Differentiation

Let $y = f(x)$ be a function of x. Given the value of x, we can find the corresponding value of y. If there is some change in x, then a change also occurs in y. Here, we shall discuss the change in both the variables. A small change in a variable is symbolized with the variable prefixed by delta-δ (small) or Δ (capital). Thus, δx or Δx denotes a small change in x followed by δy or Δy as the change in y.

Whenever a change occurs in any variable, we witness its initial and new values.

Let the initial value of x be x itself and that of y be y. Then the new value of x can be taken as $x + \delta x$ and the new value of y as $y + \delta y$. Clearly, the initial value of y is

$$y = f(x)$$

and the new value of y *is* $\quad y + \delta y = f(x + \delta x)$

Now, the change in y, i.e., $\delta y = (y + \delta y) - y = f(x + \delta x) - f(x)$.

Our concern is for the change in y and more importantly for the rate of change of y with respect to x. Obviously, the average rate of change of y with respect to x is the amount of change in y with regard to a unit change in x. So, the average rate of change of y with respect to x can be given by

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Taking limit both sides, as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

If the above limit exists, it is called the derivative or the differential co-efficient of y with respect to x and is usually denoted by $\frac{dy}{dx}$. Thus,

$$\boldsymbol{\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}} \quad \ldots\ldots\ldots\ldots (\mathbf{1})$$

At times, δx is denoted by h, then the above formula can be written as

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \quad \ldots\ldots\ldots\ldots (2)$$

Remarks: (1) The $\frac{dy}{dx}$ represents the derivative of y with respect to x. The symbol $\frac{d}{dx}$ is an operator which is read as 'the derivative with respect to x'. When $\frac{d}{dx}$ is operated over a function, it gives the derivative of that function. So, $\frac{dy}{dx}$ $means$ $\frac{d}{dx}(y)$ or $\frac{d}{dx}\{f(x)\}$. The other notations of this derivative are $f'(x), y_1, Dy$ etc.

(2) The derivative $\frac{dy}{dx}$ is also called the instantaneous rate of change of y w.r.t. x.

(3) The process of finding the derivative is called the differentiation.

(4) The process of finding the derivative using the above formula is referred to as the *differentiation from the first principles, by definition, by delta method, by ab-initio method* etc.

(5) The derivative $\frac{dy}{dx}$ at a particular point, say $x = a$, is denoted by $\left(\frac{dy}{dx}\right)_{x=a}$ or $f'(a)$. Clearly,

$$\left(\frac{dy}{dx}\right)_{x=a} = f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \quad \ldots\ldots\ldots\ldots (3).$$

The formula (3) can also be given as

$$\left(\frac{dy}{dx}\right)_{x=a} = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \ldots\ldots\ldots\ldots (4).$$

We can consider one-sided limits associated with formula (3) or (4) which give rise to one sided derivatives namely left-hand derivative and the right-hand derivative at $x = a$, as given below:

$$The\ left - hand\ derivative\ = \lim_{h \to 0} \frac{f(a - h) - f(a)}{h}\ \ ;\ h\ being\ a\ small\ positive\ real\ number.$$

$$The\ right - hand\ derivative$$
$$= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}\ \ ;\ h\ being\ a\ small\ positive\ real\ number.$$

If the left-hand derivative is equal to the right-hand derivative, the function $f(x)$ is differentiable at $x = a$ and $f'(a)$ is given by (3) or (4).

Derivatives of some elementary functions

Formulae:

1. $\dfrac{d}{dx}(x^n) = nx^{n-1}$
2. $\dfrac{d}{dx}(e^x) = e^x$
3. $\dfrac{d}{dx}(c) = 0, \quad c \text{ being a constant}$
4. $\dfrac{d}{dx}(\sin x) = \cos x$
5. $\dfrac{d}{dx}(\cos x) = -\sin x$
6. $\dfrac{d}{dx}(\log_e x) = \dfrac{1}{x}$

Proof (1) Let $y = f(x) = x^n$

$$\Rightarrow \text{New value of } y = y + \delta y = f(x+\delta x) = (x+\delta x)^n$$

$$\text{Change in } y = \delta y = (y+\delta y) - y = f(x+\delta x) - f(x) = (x+\delta x)^n - x^n$$

$$\text{From the first principles,} \quad \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{(x+\delta x)^n - x^n}{\delta x} = \lim_{\delta x\to 0}\frac{(x+\delta x)^n - x^n}{(x+\delta x)-x} = nx^{n-1}$$

$$\Rightarrow \frac{d}{dx}(x^n) = nx^{n-1} \qquad \left(\text{Using the formula } \lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}\right)$$

(2) Let $y = f(x) = e^x$

$$\Rightarrow \text{New value of } y = y + \delta y = f(x+\delta x) = e^{x+\delta x} = e^x . e^{\delta x}$$

$$\text{Change in } y = \delta y = (y+\delta y) - y = f(x+\delta x) - f(x) = e^x . e^{\delta x} - e^x = e^x\left(e^{\delta x} - 1\right)$$

$$\text{By definition,} \quad \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{e^x\left(e^{\delta x}-1\right)}{\delta x} = e^x \lim_{\delta x\to 0}\frac{e^{\delta x}-1}{\delta x} = e^x . 1 = e^x.$$

$$\Rightarrow \frac{d}{dx}(e^x) = e^x \qquad \left(\text{Using the formula } \lim_{x\to 0}\frac{e^x - 1}{x} = 1\right)$$

(3) Let $y = f(x) = \boldsymbol{c}$

$$\Rightarrow \text{New value of } y = y + \delta y = f(x+\delta x) = c$$

$$\text{Change in } y = \delta y = (y+\delta y) - y = f(x+\delta x) - f(x) = c - c = 0$$

$$\text{By delta method,} \quad \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{0}{\delta x} = 0$$

$$\Rightarrow \frac{d}{dx}(c) = 0\,.$$

(4) Let $y = f(x) = \sin x$

$$\Rightarrow \text{New value of } y = y + \delta y = f(x+\delta x) = \sin(x+\delta x)$$

$$\text{Change in } y = \delta y = (y+\delta y) - y = f(x+\delta x) - f(x) = \sin(x+\delta x) - \sin x$$

$$= 2\cos\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

$$\text{By ab-initio method,} \quad \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{2\cos\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x} = \lim_{\delta x \to 0}\left[\cos\left(x+\frac{\delta x}{2}\right)\left\{\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}\right\}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \cos x.1 = \cos x$$

$$\Rightarrow \quad \frac{d}{dx}(\sin x) = \cos x$$

(**5**) Let $y = f(x) = \cos x$

$$\Rightarrow \text{ New value of } y = y + \delta y = f(x+\delta x) = \cos(x+\delta x)$$

$\text{Change in } y = \delta y = (y+\delta y) - y = f(x+\delta x) - f(x) = \cos(x+\delta x) - \cos x$

$$= -2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

$$\text{From the first principles,} \quad \frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{-2\sin\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x} = -\lim_{\delta x \to 0}\left[\sin\left(x+\frac{\delta x}{2}\right)\left\{\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}\right\}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = -\sin x.1 = -\sin x$$

$$\Rightarrow \quad \frac{d}{dx}(\cos x) = -\sin x$$

(6) Let $y = f(x) = \log_e x$

$\Rightarrow \text{ New value of } y = y + \delta y = f(x+\delta x) = \log_e(x+\delta x)$

$\text{Change in } y = \delta y = (y+\delta y) - y = f(x+\delta x) - f(x) = \log_e(x+\delta x) - \log_e x$

$= \log_e\left(\frac{x+\delta x}{x}\right) = \log_e\left(1+\frac{\delta x}{x}\right)$

$$\text{By definition,} \quad \frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\log_e\left(1+\frac{\delta x}{x}\right)}{\delta x} = \lim_{\delta x \to 0}\left[\left\{\frac{\log_e\left(1+\frac{\delta x}{x}\right)}{\frac{\delta x}{x}}\right\}.\frac{1}{x}\right] = 1.\frac{1}{x} = \frac{1}{x}.$$

$$\left\{\text{Using } \lim_{x \to 0}\frac{\log_e(1+x)}{x} = 1\right\}$$

$$\Rightarrow \frac{d}{dx}(\log_e x) = \frac{1}{x}\ .$$

Fundamental rules of differentiation

(1) *The derivative of the sum of two functions is equal to the sum of their derivatives.*

$$\boldsymbol{\text{If } u \text{ and } v \text{ be two functions of } x, \quad \text{then } \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.}$$

Proof: ***Let*** $y = u + v$. *Let, for a small change* δx *in* x, *the corresponding changes in* u, v *and* y *be* $\delta u, \delta v$ *and* δy *respectively.*

$\text{New value of } y = y + \delta y = (u+\delta u) + (v+\delta v) = u + v + \delta u + \delta v$

$\text{Change in } y = \delta y = (y+\delta y) - y = u + v + \delta u + \delta v - (u+v)$

$\Rightarrow \quad \delta y = \delta u + \delta v$

$\text{By definition,} \quad \frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x}$

$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta u + \delta v}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}\right)$

$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta u}{\delta x} + \lim_{\delta x \to 0}\frac{\delta v}{\delta x} = \frac{du}{dx} + \frac{dv}{dx}\ .$

$\Rightarrow \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}\ .$

(2) *The derivative of the product of a constant and a function is equal to the product of the constant and the derivative of the function.*

If k be a constant and u be a function of x, ***then*** $\boldsymbol{\frac{d}{dx}(ku) = k\frac{du}{dx}}$

Proof: *Let* $y = ku$.

$$Then\ \ y + \delta y = k(u + \delta u) = ku + k\delta u$$

Change in $y = \delta y = (y + \delta y) - y = ku + k\delta u - ku = k.\delta u.$

$$By\ definition,\quad \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{k.\delta u}{\delta x}\right) = k\lim_{\delta x\to 0}\frac{\delta u}{\delta x}\qquad \left[Using\ \ \lim_{x\to a}\left\{k.f(x) = k\lim_{x\to a}f(x)\right\}\right]$$

$$\Rightarrow \frac{dy}{dx} = k\frac{du}{dx}$$

$$\Rightarrow \frac{d}{dx}(ku) = k\frac{du}{dx}$$

Generalising the above rules, we have

$$\frac{d}{dx}(au \pm bv \pm cw \pm\ \ldots..) = a\frac{du}{dx} \pm b\frac{dv}{dx} \pm c\frac{dw}{dx} \pm \ldots\ldots.$$

where $a, b, c, \ldots..$ *are constants and* $u, v, w, \ldots..$ *are functions of* x.

Solved Examples

Example 1: Differentiate with respect to x:

$(i)\ \sqrt{x} - \frac{1}{\sqrt{x}}$ $\quad(ii)\ 3\sin x - 2e^x$ $\quad(iii)\ 6\log_e x - 2\cos x + 3$

Solution: (i) *Let* $y = \sqrt{x} - \dfrac{1}{\sqrt{x}} = x^{1/2} - x^{-1/2}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(x^{1/2} - x^{-1/2}\right) = \frac{d}{dx}\left(x^{1/2}\right) - \frac{d}{dx}\left(x^{-1/2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} - \left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} = \frac{1}{2}\left(x^{-1/2} + x^{-3/2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{\sqrt{x}} + \frac{1}{x\sqrt{x}}\right) = \frac{x+1}{2x\sqrt{x}}.$$

$(ii)\ \dfrac{d}{dx}(3\sin x - 2e^x) = 3\dfrac{d}{dx}(\sin x) - 2\dfrac{d}{dx}(e^x) = 3\cos x - 2e^x$.

$(iii)\ \dfrac{d}{dx}(6\log_e x - 2\cos x + 3) = 6\dfrac{d}{dx}(\log_e x) - 2\dfrac{d}{dx}(\cos x) + \dfrac{d}{dx}(3)$

$= 6.\dfrac{1}{x} - 2.(-\sin x) + 0 = \dfrac{6}{x} + 2\sin x$.

Example 2: Find $(i)\ \dfrac{d}{dx}(|x|),\quad if\ x < 0\qquad (ii)\ \dfrac{d}{dx}\left(5^{2\log_5 x}\right)$

Solution: (i) *We know* $|x| = -x,\quad$ if $x < 0$

$$\Rightarrow \frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = -\frac{d}{dx}(x) = -1.x^{1-1} = -x^0 = -1$$

$(ii)\ \dfrac{d}{dx}\left(5^{2\log_5 x}\right) = \dfrac{d}{dx}\left(5^{\log_5 x^2}\right)$

$= \dfrac{d}{dx}(x^2) = 2x\quad \left(Using\!: \ p = q^{\log_q p}\right)$

Example 3: If $f(x) = \sin x + \cos x$, $\quad find\ f'\left(-\dfrac{\pi}{3}\right)$.

