)\ \lim\limits_{x \to -1} (2x^2 - 3x + 4) = [2.(-1)^2 - 3.(-1) + 4] = 9$

$$(ii)\ \lim_{x \to \pi/4} (3\sin^2 x - 2\tan x) = 3\sin^2 \pi/4 - 2\tan \pi/4$$

$$= 3.\left(\frac{1}{\sqrt{2}}\right)^2 - 2.1 = \frac{3}{2} - 2 = -\frac{1}{2}.$$

In real situations, we deal with the functions for which the value of the function $f(a)$ is not defined or indeterminate. In such cases, we have special methods of finding limits as discussed below:

Methods of finding limits when function takes $\frac{0}{0}$ form

If a function takes $\frac{0}{0}$ form, it is in the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are poynomial functions both of which become zero at the given value of $x = a$. Clearly, $(x - a)$ is a factor common to both. We may cancel out this factor and then the limit can be obtained by putting $x = a$ in the remaining function. This method of finding limit is termed as ***factorization method***.

Solved Examples

Example 1: Evaluate (*i*) $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}$ (*ii*) $\lim\limits_{x\to 3}\dfrac{x^3-27}{x^2-9}$

Solution: (*i*) $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}\left(\dfrac{0}{0}\ form\right)=\lim\limits_{x\to 2}\dfrac{(x-2)(x+2)}{x-2}=\lim\limits_{x\to 2}(x+2)=4$

Alternatively, $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}=\lim\limits_{h\to 0}\dfrac{(2+h)^2-4}{(2+h)-2}=\lim\limits_{h\to 0}\dfrac{(4+h^2+4h)-4}{h}=\lim\limits_{h\to 0}\dfrac{h(h+4)}{h}$
$=\lim\limits_{h\to 0}(h+4)=4.$

(*ii*) $\lim\limits_{x\to 3}\dfrac{x^3-27}{x^2-9}\left(\dfrac{0}{0}\ form\right)=\lim\limits_{x\to 3}\dfrac{(x-3)(x^2+3x+9)}{(x-3)(x+3)}=\lim\limits_{x\to 3}\dfrac{x^2+3x+9}{x+3}=\dfrac{27}{6}=\dfrac{9}{2}$

Example 2: Evaluate (*i*) $\lim\limits_{x\to -2}\dfrac{2x^3-4x^2-5x+22}{x^3-3x^2-7x+6}$ (*ii*) $\lim\limits_{x\to 2}\dfrac{x^3-8x^2+7x+10}{x^4-16}$

Solution: (*i*) $\lim\limits_{x\to -2}\dfrac{2x^3-4x^2-5x+22}{x^3-3x^2-7x+6}\ \left(\dfrac{0}{0}\ form\right)$. As $(x+2)$ is a factor of Nr. and Dr., $to\ find\ the\ other\ factor, we\ divide\ Nr.\ and\ Dr.\ by\ (x+2)$:

$$\begin{array}{r|l}
 & 2x^2-8x+11 \\ \hline
x+2) & 2x^3-4x^2-5x+22 \\
 & \underline{2x^3+4x^2} \\
 & -8x^2-5x+22 \\
 & -8x^2-16x \\
 & \underline{+\qquad +} \\
 & 11x+22 \\
 & \underline{\underline{11x+22}}
\end{array}
\qquad
\begin{array}{r|l}
 & x^2-5x+3 \\ \hline
x+2) & x^3-3x^2-7x+6 \\
 & \underline{x^3+2x^2} \\
 & -5x^2-7x+6 \\
 & -5x^2-10x \\
 & \underline{+\qquad +} \\
 & 3x+6 \\
 & \underline{\underline{3x+6}}
\end{array}$$

$=\lim\limits_{x\to -2}\dfrac{(x+2)(2x^2-8x+11)}{(x+2)(x^2-5x+3)}=\lim\limits_{x\to -2}\dfrac{2x^2-8x+11}{x^2-5x+3}=\dfrac{8+16+11}{4+10+3}=\dfrac{35}{17}.$

(*ii*) $\lim\limits_{x\to 2}\dfrac{x^3-8x^2+7x+10}{x^4-16}\left(\dfrac{0}{0}\ form\right)$. $As\ (x-2)\ is\ a\ factor\ of\ Nr.,$
$to\ find\ the\ other\ factor, we\ divide\ it\ by\ (x-2)$:

$$\begin{array}{r|l}
 & x^2-6x-5 \\ \hline
x-2) & x^3-8x^2+7x+10 \\
 & x^3-2x^2 \\
 & \underline{-\qquad +} \\
 & -6x^2+7x+10 \\
 & -6x^2+12x \\
 & \underline{+\qquad -} \\
 & -5x+10 \\
 & -5x+10 \\
 & \underline{+\qquad -}
\end{array}$$

$=\lim\limits_{x\to 2}\dfrac{(x-2)(x^2-6x-5)}{(x^2-4)(x^2+4)}=\lim\limits_{x\to 2}\dfrac{(x-2)(x^2-6x-5)}{(x-2)(x+2)(x^2+4)}=\lim\limits_{x\to 2}\dfrac{(x^2-6x-5)}{(x+2)(x^2+4)}$
$=\dfrac{4-12-5}{4.8}=-\dfrac{13}{32}$

Example 3: Evaluate (*i*) $\lim\limits_{x\to 2}\left(\dfrac{1}{x-2}-\dfrac{2}{x^2-2x}\right)$ (*ii*) $\lim\limits_{x\to 1}\left(\dfrac{3}{x^2+x-2}-\dfrac{1}{x-1}\right)$

Solution: (*i*) $\lim\limits_{x\to 2}\left(\dfrac{1}{x-2}-\dfrac{2}{x^2-2x}\right)$ $(\infty-\infty\quad form)$

$=\lim\limits_{x\to 2}\left\{\dfrac{1}{x-2}-\dfrac{2}{(x-2)x}\right\}=\lim\limits_{x\to 2}\dfrac{x-2}{(x-2)x}\quad\left(\dfrac{0}{0}\quad form\right)$

$$= \lim_{x\to 2} \frac{1}{x} = \frac{1}{2}.$$

(ii) $\lim_{x\to 1}\left(\frac{3}{x^2+x-2} - \frac{1}{x-1}\right)$ $\quad (\infty - \infty \quad form)$

$$= \lim_{x\to 1}\left\{\frac{3}{(x-1)(x+2)} - \frac{1}{x-1}\right\} = \lim_{x\to 1}\frac{3-x-2}{(x-1)(x+2)} = \lim_{x\to 1}\frac{-(x-1)}{(x-1)(x+2)} \quad \left(\frac{0}{0} \quad form\right)$$

$$= -\lim_{x\to 1}\frac{1}{x+2} = -\frac{1}{3}.$$

Assignment No. 1

1. Write the domain and range of the functions (i) $\frac{x^2-4}{x-2}$ $\quad (ii)\ \sqrt{9-x^2}$ $\quad (iii)\ \log_e x$.

$\langle$ ***Answer***: $(i)\ R-\{2\}, \quad R-\{4\} \quad (ii)\ [-3,3],\ [0,3] \quad (iii)\ (0,\infty), \quad R\ \rangle$

Evaluate the following limits:

2. $\lim_{x\to -1}(4x^3-5x+3)$ $\quad (Answer: 4)$ $\quad$ 3. $\lim_{x\to\pi/3}(3\sec^2 x - \tan^2 x)$ $\quad (Answer: 9)$

4. $\lim_{x\to 5}\frac{x^2-25}{x-5}$ $(Answer: 10)$ 5. $\lim_{x\to 3}\frac{x^3-9x}{x-3}$ $(Answer: 18)$ 6. $\lim_{x\to\sqrt{3}}\frac{x^2-3}{x^3-3\sqrt{3}}$ $\left(Answer: \frac{2\sqrt{3}}{9}\right)$

7. $\lim_{x\to 1}\frac{3x^3+4x^2-x-6}{x^3-2x^2-5x+6}$ $\left(Answer: -\frac{8}{3}\right)$ $\quad$ 8. $\lim_{x\to -2}\frac{x^3-2x^2-3x+10}{3x^3-4x+16}$ $\left(Answer: \frac{17}{32}\right)$

9. $\lim_{x\to 3}\left(\frac{3}{x^2-3x} - \frac{1}{x-3}\right)$ $\left(Answer: -\frac{1}{3}\right)$ $\quad$ 10. $\lim_{x\to 3}\left(\frac{2}{x^2-4x+3} - \frac{1}{x-3}\right)$ $\left(Answer: -\frac{1}{2}\right)$

11. $\lim_{x\to 2}\frac{x^3-3x^2+5x-6}{3x^3-2x^2-4x-8}$ $\left(Answer: \frac{5}{24}\right)$ $\quad$ 12. $\lim_{x\to -2\sqrt{2}}\frac{x^4-64}{x^3+16\sqrt{2}}$ $\quad (Answer: -2\sqrt{2})$.

xxxxxxxxxxx

Rationalisation method of finding limit

If a function takes $\frac{0}{0}$ form and is of the type $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ involve radicals, we use this method to find the limit.

