Applied Combinatorial Optimization

Exercise sheet 2

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Max-Weight Independent Set (MWIS) Problem

Notation

$[n] = \{1, \dots, n\}$

$$[\![A]\!] = \begin{cases} 1, & A = \text{True} \\ 0, & \text{otherwise} \end{cases}$$

MWIS problem definition and inequality representation

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a undirected graph with a *node set* $\mathcal{V}$ and an *edge set* $\mathcal{E} \subseteq \binom{\mathcal{V}}{2}$. An *independent* or *stable set* is the subset of mutually non-adjacent vertices $\mathcal{V}' \subseteq \mathcal{V}$, *e.g.*, for any $i, j \in \mathcal{V}'$ it holds $\{i, j\} \notin \mathcal{E}$. Given the costs c_i, $i \in \mathcal{V}$, the *maximum weight independent set* (*MWIS*) problem consists in finding an independent set with the maximum total cost.

ILP formulation and edge relaxation. Let $[n]$ stand for $\{1, 2, \dots, n\}$ and assume that the set of graph nodes has n elements, *i.e.*, $\mathcal{V} = [n]$. Introduce a binary variable $x_i \in \{0, 1\}$ for each node $i \in \mathcal{V}$. Its value 1 denotes that the respective node belongs to the maximum-weight independent set. This leads to a natural integer linear program (ILP) formulation of the MWIS problem:

$$\begin{aligned} &\max_{\mathbf{x} \in \{0,1\}^n} \langle \mathbf{c}, \mathbf{x} \rangle \\ &\text{s.t. } x_i + x_j \leq 1\,,\ \{i, j\} \in \mathcal{E}\,. \end{aligned} \tag{1}$$

The respective LP relaxation, obtained by substituting the integer constraints $\mathbf{x} \in \{0, 1\}^n$ with the box constraints $\mathbf{x} \in [0, 1]^n$, is referred to as the *edge relaxation*. This relaxation has several nice properties like *half-integrality*(See Exercise Sheet 1), *persistency* (considered later in the course), and is reducible to the max-low problem. However, it is typically very loose for dense graph, as the following proposition (proven in Exercise Sheet 1) shows:

Proposition 0.1. Let G be fully-connected, all costs positive, $c_j > 0$, $j \in \mathcal{V}$, and $c_i < \sum_{j \in \mathcal{V} \setminus \{i\}} c_j$ for $i = \operatorname{argmax}_{j \in \mathcal{V}} c_j$. Then the edge relaxation has a unique solution $\underbrace{(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})}_{n}$.

To address this limitation, the *clique constraints* and the respective *clique relaxation* are considered for the MWIS problem in the literature. Before defining it, we first introduce a family of relaxations that includes both the edge and the clique relaxations.

Clique cover relaxation family. Let $K_j \subseteq \mathcal{V}$, $j \in [m]$, be subsets of the nodes such, that the respective induced subgraphs of $\mathcal{G}$ are cliques. Abusing notation, we will refer to K_j themselves as *cliques*. We assume that K_j, $j \in [m]$, cover all edges and nodes of the graph, *e.g.* $\mathcal{E} = \bigcup_{j=1}^{m}\{\{i,i'\} \colon i,i' \in K_j\}$, $\mathcal{V} = \bigcup_{j=1}^{m}\{i \colon i \in K_j\}$, and, therefore, are referred to as a *clique cover*.[1] We define the *clique cover ILP formulation* of the MWIS problem as

$$\begin{aligned} &\max_{\mathbf{x}\in\{0,1\}^n} \langle \mathbf{c}, \mathbf{x}\rangle \\ &\text{s.t.} \sum_{i\in K_j} x_i \le 1\,,\, j \in [m]\,. \end{aligned} \tag{2}$$

Its natural LP relaxation, further referred to as *clique cover relaxation* is obtained by substituting integer constraints $\mathbf{x} \in \{0,1\}^n$ with the respective box constraints $\mathbf{x} \in [0,1]^n$. The relaxation is called *clique relaxation*, if the set $\{K_j \colon j \in [m]\}$, consists of *all* cliques of G.

Exercise 2.1 [0.25 Points] Show that all MWIS ILP formulations from the clique cover family are equivalent, *i.e.*, their sets of feasible solutions and the respective soluiton costs coincide.