Solution: *Given* $f(x) = \sin x + \cos x$

Differentiating with respect to x, we get

$$\frac{d}{dx}\{f(x)\} = \frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos x) \Rightarrow f'(x) = \cos x - \sin x$$

$$\Rightarrow f'\left(-\frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3}\right) - \sin\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} - \left(-\sin\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1}{2}\left(\sqrt{3} + 1\right).$$

Assignment No. 1

1. Differentiate with respect to x:
   $(i)\ ax^2 + bx + c \quad (ii)\ 4\sin x - 7\cos x \quad (iii)\ 5e^x - 2\log_e x - \cos x$
   $[Answer: (i)\ 2ax + b \quad (ii)\ 4\cos x + 7\sin x \quad (iii)\ 5e^x - 2/x + \sin x]$

2. Find $(i)\ \frac{d}{dx}(|x|), \quad if\ x < 0 \quad (ii)\ \frac{d}{dx}\left(3^{\log_3 \sin x}\right) \quad (iii)\ \frac{d}{dx}\left(\frac{2}{x\sqrt{x}}\right)$

$[Answer: (i)\ -1 \quad (ii)\ \cos x \quad (iii)\ -3x^{-5/2}]$

3. Find the derivative with respect to x from the first priciples of
$(i)\ x^3 \quad (ii)\ \sin x \quad (iii)\ \log_e x$

$[Answer: (i)\ 3x^2 \quad (ii)\ \cos x \quad (iii)\ 1/x]$

4. $(i)\ Find\ \frac{d}{dx}(x|x|), \quad given\ that\ x < 0 \quad (ii)\ If\ f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}, \quad find\ f'(1)$.

$[Answer: (i)\ -2x \quad (ii)\ 0]$

5. (i) If $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ to ∞ , show that $\frac{dy}{dx} = e^x$ (ii) Find $\frac{d}{dx}\left(5^{3\log_5 x}\right)$.

$[Answer: \quad (ii)\ 3x^2]$

Product rule and quotient rule of differentiation

Let u and v be two functions of x, then

(1) ***Product rule***: $$\frac{d}{dx}(u.v) = \frac{du}{dx}.v + u.\frac{dv}{dx}$$

(2) ***Quotient rule***: $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}.v - u.\frac{dv}{dx}}{v^2}$$

Proof for the product rule:

Let $y = u.v$

Let, for a small change δx in x, the corresponding changes in u, v and y be $\delta u, \delta v$ and δy respectively.

New value of $y = y + \delta y = (u + \delta u).(v + \delta v) = u.v + \delta u.v + u.\delta v + \delta u.\delta v$

Change in $y = \delta y = (y + \delta y) - y = u.v + \delta u.v + u.\delta v + \delta u.\delta v - u.v$

$\Longrightarrow \quad \delta y = \delta u.v + u.\delta v + \delta u.\delta v$

By definition, $\quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

$$\Longrightarrow \frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta u.v + u.\delta v + \delta u.\delta v}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}.v + u.\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x}.\delta v\right)$$

$$\Longrightarrow \frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta u}{\delta x}.v + u.\lim_{\delta x \to 0}\frac{\delta v}{\delta x} + \lim_{\delta x \to 0}\frac{\delta u}{\delta x}.\lim_{\delta x \to 0}\delta v$$

$$\Longrightarrow \frac{dy}{dx} = \frac{du}{dx}.v + u.\frac{dv}{dx} + \frac{du}{dx}.0 \quad \left(As\ \delta x \to 0, \quad \delta v \to 0 \Longrightarrow \lim_{\delta x \to 0}\delta v = 0\right)$$

$$\Longrightarrow \quad \frac{d}{dx}(u.v) = \frac{du}{dx}.v + u.\frac{dv}{dx}$$

Proof for the quotient rule:

Let $y = \frac{u}{v}$

Let, for a small change δx in x, the corresponding changes in u, v and y be $\delta u, \delta v$ and δy respectively.

New value of $y = y + \delta y = \frac{u + \delta u}{v + \delta v}$

Change in $y = \delta y = (y + \delta y) - y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v} = \frac{(u + \delta u)v - u(v + \delta v)}{v(v + \delta v)}$

$$\Longrightarrow \quad \delta y = \frac{uv + \delta u.v - uv - u\delta v}{v(v + \delta v)} = \frac{\delta u.v - u\delta v}{v(v + \delta v)}$$

$By\ definition, \qquad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \left\{ \frac{\frac{\delta u . v - u \delta v}{v(v+\delta v)}}{\delta x} \right\} = \lim_{\delta x \to 0} \left(\frac{\delta u . v - u\delta v}{\delta x . v(v+\delta v)} \right)$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \left\{ \left(\frac{\delta u}{\delta x} . v - u . \frac{\delta v}{\delta x} \right) . \frac{1}{v(v+\delta v)} \right\}$$

$$\Rightarrow \frac{dy}{dx} = \left(\lim_{\delta x \to 0} \frac{\delta u}{\delta x} . v - u . \lim_{\delta x \to 0} \frac{\delta v}{\delta x} \right) . \frac{1}{v} . \lim_{\delta x \to 0} \frac{1}{(v+\delta v)}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{du}{dx} . v - u . \frac{dv}{dx} \right) \frac{1}{.v} . \frac{1}{(v+0)} \qquad (As\ \delta x \to 0, \qquad \delta v \to 0)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{du}{dx} . v - u . \frac{dv}{dx} \right) \frac{1}{.v^2}$$

$$\Rightarrow \boldsymbol{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx} . v - u . \frac{dv}{dx}}{v^2}}$$

Remark:

Product rule for three functions:

Let u, v and w be three functions of x, then

$$\frac{d}{dx}(uvw) = \frac{du}{dx} vw + u \frac{dv}{dx} w + uv \frac{dw}{dx}$$

$$Proof: \frac{d}{dx}(uvw) = \frac{d}{dx}\{(uv)w\} = \frac{d}{dx}(uv) . w + (uv)\frac{dw}{dx}$$

$$\Rightarrow \frac{d}{dx}(uvw) = \left(\frac{du}{dx} v + u \frac{dv}{dx} \right) w + uv \frac{dw}{dx}$$

$$\Rightarrow \frac{d}{dx}(uvw) = \frac{du}{dx} vw + u \frac{dv}{dx} w + uv \frac{dw}{dx}$$

Solved examples

Example 1: ***Find the derivative with respect to*** x:

$(i)\ x^2 \sin x \qquad (ii)\ \sin 2x \qquad (iii)\ e^x \log x \qquad (iv)\ x^3\ e^x \cos x$

Solution: $(i) \frac{d}{dx}(x^2 \sin x) = \frac{d}{dx}(x^2) . \sin x + x^2 . \frac{d}{dx}(\sin x) = 2x \sin x + x^2 \cos x$.

$(ii) \frac{d}{dx}(\sin 2x) = \frac{d}{dx}(2 \sin x \cos x) = \mathbf{2} \frac{d}{dx}(\sin x \cos x)$

$= 2\left[\frac{d}{dx}(\sin x) . \cos x + \sin x . \frac{d}{dx}(\cos x) \right] = 2[\cos x \cos x + \sin x (-\sin x)]$

$= 2(\cos^2 x - \sin^2 x) = 2 \cos 2x$.

$(iii) \quad \frac{d}{dx}(e^x \log x) = \frac{d}{dx}(e^x) . \log x + e^x . \frac{d}{dx}(\log x) = e^x \log x + e^x . \frac{1}{x}$

$= e^x \left(\log x + \frac{1}{x} \right)$.

$(iv) \quad \frac{d}{dx}(x^3\ e^x \cos x) = \frac{d}{dx}(x^3) . e^x \cos x + x^3 \frac{d}{dx}(e^x) \cos x + x^3 e^x \frac{d}{dx}(\cos x)$

$= 3x^2 e^x \cos x + x^3 e^x \cos x + x^3 e^x (-\sin x) = x^2 e^x (3 \cos x + x \cos x - x \sin x)$.

Example 2: ***Find the derivative with respect to*** x:

$(i)\ \frac{x^2}{\cos x} \qquad (ii)\ \frac{\log x}{x} \qquad (iii)\ \frac{x + \sin x}{x - \sin x} \qquad (iv)\ \frac{x^2 + 2x - 3}{2x - 1}$

Solution: $(i) \frac{d}{dx}\left(\frac{x^2}{\cos x} \right) = \frac{\frac{d}{dx}(x^2) \cos x - x^2 . \frac{d}{dx}(\cos x)}{\cos^2 x} = \frac{2x \cos x - x^2(-\sin x)}{\cos^2 x}$

$= \frac{2x \cos x + x^2 \sin x}{\cos^2 x}$

$(ii)\ \frac{d}{dx}\left(\frac{\log x}{x}\right) = \frac{\frac{d}{dx}(\log x)x - \log x.\frac{d}{dx}(x)}{x^2} = \frac{\frac{1}{x}.x - \log x.1}{x^2} = \frac{1-\log x}{x^2}$

$(iii)\ \frac{d}{dx}\left(\frac{x+\sin x}{x-\sin x}\right) = \frac{\frac{d}{dx}(x+\sin x).(x-\sin x) - (x+\sin x)\frac{d}{dx}(x-\sin x)}{(x-\sin x)^2}$

$= \frac{(1+\cos x).(x-\sin x) - (x+\sin x)(1-\cos x)}{(x-\sin x)^2}$

$= \frac{x + x\cos x - \sin x - \sin x\cos x - x + x\cos x - \sin x + \sin x\cos x}{(x-\sin x)^2}$

$= \frac{2x\cos x - 2\sin x}{(x-\sin x)^2}$

$(iv)\ \frac{d}{dx}\left(\frac{x^2+2x-3}{2x-1}\right) = \frac{\frac{d}{dx}(x^2+2x-3)(2x-1) - (x^2+2x-3)\frac{d}{dx}(2x-1)}{(2x-1)^2}$

$= \frac{(2x+2)(2x-1) - (x^2+2x-3).2}{(2x-1)^2} = \frac{2(2x^2+2x-x-1-x^2-2x+3)}{(2x-1)^2}$

$= \frac{2(x^2-x+2)}{(2x-1)^2}$

Example 3: If $y = \frac{1-\sin x}{1+\sin x}, \quad find\ \frac{dy}{dx}.$

Solution: ***Given that*** $y = \frac{1-\sin x}{1+\sin x}$

$\Rightarrow \frac{dy}{dx} = \frac{\frac{d}{dx}(1-\sin x)(1+\sin x) - (1-\sin x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{(-\cos x)(1+\sin x) - (1-\sin x).\cos x}{(1+\sin x)^2} = \frac{(-\cos x)(1+\sin x+1-\sin x)}{(1+\sin x)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{-2\cos x}{(1+\sin x)^2}.$

Derivatives of $\boldsymbol{\tan x, \cot x, \sec x}$ & $\boldsymbol{cosec\ x}$

Formulae:

$(\boldsymbol{i})\ \boldsymbol{\frac{d}{dx}(\tan x) = \sec^2 x}$

$(\boldsymbol{ii})\ \boldsymbol{\frac{d}{dx}(\cot x) = -cosec^2\, x}$

$(\boldsymbol{iii})\ \boldsymbol{\frac{d}{dx}(\sec x) = \sec x \tan x}$

$(\boldsymbol{iv})\ \boldsymbol{\frac{d}{dx}(cosec\, x) = -cosec\, x \cot x}$

Proof: $(i)\ \frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\frac{d}{dx}(\sin x).\cos x - \sin x.\frac{d}{dx}(\cos x)}{\cos^2 x}$

$= \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

Hence, $\boldsymbol{\frac{d}{dx}(\tan x) = \sec^2 x}$

$(ii)\ \frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\frac{d}{dx}(\cos x).\sin x - \cos x.\frac{d}{dx}(\sin x)}{\sin^2 x}$

$\frac{-\sin x\sin x - \cos x\cos x}{\sin^2 x} = -\left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\right) = -\frac{1}{\sin^2 x} = -\,cosec^2\, x$

$$\boldsymbol{Hence,} \quad \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$(iii)\ \frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\frac{d}{dx}(1).\cos x - 1.\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{0.\cos x - 1.(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}.\frac{\sin x}{\cos x} = \sec x \tan x$$

$$\boldsymbol{Hence,} \quad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$(iv)\ \frac{d}{dx}(\operatorname{cosec} x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{\frac{d}{dx}(1).\sin x - 1.\frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$\frac{0.\sin x - 1.\cos x}{\sin^2 x} = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x}.\frac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x$$

$$\boldsymbol{Hence,} \quad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

Assignment No. 2

1. Differentiate with respect to x:
   (i) $x \sin x$ $\quad$ (ii) $x^3 \log x$ $\quad$ (iii) $e^x \cos x$
   [$Answer$: (i) $\sin x + x\cos x$ (ii) $x^2(3\log x + 1)$ (iii) $e^x(\cos x - \sin x)$]