Solved Examples

Example 1: Evaluate $\quad (i)\ \lim_{x\to 1}\frac{\sqrt{3x+1}-2}{x-1}$ $\quad (ii)\ \lim_{x\to 2}\frac{x^2-4}{2-\sqrt{6-x}}$

Solution: (i) $\lim_{x\to 1}\frac{\sqrt{3x+1}-2}{x-1}$ $\quad \left(\frac{0}{0}\ form\right)$

$$= \lim_{x\to 1}\frac{(\sqrt{3x+1}-2)(\sqrt{3x+1}+2)}{(x-1)(\sqrt{3x+1}+2)} \qquad (Rationalysing\ Nr.)$$

$$= \lim_{x\to 1}\frac{(3x+1)-4}{(x-1)(\sqrt{3x+1}+2)} = \lim_{x\to 1}\frac{3x-3}{(x-1)(\sqrt{3x+1}+2)}$$

$$= \lim_{x\to 1}\frac{3(x-1)}{(x-1)(\sqrt{3x+1}+2)} = \lim_{x\to 1}\frac{3}{(\sqrt{3x+1}+2)} = \frac{3}{4}.$$

(ii) $\lim_{x\to 2}\frac{x^2-4}{2-\sqrt{6-x}}$ $\quad \left(\frac{0}{0}\ form\right)$

$$= \lim_{x\to 2}\frac{x^2-4}{2-\sqrt{6-x}} = \lim_{x\to 2}\frac{(x^2-4)(2+\sqrt{6-x})}{(2-\sqrt{6-x})(2+\sqrt{6-x})} \qquad (Rationalysing\ Dr.)$$

$$= \lim_{x\to 2}\frac{(x^2-4)(2+\sqrt{6-x})}{4-(6-x)} = \lim_{x\to 2}\frac{(x-2)(x+2)(2+\sqrt{6-x})}{(x-2)}$$

$$= \lim_{x\to 2}(x+2)(2+\sqrt{6-x}) = 4.4 = 16$$

Example 2: Evaluate $\lim\limits_{x\to 2}\dfrac{\sqrt{2x+5}-3}{2-\sqrt{10-3x}}$

Solution: $\lim\limits_{x\to 2}\dfrac{\sqrt{2x+5}-3}{2-\sqrt{10-3x}}$ $\quad\left(\dfrac{0}{0}\ form\right)$

$$=\lim_{x\to 2}\frac{\sqrt{2x+5}-3}{2-\sqrt{10-3x}}=\lim_{x\to 2}\frac{\left(\sqrt{2x+5}-3\right)\left(\sqrt{2x+5}+3\right)\left(2+\sqrt{10-3x}\right)}{\left(2-\sqrt{10-3x}\right)\left(2+\sqrt{10-3x}\right)\left(\sqrt{2x+5}+3\right)}$$

$$(Rationalysing\ Nr.\,\&\,Dr.)$$

$$=\lim_{x\to 2}\frac{\{(2x+5)-9\}\left(2+\sqrt{10-3x}\right)}{\{4-(10-3x)\}\left(\sqrt{2x+5}+3\right)}=\lim_{x\to 2}\frac{(2x-4)\left(2+\sqrt{10-3x}\right)}{(3\mathrm{x}-6)\left(\sqrt{2x+5}+3\right)}$$

$$=\lim_{x\to 2}\frac{2(x-2)\left(2+\sqrt{10-3x}\right)}{3(x-2)\left(\sqrt{2x+5}+3\right)}=\lim_{x\to 2}\frac{2\left(2+\sqrt{10-3x}\right)}{3\left(\sqrt{2x+5}+3\right)}=\frac{2.4}{3.6}=\frac{4}{9}.$$

Example 3: Evaluate $\lim\limits_{x\to\sqrt{10}}\dfrac{\sqrt{7-2x}-\left(\sqrt{5}-\sqrt{2}\right)}{x^2-10}$

Solution: $\lim\limits_{x\to\sqrt{10}}\dfrac{\sqrt{7-2x}-\left(\sqrt{5}-\sqrt{2}\right)}{x^2-10}$ $\quad\left(\dfrac{0}{0}\ form\right)$ $\quad\left\{\begin{aligned}&7-2\sqrt{10}=5+2-2\sqrt{5.2}\\&=\left(\sqrt{5}\right)^2+\left(\sqrt{2}\right)^2-2\sqrt{5}\sqrt{2}\\&\qquad=\left(\sqrt{5}-\sqrt{2}\right)^2\end{aligned}\right.$

$$=\lim_{x\to\sqrt{10}}\frac{\sqrt{7-2x}-\left(\sqrt{5}-\sqrt{2}\right)}{x^2-10}=\lim_{x\to\sqrt{10}}\frac{\left\{\sqrt{7-2x}-\left(\sqrt{5}-\sqrt{2}\right)\right\}\left\{\sqrt{7-2x}+\left(\sqrt{5}-\sqrt{2}\right)\right\}}{(x^2-10)\left\{\sqrt{7-2x}+\left(\sqrt{5}-\sqrt{2}\right)\right\}}$$

$$(Rationalysing\ Nr.)$$

$$=\lim_{x\to\sqrt{10}}\frac{\left\{(7-2x)-\left(\sqrt{5}-\sqrt{2}\right)^2\right\}}{(x^2-10)\left\{\sqrt{7-2x}+\left(\sqrt{5}-\sqrt{2}\right)\right\}}=\lim_{x\to\sqrt{10}}\frac{\left\{(7-2x)-\left(5+2-2\sqrt{10}\right)\right\}}{(x^2-10)\left\{\sqrt{7-2x}+\left(\sqrt{5}-\sqrt{2}\right)\right\}}$$

$$=\lim_{x\to\sqrt{10}}\frac{-2\left(x-\sqrt{10}\right)}{\left(x-\sqrt{10}\right)\left(x+\sqrt{10}\right)\left\{\sqrt{7-2x}+\left(\sqrt{5}-\sqrt{2}\right)\right\}}$$

$$=\lim_{x\to\sqrt{10}}\frac{-2}{\left(x+\sqrt{10}\right)\left\{\sqrt{7-2x}+\left(\sqrt{5}-\sqrt{2}\right)\right\}}=\frac{-2}{\left(\sqrt{10}+\sqrt{10}\right)\left\{\sqrt{7-2\sqrt{10}}+\left(\sqrt{5}-\sqrt{2}\right)\right\}}$$

$$=\frac{-2}{2\sqrt{10}\left\{\left(\sqrt{5}-\sqrt{2}\right)+\left(\sqrt{5}-\sqrt{2}\right)\right\}}=\frac{-1}{\sqrt{10}.2\left(\sqrt{5}-\sqrt{2}\right)}=\frac{-1\left(\sqrt{5}+\sqrt{2}\right)}{\sqrt{10}.2\left(\sqrt{5}-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{2}\right)}$$

$$=-\frac{\sqrt{5}+\sqrt{2}}{2\sqrt{10}(5-2)}=-\frac{1}{6\sqrt{10}}\left(\sqrt{5}+\sqrt{2}\right).$$

Assignment No. 2

Evaluate the following limits:

1. $\lim\limits_{x\to 1}\dfrac{\sqrt{x+1}-\sqrt{2}}{x^2-1}$ $\left(Answer\colon \dfrac{1}{4\sqrt{2}}\right)$
2. $\lim\limits_{x\to -1}\dfrac{x^2-1}{2-\sqrt{1-3x}}$ $\left(Answer\colon -\dfrac{8}{3}\right)$
3. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$ $(Answer\colon 1)$
4. $\lim\limits_{x\to 1}\dfrac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+x-3}$ $\left(Answer\colon -\dfrac{1}{10}\right)$
5. $\lim\limits_{x\to -1}\dfrac{\sqrt{4-5x}-3}{1-\sqrt{2+x}}$ $\left(Answer\colon \dfrac{5}{3}\right)$
6. $\lim\limits_{x\to 3}\dfrac{\sqrt{x-2}-\sqrt{4-x}}{x-3}$ $(Answer\colon 1)$
7. $\lim\limits_{x\to -2\sqrt{3}}\dfrac{5-\sqrt{`1+2x^2}}{x^2-12}$ $\left(Ans\colon -\dfrac{1}{5}\right)$
8. $\lim\limits_{x\to\sqrt{10}}\dfrac{\sqrt{7+2x}-\left(\sqrt{5}+\sqrt{2}\right)}{x^2-10}$ $\left(Ans\colon \dfrac{1}{6\sqrt{10}}\left(\sqrt{5}-\sqrt{2}\right)\right)$
9. $\lim\limits_{x\to\sqrt{6}}\dfrac{\sqrt{5-2x}-\left(\sqrt{3}-\sqrt{2}\right)}{x^2-6}$ $\left(Answer\colon -\dfrac{1}{2\sqrt{6}}\left(\sqrt{3}+\sqrt{2}\right)\right).$
10. $\lim\limits_{x\to\sqrt{14}}\dfrac{\sqrt{9+2x}-\left(\sqrt{7}+\sqrt{2}\right)}{x^2-14}$ $\left(Answer\colon \dfrac{1}{10\sqrt{14}}\left(\sqrt{7}-\sqrt{2}\right)\right).$

Properties of limits

The following properties of limits are helpful while finding the limits of functions:

$$(i)\ \lim_{x\to a}\{f(x)\pm g(x)\} = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$(ii)\ \lim_{x\to a}\{f(x).g(x)\} = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$$

$$(iii)\ \lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)};\ provided\ \lim_{x\to a} g(x) \neq 0.$$

$$(iv)\ \lim_{x\to a}\{kf(x)\} = k\lim_{x\to a} f(x);\ k\ being\ a\ constant.$$

An Important limit

Formula: If n is a rational number, then

$$\boldsymbol{\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}}$$

Proof: $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}\quad \left(\frac{0}{0}\quad form\right)$

$$= \lim_{h\to 0}\frac{(a+h)^n-a^n}{a+h-a}\qquad \left[Using\quad \lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)\right]$$

$$= \lim_{h\to 0}\frac{(a+h)^n-a^n}{a+h-a} = \lim_{h\to 0}\frac{a^n\left(1+\frac{h}{a}\right)^n - a^n}{h} = \lim_{h\to 0}\frac{a^n\left\{\left(1+\frac{h}{a}\right)^n - 1\right\}}{h}$$

$$= \lim_{h\to 0}\frac{a^n\left[\left\{1+n\frac{h}{a}+\frac{n(n-1)}{2!}\left(\frac{h}{a}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{h}{a}\right)^3+\dots.to\ \infty\right\}-1\right]}{h}$$

$$\boxed{Using\ binomial\ theorem:\quad (1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots.to\ \infty}$$

$provided\ \ |x|<1$

$$= \lim_{h\to 0}\frac{a^n.n\frac{h}{a}\left[1+\frac{(n-1)}{2!}\left(\frac{h}{a}\right)+\frac{(n-1)(n-2)}{3!}\left(\frac{h}{a}\right)^2+\dots.to\ \infty\right]}{h}$$

$$= \lim_{h\to 0} a^n.n\frac{1}{a}\left[1+\frac{(n-1)}{2!}\left(\frac{h}{a}\right)+\frac{(n-1)(n-2)}{3!}\left(\frac{h}{a}\right)^2+\dots.to\ \infty\right]$$

$$= a^n.n\frac{1}{a}.1 = na^{n-1}.$$

Solved Examples

Example 1: Evaluate $\quad (i)\ \lim\limits_{x\to 2}\frac{x^3-8}{x-2}\qquad (ii)\ \lim\limits_{x\to -1}\frac{x^5+1}{x+1}$