Answer. It is sufficient to show that each clique cover relaxation is equivalent to the edge relaxation. Indeed, for $\mathbf{x} \in \{0,1\}^n$ and any $\{i,l\} \in \mathcal{E}$ there is $j \in [m]$ such that $i,l \in K_j$.

The clique inequality $\sum_{i\in K_j} x_i \le 1$ is equivalent to the condition that there is at most one $i \in K_j$ such that $x_i = 1$. Assume that $x_i = x_l = 1$ for some $i,l \in K_j$, $i \ne l$. Then they would violate the inequality $x_i + x_j \le 1$, as by definition of a clique $\{i,l\} \in K_j$.

On the other side, the condition that there is at most one $i \in K_j$ such that $x_i = 1$ is imposed by all edge inequalities $x_i + x_j \le 1$ such that $i,j \in K_j$ and implies the respective clique inequality $\sum_{i\in K_j} x_i \le 1$.

Hence the feasible set of both formulations coincide. The costs of each independent set is a sum of the respective node costs and is independent on the respective constraint representation. □

Whereas the solutions of the non-relaxed problems (1) and (2) coincide, their relaxations may differ a lot. In particular, the clique relaxation possess neither the half-integrality nor the persistency property in general. Also, no reduction to any efficiently solvable LP subclass like the min-cost-flow problem is possible for it, as this relaxation is as difficult as linear programs in general. However, it is tighter in general, as shown by the following statement:

Proposition 0.2. Let K_j, $j \in [m]$, and K'_j, $j \in [m']$, be two clique covers of the graph G such that for every K'_j, $j \in [m']$, there is $l \in [m] \colon K'_j \subseteq K_l$. Then the relaxation determined by K_j, $j \in [m]$, is at least as tight as the relaxation determined by K'_j, $j \in [m']$.

[1]The covering of nodes in addition to the covering of edges is required to deal with isolated nodes.

Exercise 2.2 [0.25 Points] Prove Proposition 0.2.

Answer. The proof is a direct consequence of the fact that for any $\mathbf{x} \in [0,1]^n$ an inequality $\sum_{i\in K_l} x_i \leq 1$ implies $\sum_{i\in K'_j} x_i \leq 1$ as long as $K'_j \subseteq K_l$. □

Proposition 0.2 defines a partial order on the set of clique cover relaxations with respect to their tightness. The edge relaxation is the least tight and constitutes the minimum with respect to this partial order. The maximum is determined by the clique relaxation. According to Proposition 0.2, the latter is equivalent to the relaxation, where K_j, $j \in [m]$, consists of maximal cliques (*i.e.*, those that are not subgraphs of any other clique) only. It is also known that clique inequalities are facet-defining for the stable set polytope[2] if and only if the respective clique is maximal.

Exercise 2.3 Considering the same setting as in Proposition 0.1, prove the following result:

Proposition 0.3 (0.5 Points)**.** The maximal clique relaxation, defined by the single maximal clique, is tight for complete graphs.

Answer. It is sufficient to prove that the feasible set being a polytope has only integer vertices. Indeed, assume $\mathbf{x}' \in [0,1]^n$ is a vertex. Then there is $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{x}'$ is a unique solution of $\{\max_{\mathbf{x}\in[0,1]^n} \langle \mathbf{c}, \mathbf{x}\rangle$, s.t. $\sum_{i=1}^n x_i \leq 1\}$. Assume that $\mathbf{x}'$ is fractional and w.l.o.g. $c_1 \geq c_2 > 0$ and $x'_1, x'_2 > 0$. Then

$$\langle \mathbf{c}, \mathbf{x}'\rangle = c_1 x'_1 + c_2 x'_2 + \sum_{i=3}^{n} c_i x'_i \leq c_1(x'_1 + x'_2) + c_2 \cdot 0 + \sum_{i=3}^{n} c_i x'_i = \langle \mathbf{c}, \mathbf{x}''\rangle\,, \tag{3}$$

where $\mathbf{x}''$ such that $x''_1 = x'_1 + x'_2$, $x''_2 = 0$ and $x''_i = x'_i$ for $i = 3 \dots n$. $\mathbf{x}''$ is feasible since $\mathbf{x}'$ is feasible as vertex. If the inequality in (3) holds strictly, then $\mathbf{x}'$ is not a solution and if it holds as equality, then $\mathbf{x}'$ is not a unique one. So we obtained a contradiction. □

Unfortunately, the number of maximal cliques may grow exponentially with the graph size in general. Hence in practice one has to ideally consider a clique cover consisting of a subset of the maximal cliques.