2. Find the derivative with respect to x:
   (i) $(3x-1)e^x$ $\quad$ (ii) $(1+\sin x)(1-\sin x)$ $\quad$ (iii) $x\cos x \log x$
   [$Answer$: (i) $(3x+2)e^x$ (ii) $-\sin 2x$ (iii) $\cos x \log x - x\sin x \log x + \cos x$]

3. Find the derivative with respect to x from the first priciples of
   (i) $\tan x$ $\quad$ (ii) $\cot x$ $\quad$ (iii) $\sec x$
   [$Answer$: (i) $\sec^2 x$ (ii) $-\operatorname{cosec}^2 x$ (iii) $\sec x \tan x$]

4. Differentiate with respect to x:
   (i) $\frac{\sin x}{x}$ $\quad$ (ii) $\frac{1-\cos x}{1+\cos x}$ $\quad$ (iii) $\frac{x+\operatorname{cosec} x}{x-\operatorname{cosec} x}$
   $\left[Answer: (i)\ \frac{x\cos x - \sin x}{x^2}\quad (ii)\ \frac{2\sin x}{(1+\cos x)^2}\quad (iii)\ \frac{-2\operatorname{cosec} x(1+x\cot x)}{(x-\operatorname{cosec} x)^2}\right]$

5. Differentiate with respect to x:
   (i) $\frac{1+\sin x}{1-\sin x}$ $\quad$ (ii) $\frac{1+\tan x}{1-\tan x}$ $\quad$ (iii) $\frac{1+\sin x}{\cos x}$
   $\left[Answer: (i)\ \frac{2\cos x}{(1-\sin x)^2}\quad (ii)\ \frac{2\sec^2 x}{(1-\tan x)^2}\quad (iii)\ \frac{1}{(1-\sin x)}\right]$

Differentiation of a function of a function

Let us consider the function $y = \sin 2x$. In order to find the value of y *corresponding to a given value of* x, *we need to find* $2x$ *first*, after *that sine* $2x$ can be found. We can say that $\sin 2x$ depends on $2x$ and $2x$ *depends on* x. *In other words, we may also say that* $\sin 2x$ is a function of $2x$ *and* $2x$ *is a function of* x. *Thus, it is an example of a function of a function.*

Similarly, the function $y = \log \sin 3x$ *can be regarded as a function of* $\sin 3x$, *then* $\sin 3x$ *as a function of* $3x$ *and lastly* $3x$ *as a function of* x. *This way* $\log \sin 3x$ *is an example of a* function of a function of a function.

Clealy, the above examples of functions involve a chain of functions.

Further, in case of the function $y = \sin 2x$, if we take $2x = u$, then $y = \sin u$. *Now, we can say that* y *is a function of* u *and* u *is a function of* x. *Obviously, any change in* x, *is bound to bring changes in* u *and* y.

To generalise the concept, let for a small change δx in x, the corresponding changes in u and y be δu and δv respectively. From algebra, we have

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta u}.\frac{\delta u}{\delta x}.$$

$Taking\ limit\ both\ sides\ as\ \delta x\to 0, we\ have$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta u}.\frac{\delta u}{\delta x}\right)=\lim_{\delta x\to 0}\frac{\delta y}{\delta u}.\lim_{\delta x\to 0}\frac{\delta u}{\delta x}=\lim_{\delta u\to 0}\frac{\delta y}{\delta u}.\lim_{\delta x\to 0}\frac{\delta u}{\delta x}\quad (As\ \delta x\to 0,\qquad \delta u\to 0)$$

$$Hence,\qquad \frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$$

$Thus,$ if $y\ be\ a\ function\ of\ u\ and\ u\ be\ a\ function\ of\ x, then,$

$$\boxed{\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}}\quad \boldsymbol{which\ is\ known\ as\ the\ chain\ rule\ of\ differentiation}$$

The above rule of differentiation can be generalized for a long chain of functions as follows:

$If\ y\ be\ a\ function\ of\ u, u\ a\ function\ of\ v, v\ a\ function\ of\ w,\ldots, z\ a\ function\ of\ x, then,$

$$\boxed{\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dw}\ldots\ldots\ldots\frac{dz}{dx}}$$

Solved Examples

Example 1: **Differentiate w.r.t.** x:

$(i)\ \sin 2x\qquad (ii)\ \sin x^2\qquad (iii)\ \log\tan\frac{x}{2}\qquad (iv)\ \sqrt{\cos x}\qquad (v)\ \sqrt{x^2+a^2}$

$Solution:\ Let\ \ y=sin\ 2x\quad\Longrightarrow\ y=sin\,u,\quad where\ \ u=2x$

$$\Longrightarrow\quad \frac{dy}{du}=\frac{d}{du}(sin\,u)=cos\,u=cos\ 2x\qquad \&\ \frac{du}{dx}=\frac{d}{dx}(2x)=2.1=2$$

$$By\ chain\ rule\quad \frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}=cos\ 2x\,.\,2=2\,cos\ 2x$$

$$Alternatively,\ \frac{d}{dx}(sin\ 2x)=\frac{d(sin\ 2x)}{d(2x)}.\frac{d(2x)}{dx}=cos\ 2x\,.\,2.1=2\,cos\ 2x$$

$$(ii)\ \frac{d}{dx}(sin\,x^2)=\frac{d(sin\,x^2)}{d(x^2)}.\frac{d(x^2)}{dx}=cos\,x^2\,.\,2x=2x\,cos\,x^2$$

$$(iii)\ \frac{d}{dx}\left(log\ tan\frac{x}{2}\right)=\frac{d\left(log\ tan\frac{x}{2}\right)}{d\left(tan\frac{x}{2}\right)}.\frac{d\left(tan\frac{x}{2}\right)}{d\left(\frac{x}{2}\right)}.\frac{d}{dx}\left(\frac{x}{2}\right)=\frac{1}{tan\frac{x}{2}}.sec^2\frac{x}{2}.\frac{1}{2}$$

$$=\frac{cos\frac{x}{2}}{2\,sin\frac{x}{2}}.\frac{1}{cos^2\frac{x}{2}}=\frac{1}{2\,sin\frac{x}{2}}.\frac{1}{cos\frac{x}{2}}=\frac{1}{sin\,x}=cosec\,x\,.$$

$$(iv)\ \ \frac{d}{dx}\left(\sqrt{cos\,x}\right)=\frac{d\left(\sqrt{cos\,x}\right)}{d(cos\,x)}.\frac{d}{dx}(cos\,x)=\frac{1}{2\sqrt{cos\,x}}.(-\,sin\,x)=-\frac{sin\,x}{2\sqrt{cos\,x}}$$

$$(v)\ \ \frac{\mathrm{d}}{\mathrm{dx}}\left(\sqrt{x^2+a^2}\right)=\frac{d\left(\sqrt{x^2+a^2}\right)}{d(x^2+a^2)}.\frac{d}{dx}(x^2+a^2)=\frac{1}{2\sqrt{x^2+a^2}}(2x)=\frac{x}{\sqrt{x^2+a^2}}$$

Example 2: Differentiate w.r.t. x: $(i)\log\left(x+\sqrt{x^2+a^2}\right)\qquad (ii)\ \ \log(\sec x+\tan x)$

Solution: (i)
$$\frac{d}{dx}\left\{\log\left(x+\sqrt{x^2+a^2}\right)\right\}=\frac{1}{x+\sqrt{x^2+a^2}}\ .\frac{d}{dx}\left(x+\sqrt{x^2+a^2}\right)$$

$$=\frac{1}{x+\sqrt{x^2+a^2}}\ .\left(1+\frac{1}{2\sqrt{x^2+a^2}}.2x\right)=\frac{1}{x+\sqrt{x^2+a^2}}\ .\left(1+\frac{x}{\sqrt{x^2+a^2}}\right)$$

$$=\frac{1}{x+\sqrt{x^2+a^2}}\ .\left(\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right)$$

$$=\frac{1}{\sqrt{x^2+a^2}}$$

$$(ii)\ \frac{d}{dx}\ \{log(sec\,x+tan\,x)\}=\frac{1}{(sec\,x+tan\,x)}.\frac{d}{dx}(sec\,x+tan\,x)$$

$$= \frac{1}{(\sec x + \tan x)}.(\sec x \tan x + \sec^2 x)$$
$$= \frac{\sec x}{(\sec x + \tan x)}.(\sec x + \tan x) = \sec x$$

Example 3: Differentiate $log\sqrt{\frac{1-cos\,x}{1+cos\,x}}$ $w.r.t.x.$

Solution: (i) $\frac{d}{dx}\left(log\sqrt{\frac{1-cos\,x}{1+cos\,x}}\right) = \frac{1}{\sqrt{\frac{1-cos\,x}{1+cos\,x}}}.\frac{1}{2\sqrt{\frac{1-cos\,x}{1+cos\,x}}}.\frac{d}{dx}\left(\frac{1-cos\,x}{1+cos\,x}\right)$

$$= \frac{1}{2\left(\frac{1-cos\,x}{1+cos\,x}\right)}.\left[\frac{sin\,x.(1+cos\,x)-(1-cos\,x).(-sin\,x)}{(1+cos\,x)^2}\right]$$
$$= \frac{2\,sin\,x}{2(1-cos\,x)(1+cos\,x)} = \frac{sin\,x}{1-cos^2 x} = \frac{sin\,x}{sin^2 x} = \frac{1}{sin\,x} = cosec\,x.$$

$$Alternatively,\ \frac{d}{dx}\left(log\sqrt{\frac{1-cos\,x}{1+cos\,x}}\right) = \frac{d}{dx}\left(log\sqrt{\frac{2\,sin^2\frac{x}{2}}{2\,cos^2\frac{x}{2}}}\right) = \frac{d}{dx}\left(log\,tan\frac{x}{2}\right)$$
$$= \frac{1}{tan\frac{x}{2}}.sec^2\frac{x}{2}.\frac{1}{2} = \frac{cos\frac{x}{2}}{sin\frac{x}{2}}.\frac{1}{cos^2\frac{x}{2}}.\frac{1}{2} = \frac{1}{2\,sin\frac{x}{2}cos\frac{x}{2}} = \frac{1}{sin\,x} = cosec\,x.$$

Yet again as an alternative method, we can apply properties of logrithms as follows:

$Let\ \ y = log\sqrt{\frac{1-cos\,x}{1+cos\,x}} = log\left(\frac{1-cos\,x}{1+cos\,x}\right)^{1/2} = \frac{1}{2}log\left(\frac{1-cos\,x}{1+cos\,x}\right)$

$\Rightarrow\quad y = \frac{1}{2}\{log(1-cos\,x) - log(1+cos\,x)\}$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{1}{2}.\frac{d}{dx}[log(1-cos\,x) - log(1+cos\,x)] = \frac{1}{2}\left[\frac{d}{dx}\{log(1-cos\,x)\} - \frac{d}{dx}\{log(1+cos\,x)\}\right]$

$\Rightarrow\quad \frac{dy}{dx} = \frac{1}{2}.\left[\frac{1}{(1-cos\,x)}.sin\,x - \frac{1}{(1+cos\,x)}.(-sin\,x)\right]$

$\Rightarrow\quad \frac{dy}{dx} = \frac{1}{2}\left[\frac{sin\,x(1+cos\,x)+sin\,x(1-cos\,x)}{(1-cos\,x)(1+cos\,x)}\right] = \frac{sin\,x\{(1+cos\,x)+(1-cos\,x)\}}{2(1-cos\,x)(1+cos\,x)}$

$\Rightarrow\quad \frac{dy}{dx} = \frac{2\,sin\,x}{2(1-cos\,x)(1+cos\,x)} = \frac{sin\,x}{1-cos^2 x} = \frac{sin\,x}{sin^2 x} = \frac{1}{sin\,x} = cosec\,x.$

Example 4: $If\ \ y = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2\,log\left(x+\sqrt{x^2+a^2}\right),\quad show\ that\ \frac{dy}{dx} = \sqrt{x^2+a^2}.$

Solution: $We\ have\ \ y = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2\,log\left(x+\sqrt{x^2+a^2}\right)$

$\Rightarrow\quad y = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2\,log\left(x+\sqrt{x^2+a^2}\right)\right]$

Differentiating with respect to x *both sides*, we get

$\frac{dy}{dx} = \frac{1}{2}.\frac{d}{dx}\left[x\sqrt{x^2+a^2} + a^2\,log\left(x+\sqrt{x^2+a^2}\right)\right]$

$\Rightarrow\quad \frac{dy}{dx} = \frac{1}{2}.\left[\frac{d}{dx}\left(x\sqrt{x^2+a^2}\right) + a^2.\frac{d}{dx}\left\{log\left(x+\sqrt{x^2+a^2}\right)\right\}\right]$

$\Rightarrow\quad \frac{dy}{dx} = \frac{1}{2}.\left[\left\{\frac{d}{dx}(x).\sqrt{x^2+a^2} + x.\frac{d}{dx}\left(\sqrt{x^2+a^2}\right)\right\} + a^2.\frac{1}{x+\sqrt{x^2+a^2}}.\frac{d}{dx}\left(x+\sqrt{x^2+a^2}\right)\right]$

$\Rightarrow\quad \frac{dy}{dx} = \frac{1}{2}.\left[\left\{1.\sqrt{x^2+a^2} + x.\frac{1}{2\sqrt{x^2+a^2}}.2x\right\} + a^2.\frac{1}{x+\sqrt{x^2+a^2}}.\left(1+\frac{1}{2\sqrt{x^2+a^2}}.2x\right)\right]$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{2}.\left[\left\{\sqrt{x^2+a^2}+x.\frac{1}{\sqrt{x^2+a^2}}x\right\}+\frac{a^2}{x+\sqrt{x^2+a^2}}.\left(1+\frac{x}{\sqrt{x^2+a^2}}\right)\right]$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{2}.\left[\frac{(x^2+a^2)+x^2}{\sqrt{x^2+a^2}}+\frac{a^2}{x+\sqrt{x^2+a^2}}.\left(\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right)\right]$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{2}.\left[\frac{2x^2+a^2}{\sqrt{x^2+a^2}}+\frac{a^2}{\sqrt{x^2+a^2}}\right]=\frac{1}{2}.\left[\frac{2x^2+2a^2}{\sqrt{x^2+a^2}}\right]=\frac{x^2+a^2}{\sqrt{x^2+a^2}}$$

$$\Rightarrow \quad \frac{dy}{dx}=\sqrt{x^2+a^2}.$$

Example 5: If $y=\dfrac{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}$, show that $\dfrac{dy}{dx}=-\dfrac{2a^2}{x^3}\left(1-\dfrac{a^2}{\sqrt{a^4-x^4}}\right)$.