Solution: $(i)\ \lim\limits_{x\to 2}\frac{x^3-8}{x-2}\quad \left(\frac{0}{0}\ form\right)$

$$= \lim_{x\to 2}\frac{x^3-2^3}{x-2} = 3.2^{3-1} = 12\qquad \left(Using\ the\ formula\quad \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right).$$

$(ii)\ \lim\limits_{x\to -1}\frac{x^5+1}{x+1}\quad \left(\frac{0}{0}\ form\right)$

$$= \lim_{x\to -1}\frac{x^5-(-1)^5}{x-(-1)} = 5.(-1)^{5-1} = 5\qquad \left(By\ formula\quad \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right).$$

Example 2: Evaluate $\quad (i)\ \lim\limits_{x\to 9}\frac{x^{3/2}-27}{x-9}\qquad (ii)\ \lim\limits_{x\to -2}\frac{x^7+128}{x^5+32}$

Solution: $(i)\ \lim\limits_{x\to 9}\frac{x^{3/2}-27}{x-9}\quad \left(\frac{0}{0}\ form\right)$

$$= \lim_{x\to 9} \frac{x^{3/2} - 9^{3/2}}{x - 9} = (3/2).9^{(3/2)-1} = \frac{3}{2}.9^{1/2} = \frac{9}{2}$$

$$\left(Using\ the\ formula\quad \lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right)$$

$(ii)\ \lim_{x\to -2} \frac{x^7 + 128}{x^5 + 32} \quad \left(\frac{0}{0}\ form\right)$

$$\lim_{x\to -2} \frac{x^7 - (-2)^7}{x^5 - (-2)^5} = \lim_{x\to -2} \frac{\left(\frac{x^7 - (-2)^7}{x - (-2)}\right)}{\left(\frac{x^5 - (-2)^5}{x - (-2)}\right)} \qquad (Dividing\ Nr.\ \&\ Dr.\ by\ x + 2)$$

$$= \frac{\lim_{x\to -2}\left(\frac{x^7 - (-2)^7}{x - (-2)}\right)}{\lim_{x\to -2}\left(\frac{x^5 - (-2)^5}{x - (-2)}\right)} \qquad \left[Using:\ \lim_{x\to a}\left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\right.$$

$$= \frac{7.(-2)^{7-1}}{5.(-2)^{5-1}} = \frac{7.4}{5} = \frac{28}{5}.$$

Example 3: If $\lim_{x\to 2} \frac{x^n - 2^n}{x - 2} = 192, \quad n \in N, \quad find\ n.$

Solution: $Given\ \lim_{x\to 2} \frac{x^n - 2^n}{x - 2} = 192 \quad \Rightarrow n.2^{n-1} = 3.2.2^5 = 6.2^{6-1} \quad \Rightarrow n = 6.$

Example 4: Evaluate $\lim_{x\to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x - a}$

Solution: $\lim_{x\to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x - a} = \lim_{z\to A} \frac{z^{5/2} - A^{5/2}}{z - A}.$

$$\begin{cases} Taking\ \ z = x + 2\ \ and\ \ A = a + 2 \\ x \to a \ \Rightarrow\ x + 2 \longrightarrow a + 2 \\ \Rightarrow\ z \longrightarrow A.\ Also, z - A = x - a \end{cases}$$

$$= \lim_{z\to A} \frac{z^{5/2} - A^{5/2}}{z - A} = \frac{5}{2}.A^{(5/2)-1} = \frac{5}{2}(a+2)^{3/2}$$

Assignment No. 3

Evaluate the following limits:

1. $\lim_{x\to 1} \frac{x^5 - 1}{x - 1}$ $(Answer: 5)$
2. $\lim_{x\to 2} \frac{x^4 - 16}{x - 2}$ $(Answer: 32)$
3. $\lim_{x\to -2} \frac{x^5 + 32}{x + 2}$ $(Answer: 80)$
4. $\lim_{x\to 3} \frac{x^6 - 729}{x - 3}$ $(Answer: 1458)$
5. $\lim_{x\to 2} \frac{\mathrm{x}^5 - 32}{\mathrm{x}^7 - 128}$ $\left(Answer: \frac{5}{28}\right)$
6. $\lim_{x\to \sqrt{3}} \frac{x^5 - 9\sqrt{3}}{x^3 - 3\sqrt{3}}$ $(Answer: 5)$
7. $\lim_{x\to 27} \frac{\mathrm{x}^{5/3} - 243}{\mathrm{x} - 27}$ $(Answer: 15)$
8. $\lim_{x\to 16} \frac{x^{3/4} - 8}{x - 16}$ $\left(Answer: \frac{3}{8}\right)$
9. If $\lim_{x\to 5} \frac{x^n - 5^n}{x - 5} = 500$; $n \in N$, determine the value of n. $(Answer: 4)$
10. If $\lim_{x\to 2} \frac{x^n - 2^n}{x - 2} = 448$; $n \in N$, determine the value of n. $(Answer: 7)$
11. Evaluate $\lim_{x\to a} \frac{(x+2)^{7/2} - (a+2)^{7/2}}{x - a}$ $\left[Answer: \frac{7}{2}(a+2)^{5/2}\right]$

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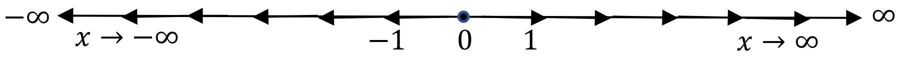
Finding limits when function takes $\frac{\infty}{\infty}$ form

A function generally takes $\frac{\infty}{\infty}$ form when $x \to \pm\infty$. Firstly, we understand the concept of $x \to \pm\infty$.

If x takes values in such a way that it becomes larger and larger, and it gets larger than any number however large the number may be, we say that x tends to infinity and write $x \to \infty$.



Similarly, If x takes values in such a way that it becomes smaller and smaller, and it gets smaller than any number however small it may be, we say that x tends to minus infinity and write $x \to -\infty$.

We notice that $x \to \infty \implies \frac{1}{x} \longrightarrow 0, \quad \frac{k}{x^n} \longrightarrow 0\,;\ n \in N.$

$\boldsymbol{Q}.\, Discuss\ the\ existence\ of\ \lim\limits_{x\to 0}\frac{1}{x}.$

$\boldsymbol{Solution}: LHL = \lim\limits_{x\to 0^-}\frac{1}{x} = -\infty \ \ \& \ \ RHL = \lim\limits_{x\to 0^+}\frac{1}{x} = \infty.$

$Since\ \ LHL \neq RHL,\ \ \lim\limits_{x\to 0}\frac{1}{x}\ does\ not\ exist.\ However,\ \ \ \lim\limits_{x\to 0}\frac{1}{|x|} = \infty.$

Finding limit when function takes $\frac{\infty}{\infty}$ form

Let a function be in the form $\frac{p(x)}{q(x)}$, then as $x \longrightarrow \infty$, the function takes the form $\frac{\infty}{\infty}$.

In order to find the limit of the function, we divide Nr. and Dr. by the highest power of x and then apply $\frac{k}{x^n} \longrightarrow 0\,;\ n \in N\ as\ \ x \to \infty$.

Solved examples

Example 1: $Evaluate\ \ (i)\ \ \lim\limits_{x\to\infty}\frac{2x^2-3x+4}{5x^2+2x-5} \qquad (ii)\ \ \lim\limits_{x\to\infty}\frac{4x^3+6x^2-3\mathrm{x}+4}{6x^3+5x-1}$

Solution: $(i)\ \ \lim\limits_{x\to\infty}\frac{2x^2-3x+4}{5x^2+2x-5}\ \ \left(\frac{\infty}{\infty}\ form\right)$

$$= \lim_{x\to\infty}\frac{2-\frac{3}{x}+\frac{4}{x^2}}{5+\frac{2}{x}-\frac{5}{x^2}}\ \ (Dividing\ Nr\ \&\ Dr\ by\ \ x^2)$$

$$= \frac{\lim\limits_{x\to\infty}\left(2-\frac{3}{x}+\frac{4}{x^2}\right)}{\lim\limits_{x\to\infty}\left(5+\frac{2}{x}-\frac{5}{x^2}\right)} = \frac{2-0+0}{5+0-0} = \frac{2}{5}.$$

$(ii)\ \ \lim\limits_{x\to\infty}\frac{4x^3+6x^2-3\mathrm{x}+4}{6x^3+5x-1} \qquad \left(\frac{\infty}{\infty}\ form\right)$

$$= \lim_{x\to\infty}\frac{4+\frac{6}{x}-\frac{3}{x^2}+\frac{4}{x^3}}{6+\frac{5}{x^2}-\frac{1}{x^3}} \qquad (Dividing\ Nr.\ \&\ Dr.\ by\ x^3)$$

$$= \frac{\lim\limits_{x\to\infty}\left(4+\frac{6}{x}-\frac{3}{x^2}+\frac{4}{x^3}\right)}{\lim\limits_{x\to\infty}\left(6+\frac{5}{x^2}-\frac{1}{x^3}\right)} = \frac{4+0-0+0}{6+0-0} = \frac{4}{6} = \frac{2}{3}.$$

Example 2: $Evaluate\ \ (i)\ \ \lim\limits_{x\to\infty}\frac{\sqrt{9x^2-3x}+4}{2x-5} \qquad (ii)\ \ \lim\limits_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)$

Solution: $(i)\ \ \lim\limits_{x\to\infty}\frac{\sqrt{9x^2-3x}+4}{2x-5}\ \left(\frac{\infty}{\infty}\ form\right)$

$$= \lim_{x\to\infty} \frac{\sqrt{9-\frac{3}{x}+\frac{4}{x}}}{2-\frac{5}{x}} \qquad (Dividing\ Nr.\ \&\ Dr.\ by\ x)$$

$$= \frac{\lim\limits_{x\to\infty}\left(\sqrt{9-\frac{3}{x}+\frac{4}{x}}\right)}{\lim\limits_{x\to\infty}\left(2-\frac{5}{x}\right)} = \frac{\sqrt{9}}{2} = \frac{3}{2}.$$