Solution: Given $y=\dfrac{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}$

$$\Rightarrow \quad y=\frac{\left(\sqrt{a^2+x^2}-\sqrt{a^2-x^2}\right)\left(\sqrt{a^2+x^2}-\sqrt{a^2-x^2}\right)}{\left(\sqrt{a^2+x^2}+\sqrt{a^2-x^2}\right)\left(\sqrt{a^2+x^2}-\sqrt{a^2-x^2}\right)} \quad (Rationalising\ \ Dr.)$$

$$\Rightarrow \quad y=\frac{\left\{(a^2+x^2)+(a^2-x^2)-2\sqrt{a^2+x^2}.\sqrt{a^2-x^2}\right\}}{\left\{(a^2+x^2)-(a^2-x^2)\right\}}=\frac{2\left(a^2-\sqrt{a^4-x^4}\right)}{2x^2}$$

$$\Rightarrow \quad y=\frac{\left(a^2-\sqrt{a^4-x^4}\right)}{x^2}$$

Differentiating with respect to x both sides, we get

$$\frac{dy}{dx}=\frac{\frac{d}{dx}\left(a^2-\sqrt{a^4-x^4}\right).x^2-\left(a^2-\sqrt{a^4-x^4}\right).\frac{d}{dx}(x^2)}{x^4}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{\left\{0-\frac{1}{2\sqrt{a^4-x^4}}.(-4x^3)\right\}.x^2-\left(a^2-\sqrt{a^4-x^4}\right).2x}{x^4}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{\frac{2x^5}{\sqrt{a^4-x^4}}-2x.\left(a^2-\sqrt{a^4-x^4}\right)}{x^4}=\frac{2x}{x^4}\left\{\frac{x^4}{\sqrt{a^4-x^4}}-\left(a^2-\sqrt{a^4-x^4}\right)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2}{x^3}\left\{\frac{x^4-\sqrt{a^4-x^4}.\left(a^2-\sqrt{a^4-x^4}\right)}{\sqrt{a^4-x^4}}\right\}=\frac{2}{x^3}\left\{\frac{x^4-a^2\sqrt{a^4-x^4}+(a^4-x^4)}{\sqrt{a^4-x^4}}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{2}{x^3}\left\{\frac{-a^2\sqrt{a^4-x^4}+a^4}{\sqrt{a^4-x^4}}\right\}=\frac{2a^2}{x^3}\left\{\frac{-\sqrt{a^4-x^4}+a^2}{\sqrt{a^4-x^4}}\right\}=\frac{2a^2}{x^3}\left(-1+\frac{a^2}{\sqrt{a^4-x^4}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{2a^2}{x^3}\left(1-\frac{a^2}{\sqrt{a^4-x^4}}\right).$$

Assignment No. 3

1. Differentiate with respect to x:

   (i) $sin\sqrt{x}$ (ii) $log\, sin\, 4x$ (iii) $cos^3 2x$

   $\left[Answer: \quad (i)\ \dfrac{1}{2\sqrt{x}} cos\sqrt{x} \quad (ii)\ 4\cot 4x \quad (iii)\ -6\, sin\, 2x\, cos^2 2x\right]$

2. Find the derivative with respect to x:

   (i) $log_e(cosec\, x-cot\, x)$ (ii) $\log_e\left(x+\sqrt{x^2-a^2}\right)$ (iii) $log_e\, tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$

   $\left[Answer: (i)\ cosec\, x \quad (ii)\ \dfrac{1}{\sqrt{x^2-a^2}} \quad (iii)\ sec\, x\right]$

3. (i) Show that $\dfrac{d}{dx}\left(log\sqrt{\dfrac{1+cos\, x}{1-cos\, x}}\right)=-cosec\, x$.

   (ii) If $y=\log tan^2\dfrac{x}{2}$, show that $\dfrac{dy}{dx}=2\, cosec\, x$.

4. Differentiate with respect to x:

$(i)\ \ log_e(cosec\, x + cot\, x)$ $\quad (ii)\ log\sqrt{\dfrac{1-cos\, x}{1+cos\, x}}$ $\quad (iii)\ \ log\sqrt{\dfrac{1-sin\, x}{1+sin\, x}}$

$[Answer: (i)\ \ -cosec\, x \quad (ii)\ \ cosec\, x \quad (iii)\ \ -\ sec\, x]$

5. If $y = \dfrac{x}{2}\sqrt{x^2-a^2} - \dfrac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)$, show that $\dfrac{dy}{dx} = \sqrt{x^2-a^2}$.

6. If $y = \dfrac{x}{2}\sqrt{a^2+x^2} + \dfrac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right)$, show that $\dfrac{dy}{dx} = \sqrt{a^2+x^2}$.

7. If $y = \dfrac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}$, show that $\dfrac{dy}{dx} = -\dfrac{2a^2}{x^3}\left(1+\dfrac{a^2}{\sqrt{a^4-x^4}}\right)$.

8. If $y = \dfrac{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}$, show that $\dfrac{dy}{dx} = -\dfrac{2a^2}{x^3}\left(1-\dfrac{a^2}{\sqrt{a^4-x^4}}\right)$.

Differentiation of implicit functions

Explicit function: $It\ is\ a\ function\ in\ which\ y\ is\ clearly\ expressed\ in\ terms\ of\ x.$

Examples: $(i)\ y = 2x^2 - 3x + 3 \quad (ii)\ \ y = 4\sin x - 3e^x$

Implicit function: $It\ is\ a\ function\ in\ which\ y\ is\ not\ expressible\ in\ terms\ of\ x.$

Examples: $(i)\ x^3+y^3 = 3axy \quad (ii)\ \ sin(xy) = tan\left(\dfrac{x}{y}\right)$

Solved Examples

Example 1: If $x^3+y^3 = 3axy,\ \ find\ \ \dfrac{dy}{dx}$.

Solution: Given $\quad x^3+y^3 = 3axy$

$Differentiating\ both\ sides\ w.r.t.x, we\ get$

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(3axy)$$

$$\Rightarrow \quad \frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3a\frac{d}{dx}(xy)$$

$$\Rightarrow \quad 3x^2 + \frac{d}{dy}(y^3).\frac{dy}{dx} = 3a\left(1.y + x.\frac{dy}{dx}\right)$$

$$\Rightarrow \quad 3x^2 + 3y^2.\frac{dy}{dx} = 3\left(ay + ax\frac{dy}{dx}\right)$$

$$\Rightarrow \quad x^2 + y^2.\frac{dy}{dx} = ay + ax\frac{dy}{dx}$$

$$\Rightarrow \quad y^2.\frac{dy}{dx} - ax\frac{dy}{dx} = ay - x^2$$

$$\Rightarrow \quad (y^2-ax)\frac{dy}{dx} = ay - x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}$$

Example 2: If $ax^2+by^2+2hxy+2gx+2fy+c = 0,\ \ find\ \ \dfrac{dy}{dx}$.

Solution: Given $\quad ax^2+by^2+2hxy+2gx+2fy+c = 0$

$Differentiating\ both\ sides\ w.r.t.x, we\ get$

$$\frac{d}{dx}\ (ax^2+by^2+2hxy+2gx+2fy+c) = \mathbf{0}$$

$$\Rightarrow\ 2ax + 2by\frac{dy}{dx} + 2h\left(1.y + x\frac{dy}{dx}\right) + 2g.1 + 2f.\frac{dy}{dx} + 0 = 0$$

$$\Rightarrow \quad \frac{dy}{dx}(by+hx+f) = -(ax+hy+g)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{ax + hy + g}{by + hx + f}\right)$$

Example 3: If $x \sin(a+y) = \sin y$, show that $\frac{dy}{dx} = \frac{sin^2(a+y)}{\sin a}$.

Solution: Given that $x\, sin\,(a+y) = sin\, y$

$$\Rightarrow \quad x = \frac{sin\, y}{sin(a+y)}.$$

Differentiating with respect to *x* both sides, we get

$$1 = \frac{\frac{d}{dx}(sin\, y).sin(a+y) - sin\, y.\frac{d}{dx}\{sin(a+y)\}}{sin^2(a+y)}$$

$$\Rightarrow \quad 1 = \frac{cos\, y.\frac{dy}{dx}.sin(a+y) - sin\, y.cos(a+y).\frac{dy}{dx}}{sin^2(a+y)}$$

$$\Rightarrow \quad 1 = \frac{dy}{dx}\left\{\frac{cos\, y.sin(a+y) - sin\, y.cos(a+y)}{sin^2(a+y)}\right\}$$

$$\Rightarrow \quad 1 = \frac{dy}{dx}\left\{\frac{sin(a+y).cos\, y - cos(a+y).sin\, y}{sin^2(a+y)}\right\}$$

$$\Rightarrow \quad 1 = \frac{dy}{dx}\left[\frac{sin\{(a+y)-y\}}{sin^2(a+y)}\right]$$

$$\Rightarrow \quad 1 = \frac{dy}{dx}.\left[\frac{sin\, a}{sin^2(a+y)}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{sin^2(a+y)}{\sin a}$$

Alternatively, given that $x\, sin\,(a+y) = sin\, y$ …… (1)

Differentiating with respect to *x* both sides, we get

$$1.\sin(a+y) + x.\cos(a+y).\frac{dy}{dx} = \cos y.\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}.\{x\, cos(a+y) - cos\, y\} = -\,sin(a+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\,sin(a+y)}{\{x\, cos(a+y) - cos\, y\}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\,sin^2(a+y)}{\{x\, cos(a+y) - cos\, y\}.\sin(a+y)} \quad [Multiplying\ Nr\ \&\ Dr\ by\ \ \sin(a+y)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\,sin^2(a+y)}{x\sin(a+y)\, cos(a+y) - cos\, y.\sin(a+y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\,sin^2(a+y)}{sin\, y\, cos(a+y) - cos\, y.sin(a+y)} = \frac{-\,sin^2(a+y)}{-\{sin(a+y).cos\, y - cos(a+y).sin\, y\}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{sin^2(a+y)}{sin\{(a+y)-y\}} = \frac{sin^2(a+y)}{\sin a}.$$

Example 4: If $x \sin(a+y) + \sin a \cos(a+y) = 0$, show that $\frac{dy}{dx} = \frac{sin^2(a+y)}{\sin a}$.