(ii) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)$ $[\infty-\infty\ form]$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+5x}-x\right)\left(\sqrt{x^2+5x}+x\right)}{\left(\sqrt{x^2+5x}+x\right)}$$

$$= \lim_{x\to\infty} \frac{(x^2+5x-x^2)}{\left(\sqrt{x^2+5x}+x\right)} = \lim_{x\to\infty} \frac{5x}{\left(\sqrt{x^2+5x}+x\right)} \quad \left(\frac{\infty}{\infty}\ form\right)$$

$$= \lim_{x\to\infty} \frac{5}{\left(\sqrt{1+\frac{5}{x}}+1\right)} \qquad (Dividing\ Nr.\ \&\ Dr.\ by\ x)$$

$$= \frac{5}{1+1} = \frac{5}{2}.$$

Example 3: $If\ f(x) = \frac{ax+2}{bx-3}$, $\lim\limits_{x\to\infty} f(x) = 4\ \ and\ \ f(-1) = 2, find\ a\ and\ b.$

Solution: Given $\lim\limits_{x\to\infty} f(x) = 4 \implies \lim\limits_{x\to\infty} \frac{ax+2}{bx-3} = 4 \implies \lim\limits_{x\to\infty} \frac{a+\frac{2}{x}}{b-\frac{3}{x}} = 4$

$$\implies \frac{a}{b} = 4 \implies a = 4b \quad -------------- (1)$$

$$Also\ given\ that\quad f(-1) = 2 \implies \frac{a(-1)+2}{b(-1)-3} = 2 \implies -a+2 = -2b-6$$

$$\implies -4b+2 = -2b-6 \quad [from\ (1)]$$

$$\implies b = 4 \implies a = 4b = 16$$

Example 4: $If\ f(x) = \frac{ax+2}{bx-3}$, $\lim\limits_{x\to\infty} f(x) = 4\ \ and\ \ f(-1) = 2, find\ a\ and\ b.$

Solution: Given $\lim\limits_{x\to\infty} f(x) = 4 \implies \lim\limits_{x\to\infty} \frac{ax+2}{bx-3} = 4 \implies \lim\limits_{x\to\infty} \frac{a+\frac{2}{x}}{b-\frac{3}{x}} = 4$

Example 5: $If\ f(x) = \frac{ax^2+b}{2x^2-3}$, $\lim\limits_{x\to\infty} f(x) = 3\ \ and\ \ \lim\limits_{x\to 0} f(x) = 2, find\ a\ and\ b.$

Solution: Given $\lim\limits_{x\to\infty} f(x) = 3 \implies \lim\limits_{x\to\infty} \frac{ax^2+b}{2x^2-3} = 3 \implies \lim\limits_{x\to\infty} \frac{a+\frac{b}{x^2}}{2-\frac{3}{x^2}} = 3$

$$\implies \frac{a}{2} = 3 \implies a = 6.$$

Again given that $\lim\limits_{x\to 0} f(x) = 2 \implies \lim\limits_{x\to 0} \frac{ax^2+b}{2x^2-3} = 2 \implies \frac{b}{-3} = 2 \implies b = -6.$

Example 6: $Evaluate$ (i) $\lim\limits_{x\to\infty} \frac{(2x^2+3)(3x+2)(2x-4)}{(2x^2-3)(5x^2-4x+2)}$ (ii) $\lim\limits_{n\to\infty} \frac{3^n-2^n}{3^n+2^n}$ $; n \in N.$

$$(iii)\ \lim_{n\to\infty} \frac{1^2+2^2+3^2+\dots+n^2}{(2n-3)(4n+5)(2n+7)}\ ;\ \ n \in N.$$

Solution: (i) $$\lim_{x\to\infty} \frac{(2x^2+3)(3x+2)(2x-4)}{(2x^2-3)(5x^2-4x+2)} \qquad \left(\frac{\infty}{\infty}\ form\right)$$

$$= \lim_{x\to\infty} \frac{(2x^2+3)(3x+2)(2x-4)}{(2x^2-3)(5x^2-4x+2)}$$

$$= \lim_{x\to\infty} \frac{\left(2+\frac{3}{x^2}\right)\left(3+\frac{2}{x}\right)\left(2-\frac{4}{x}\right)}{\left(2-\frac{3}{x^2}\right)\left(5-\frac{4}{x}+\frac{2}{x^2}\right)} \quad (Dividing\ Nr.\ \&\ Dr.\ by\ x^4)$$

$$= \frac{2.3.2}{2.5} = \frac{6}{5}.$$

$(ii)\ \lim_{n\to\infty} \frac{3^n-2^n}{3^n+2^n}\left(\frac{\infty}{\infty}\ form\right); n \in N.$ Dividing Nr. and Dr. by 3^n, we get

$$\lim_{n\to\infty} \frac{3^n-2^n}{3^n+2^n} = \lim_{n\to\infty} \frac{1-\left(\frac{2}{3}\right)^n}{1+\left(\frac{2}{3}\right)^n} = \frac{1-0}{1+0} = \frac{1}{1} = 1.\ (Using\ \lim_{x\to\infty} a^x = 0, if\ |a| < 1)$$

$$(ii)\ \lim_{n\to\infty} \frac{1^2+2^2+3^2+\ldots.+n^2}{(2n-3)(4n+5)(2n+7)}, \quad n \in N \quad \left(\frac{\infty}{\infty}\ form\right)$$

$$= \lim_{n\to\infty} \frac{\frac{n(n+1)(2n+1)}{6}}{(2n-3)(4n+5)(2n+7)} \quad \left\{\begin{array}{l} Using\ the\ sum\ of\ squares\ of\ first\ n\ natural\ numbers: \\ \qquad 1^2+2^2+3^2+\ldots.+n^2 = \frac{n(n+1)(2n+1)}{6} \end{array}\right.$$

$$= \frac{1}{6} \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{(2n-3)(4n+5)(2n+7)} = \frac{1}{6} \lim_{n\to\infty} \frac{1.\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{\left(2-\frac{3}{n}\right)\left(4+\frac{5}{n}\right)\left(2+\frac{7}{n}\right)}$$

$(Dividing\ each\ factor\ by\ n\ or\ dividing\ Nr. and\ Dr. by\ n^3)$

$$= \frac{1}{6}.\frac{1.1.2}{2.4.2} = \frac{1}{48}.$$

Assignment No. 4

Evaluate the following limits:

1. $\lim_{x\to\infty} \frac{x^2-3x+2}{3x^2+2x-6} \quad \left(Answer: \frac{1}{3}\right)$ 2. $\lim_{x\to\infty} \frac{5x^3+6x^2-x+4}{6x^3+4x^2-5x-1} \left(Answer: \frac{5}{6}\right)$

3. $\lim_{x\to\infty} \frac{\sqrt{4x^2-3x}+4x}{2x-5} \quad (Answer: 3)$ 4 $\lim_{x\to\infty}\left(\sqrt{x^2+7x}-x\right) \quad \left(Answer: \frac{7}{2}\right)$

5. $If\ f(x) = \frac{ax+b}{2x-3}, \quad \lim_{x\to\infty} f(x) = 1\ \ and\ \ \lim_{x\to 0} f(x) = 2, \quad find\ a\ and\ b.$

$(Answer: a = 2,\ b = -6)$

6. $If\ f(x) = \frac{ax^2+2}{2x^2-b}, \quad \lim_{x\to\infty} f(x) = 3\ \ and\ \ f(-3) = 2, \quad find\ a\ and\ b.$

$(Answer: a = 6,\ b = -10)$

7. $Evaluate\ \lim_{x\to\infty} \frac{(2x^2+1)(3x-2)(2x-5)}{(4x^2-3)(3x-1)(5x+2)} \quad \left(Answer: \frac{1}{5}\right)$

8. $Evaluate\ \lim_{n\to\infty} \frac{1+2+3+\ldots.+n}{(n-1)(3n+2)};\ \ n \in N. \quad \left(Answer: \frac{1}{6}\right)$

9. $Evaluate\ \lim_{n\to\infty} \frac{1^3+2^3+3^3+\ldots.+n^3}{(2n^2-3)(4n+5)(2n+7)};\ \ n \in N. \quad \left(Answer: \frac{1}{64}\right)$

10. Evaluate $\lim_{n\to\infty} \frac{5^n-7^n}{5^n+7^n};\ n \in N. \quad (Answer: -1)$

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Trigonometric Limits

Formulae:

$(i)\ \lim_{x\to 0} \sin x = 0$ $\qquad$ $(ii)\ \lim_{x\to 0} \cos x = 1$ $\qquad$ $(iii)\ \lim_{x\to 0} \tan x = 0$

$(iv)\ \lim_{x\to 0} \frac{\sin x}{x} = 1$ $\qquad$ $(v)\ \lim_{x\to 0} \frac{\tan x}{x} = 1$ $\qquad$ $(vi)\ \lim_{x\to 0} \frac{x}{\sin x} = 1$

$(vii)\ \lim_{x\to 0} \frac{x}{\tan x} = 1.$

Proof: Formulae (i), (ii), (iii) are self − evident.

$$(iv)\ \lim_{x\to 0} \frac{\sin x}{x} \left(\frac{0}{0}\ form\right) = \lim_{x\to 0} \left(\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots\ldots to\ \infty}{x} \right)$$

$$\left\{ \begin{aligned} Using: \qquad & \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots\ldots to\ \infty, \\ & \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots\ldots to\ \infty \end{aligned} \right.$$

$$\Rightarrow \qquad \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \left\{ \frac{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \ldots\ldots to\ \infty\right)}{x} \right\}$$

$$\Rightarrow \qquad \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} - \frac{x^{10}}{11!} + \ldots\ldots to\ \infty\right) = 1.$$

$$(v)\ \lim_{x\to 0} \frac{\tan x}{x} \left(\frac{0}{0}\ form\right) = \lim_{x\to 0} \frac{\left(\frac{\sin x}{\cos x}\right)}{x} = \lim_{x\to 0} \frac{\sin x}{x.\cos x} = \lim_{x\to 0} \left\{ \left(\frac{\sin x}{x}\right) \left(\frac{1}{\cos x}\right) \right\}$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x}\right) . \lim_{x\to 0} \left(\frac{1}{\cos x}\right) \qquad \left[Using\ \ \lim_{x\to a} \{f(x).g(x)\} = \lim_{x\to a} f(x) . \lim_{x\to a} g(x) \right]$$

$$= 1.\frac{1}{1} = 1 \qquad \left(Using\ \ \lim_{x\to 0} \frac{\sin x}{x} = 1 \right)$$

Remark: We can easily deduce that

$$\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^n = \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^n = \lim_{x\to 0} \left(\frac{x}{\sin x}\right)^n = \lim_{x\to 0} \left(\frac{x}{\tan x}\right)^n = 1;\ n \in I.$$