Solution: Given $x \sin(a+y) + \sin a \cos(a+y) = 0$

$$\Rightarrow \quad x = -\,sin\, a.\frac{cos(a+y)}{sin(a+y)} = -\,sin\, a.cot(a+y)$$

$$\Rightarrow \quad x = -\,sin\, a.cot(a+y)$$

Differentiating with respect to *x* both sides, we get

$$\Rightarrow \quad 1 = -\,sin\, a.\{-cosec^2(a+y)\}.\frac{dy}{dx} = \frac{sin\, a}{sin^2(a+y)}.\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{sin^2(a+y)}{\sin a}$$

Alternatively, we have $x \sin(a+y) + \sin a \cos(a+y) = 0$ ....(1)

Differentiating with respect to *x* both sides, we get

$$1. sin(a+y) + x. cos(a+y). \frac{dy}{dx} + sin\, a\{-sin(a+y)\}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}\{x\, cos(a+y) - sin(a+y)\, sin\, a\} = -sin(a+y)$$

$$\Rightarrow \frac{dy}{dx}\{sin(a+y)\, sin\, a - x\, cos(a+y)\} = sin(a+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin(a+y)}{sin(a+y)\, sin\, a - x\, cos(a+y)} = \frac{sin^2(a+y)}{\{sin(a+y)\, sin\, a - x\, cos(a+y)\}\, sin(a+y)}$$

$$[Multiplying\ Nr\ \&\ Dr\ by\ \sin(a+y)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{sin^2(a+y)}{sin^2(a+y)\, sin\, a - x\, sin(a+y)\, cos(a+y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{sin^2(a+y)}{sin^2(a+y)\, sin\, a - \{-\, sin\, a. cos(a+y). cos(a+y)\}}$$

$$[From\ (1){:}\ x\, sin(a+y) = -\, sin\, a\, cos(a+y)]$$

$$\Rightarrow \frac{dy}{dx} = \frac{sin^2(a+y)}{sin\, a. \{sin^2(a+y) + cos^2(a+y)\}} = \frac{sin^2(a+y)}{\sin a\, .1} = \frac{sin^2(a+y)}{\sin a}.$$

Example 5: If $x\sqrt{1+y} + y\sqrt{1+x} = 0$; $(x \neq y)$, show that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

Solution: Given that $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$$\Rightarrow x.\sqrt{1+y} = -y.\sqrt{1+x}$$

Squaring both sides, we get

$$x^2.(1+y) = y^2.(1+x) \Rightarrow x^2 + x^2y = y^2 + y^2x$$

$\Rightarrow x^2 - y^2 = -x^2y + y^2x \Rightarrow (x-y)(x+y) = -xy(x-y)$

Since $x \neq y$, *dividing both sides by* $(x-y)$, *we get*

$$x + y = -xy \Rightarrow y + xy = -x \Rightarrow y(1+x) = -x$$

$\Rightarrow y = -\frac{x}{1+x}$. *Differentiating w.r.t. x, we get*

$\frac{dy}{dx} = -\left[\frac{1.(1+x) - x.1}{(1+x)^2}\right] = -\frac{1}{(1+x)^2}$.

Derivatives of inverse trigonometric functions

Formulae:

(i) $\frac{d}{dx}(sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}(cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

(iii) $\frac{d}{dx}(tan^{-1} x) = \frac{1}{1+x^2}$

(iv) $\frac{d}{dx}(cot^{-1} x) = -\frac{1}{1+x^2}$

(v) $\frac{d}{dx}(sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(vi) $\frac{d}{dx}(cosec^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

Proof: (i) *Let* $y = \sin^{-1} x$

$\Rightarrow \sin y = x$

Differentiating both sides w.r.t. x, we get

$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$

$$\Rightarrow \ \cos y.\frac{dy}{dx} = 1 \quad \Rightarrow \ \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\boldsymbol{\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}}$$

(ii) Let $y = \cos^{-1} x$

$\Rightarrow \ \cos y = x$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\cos y) = \frac{d}{dx}(x)$$

$$\Rightarrow \ -\sin y.\frac{dy}{dx} = 1 \quad \Rightarrow \ \frac{dy}{dx} = \frac{1}{-\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

(iii) Let $y = \tan^{-1} x$

$\Rightarrow \ \tan y = x$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$

$$\Rightarrow \ \sec^2 y.\frac{dy}{dx} = 1 \quad \Rightarrow \ \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

(iv) Let $y = \cot^{-1} x$

$\Rightarrow \ \cot y = x$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\cot y) = \frac{d}{dx}(x)$$

$$\Rightarrow \ -\mathrm{cosec}^2 y.\frac{dy}{dx} = 1 \quad \Rightarrow \ \frac{dy}{dx} = \frac{1}{-\mathrm{cosec}^2 y} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

(v) Let $y = \sec^{-1} x$

$\Rightarrow \ \sec y = x$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\sec y) = \frac{d}{dx}(x)$$

$$\Rightarrow \ \sec y \tan y.\frac{dy}{dx} = 1 \quad \Rightarrow \ \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y\sqrt{\sec^2 y - 1}} = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

(vi) Let $y = \mathrm{cosec}^{-1} x$

$\Rightarrow \ \mathrm{cosec}\, y = x$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\mathrm{cosec}\, y) = \frac{d}{dx}(x)$$

$$\Rightarrow \ -\mathrm{cosec}\, y \cot y\frac{dy}{dx} = 1 \quad \Rightarrow \ \frac{dy}{dx} = \frac{1}{-\mathrm{cosec}\, y \cot y} = \frac{1}{\mathrm{cosec}\, y\sqrt{\mathrm{cosec}^2 y - 1}} = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(cosec^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

Solved Examples

Example 1: Find the drerivative w.r.t. $\boldsymbol{x}$:

$(i)\ sin^{-1}\left(2x\sqrt{1-x^2}\right) \qquad (ii)\ cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \qquad (iii)\ tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Solution: (i) $\frac{d}{dx}\left\{sin^{-1}\left(2x\sqrt{1-x^2}\right)\right\} = \frac{1}{\sqrt{1-\left(2x\sqrt{1-x^2}\right)^2}}.\frac{d}{dx}\left(2x\sqrt{1-x^2}\right)$

$$= \frac{2\left[1.\sqrt{1-x^2} + x.\frac{1}{2\sqrt{1-x^2}}.(-2x)\right]}{\sqrt{1-4x^2(1-x^2)}} = \frac{2\left[\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right]}{\sqrt{1-4x^2(1-x^2)}} = \frac{\frac{2\{(1-x^2)-x^2\}}{\sqrt{1-x^2}}}{\sqrt{1-4x^2+4x^4}}$$

$$= \frac{2(1-2x^2)}{\sqrt{(1-2x^2)^2}\sqrt{1-x^2}} = \frac{2(1-2x^2)}{(1-2x^2)\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}.$$

Alternatively, let $y = sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$\implies\ y = sin^{-1}\left(2\sin\theta\sqrt{1-sin^2\theta}\right) \qquad (Taking\ \ x = sin\,\theta)$

$\implies\ y = sin^{-1}(2\,sin\,\theta\,cos\,\theta) = sin^{-1}(sin\,2\theta) = 2\theta$

$\implies\ y = 2\,sin^{-1} x$

$\implies\ \frac{dy}{dx} = 2.\frac{d}{dx}(sin^{-1} x) = 2.\frac{1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$

$(ii)\ \ Let\ y = cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\implies\ y = cos^{-1}\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right) \qquad (Taking\ substitution\ \ x = tan\,\theta)$

$\implies\ y = cos^{-1}(cos\,2\theta) = 2\theta$

$\implies\ y = 2\,tan^{-1} x$

$\implies\ \frac{dy}{dx} = 2\frac{d}{dx}(tan^{-1} x) = 2.\frac{1}{1+x^2} = \frac{2}{1+x^2}.$

Alternatively, $\ let\ y = cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$$\implies \frac{dy}{dx} = -\frac{1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}.\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\implies \frac{dy}{dx} = -\frac{1}{\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}}}.\left\{\frac{-2x.(1+x^2)-(1-x^2).2x}{(1+x^2)^2}\right\}$$

$$\implies \frac{dy}{dx} = -\frac{1}{\sqrt{\frac{4x^2}{(1+x^2)^2}}}.\left[\frac{-2x.\{(1+x^2)+(1-x^2)\}}{(1+x^2)^2}\right] = -\frac{(1+x^2)}{2x}.\left\{\frac{-2x.2}{(1+x^2)^2}\right\}$$

$$\implies \frac{dy}{dx} = \frac{2}{1+x^2}.$$

$(iii)\ Let\ \ y = tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

$$\Rightarrow\ y = tan^{-1}\left(\frac{\sqrt{1+tan^2\theta}-1}{\tan\theta}\right) \qquad (Taking\ substitution\ \ x = tan\,\theta)$$

$$\Rightarrow\ y = tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right) = tan^{-1}\left(\frac{\frac{1-\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$\Rightarrow\ y = tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right) = tan^{-1}\left(\frac{\sin\theta/2}{\cos\theta/2}\right)$$

$$\Rightarrow\ y = tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}tan^{-1}x$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{1}{2}.\frac{d}{dx}(tan^{-1}x) = \frac{1}{2}.\frac{1}{1+x^2} = \frac{1}{2(1+x^2)}.$$

Alternatively, $let\ \ y = tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

$$\Rightarrow\ \frac{dy}{dx} = \frac{1}{1+\left(\frac{\sqrt{1+x^2}-1}{x}\right)^2}.\frac{d}{dx}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{1}{\frac{x^2+\{(1+x^2)+1-2.\sqrt{1+x^2}.1\}}{x^2}}.\left\{\frac{\frac{1}{2\sqrt{1+x^2}}.2x.x-\left(\sqrt{1+x^2}-1\right).1}{x^2}\right\}$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{x^2}{2.(1+x^2)-2.\sqrt{1+x^2}}.\left\{\frac{\frac{x^2}{\sqrt{1+x^2}}-\left(\sqrt{1+x^2}-1\right)}{x^2}\right\}$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}\left(\sqrt{1+x^2}-1\right)}.\left\{\frac{x^2-\left(1+x^2-\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}\right\}$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}\left(\sqrt{1+x^2}-1\right)}.\left\{\frac{\left(\sqrt{1+x^2}-1\right)}{\sqrt{1+x^2}}\right\}$$

$$\Rightarrow\ \frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}.\sqrt{1+x^2}} = \frac{1}{2(1+x^2)}.$$

Example 2: Find the drerivative w. r. t. x:

$(i)\ \ tan^{-1}\left(\frac{\sqrt{1+sin\,x}+\sqrt{1-sin\,x}}{\sqrt{1+sin\,x}-\sqrt{1-sin\,x}}\right) \qquad (ii)\ \ tan^{-1}\left(\frac{\sqrt{1+cos\,x}+\sqrt{1-cos\,x}}{\sqrt{1+cos\,x}-\sqrt{1-cos\,x}}\right)$

Solution: $(i)\ \ Let\ \ y = tan^{-1}\left(\frac{\sqrt{1+sin\,x}+\sqrt{1-sin\,x}}{\sqrt{1+sin\,x}-\sqrt{1-sin\,x}}\right)$

$$\Rightarrow\ y = tan^{-1}\left(\frac{\sqrt{cos^2\frac{x}{2}+sin^2\frac{x}{2}+2\,sin\frac{x}{2}cos\frac{x}{2}}+\sqrt{cos^2\frac{x}{2}+sin^2\frac{x}{2}-2\,sin\frac{x}{2}cos\frac{x}{2}}}{\sqrt{cos^2\frac{x}{2}+sin^2\frac{x}{2}+2\,sin\frac{x}{2}cos\frac{x}{2}}-\sqrt{cos^2\frac{x}{2}+sin^2\frac{x}{2}-2\,sin\frac{x}{2}cos\frac{x}{2}}}\right)$$

$$\Rightarrow\ y = tan^{-1}\left\{\frac{\sqrt{\left(cos\frac{x}{2}+sin\frac{x}{2}\right)^2}+\sqrt{\left(cos\frac{x}{2}-sin\frac{x}{2}\right)^2}}{\sqrt{\left(cos\frac{x}{2}+sin\frac{x}{2}\right)^2}-\sqrt{\left(cos\frac{x}{2}-sin\frac{x}{2}\right)^2}}\right\}$$

$$\Rightarrow\ y = tan^{-1}\left\{\frac{\left(cos\frac{x}{2}+sin\frac{x}{2}\right)+\left(cos\frac{x}{2}-sin\frac{x}{2}\right)}{\left(cos\frac{x}{2}+sin\frac{x}{2}\right)-\left(cos\frac{x}{2}-sin\frac{x}{2}\right)}\right\} = tan^{-1}\left(\frac{2\,cos\frac{x}{2}}{2\,sin\frac{x}{2}}\right) = tan^{-1}\left(cot\frac{x}{2}\right)$$

$$\Rightarrow\ y = \frac{\pi}{2}-cot^{-1}\left(cot\frac{x}{2}\right) = \frac{\pi}{2}-\frac{x}{2} \qquad \left[Using\quad tan^{-1}x+cot^{-1}x = \frac{\pi}{2}\right]$$

$$\boxed{\begin{array}{c} Learn\ the\ formulae: \\ (i)\ sin^{-1}x + cos^{-1}x = \frac{\pi}{2} \\ (ii)\ tan^{-1}x + cot^{-1}x = \frac{\pi}{2} \\ (iii) sec^{-1}x + cosec^{-1}x = \frac{\pi}{2} \end{array}}$$

$$\Rightarrow \ \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - \frac{x}{2}\right) = -\frac{1}{2}.$$

$$(ii)\ Let\ \ y = tan^{-1}\left(\frac{\sqrt{1+cos\,x} + \sqrt{1-cos\,x}}{\sqrt{1+cos\,x} - \sqrt{1-cos\,x}}\right)$$

$$\Rightarrow \ y = tan^{-1}\left(\frac{\sqrt{2}\,cos\,x/2 + \sqrt{2}\,sin\,x/2}{\sqrt{2}\,cos\,x/2 - \sqrt{2}\,sin\,x/2}\right) = \ tan^{-1}\left(\frac{1+tan\,x/2}{1-tan\,x/2}\right)$$

$$\Rightarrow \ y = tan^{-1}\left(\frac{\tan\pi/4 + tan\,x/2}{1-\tan\pi/4.\,tan\,x/2}\right) = tan^{-1}\left\{tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\} = \frac{\pi}{4} + \frac{x}{2}$$

$$\Rightarrow \ \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{1}{2}.$$

Example 3: If $y = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$, show that $\frac{dy}{dx} = \sqrt{a^2 - x^2}$.