Solved Examples

Example 1: *Evaluate* $\quad (i)\ \lim_{x\to 0} \frac{\sin 2x}{x} \quad (ii)\ \lim_{x\to 0} \frac{\tan 4x}{x} \quad (iii)\ \lim_{x\to 0} \frac{x \tan 5x}{\sin^2 3x}$

Solution: $(i)\ \lim_{x\to 0} \frac{\sin 2x}{x} \left(\frac{0}{0}\ form\right) = \lim_{x\to 0} \left\{ \left(\frac{\sin 2x}{2x}\right) . 2 \right\} = 2 \lim_{x\to 0} \frac{\sin 2x}{2x} = 2.1 = 2.$

$(ii)\ \lim_{x\to 0} \frac{\tan 4x}{x} \left(\frac{0}{0}\ form\right) = \lim_{x\to 0} \left\{ \left(\frac{\tan 4x}{4x}\right) 4 \right\} = 4 \lim_{x\to 0} \left(\frac{\tan 4x}{4x}\right) = 4.1 = 4.$

$$(iii)\ \lim_{x\to 0} \frac{x \tan 5x}{\sin^2 3x} \left(\frac{0}{0}\ form\right) = \lim_{x\to 0} \left\{ \left(\frac{3x}{\sin 3x}\right)^2 \left(\frac{\tan 5x}{5x}\right) . \frac{5}{3^2} \right\} = \frac{5}{9} \lim_{x\to 0} \left\{ \left(\frac{3x}{\sin 3x}\right)^2 \left(\frac{\tan 5x}{5x}\right) \right\}$$

$$= \frac{5}{9} . 1^2 . 1 = \frac{5}{9} .$$

Example 2: *Evaluate* $\quad (i)\ \lim_{x\to 0} \frac{\sin x^0}{x} \quad (ii)\ \lim_{x\to 0} \frac{1 - \cos 2x}{x^2} \quad (iii)\ \lim_{x\to 0} \frac{1 - \cos mx}{1 - \cos nx}$

Solution: $(i)\ \lim_{x\to 0} \frac{\sin x^0}{x} \quad \left(\frac{0}{0}\ form\right) = \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x} = \lim_{x\to 0} \left[\left\{ \frac{\sin\left(\frac{\pi x}{180}\right)}{\left(\frac{\pi x}{180}\right)} \right\} \left(\frac{\pi}{180}\right) \right]$

$$= \frac{\pi}{180} \lim_{x\to 0} \left\{ \frac{\sin\left(\frac{\pi x}{180}\right)}{\left(\frac{\pi x}{180}\right)} \right\} = \frac{\pi}{180} . 1 = \frac{\pi}{180} .$$

$(ii)\ \lim_{x\to 0}\frac{1-\cos 2x}{x^2}\left(\frac{0}{0}\ form\right)$

$= \lim_{x\to 0}\frac{2\sin^2 x}{x^2}$ $\quad\{Using\ the\ formula:\quad 1-\cos 2x = 2\sin^2 x$

$= 2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = 2.1^2 = 2.$

(iii) $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}\quad\left(\frac{0}{0}\ form\right)$

$$= \lim_{x\to 0}\frac{2\sin^2\frac{mx}{2}}{2\sin^2\frac{nx}{2}} = \lim_{x\to 0}\left[\frac{\sin^2\left(\frac{mx}{2}\right)}{\sin^2\left(\frac{nx}{2}\right)}\right] = \lim_{x\to 0}\left[\frac{\left\{\frac{\sin\left(\frac{mx}{2}\right)}{\left(\frac{mx}{2}\right)}\right\}^2}{\left\{\frac{\sin\left(\frac{nx}{2}\right)}{\left(\frac{nx}{2}\right)}\right\}^2}.\left\{\frac{\left(\frac{mx}{2}\right)}{\left(\frac{nx}{2}\right)}\right\}^2\right]$$

$= \frac{1^2}{1^2}.\left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}.$

Example 3: *Evaluate* $(i)\ \lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin 7x+\sin 5x}\qquad (ii)\ \lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$

Solution: $(i)\ \lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin 7x+\sin 5x}\ \left(\frac{0}{0}\ form\right)$ *.The following trigonometric* identities need to be memorized:

$$\boxed{\begin{aligned}\sin C+\sin D &= 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\\ \sin C-\sin D &= 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\\ \cos C+\cos D &= 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)\\ \cos C-\cos D &= 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)\\ &= -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\end{aligned}}$$

Now, $\lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin 7x+\sin 5x} = \lim_{x\to 0}\frac{2\cos 4x\,.\sin x}{2\sin 6x\,.\cos x} = \lim_{x\to 0}\frac{\cos 4x\,.\sin x}{\sin 6x\,.\cos x}$

$= \lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{6x}{\sin 6x}\right).\frac{1}{6}.\left(\frac{\cos 4x}{\cos x}\right)\right] = \frac{1}{6}.1.1.\frac{1}{1} = \frac{1}{6}.$

Alternatively, $\lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin 7x+\sin 5x} = \lim_{x\to 0}\frac{\left(\frac{\sin 5x}{5x}\right).5x-\left(\frac{\sin 3x}{3x}\right).3x}{\left(\frac{\sin 7x}{7x}\right).7x+\left(\frac{\sin 5x}{5x}\right).5x}$

$$= \lim_{x\to 0}\frac{x\left\{\left(\frac{\sin 5x}{5x}\right).5-\left(\frac{\sin 3x}{3x}\right).3\right\}}{x\left\{\left(\frac{\sin 7x}{7x}\right).7+\left(\frac{\sin 5x}{5x}\right).5\right\}} = \lim_{x\to 0}\left\{\frac{\left(\frac{\sin 5x}{5x}\right).5-\left(\frac{\sin 3x}{3x}\right).3}{\left(\frac{\sin 7x}{7x}\right).7+\left(\frac{\sin 5x}{5x}\right).5}\right\} = \frac{1.5-1.3}{1.7+1.5} = \frac{2}{12} = \frac{1}{6}.$$

$(ii)\ \lim_{x\to 0}\frac{\tan x-\sin x}{x^3}\qquad\left(\frac{0}{0}\ form\right)$

$$= \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3} = \lim_{x\to 0}\frac{\left(\frac{\sin x-\sin x\cos x}{\cos x}\right)}{x^3} = \lim_{x\to 0}\frac{(\sin x-\sin x\cos x)}{x^3\cos x}$$

$$= \lim_{x\to 0}\frac{\sin x\,(1-\cos x)}{x^3 \cos x} = \lim_{x\to 0}\frac{\sin x\,.\,2\sin^2\frac{x}{2}}{x^3\cos x} = 2.\lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\frac{1}{4\cos x}\right]$$

$$= \frac{2}{4}.1.1^2.\frac{1}{1} = \frac{1}{2}.$$

Example 4: *Evaluate* $(i)\ \lim_{x\to 0}\frac{\sin^2 5x - 4x\sin 3x}{2x^2+3\tan^2 4x}\quad (ii)\ \lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{x^2}$

Solution: $(i)\ \lim_{x\to 0}\frac{\sin^2 5x - 4x\sin 3x}{2x^2+3\tan^2 4x}\quad \left(\frac{0}{0}\ form\right).$

$$= \lim_{x\to 0}\left(\frac{\frac{\sin^2 5x}{x^2} - 4\frac{x\sin 3x}{x^2}}{\frac{2x^2}{x^2}+3\frac{\tan^2 4x}{x^2}}\right)\quad (Dividing\ Nr.\ \&\ Dr.\ by\ x^2)$$

$$= \lim_{x\to 0}\left\{\frac{\left(\frac{\sin 5x}{5x}\right)^2.5^2 - 4\left(\frac{\sin 3x}{3x}\right).3}{2+3\left(\frac{\tan 4x}{4x}\right)^2.4^2}\right\} = \frac{25.1^2-12.1}{2+48.1^2} = \frac{13}{50}.$$

$(ii)\ \lim_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{x^2}\quad \left(\frac{0}{0}\ form\right)$

$$= \lim_{x\to 0}\frac{\sqrt{2}-\sqrt{2\cos^2\frac{x}{2}}}{x^2} = \lim_{x\to 0}\frac{\sqrt{2}-\sqrt{2}\cos\frac{x}{2}}{x^2} = \lim_{x\to 0}\frac{\sqrt{2}\left(1-\cos\frac{x}{2}\right)}{x^2}$$

$$= \lim_{x\to 0}\frac{\sqrt{2}.2\sin^2\frac{x}{4}}{x^2} = 2\sqrt{2}.\lim_{x\to 0}\left\{\left(\frac{\sin\frac{x}{4}}{\frac{x}{4}}\right)^2.\left(\frac{1}{4}\right)^2\right\} = 2\sqrt{2}.\frac{1}{16}.1^2 = \frac{\sqrt{2}}{8}.$$

Assignment No. 5

Evaluate:

1. $(i)\ \lim_{x\to 0}\frac{\sin^2 ax}{\sin^2 bx}\quad (ii)\ \lim_{x\to 0}\frac{\tan mx}{\tan nx}\quad (iii)\ \lim_{x\to 0}\frac{x}{\tan x^0}\quad (iv)\ \lim_{x\to 0}\frac{\sin^2 3x}{x\tan 4x}$

$$\left[Answer: (i)\ \frac{a^2}{b^2}\quad (ii)\ \frac{m}{n}\quad (iii)\ \frac{180}{\pi}\quad (iv)\ \frac{9}{4}\right]$$

2. $(i)\ \lim_{x\to 0}\frac{\sin x^3}{x\sin 5x\tan 3x}\quad (ii)\ \lim_{x\to 0}\frac{1-\cos 2x}{x\sin 3x}\quad (iii)\ \lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}$

$$\left[Answer: (i)\ \frac{1}{15}\quad (ii)\ \frac{2}{3}\quad (iii)\ \frac{m^2}{n^2}\right]$$

3. $(i)\ \lim_{x\to 0}\frac{\sin 4x+\sin 2x}{\sin 3x+\sin x}\quad (ii)\ \lim_{x\to 0}\frac{\sin 7x-\sin 2x}{\sin 5x-\sin x}\quad (iii)\ \lim_{x\to 0}\frac{\cos 7x-\cos x}{\cos 5x-\cos 9x}$

$$\left[Answer: (i)\ \frac{3}{2}\quad (ii)\ \frac{5}{4}\quad (iii)\ -\frac{6}{7}\right]$$