Solution: We have $y = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

$$\Rightarrow \quad y = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2\,sin^{-1}\frac{x}{a}\right]$$

Differentiating with respect to x both sides, we get

$$\frac{dy}{dx} = \frac{1}{2}.\frac{d}{dx}\left[x\sqrt{a^2 - x^2} + a^2\,sin^{-1}\frac{x}{a}\right] = \frac{1}{2}\left[\frac{d}{dx}\left(x\sqrt{a^2 - x^2}\right) + \frac{d}{dx}\left(a^2\,sin^{-1}\frac{x}{a}\right)\right]$$

$$\Rightarrow \ \frac{dy}{dx} = \frac{1}{2}.\left[\left\{\frac{d}{dx}(x).\sqrt{a^2 - x^2} + x.\frac{d}{dx}\left(\sqrt{a^2 - x^2}\right)\right\} + a^2.\frac{d}{dx}\left(sin^{-1}\frac{x}{a}\right)\right]$$

$$\Rightarrow \ \frac{dy}{dx} = \frac{1}{2}.\left[\left\{\frac{d}{dx}(x).\sqrt{a^2 - x^2} + x.\frac{d}{dx}\left(\sqrt{a^2 - x^2}\right)\right\} + a^2.\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}.\frac{d}{dx}\left(\frac{x}{a}\right)\right]$$

$$\Rightarrow \ \frac{dy}{dx} = \frac{1}{2}.\left[\left\{1.\sqrt{a^2 - x^2} + x.\frac{1}{2\sqrt{a^2 - x^2}}.(-2x)\right\} + a^2.\frac{1}{\sqrt{\frac{a^2 - x^2}{a^2}}}.\frac{1}{a}.1\right]$$

$$\Rightarrow \ \frac{dy}{dx} = \frac{1}{2}.\left[\left(\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}}\right) + \frac{a^2}{\sqrt{a^2 - x^2}}\right]$$

$$\Rightarrow \ \frac{dy}{dx} = \frac{1}{2}.\left[\frac{(a^2 - x^2) - x^2}{\sqrt{a^2 - x^2}} + \frac{a^2}{\sqrt{a^2 - x^2}}\right]$$

$$\Rightarrow \ \frac{dy}{dx} = \frac{1}{2}.\left(\frac{2a^2 - 2x^2}{\sqrt{a^2 - x^2}}\right) = \frac{1}{2}.2\left[\frac{a^2 - x^2}{\sqrt{a^2 - x^2}}\right] = \sqrt{a^2 - x^2}$$

Assignment No. 4

1. If $x^3 + y^3 = (x - y)^2$, $find\ \frac{dy}{dx}$. $\left[Answer: \frac{2(x-y) - 3x^2}{2(x-y) + 3y^2}\right]$

2. If $x^3 + y^3 = 3axy,$ $find\ \frac{dy}{dx}$ $\left[Answer: \frac{ay - x^2}{y^2 - ax}\right]$

3. If $x\ \sin(a + y) = \sin y$, show that $\frac{dy}{dx} = \frac{sin^2(a+y)}{\sin a}$.

4. If $y=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$, show that $\frac{dy}{dx}=\sqrt{a^2-x^2}$

5. Differentiate with respect to x :

$$(i)\ \sin(m\ \sin^{-1}x)\quad (ii)\ \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\ (iii)\ \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)\quad (iv)\ \tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$$

$$\left[Answer: (i)\ \frac{m\cos(m\sin^{-1}x)}{\sqrt{1-x^2}}\quad (ii)\ \frac{2}{1+x^2}\quad (iii)\ \frac{2}{1+x^2}\quad (iv)\ \frac{1}{2}\right]$$

6. Differentiate with respect to x :

$$(i)\ tan^{-1}\left(\sqrt{\frac{1-sin\,x}{1+sin\,x}}\right)\quad (ii)\ tan^{-1}\left(\frac{\sqrt{1+x^2}+1}{x}\right)\quad (iii)\ tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right).$$

$$\left[Answer: (i)\ -\frac{1}{2}\quad (ii)\ -\frac{1}{2(1+x^2)}\quad (iii)\ \frac{1}{2(1+x^2)}\ \right]$$

7. Differentiate with respect to x :

$$(i)\ \tan^{-1}\left(\sqrt{1+x^2}+x\right)\quad (ii)\ \tan^{-1}\left(\sqrt{1+x^2}-x\right)\quad (iii)\ \cot^{-1}\left(\frac{1-x}{1+x}\right).$$

$$\left[Answer: (i)\ \frac{1}{2(1+x^2)}\quad (ii)\ -\frac{1}{2(1+x^2)}\quad (iii)\ \frac{1}{1+x^2}\ \right]$$

8. If $y=\tan^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$, show that $\frac{dy}{dx}=-\frac{1}{2}$.

9. If $y=\tan^{-1}\left(\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right)$, show that $\frac{dy}{dx}=\frac{1}{2}$.

10. Find $\frac{dy}{dx}$ if $(i)\ y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)\quad (ii)\ y=sin\left\{2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right\}$.

$$\left[Answer:\quad (i)\ -\frac{x}{\sqrt{1-x^2}}\qquad (iii)\ -\frac{x}{\sqrt{1-x^2}}\ \right]$$

Parametric differentiation

If x and y both are expressed in terms of a single variable, then such functions are called parametric functions with the single variable as a parameter.

$Let\ \ x=f(t)\ldots\ldots(1)\quad and$

$y=g(t)\ \ldots\ldots(2)$ be functions of $'t'$. Then (1) and (2)

are parametric functions and ‘t’ is a parameter.

Let a small change δt in t, cause the corresponding changes δx and δy in x and y respectively. Clearly, we have

$$\frac{\delta y}{\delta x}=\frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}}\ \Rightarrow\ \lim_{\delta t\to 0}\frac{\delta y}{\delta x}=\lim_{\delta t\to 0}\frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}}\ \ (Taking\ limit\ both\ sides\ as\ \delta t\to 0)$$

$$\Rightarrow \lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{\lim_{\delta t\to 0}\frac{\delta y}{\delta t}}{\lim_{\delta t\to 0}\frac{\delta x}{\delta t}}\quad (As\ \ \delta t\to 0\ \ \Rightarrow \delta x\to 0)$$

$$\Rightarrow \boxed{\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}}$$

which is the required formula for parametric differentiation.

Solved Examples

Example 1: $If\ \ x = a\cos\theta\ \ and\ \ y = a\sin\theta\,,\qquad find\ \dfrac{dy}{dx}.$

Solution: $Given\ \ x = a\cos\theta\ \ and\ \ y = a\sin\theta$

$$\Rightarrow\ \frac{dx}{d\theta} = -\,a\,sin\,\theta\quad and\quad \frac{dy}{d\theta} = a\,cos\,\theta$$

$$By\ formula\quad \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{a\,cos\,\theta}{-a\,sin\,\theta} = -\,cot\,\theta\,.$$

Remark: The equations $x = a\cos\theta$ and $y = a\sin\theta$ are the parametric equations of the circle $x^2 + y^2 = a^2$. Differentiating w.r.t. x, we get $2x + 2y\dfrac{dy}{dx} = 0$

$$\Rightarrow\ \frac{dy}{dx} = -\frac{x}{y} = -\frac{a\cos\theta}{a\sin\theta} = -\,cot\,\theta.$$

Example 2: If $x = at^2\ \ and\ \ y = 2at,\qquad find\ \dfrac{dy}{dx}.$

Solution: $Given\ \ x = at^2\ \ and\ \ y = 2at$

$$\Rightarrow\ \frac{dx}{dt} = 2at\quad and\quad \frac{dy}{dt} = 2a$$

$$By\ formula\quad \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2a}{2at} = \frac{1}{t}\,.$$

Remark: $The\ equations\ \ x = at^2\ \ and\ \ y = 2at\ are\ the\ parametric\ equations\ of\ the\ parabola\ \ y^2 = 4ax.\ Differentiating\ w.r.t.\,x, we\ get\ \ 2y\dfrac{dy}{dx} = 4a \Rightarrow \dfrac{dy}{dx} = \dfrac{2a}{y} = \dfrac{2a}{2at} = \dfrac{1}{t}.$

Example 3: $If\ \ x = a\left(cos\,t + \dfrac{1}{2}log\,tan^2\dfrac{t}{2}\right)\ \ and\ \ y = a\,sin\,t\,,\qquad show\ that\ \dfrac{dy}{dx} = tan\,t.$

Solution: $Given\ \ x = a\left(cos\,t + \dfrac{1}{2}log\,tan^2\dfrac{t}{2}\right)$

$$\Rightarrow\ \frac{dx}{dt} = a\left(-\,sin\,t + \frac{1}{2}.\frac{1}{tan^2\frac{t}{2}}.2\,tan\frac{t}{2}.sec^2\frac{t}{2}.\frac{1}{2}\right) = a\left(-\,sin\,t + \frac{1}{tan\frac{t}{2}}.sec^2\frac{t}{2}.\frac{1}{2}\right)$$

$$\Rightarrow\ \frac{dx}{dt} = a\left(-\,sin\,t + \frac{\cos\frac{t}{2}}{sin\frac{t}{2}}.\frac{1}{\cos^2\frac{t}{2}}.\frac{1}{2}\right) = a\left(-\,sin\,t + \frac{1}{sin\frac{t}{2}}.\frac{1}{\cos\frac{t}{2}}.\frac{1}{2}\right)$$

$$\Rightarrow\ \frac{dx}{dt} = a\left(-\,sin\,t + \frac{1}{2\,sin\frac{t}{2}\cos\frac{t}{2}}\right) = a\left(-\,sin\,t + \frac{1}{\sin t}\right) = a\left(\frac{-\,sin^2\,t + 1}{sin\,t}\right)$$

$$\Rightarrow\ \frac{dx}{dt} = a\frac{cos^2\,t}{sin\,t}$$

$$Also,\qquad given\ that\quad y = a\,sin\,t\quad \Rightarrow \frac{dy}{dt} = a\,cos\,t$$

$$By\ formula\quad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{(a\,cos\,t)}{\left(a\frac{cos^2\,t}{sin\,t}\right)} = tan\,t\,.$$

Example 4: $If\ \ x = a\sec^2\theta\ \ and\ \ y = b\,\mathrm{cosec}^2\,\theta\,,\qquad find\ \left(\dfrac{dy}{dx}\right)_{\theta=\pi/3}.$

Solution: $Given\ \ x = a\sec^2\theta\ \ and\ \ y = b\,\mathrm{cosec}^2\,\theta$

$$\Rightarrow\ \frac{dx}{d\theta} = \ a.2\,sec\,\theta.sec\,\theta\,tan\,\theta = 2a\,sec^2\,\theta\,tan\,\theta$$

$$\&\quad \frac{dy}{d\theta} = b.2\,cosec\,\theta\,(-\,cosec\,\theta\,cot\,\theta) = -2b\,cosec^2\,\theta\,cot\,\theta$$

$$By\ formula\quad \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{-2b\,cosec^2\,\theta\,cot\,\theta}{2a\,sec^2\,\theta\,tan\,\theta} = -\frac{b\,cos^4\,\theta}{a\,sin^4\,\theta}\,.$$

$$Now, \quad \left(\frac{dy}{dx}\right)_{\theta=\pi/3} = -\frac{b\cos^4\pi/3}{a\sin^4\pi/3} = -\frac{b}{a}.\frac{1}{2^4}.\frac{2^4}{3^2} = -\frac{b}{9a}.$$

Assignment No. 5

1. If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, find $\frac{dy}{dx}$. $\left(Answer:\ tan\frac{\theta}{2}\right)$
2. If $x = a(\cos\theta + \theta\sin\theta)$ *and* $y = a(\sin\theta - \theta\cos\theta)$, find $\frac{dy}{dx}$. $(Answer:\ tan\,\theta)$
3. If $x = a\left(\cos t + \frac{1}{2}\log tan^2\frac{t}{2}\right)$ *and* $y = a\sin t$, show that $\frac{dy}{dx} = \tan t$.
4. If $x = a(\theta - sin\,\theta)$ and $y = a(1 - cos\,\theta)$, determine the value of $\left(\frac{dy}{dx}\right)_{\theta=\pi/6}$.
$(Answer:\ 2 + \sqrt{3})$
5. If $x = \cos 3\theta - 3\cos\theta$ and $y = \sin 3\theta - 3\sin\theta$, show that $\frac{dy}{dx} = \tan 2\theta$.
6. If $x = \cos n\theta - n\cos\theta$ and $y = \sin n\theta - n\sin\theta$, show that $\frac{dy}{dx} = \tan\left\{\left(\frac{n+1}{2}\right)\theta\right\}$.
7. *If* $x = a\,sec\,t$ and $y = b\,tan\,t$, find $\left(\frac{dy}{dx}\right)_{t=\pi/3}$. $\left(Answer:\ \frac{2\sqrt{3}\,b}{3a}\right)$

Differentiation of a function with respect to another function

Let $u = f(x)$ and $v = g(x)$ be two functions of x. We may require the derivative of u with respect to v, i.e., $\frac{du}{dv}$. Using the concept of parametric differentiation, we get

$$\frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}$$

$$Also, \quad \frac{dv}{du} = \frac{\left(\frac{dv}{dx}\right)}{\left(\frac{du}{dx}\right)}$$

Solved Examples

Example 1: Differentiate $log\,sin\,x$ *with respect to* $tan^3 x$.