4. $(i)\ \lim_{x\to 0}\frac{x\sin 4x+3\sin^2 2x}{\tan 2x\sin 3x+4x^2}\quad (ii)\ \lim_{x\to 0}\frac{3\sin^2 x-2\sin x^2}{\tan^2 5x-3x\sin x}$

$$\left[Answer: (i)\ \frac{8}{5}\quad (ii)\ \frac{1}{22}\right]$$

5. $(i)\ \lim_{x\to 0}\frac{\tan x-\sin x}{x^3}\quad (ii)\ \lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 2x}\quad (iii)\ \lim_{x\to 0}\frac{\tan x-\sin x}{x(1-\cos 2x)}$

$$\left[Answer: (i)\ \frac{1}{2}\quad (ii)\ \frac{1}{16}\quad (iii)\ \frac{1}{4}\right]$$

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Trigonometric limits when x approaches a non − zero quantity

Such limits can be evaluated by simplification or by using the formula

$$\lim_{x\to a} f(x) = \lim_{h\to 0} f(a\pm h)$$

Solved Examples

Example 1: Evaluate $(i)\ \lim\limits_{x\to\pi/2} \dfrac{1-\sin x}{\cos^2 x}$ $\ (ii)\ \lim\limits_{x\to\pi/2}(\sec x-\tan x)$ $\ (iii)\ \lim\limits_{x\to\pi} \dfrac{1+\cos x}{(\pi-x)^2}$

Solution: $(i)\ \lim\limits_{x\to\pi/2} \dfrac{1-\sin x}{\cos^2 x}\quad \left(\dfrac{0}{0}\ form\right) = \lim\limits_{x\to\pi/2}\dfrac{1-\sin x}{1-\sin^2 x} = \lim\limits_{x\to\pi/2}\dfrac{1-\sin x}{(1-\sin x)(1+\sin x)}$

$= \lim\limits_{x\to\pi/2}\dfrac{1}{1+\sin x} = \dfrac{1}{1+\sin \pi/2} = \dfrac{1}{1+1} = \dfrac{1}{2}.$

Alternatively, $\lim\limits_{x\to\pi/2} \dfrac{1-\sin x}{\cos^2 x} = \lim\limits_{h\to 0}\dfrac{1-\sin\left(\frac{\pi}{2}-h\right)}{\cos^2\left(\frac{\pi}{2}-h\right)} = \lim\limits_{h\to 0}\dfrac{1-\cos h}{\sin^2 h} = \lim\limits_{h\to 0}\dfrac{1-\cos h}{1-\cos^2 h}$

$= \lim\limits_{h\to 0}\dfrac{1-\cos h}{(1-\cos h)(1+\cos h)} = \lim\limits_{h\to 0}\dfrac{1}{1+\cos h} = \dfrac{1}{1+1} = \dfrac{1}{2}.$

$(ii)\ \lim\limits_{x\to\pi/2}(\sec x-\tan x)\quad (\infty-\infty\ form) = \lim\limits_{x\to\pi/2}\dfrac{1-\sin x}{\cos x}\left(\dfrac{0}{0}\ form\right)$

$= \lim\limits_{x\to\pi/2}\dfrac{(1-\sin x)(1+\sin x)}{\cos x\,(1+\sin x)} = \lim\limits_{x\to\pi/2}\dfrac{1-\sin^2 x}{\cos x\,(1+\sin x)} = \lim\limits_{x\to\pi/2}\dfrac{\cos^2 x}{\cos x\,(1+\sin x)}$

$\lim\limits_{x\to\pi/2}\dfrac{\cos x}{1+\sin x} = \dfrac{0}{1+1} = \dfrac{0}{2} = 0.$

$(iii)\ \lim\limits_{x\to\pi}\dfrac{1+\cos x}{(\pi-x)^2}\left(\dfrac{0}{0}\ form\right) = \lim\limits_{h\to 0}\left[\dfrac{1+\cos(\pi-h)}{\{\pi-(\pi-h)\}^2}\right] = \lim\limits_{h\to 0}\left[\dfrac{1-\cos h}{h^2}\right]$

$= \lim\limits_{h\to 0}\left[\dfrac{2\sin^2\frac{h}{2}}{h^2}\right] = 2\lim\limits_{h\to 0}\left[\left(\dfrac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2.\dfrac{1}{4}\right] = 2.1^2.\dfrac{1}{4} = \dfrac{1}{2}.$

Example 2: *Evaluate* $(i)\ \lim\limits_{x\to\pi/4}\dfrac{\cot x-1}{\cos 2x}$ $\ (ii)\ \lim\limits_{x\to\pi/4}\dfrac{\sec^2 x-2}{1-\tan x}$ $\ (iii)\ \lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}$

Solution: $(i)\ \lim\limits_{x\to\pi/4}\dfrac{\cot x-1}{\cos 2x}\left(\dfrac{0}{0}\ form\right) = \lim\limits_{x\to\pi/4}\dfrac{\frac{\cos x}{\sin x}-1}{\cos^2 x-\sin^2 x} = \lim\limits_{x\to\pi/4}\dfrac{\frac{\cos x-\sin x}{\sin x}}{\cos^2 x-\sin^2 x}$

$= \lim\limits_{x\to\pi/4}\dfrac{\cos x-\sin x}{\sin x\,(\cos x-\sin x)(\cos x+\sin x)} = \lim\limits_{x\to\pi/4}\dfrac{1}{\sin x\,(\cos x+\sin x)} = \dfrac{1}{\frac{1}{\sqrt{2}}.\frac{2}{\sqrt{2}}} = 1$

$(ii)\ \lim\limits_{x\to\pi/4}\dfrac{\sec^2 x-2}{1-\tan x}\left(\dfrac{0}{0}\ form\right) = \lim\limits_{x\to\pi/4}\dfrac{1+\tan^2 x-2}{1-\tan x} = \lim\limits_{x\to\pi/4}\dfrac{\tan^2 x-1}{1-\tan x}$

$= -\lim\limits_{x\to\pi/4}\dfrac{1-\tan^2 x}{1-\tan x} = -\lim\limits_{x\to\pi/4}\dfrac{(1-\tan x)(1+\tan x)}{1-\tan x} = -\lim\limits_{x\to\pi/4}(1+\tan x) = -2.$

$(iii)\ \lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}\left(\dfrac{0}{0}\ form\right) = \lim\limits_{h\to 0}\dfrac{\sin(a+h)-\sin a}{(a+h)-a}$

$= \lim\limits_{h\to 0}\dfrac{\sin a\cos h+\cos a\sin h-\sin a}{(a+h)-a} = -\lim\limits_{h\to 0}\dfrac{\sin a(1-\cos h)-\cos a.\sin h}{h}$

$= -\lim\limits_{h\to 0}\dfrac{\sin a.2\sin^2\frac{h}{2}-\cos a.\sin h}{h} = -\lim\limits_{h\to 0}\left[2\sin a.\left(\dfrac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2.\dfrac{h}{4}-\cos a.\left(\dfrac{\sin h}{h}\right)\right]$

$= -\left(2\sin a.1^2.\dfrac{0}{4}-\cos a.1\right) = \cos a.$

Alternatively, $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a} = \lim\limits_{x\to a}\dfrac{2\cos\frac{x+a}{2}.\sin\frac{x-a}{2}}{x-a}$

$= \lim\limits_{x\to a}\left[\cos\left(\dfrac{x+a}{2}\right).\left\{\dfrac{\sin\left(\frac{x-a}{2}\right)}{\left(\frac{x-a}{2}\right)}\right\}\right] = \cos\left(\dfrac{a+a}{2}\right).1 = \cos a.$

Assignment No. 6

Evaluate the following limits:

1. $(i)\ \lim\limits_{x\to\pi}\dfrac{1+\cos x}{\sin^2 x}$ $\quad (ii)\ \lim\limits_{x\to\pi/2}(\sec x-\tan x)$ $\quad (iii)\ \lim\limits_{x\to\pi}\dfrac{1+\cos x}{(\pi-x)^2}$

$$\left[Answer: (i)\ \frac{1}{2}\quad (ii)\ 0\quad (iii)\ \frac{1}{2}\ \right]$$

2. $(i)\ \lim\limits_{x\to\pi/2}\dfrac{1-\sin^3 x}{\cos^2 x}$ $\quad (ii)\ \lim\limits_{x\to\pi/4}\dfrac{2-\mathrm{cosec}^2 x}{1-\cot x}$ $\quad (iii)\ \lim\limits_{x\to\pi}\dfrac{1+\cos^3 x}{(\pi-x)^2}$

$$\left[Answer: (i)\ \frac{3}{2}\quad (ii)\ 2\quad (iii)\ \frac{3}{2}\ \right]$$

3. $(i)\ \lim\limits_{x\to\pi/4}\dfrac{\sin x-\cos x}{\pi-4x}$ $\quad (ii)\ \lim\limits_{x\to\pi/2}\dfrac{1-\sin^4 x}{\cos^2 x}$ $\quad (iii)\ \lim\limits_{x\to\pi}\dfrac{1+\sec x}{(\pi-x)^2}$

$$\left[Answer: (i)\ -\frac{\sqrt{2}}{4}\quad (ii)\ 2\quad (iii)\ -\frac{1}{2}\ \right]$$

4. $(i)\ \lim\limits_{x\to\pi}\dfrac{1+\cos^3 x}{\sin^2 x}$ $\quad (ii)\ \lim\limits_{x\to\pi/4}\dfrac{\cot x-1}{\cos 2x}$ $\quad (iii)\ \lim\limits_{x\to\pi/4}\dfrac{1-\sin 2x}{\left(\frac{\pi}{4}-x\right)^2}$

$$\left[Answer: (i)\ \frac{3}{2}\quad (ii)\ 1\quad (iii)\ 2\ \right]$$

Limits of inverse trigonometric functions

Such limits are usually evaluated by suitable substitutions which convert the function into non-inverse form.

Formulae:

$(\boldsymbol{i})\ \boldsymbol{\lim\limits_{x\to0}\sin^{-1}x=0}$ $\quad (\boldsymbol{ii})\ \boldsymbol{\lim\limits_{x\to0}\cos^{-1}x=\dfrac{\pi}{2}}$ $\quad (\boldsymbol{iii})\ \boldsymbol{\lim\limits_{x\to0}\tan^{-1}x=0}$

$(\boldsymbol{iv})\ \boldsymbol{\lim\limits_{x\to0}\dfrac{\sin^{-1}x}{x}=1}$ $\quad (\boldsymbol{v})\ \boldsymbol{\lim\limits_{x\to0}\dfrac{\tan^{-1}x}{x}=1}$ $\quad (\boldsymbol{vi})\ \boldsymbol{\lim\limits_{x\to0}\dfrac{x}{\sin^{-1}x}=1}$

$(\boldsymbol{vii})\ \boldsymbol{\lim\limits_{x\to0}\dfrac{x}{\tan^{-1}x}=1}$.

Proof: Formulae (i), (ii), (iii) are self – evident.

$(\boldsymbol{iv})\ \boldsymbol{\lim\limits_{x\to0}\dfrac{\sin^{-1}x}{x}=\lim\limits_{t\to0}\dfrac{t}{\sin t}=1}$ $\quad \begin{cases}\boldsymbol{Taking\ \ \sin^{-1}x=t\ \Rightarrow\ \sin t=x}.\\ \boldsymbol{As\ x\longrightarrow0,\ t\longrightarrow0}\end{cases}$

Similarly, we can prove (v), (vi), (vii).

Solved Examples

Example 1: *Evaluate* $\quad (i)\ \lim\limits_{x\to1^-}\dfrac{1-x}{(\cos^{-1}x)^2}$ $\quad (ii)\ \lim\limits_{x\to0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}$

Solution: $(i)\ \lim\limits_{x\to1^-}\dfrac{1-x}{(\cos^{-1}x)^2}\ \left(\dfrac{0}{0}\ form\right)$

$=\lim\limits_{x\to1^-}\dfrac{1-x}{(\cos^{-1}x)^2}=\lim\limits_{t\to0}\dfrac{1-\cos t}{t^2}$ $\quad \begin{cases}Taking\ \ \cos^{-1}x=t\ \Rightarrow\ \cos t=x.\\ As\ x\longrightarrow1,\ t\longrightarrow0\end{cases}$

$$=\lim_{t\to0}\frac{2\sin^2\frac{t}{2}}{t^2}=2\lim_{t\to0}\left\{\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2.\frac{1}{4}\right\}=2.\frac{1}{4}=\frac{1}{2}.$$

$$(ii)\ \lim_{x\to0}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}\ \left(\frac{0}{0}\ form\right)=\lim_{x\to0}\frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)\left(\sqrt{1+x}+\sqrt{1-x}\right)}{\sin^{-1}x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$$

$$=\lim_{x\to0}\frac{(1+x)-(1-x)}{\sin^{-1}x\left(\sqrt{1+x}+\sqrt{1-x}\right)}=2\lim_{x\to0}\left(\frac{x}{\sin^{-1}x}\right)\left(\frac{1}{\sqrt{1+x}+\sqrt{1-x}}\right)=2.1.\frac{1}{2}=1.$$

Example 2: *Evaluate* $\quad (i)\ \lim\limits_{x\to0}\dfrac{3x-\sin^{-1}x}{4x+\tan^{-1}x}$ $\quad (ii)\ \lim\limits_{x\to0}\dfrac{1}{x}\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

Solution: $(i)\ \lim_{x\to 0}\dfrac{3x-\sin^{-1}x}{4x+\tan^{-1}x}\ \left(\dfrac{0}{0}\ form\right)$

$$= \lim_{x\to 0}\frac{3-\dfrac{\sin^{-1}x}{x}}{4+\dfrac{\tan^{-1}x}{x}} = \frac{3-1}{4+1}=\frac{2}{5}.$$

$(ii)\ \lim_{x\to 0}\dfrac{1}{x}\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)\quad [0\times\infty\ \ form]$

$$= \lim_{\theta\to 0}\frac{1}{\tan\theta}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)\quad \begin{cases}Taking\ \ x=\tan\theta\\ As\ \ x\longrightarrow 0,\ \ \theta\longrightarrow 0\end{cases}$$

$$= \lim_{\theta\to 0}\frac{1}{\tan\theta}\cos^{-1}(\cos 2\theta) = \lim_{\theta\to 0}\frac{2\theta}{\tan\theta}\left(\frac{0}{0}\ form\right)\quad \begin{cases}Using\ \ \cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\end{cases}$$

$$= 2\lim_{\theta\to 0}\frac{\theta}{\tan\theta}=2.1=2.$$

Assignment No. 7

Evaluate the following limits:

1. $\lim_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1}x}$ $(Answer: 1)$ 2. $\lim_{x\to 1^-}\dfrac{1-x}{(\cos^{-1}x)^2}$ $(Answer: 1/2)$

3. $\lim_{x\to 0}\dfrac{4x-\sin^{-1}x}{4x+3\tan^{-1}x}$ $\left(Answer: \dfrac{3}{7}\right)$ 4. $\lim_{x\to 0}\dfrac{1}{x}\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ $(Answer: 2)$

5. $\lim_{x\to 0}\dfrac{1}{x}\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ $(Answer: 2)$ 6. $\lim_{x\to 0}\dfrac{1}{x}\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ $(Answer: 2)$

Other important limits

Formulae:

$(\boldsymbol{i})\ \lim_{x\to 0}\dfrac{e^x-1}{x}=1$ $\quad(\boldsymbol{ii})\ \lim_{x\to 0}\dfrac{a^x-1}{x}=\log_e a$ $\quad(\boldsymbol{iii})\ \lim_{x\to 0}\dfrac{\log_e(1+x)}{x}=1$

$(\boldsymbol{iv})\ \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$ $\quad(\boldsymbol{v})\ \lim_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=e$

Proof: $(i)\ \lim_{x\to 0}\dfrac{e^x-1}{x}\left(\dfrac{0}{0}\ \ form\right)=\lim_{x\to 0}\dfrac{\left(1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots..\right)-1}{x}$

$$\left[\,Using\ \ e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots..\right.$$

$$= \lim_{x\to 0}\frac{\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots..}{x}=\lim_{x\to 0}\frac{x\left(\dfrac{1}{1!}+\dfrac{x}{2!}+\dfrac{x^2}{3!}+\ldots..\right)}{x}$$

$$= \lim_{x\to 0}\left(\frac{1}{1!}+\frac{x}{2!}+\frac{x^2}{3!}+\ldots..\right)=\frac{1}{1!}+0=1$$

$(ii)\ \lim_{x\to 0}\dfrac{a^x-1}{x}\ \left(\dfrac{0}{0}\ \ form\right)=\lim_{x\to 0}\dfrac{\left(1+\dfrac{x\log_e a}{1!}+\dfrac{x^2(\log_e a)^2}{2!}+\dfrac{x^3(\log_e a)^3}{3!}+\ldots..\right)-1}{x}$

$$\begin{cases}\boldsymbol{Remember}: Any\ number\ \boxed{p=q^{\log_q p}}\ \Longrightarrow\ a^x=e^{\log_e a^x}=e^{x\log_e a}\\ \Longrightarrow\ a^x=e^{x\log_e a}=1+\dfrac{x\log_e a}{1!}+\dfrac{x^2(\log_e a)^2}{2!}+\dfrac{x^3(\log_e a)^3}{3!}+\ldots..\end{cases}$$

$$= \lim_{x\to 0}\frac{\dfrac{x\log_e a}{1!}+\dfrac{x^2(\log_e a)^2}{2!}+\dfrac{x^3(\log_e a)^3}{3!}+\ldots.}{x}$$

$$= \lim_{x\to 0}\frac{x\log_e a\left(\dfrac{1}{1!}+\dfrac{x\log_e a}{2!}+\dfrac{x^2(\log_e a)^2}{3!}+\ldots..\right)}{x}$$

$$= log_e a . \lim_{x\to 0}\left(\frac{1}{1!}+\frac{x\, log_e a}{2!}+\frac{x^2(log_e a)^2}{3!}+\ldots\right) = log_e a . \frac{1}{1!}+0 = log_e a .$$

(iii) $\lim_{x\to 0}\frac{\log_e(1+x)}{x}$ $\left(\frac{0}{0}\ form\right) = \lim_{x\to 0}\frac{x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots}{x}$

$$[Using\quad log_e(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots;\ if\ \ |x|<1.$$

$$= \lim_{x\to 0}\frac{x\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots\ldots\right)}{x}$$

$$= \lim_{x\to 0}\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots\right) = 1-0 = 1$$

(iv) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$ $(1^\infty\ form)$ $= \lim_{x\to 0}\left[1+\frac{\frac{1}{x}}{1!}.x+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}.x^2+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}x^3+\ldots\ldots\right]$

$$\boxed{Using\quad (1+x)^n = 1+\frac{n}{1!}x+\frac{n(n-1)}{2!}\,x^2+\frac{n(n-1)(n-2)}{3!}\,x^3+\ \ \ldots\ to\ \infty}$$

$$= \lim_{x\to 0}\left[1+\frac{1}{1!}+\frac{1.(1-x)}{2!}.1+\frac{1.(1-x)(1-2x)}{3!}.1+\ldots\ldots\right]$$

$$= 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots = e$$

(v) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ $(1^\infty\ form) = \lim_{t\to 0}(1+t)^{\frac{1}{t}} = e$ $\begin{cases} Taking\ \ \frac{1}{x}=t\ \ \Longrightarrow\ \ x=\frac{1}{t} \\ As\ \ x\longrightarrow\infty,\qquad t\longrightarrow 0\end{cases}$

Solved Examples

Example 1: *Evaluate* (i) $\lim_{x\to 0}\frac{e^{2x}-1}{x}$ (ii) $\lim_{x\to 0}\frac{e^{sin\,5x}-1}{x}$ (iii) $\lim_{x\to 0}\frac{e^{sin\,3x}-1}{tan\ 4x}$

Solution: (i) $\lim_{x\to 0}\frac{e^{2x}-1}{x}$ $\left(\frac{0}{0}\,form\right)$ $= \lim_{x\to 0}\left(\frac{e^{2x}-1}{2x}\right).2 = 1.2 = 2.$

(ii) $\lim_{x\to 0}\frac{e^{sin\,5x}-1}{x}$ $\left(\frac{0}{0}\,form\right) = \lim_{x\to 0}\left(\frac{e^{sin\,5x}-1}{sin\ 5x}\right).\left(\frac{\sin 5x}{5x}\right).5 = 1.1.5 = 5.$