Solution: *Let* $u = log\,sin\,x$ *and* $v = tan^3 x$. *We need to find* $\frac{du}{dv}$.

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{sin\,x}.cos\,x = cot\,x \quad \& \quad \frac{dv}{dx} = 3\,tan^2 x.sec^2 x$$

$$By\ formula\ \frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)} = \frac{cot\,x}{3\,tan^2 x.sec^2 x} = \frac{1}{3}cot^3 x\,cos^2 x.$$

Relation between $\frac{dy}{dx}$ and $\frac{dx}{dy}$

Let $y = f(x)$ be a function of x. Let a small change δx in x causes the corresponding change δy in y.

We know that $\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\frac{\delta x}{\delta y}} = \frac{\lim_{\delta x\to 0} 1}{\lim_{\delta x\to 0}\frac{\delta x}{\delta y}} = \frac{1}{\lim_{\delta y\to 0}\frac{\delta x}{\delta y}} = \frac{1}{\left(\frac{dx}{dy}\right)}$ $(As\ \delta x \to 0, \delta y \to 0)$

$$\boxed{\Rightarrow \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} \quad \Rightarrow \left(\frac{dy}{dx}\right)\left(\frac{dx}{dy}\right) = 1 \quad \Rightarrow \frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}}$$

Logarithmic differentiation

If a function involves multiplication, division, exponents, radicals etc., it can easily be differentiated taking logarithms. The following properties of logarithms are helpful for the purpose:

(1) $log_a(mn) = log_a m + log_a n$ (2) $log_a\left(\frac{m}{n}\right) = log_a m - log_a n$

(3) $log_a m^n = n\, log_a m$ (4) $log_a 1 = 0$ (5) $log_a a = 1$

(6) $log_a b = \frac{1}{log_b a}$ (7) $log_b a = \frac{log_c a}{log_c b}$ (8) $p = q^{log_q p}$

Solved examples

Example 1: Differentiate w.r.t. x $(i)\ x^x$ $(ii)\ x^{\sin x}$ $(iii)\ (\sin x)^{\cos x}$

Solution: (i) *Let* $y = x^x$ *. Taking log both sides, we get*

$log_e y = log_e x^x \implies log_e y = x\, log_e x$

Differentiating w.r.t. x both sides, we get

$$\frac{1}{y}\frac{dy}{dx} = 1.\, log_e x + x.\frac{1}{x} \implies \frac{dy}{dx} = y(log_e x + 1) = x^x(log_e x + 1).$$

$$Alternatively, \frac{d}{dx}(x^x) = \frac{d}{dx}\left(e^{log_e x^x}\right) = \frac{d}{dx}\left(e^{x\, log_e x}\right) = e^{x\, log_e x}\frac{d}{dx}(x\, log_e x)$$

$$\implies \frac{d}{dx}(x^x) = e^{x\, log_e x}\left(1.\, log_e x + x.\frac{1}{x}\right) = e^{log_e x^x}(log_e x + 1) = x^x(log_e x + 1).$$

(ii) *Let* $y = x^{\sin x}$ *. Taking log both sides, we get*

$log_e y = log_e x^{\sin x} \implies log_e y = \sin x\, log_e x$

Differentiating w.r.t. x both sides, we get

$$\frac{1}{y}\frac{dy}{dx} = \cos x\, .\, log_e x + \sin x\, .\frac{1}{x} \implies \frac{dy}{dx} = \frac{y}{x}(x \cos x\, log_e x + \sin x)$$

$$\implies \frac{dy}{dx} = x^{\sin x}\left(\frac{x \cos x\, log_e x + \sin x}{x}\right)$$

$$(iii)\ \frac{d}{dx}\{(\sin x)^{\cos x}\} = \frac{d}{dx}\left\{e^{log_e(\sin x)^{\cos x}}\right\} = \frac{d}{dx}\left(e^{\cos x\, log_e \sin x}\right)$$

$$\implies \frac{d}{dx}\{(\sin x)^{\cos x}\} = e^{\cos x\, log_e \sin x}\frac{d}{dx}(\cos x\, log_e \sin x)$$

$$\implies \frac{d}{dx}\{(\sin x)^{\cos x}\} = e^{log_e(\sin x)^{\cos x}}\left[-\sin x\, log_e \sin x + \cos x\, .\frac{1}{\sin x}.\cos x\right]$$

$$\implies \frac{d}{dx}\{(\sin x)^{\cos x}\} = (\sin x)^{\cos x}(\cos x \cot x - \sin x\, log_e \sin x)$$

Example 2: If $x^y + y^x = a,$ find $\frac{dy}{dx}$.

Solution: *Given* $x^y + y^x = a$

Differentiating w.r.t. x both sides, we get

$$\frac{d}{dx}(x^y + y^x) = 0 \implies \frac{d}{dx}(x^y) + \frac{d}{dx}(y^x) = 0 \implies \frac{d}{dx}\left(e^{log_e x^y}\right) + \frac{d}{dx}\left(e^{log_e y^x}\right) = 0$$

$$\implies \frac{d}{dx}\left(e^{y log_e x}\right) + \frac{d}{dx}\left(e^{x log_e y}\right) = 0$$

$$\implies e^{y\, log_e x}\frac{d}{dx}(y\, log_e x) + e^{x log_e y}\frac{d}{dx}(x\, log_e y) = 0$$

$$\implies e^{log_e x^y}\left(\frac{dy}{dx}log_e x + y.\frac{1}{x}\right) + e^{log_e y^x}\left(1.\, log_e y + x.\frac{1}{y}.\frac{dy}{dx}\right) = 0$$

$$\implies x^y\left(\frac{dy}{dx}log_e x + \frac{y}{x}\right) + y^x\left(1.\, log_e y + \frac{x}{y}.\frac{dy}{dx}\right) = 0$$

$$\implies \frac{dy}{dx}\left(x^y\, log_e x + y^x.\frac{x}{y}\right) = -\left(x^y.\frac{y}{x} + y^x.\, log_e y\right)$$

$$\implies \frac{dy}{dx} = -\left(\frac{y.\, x^{y-1} + y^x.\, log_e y}{x^y\, log_e x + x.\, y^{x-1}}\right)$$

Example 3: If $x^y = e^{x-y},$ show that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$.

Solution: $Given \quad x^y = e^{x-y}$

$Taking\ logarithms\ both\ sides, we\ get$

$y\, log_e\, x = (x-y).log_e\, e \quad \Rightarrow \quad y\, log_e\, x = (x-y).1 \quad \Rightarrow\ y\, log_e\, x + y = x$

$\Rightarrow \quad y(log_e\, x + 1) = x \quad \Rightarrow \quad y = \dfrac{x}{1 + log\, x}$

$Differentiating\ w.r.t.x\ both\ sides, we\ get$

$$\frac{dy}{dx} = \frac{1.(1+log\,x) - x.\frac{1}{x}}{(1+log_e\,x)^2} = \frac{1 + log\,x - 1}{(1+log\,x)^2} = \frac{log\,x}{(1+log\,x)^2}.$$

Example 4: $If \quad x^m y^n = (x+y)^{m+n}, \qquad show\ that\ \dfrac{dy}{dx} = \dfrac{y}{x}.$

Solution: $Given \quad x^m y^n = (x+y)^{m+n}$

$Taking\ logarithms\ both\ sides, we\ get$

$log(x^m y^n) = log\,(x+y)^{m+n} \quad \Rightarrow \quad m\,log\,x + n\,log\,y = (m+n)\,log(x+y)$

$Differentiating\ w.r.t.x\ both\ sides, we\ get$

$$m.\frac{1}{x} + n.\frac{1}{y}.\frac{dy}{dx} = (m+n).\frac{1}{x+y}.\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \left(\frac{m+n}{x+y}\right)\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}\left\{\frac{n}{y} - \left(\frac{m+n}{x+y}\right)\right\} = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left\{\frac{n(x+y) - (m+n)y}{y(x+y)}\right\} = \frac{(m+n)x - m(x+y)}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{nx + ny - my - ny}{y}\right) = \frac{mx + nx - mx - my}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{nx - my}{y}\right) = \frac{nx - my}{x} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}.$$

Example 5: If $y = \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \dfrac{bx}{(x-b)(x-c)} + \dfrac{c}{x-c} + 1$

$show\ that \quad \dfrac{dy}{dx} = \dfrac{y}{x}\left(\dfrac{a}{a-x} + \dfrac{b}{b-x} + \dfrac{c}{c-x}\right).$

Solution: $Given \quad y = \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \dfrac{bx}{(x-b)(x-c)} + \left(\dfrac{c}{x-c} + 1\right)$

$$\Rightarrow \quad y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c + x - c}{x - c}$$

$$\Rightarrow \quad y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \left\{\frac{bx}{(x-b)(x-c)} + \frac{x}{x-c}\right\}$$

$$\Rightarrow \quad y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx + x(x-b)}{(x-b)(x-c)}$$

$$\Rightarrow \quad y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{x^2}{(x-b)(x-c)}$$

$$\Rightarrow \quad y = \frac{ax^2 + x^2(x-a)}{(x-a)(x-b)(x-c)}$$

$$\Rightarrow \quad y = \frac{x^3}{(x-a)(x-b)(x-c)}$$

Taking logarithms both sides, we get

$$log\,y = log\left\{\frac{x^3}{(x-a)(x-b)(x-c)}\right\} = log\,x^3 - log\{(x-a)(x-b)(x-c)\}$$

$$\Rightarrow \quad log\,y = 3\,log\,x - log(x-a) - log(x-b) - log(x-c)$$

$Differentiating\ both\ sides\ with\ respect\ to\ x, we\ get$

$$\frac{1}{y}\frac{dy}{dx} = 3.\frac{1}{x} - \frac{1}{x-a}.1 - \frac{1}{x-b}.1 - \frac{1}{x-c}.1$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left(\frac{3}{x} - \frac{1}{x-a} - \frac{1}{x-b} - \frac{1}{x-c}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left\{\left(\frac{1}{x} - \frac{1}{x-a}\right) + \left(\frac{1}{x} - \frac{1}{x-b}\right) + \left(\frac{1}{x} - \frac{1}{x-c}\right)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left\{\frac{x-a-x}{x(x-a)} + \frac{x-b-x}{x(x-b)} + \frac{x-c-x}{x(x-c)}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\left(\frac{-a}{x-a} + \frac{-b}{x-b} + \frac{-c}{x-c}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right).$$

Assignment No. 6

1. Differentiate with respect to x : $(i)\ x^x$ $\quad (ii)\ (\sin x)^{\cos x}$ $\quad (iii)\ (\sin^{-1} x)^x$.

   Answer: $(i)\ x^x(1 + \log x) \quad (ii) (\sin x)^{\cos x}(\cos x \cot x - \sin x \log_e \sin x)$

   $(iii)\ (\sin^{-1} x)^x\left(\log \sin^{-1} x + \frac{x}{\sqrt{1-x^2}.\sin^{-1} x}\right)$

2. Find $\frac{dy}{dx}$ *if* $\quad (i)\ x^y = y^x \quad (ii)\ x^y + y^x = a \quad (iii)\ x^m y^n = 1.$

   Answer: $(i)\ \frac{y}{x}\left(\frac{x \log y - y}{y \log x - x}\right) \quad (ii)\ -\left(\frac{y.x^{y-1} + y^x.\log_e y}{x^y \log_e x + x.y^{x-1}}\right) \quad (iii)\ -\frac{my}{nx}$

3. If $x^y = e^{x-y}$, $\quad$ show that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$.

4. If $x^m y^n = (x+y)^{m+n}$, $\quad$ show that $\frac{dy}{dx} = \frac{y}{x}$.

5. If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1 = 0,$ $\quad$ show that $\frac{dy}{dx} = \frac{y}{x}\left(\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right)$.