(iii) $\lim_{x\to 0}\frac{e^{sin\,3x}-1}{tan\ 4x}$ $\left(\frac{0}{0}\,form\right) = \lim_{x\to 0}\left\{\left(\frac{e^{sin\,3x}-1}{sin\ 3x}\right)\left(\frac{\sin 3x}{3x}\right)\left(\frac{4x}{\tan 4x}\right).\frac{3}{4}\right\}$

$= 1.1.1.\frac{3}{4} = \frac{3}{4}.$

Example 2: *Evaluate* (i) $\lim_{x\to 0}\frac{3^{\tan 2x}-1}{x}$ (ii) $\lim_{x\to 0}\frac{5^x-1}{\sqrt{1+x}-1}$ (iii) $\lim_{x\to 0}\frac{8^x-5^x}{sin\ 2x}$

Solution: (i) $\lim_{x\to 0}\frac{3^{\tan 2x}-1}{x}$ $\left(\frac{0}{0}\,form\right)$ $= \lim_{x\to 0}\left(\frac{3^{tan\,2x}-1}{tan\ 2x}\right).\left(\frac{\tan 2x}{2x}\right).2 = (\log_e 3).1.2.$

$= 2\log_e 3.$

(ii) $\lim_{x\to 0}\frac{5^x-1}{\sqrt{1+x}-1}$ $\left(\frac{0}{0}\,form\right) = \lim_{x\to 0}\frac{(5^x-1)(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}$

$= \lim_{x\to 0}\frac{(5^x-1)(\sqrt{1+x}+1)}{1+x-1} = \lim_{x\to 0}\frac{(5^x-1)(\sqrt{1+x}+1)}{x} = \lim_{x\to 0}\left\{\left(\frac{5^x-1}{x}\right)(\sqrt{1+x}+1)\right\}$

$= (\log_e 5).(1+1) = (\log_e 5).2 = 2\log_e 5.$

(iii) $\lim_{x\to 0}\frac{8^x-5^x}{sin\ 2x}$ $\left(\frac{0}{0}\,form\right) = \lim_{x\to 0}\left[\left\{\frac{(8^x-1)-(5^x-1)}{x}\right\}\left(\frac{2x}{sin\ 2x}\right).\frac{1}{2}\right]$

$= \lim_{x\to 0}\left[\left\{\left(\frac{8^x-1}{x}\right)-\left(\frac{5^x-1}{x}\right)\right\}\left(\frac{2x}{sin\ 2x}\right).\frac{1}{2}\right] = (log_e 8 - log_e 5).1.\frac{1}{2}$

$$= \frac{1}{2}\log_e\left(\frac{8}{5}\right). \quad \left\{Using \quad log_c\, a - log_c\, b = log_c\left(\frac{a}{b}\right)\right\}$$

Example 3: $Evaluate \quad (i)\ \lim_{x\to 0}\frac{\log_e(1+\sin^{-1}3x)}{x} \quad (ii)\ \lim_{x\to 0}(1+2x)^{1/x}\ \ (iii)\ \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}$

Solution: $(i)\ \lim_{x\to 0}\frac{log_e(1+sin^{-1}3x)}{x}\ \left(\frac{0}{0}\ form\right) = \lim_{x\to 0}\left\{\frac{\log_e(1+\sin^{-1}3x)}{sin^{-1}3x}\right\}.\left(\frac{\sin^{-1}3x}{3x}\right).3$

$= 1.1.3 = 3.$

$(ii)\ \lim_{x\to 0}(1+2x)^{1/x}\ \ (1^{\infty}\ form) = \lim_{x\to 0}\left\{(1+2x)^{\frac{1}{2x}}\right\}^2 = \left\{\lim_{x\to 0}(1+2x)^{\frac{1}{2x}}\right\}^2 = (e)^2 = e^2.$

$(iii)\ \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}\ (1^{\infty}\ form) = \lim_{x\to\infty}\left\{\left(1+\frac{3}{x}\right)^{\frac{x}{3}}\right\}^6 = e^6.$

Example 4: $(i)\ \lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}\ \ (ii)\ \lim_{x\to 0}\left(\frac{a^x+b^x+c^x-3}{x}\right)\ \ (iii)\ \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}.$

Solution: $(i)\ \lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}\ \left(\frac{0}{0}\ form\right) = \lim_{x\to 0}\frac{e^{log_e(1+x)^{1/x}}-e}{x} \quad [Using\ \ p = q^{log_q p}]$

$$= \lim_{x\to 0}\frac{e^{\frac{1}{x}log_e(1+x)}-e}{x} = \lim_{x\to 0}\frac{e^{\frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)}-e}{x} = \lim_{x\to 0}\frac{e^{\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)}-e}{x}$$

$$= \lim_{x\to 0}\frac{e.e^{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)}-e}{x} = \lim_{x\to 0}\frac{e\left\{e^{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)}-1\right\}}{x}$$

$$= e.\lim_{x\to 0}\left[\left\{\frac{e^{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)}-1}{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)}\right\}\left(\frac{-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots}{x}\right)\right]$$

$$= e.\lim_{x\to 0}\left[\left\{\frac{e^{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)}-1}{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)}\right\}\left\{\frac{x\left(-\frac{1}{2}+\frac{x}{3}-\frac{x^2}{4}+\dots\right)}{x}\right\}\right]$$

$$= e.\lim_{x\to 0}\left[\left\{\frac{e^{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)}-1}{\left(-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)}\right\}\left(-\frac{1}{2}+\frac{x}{3}-\frac{x^2}{4}+\dots\right)\right] = e.1.\left(-\frac{1}{2}\right) = -\frac{1}{2}e.$$

$(ii)\ \lim_{x\to 0}\left(\frac{a^x+b^x+c^x-3}{x}\right)\left(\frac{0}{0}\ form\right) = \lim_{x\to 0}\left\{\frac{(a^x-1)+(b^x-1)+(c^x-1)}{x}\right\}$

$= \lim_{x\to 0}\left\{\frac{(a^x-1)}{x}+\frac{(b^x-1)}{x}+\frac{(c^x-1)}{x}\right\} = log_e\, a + log_e\, b + log_e\, c = log_e(abc).$

$(iii)\ \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}\ (1^{\infty}\ form) = \lim_{x\to 0}\left\{1+\left(\frac{a^x+b^x+c^x}{3}-1\right)\right\}^{1/x}$

$$= \lim_{x\to 0}\left\{1+\left(\frac{a^x+b^x+c^x-3}{3}\right)\right\}^{1/x} = \lim_{x\to 0}\left[\left\{1+\left(\frac{a^x+b^x+c^x-3}{3}\right)\right\}^{\frac{3}{a^x+b^x+c^x-3}}\right]^{\left(\frac{a^x+b^x+c^x-3}{3x}\right)}$$

$$= \left[\lim_{x\to 0}\left\{1+\left(\frac{a^x+b^x+c^x-3}{3}\right)\right\}^{\frac{3}{a^x+b^x+c^x-3}}\right]^{\frac{1}{3}\lim_{x\to 0}\left(\frac{a^x+b^x+c^x-3}{x}\right)} = e^{\frac{1}{3}log_e(abc)} = e^{log_e(abc)^{1/3}}$$

$[Using\ \ solved\ example\ \ 4(ii)]$

$$= (abc)^{1/3}.$$

Assignment No. 8

Evaluate the following limits:

1. $(i)\ \lim_{x\to 0}\frac{e^{x/2}-1}{x}$ $\quad(ii)\ \lim_{x\to 0}\frac{e^{\tan 3x}-1}{x}$ $\quad(iii)\ \lim_{x\to 0}\frac{e^{\sin^{-1}2x}-1}{\tan 3x}$

$\left[Answer: (i)\ \frac{1}{2}\quad (ii)\ 3\quad (iii)\ \frac{2}{3}\right]$

2. $(i)\ \lim_{x\to 0}\frac{3^{\tan\frac{x}{2}}-1}{x}$ $\quad(ii)\ \lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}$ $\quad(iii)\ \lim_{x\to 0}\frac{7^x-4^x}{\tan 3x}$

$\left[Answer: (i)\ \frac{1}{2}\log_e 3\quad (ii)\ \log_e 4\quad (iii)\ \frac{1}{3}\log_e\left(\frac{7}{4}\right)\right]$

3. $(i)\ \lim_{x\to 0}\frac{e^{\sin 2x}-1}{\tan 3x}$ $\quad(ii)\ \lim_{x\to 0}\frac{5^x-3^x}{\sin 2x}$ $\quad(iii)\ \lim_{x\to e}\frac{\log_e x-1}{x-e}$

$\left[Answer: (i)\ \frac{2}{3}\quad (ii)\ \frac{1}{2}\log_e 4\quad (iii)\ \frac{1}{e}\right]$

4. $(i)\ \lim_{x\to 0}\frac{\log_e(1+\sin 4x)}{\tan 2x}$ $\quad(ii)\ \lim_{x\to 0}\left(1-\frac{x}{2}\right)^{1/x}$ $\quad(iii)\ \lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x/2}$

$\left[Answer: (i)\ 2\quad (ii)\ \frac{1}{\sqrt{e}}\quad (iii)\ e\sqrt{e}\right]$

5. $(i)\ \lim_{x\to 0}\frac{\tan 3x}{3^x-1}$ $\quad(ii)\ \lim_{x\to 1}\frac{\log_e x}{x-1}$ $\quad(iii)\ \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x}$

$\left[Answer: (i)\ \frac{3}{\log_e 3}\quad (ii)\ 1\quad (iii)\ e^2\right]$

6. $(i)\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x/2}$ $\quad(ii)\ \lim_{x\to 0}\left(1-\frac{x}{2}\right)^{1/x}$ $\quad(iii)\ \lim_{x\to\infty}\left(1-\frac{3}{x}\right)^{x}$

$\left[Answer: (i)\ \sqrt{e}\quad (ii)\ 1/\sqrt{e}\quad (iii)\ 1/e^3\right]$

7. $(i)\ \lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$ $\quad(ii)\ \lim_{x\to 0}\left(\frac{a^x+b^x+c^x-3}{x}\right)$ $\quad(iii)\ \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$.

$\left[Answer: (i)\ -\frac{1}{2}e\quad (ii)\ \log_e(abc)\quad (iii)\ (abc)^{1/3}\right]$

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