Derivatives of a^x and $\log_a x$

Formulae:

$(\boldsymbol{i})$ $\boldsymbol{\frac{d}{dx}(a^x) = a^x \log_e a}$

$(\boldsymbol{ii})$ $\boldsymbol{\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a} = \frac{\log_a e}{x}}$

Proof: (i) $\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{\log_e a^x}\right) = \frac{d}{dx}\left(e^{x \log_e a}\right) = e^{x \log_e a}.\frac{d}{dx}(x \log_e a)$

$$\Rightarrow \quad \frac{d}{dx}(a^x) = e^{\log_e a^x}.\log_e a.\frac{d}{dx}(x) = a^x \log_e a.1 = a^x \log_e a.$$

Alternatively, let $y = a^x$. *Taking log both sides, we get* $\log_e y = x \log_e a$

Differentiating w.r.t. x both sides, we get

$$\frac{1}{y}\frac{dy}{dx} = 1.\log_e a \quad \Rightarrow \frac{dy}{dx} = y.\log_e a \quad \Rightarrow \quad \frac{dy}{dx} = a^x \log_e a$$

We also prove it from the first principles as follows:

$$\frac{d}{dx}(a^x) = \lim_{\delta x \to 0} \frac{a^{x+\delta x} - a^x}{\delta x} = \lim_{\delta x \to 0} \frac{a^x.a^{\delta x} - a^x}{\delta x} = \lim_{\delta x \to 0} \frac{a^x\left(a^{\delta x} - 1\right)}{\delta x}$$

$$\frac{d}{dx}(a^x) = a^x \lim_{\delta x \to 0} \frac{a^{\delta x} - 1}{\delta x} = a^x.\log_e a$$

(ii) $\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\log_e x}{\log_e a}\right) = \frac{1}{\log_e a}.\frac{d}{dx}(\log_e x) \quad \left(\text{Using} \quad \log_b a = \frac{\log_c a}{\log_c b}\right)$

$$\Rightarrow \frac{d}{dx}(log_a x) = \frac{1}{log_e a}.\frac{1}{x} = \frac{1}{x\, log_e a} = \frac{log_a e}{x}.$$

We prove it from the first principles as follows:

$$\frac{d}{dx}(log_a x) = \lim_{\delta x \to 0} \frac{log_a(x+\delta x) - log_a x}{\delta x} = \lim_{\delta x \to 0} \frac{log_a\left(\frac{x+\delta x}{x}\right)}{\delta x} = \lim_{\delta x \to 0} \frac{log_a\left(1+\frac{\delta x}{x}\right)}{\delta x}$$

$$\Rightarrow \frac{d}{dx}(log_a x) = \lim_{\delta x \to 0} \frac{\frac{log_e\left(1+\frac{\delta x}{x}\right)}{\log_e a}}{\delta x} = \frac{1}{\log_e a}.\lim_{\delta x \to 0} \frac{log_e\left(1+\frac{\delta x}{x}\right)}{\delta x} = \frac{1}{\log_e a}.\lim_{\delta x \to 0} \frac{log_e\left(1+\frac{\delta x}{x}\right)}{\delta x}$$

$$\Rightarrow \frac{d}{dx}(log_a x) = \frac{1}{log_e a}\left\{.\lim_{\delta x \to 0} \frac{log_e\left(1+\frac{\delta x}{x}\right)}{\frac{\delta x}{x}}\right\}.\frac{1}{x} = \frac{1}{x\, log_e a}.1 = \frac{1}{x\, log_e a}.$$

Solved examples

Example 1: *Find the derivative w.r.t.x*: $(i)\ 2^x$ $(ii)\ 2^{3^x}$ $(iii)\ 2^{3^{4^x}}$.

Soluion: $(i)\ \frac{d}{dx}(2^x) = 2^x\, log_e 2$

$(ii)\ \frac{d}{dx}\left(2^{3^x}\right) = 2^{3^x}\, log_e 2.\frac{d}{dx}(3^x) = 2^{3^x}\, log_e 2.3^x.log_e 3 = 2^{3^x}.3^x.log\, 2\, log\, 3.$

$(iii)\ \frac{d}{dx}\left(2^{3^{4^x}}\right) = 2^{3^{4^x}}.log\, 2.\frac{d}{dx}\left(3^{4^x}\right) = 2^{3^{4^x}}.log\, 2\, .3^{4^x}.log\, 3\, .\frac{d}{dx}(4^x)$

$\Rightarrow \frac{d}{dx}\left(2^{3^{4^x}}\right) = 2^{3^{4^x}}.log\, 2\, .3^{4^x}.log\, 3\, .4^x.log\, 4 = 2^{3^{4^x}}.3^{4^x}.4^x.log\, 2\, log\, 3\, log\, 4.$

Example 2: *Differentiate* $log_3 x$ with respect to x.

Soluion: $\frac{d}{dx}(log_3 x) = \frac{1}{x\, log_e 3}.$

Special infinite series and their differentiation

We come across functions showing the dependent or independent variable in the form of an infinite series in which the initial term/symbol repeats indefinitely. Such series can be reduced to a concise form replacing the terms beyond the first by the variable itself. Further, differentiation with respect to x both sides yields $\frac{dy}{dx}$.

Solved examples

Example 1: If $y = \cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+\ldots to\ \infty}}}}$, find $\frac{dy}{dx}$.

Solution: We can write the above function as $y = \frac{1}{x+y} \Rightarrow y(x+y) = 1$

$\Rightarrow\ xy + y^2 = 1$

Differentiating both sides with respect to x, we get

$$1.y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad (x+2y)\frac{dy}{dx} = -y \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x+2y}.$$

Example 2: If $y = x^{x^{x^{x^{\ldots to\ \infty}}}}$, find $\frac{dy}{dx}$.

Solution: We can write the above function as $y = x^y \Rightarrow log\, y = y\, log\, x$

Differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}log\, x + y.\frac{1}{x} \quad \Rightarrow \quad \left(\frac{1}{y} - log\, x\right)\frac{dy}{dx} = \frac{y}{x} \quad \Rightarrow \quad \left(\frac{1-y\, log\, x}{y}\right)\frac{dy}{dx} = \frac{y}{x}.$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2}{x(1 - y\,log\,x)}$$

Assignment No. 7

1. If $y = x^{x^{x^{\dots to\,\infty}}}$, show that $\dfrac{dy}{dx} = \dfrac{y^2}{x(1 - y\log x)}$,

2. $If\ \ y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \cdots \text{ to } \infty}}}$, $show\ that$ $\dfrac{dy}{dx} = \dfrac{cos\,y}{2y-1}$.

3. If $y = \sqrt{log\,x + \sqrt{log\,x + \sqrt{log\,x + \cdots \text{ to } \infty}}}$, show that $(2y - 1)\dfrac{dy}{dx} = \dfrac{1}{x}$.

4. Differentiate with respect to x : $(i)\ 2^{3^x}$ $(ii)\ a^{b^{c^x}}$ $(ii)\ \log_x 3$.

   $Answer$: $(i)\ 2^{3^x}.3^x.log\,2\,log\,3$ $(ii)\ a^{b^{c^x}}.b^{c^x}.c^x\,log\,a\,.log\,b.log\,c$ $(iii)\ \dfrac{1}{x\,log\,3}$

5. If $y = \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \dots to\,\infty}}}}$, prove that $\dfrac{dy}{dx} = -\dfrac{y}{x + 2y}$.

Higher order derivatives

Let $y = f(x)$ be a given function of x. Differentiating it with respect to x, we get $\dfrac{dy}{dx}$ $which\ is\ again\ a\ function\ of\ x\ and\ hence, can\ further\ be\ differentiated\ with$ $respect\ to\ x\ to\ get\ what\ we\ call\ as\ the\ second\ derivative\ to\ be\ denoted\ by$ $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$. $Continuing\ the\ process\ again\ and\ again\ n - times, we\ get\ the\ 3rd, 4th, \dots, nth$ $derivatives\ of\ 3rd, 4th, \dots, nth\ orders\ respectively. Thus, the\ derivatives\ of\ order$ $more\ than\ 1\ are\ known\ as\ higher\ derivatives\ or\ higher - order\ derivatives. We\ list\ out$ $below\ the\ derivatives\ from\ 1st\ order\ to\ nth\ order\ along\ with\ other\ common\ symbols$:

First derivative:	$\dfrac{dy}{dx}$,	$f'(x)$,	y_1,	Dy
Second derivative:	$\dfrac{d^2y}{dx^2}$,	$f''(x)$,	y_2,	D^2y
Third derivative:	$\dfrac{d^3y}{dx^3}$,	$f'''(x)$,	y_3,	D^3y
....	.....	......	....	....
nth derivative:	$\dfrac{d^ny}{dx^n}$,	$f^n(x)$,	y_n,	D^ny

Solved examples

Example 1: If $\mathrm{y} = \sin^2 x\,, find\ \dfrac{d^3y}{dx^3}$.

Solution: Given that $y = sin^2\,x$.

$Differentiating\ with\ respect\ to\ x, we\ get$

$$\frac{dy}{dx} = 2\,sin\,x\,cos\,x = sin\,2x$$

$Differentiating\ again\ with\ respect\ to\ x, we\ get$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\sin 2x) = 2\cos 2x$$

$Differentiating\ third\ time\ with\ respect\ to\ x, we\ get$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(2\,cos\,2x) = -2.2\,sin\,2x = -4\,sin\,2x.$$

Example 2: If $x = a\,sec^3 t$ and $y = b\,tan^3 t$, $find$ $\left(\dfrac{d^2y}{dx^2}\right)_{t=\pi/4}$.

Solution: Given that $x = a\,sec^3 t$ and $y = b\,tan^3 t$.

$$\Rightarrow \frac{dx}{dt} = a.3\,sec^2 t\,.sec\,t\,tan\,t = 3a\,sec^3 t\,.tan\,t$$

$$\& \quad \frac{dy}{dt} = b.3\,tan^2 t.sec^2 t = 3b\,sec^2 t.tan^2 t$$

$$By\,formula \quad \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{3b\,sec^2 t.tan^2 t}{3a\,sec^3 t\,.tan\,t} = \frac{b}{a}sin\,t$$

$$Now, \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b\,sin\,t}{a}\right) = \frac{b}{a}\frac{d}{dx}(sin\,t) = \frac{b}{a}.\frac{d}{dt}(sin\,t).\frac{dt}{dx}$$

$$= \frac{b}{a}\cos t\,.\frac{1}{3a\,sec^3 t\,.tan\,t} = \frac{b\cos^5 t}{3a^2\sin t}$$

$$\left(\frac{d^2y}{dx^2}\right)_{t=\pi/4} = \frac{b\cos^5\pi/4}{3a^2\sin\pi/4} = \frac{b}{12a^2}.$$

Example 3: If $y = e^{m\,sin^{-1}x}$, show that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - m^2y = 0.$

Solution: Given that $y = e^{m\,sin^{-1}x}$ .... (1)

$Differentiating\ both\ sides\ with\ respect\ to\ x, we\ get$

$$\frac{dy}{dx} = e^{m\,sin^{-1}x}.m.\frac{1}{\sqrt{1-x^2}} = y\frac{m}{\sqrt{1-x^2}} \quad [From\ (1)]$$

$$\Rightarrow \sqrt{1-x^2}\frac{dy}{dx} = my \quad \Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2y^2$$

$$(Squaring\ both\ sides)$$

Again differentiating both sides with respect to x, we get

$$(1-x^2).2\frac{dy}{dx}\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = m^2.2y\frac{dy}{dx}$$

$$(1-x^2).\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right) - m^2.y = 0$$

Assignment No. 8

1. If $y = e^{m\sin^{-1}x}$, show that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - m^2y = 0.$

2. If $y = \sin(m\sin^{-1}x)$, show that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + m^2y = 0.$

3. If $x = a\sec t$ and $y = b\tan t$, determine the value of $\left(\dfrac{d^2y}{dx^2}\right)_{t=\pi/6}$.

$$\left(\boldsymbol{Answer}: -\frac{3\sqrt{3}b}{a^2}\right)$$

4. If $y = \left(x + \sqrt{1+x^2}\right)^m$, show that $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - m^2y = 0.$

5. If $y = (\tan^{-1}x)^2$, show that $(1+x^2)^2\,y_2 + 2x(1+x^2)y_1 - 2 = 0.$

6. If $y = \sin^2 x$, find $\left(\dfrac{d^3y}{dx^3}\right)_{x=-\pi/4}$. $(Answer: 4)$

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