Equality representation

The clique constraints in (2) can be turned to equality by introducing *m slack variables* x_{n+j}, $j \in [m]$, and assuming $c_{n+j} = 0$ and $\bar{K}_j = K_j \cup \{n+j\}$. The ILP takes the form:

$$\max_{\mathbf{x}\in\{0,1\}^{n+m}} \langle \mathbf{c}, \mathbf{x}\rangle \tag{4}$$

$$\text{s.t.} \sum_{i\in\bar{K}_j} x_i = 1\,, j \in [m]\,. \tag{5}$$

We also introduce the constraint sets $J_i = \{j \in [m] \colon \bar{K}_j \ni i\}$ as the set of cliques containing the variable x_i. For a slack variable x_{n+j} it holds $J_{n+j} = \{j\}$.

[2] The convex hull of all feasible solutions of (1).

Exercise 2.4 [0.5 Points] Write down the Lagrange dual problem to (4) by dualizing its uniqueness constraints (5). Give an explicit expression for the reparametrized costs.

Answer. By dualizing the equality constraints (5) we obtain the Lagrange dual problem:

$$\min_{\lambda} \left[D(\lambda) := \max_{x \in \{0,1\}^{n+m}} \langle c, x \rangle + \sum_{j=1}^{m} \lambda_j \left(1 - \sum_{i \in \bar{K}_j} x_i \right) \right] \tag{6}$$

$$= \min_{\lambda} \left[\sum_{j=1}^{m} \lambda_j + \max_{x \in \{0,1\}^{n+m}} \langle c^{\lambda}, x \rangle \right] \tag{7}$$

$$= \min_{\lambda} \left[\sum_{j=1}^{m} \lambda_j + \langle c^{\lambda}, x^* \rangle \right] , \tag{8}$$

where $x^* \in \operatorname{sign}(c^{\lambda})$ and for $c \in \mathbb{R}^n$

$$\operatorname{sign}(c) := \left\{ x \in \mathbb{R}^n \colon x_i = \begin{cases} 0, & \text{if } c < 0 \\ 1, & \text{if } c > 0 \\ \in \{0,1\}, & \text{if } c = 0 \end{cases} \right\} , \tag{9}$$

where $\lambda \in \mathbb{R}^m$ are dual variables and $c_i^{\lambda} = c_i - \sum_{j \in J_i} \lambda_j$ are the *reparamerized* costs. □

Exercise 2.5 [0.25 Points] Write down the primal problem of the Lagrange relaxation. It is tighter than an LP relaxation of the problem (4) or is it equivalent to it?

Answer. The primal problem of the Lagrange relaxations reads

$$\begin{aligned} &\max_{x \in [0,1]^{n+m}} \langle c, x \rangle \\ &\text{s.t.} \sum_{i \in K_j} x_i = 1 \, , \, j \in [m] \, . \end{aligned} \tag{10}$$

It coincides with the LP relaxation of the problem (4). □

Exercise 2.6 [0.5 Points] Provide a formula for computing a subgradient for the Lagrange dual. Formulate the subgradient-based optimality condition. Compare it to the condition given by Corollary 5.17 (Dual optimality condition) given in the lecture.

Answer. Consider the Lagrange dual (6). According to Lemma 4.3.2. (Subdifferential of the maximum of a finite number of differentiable convex functions) the subdifferential of $D(\lambda)$ is equal to:

$$\partial D = \operatorname{conv}\{ g \in \mathbb{R}^m \colon g_j = \frac{\partial D}{\partial \lambda_j} = 1 - \sum_{i \in \bar{K}_j} x_i^* , \; x^* \in \operatorname{sign}(c^{\lambda}) \} \, . \tag{11}$$

According to Lemma 2.3.7. (Convex hull of a linear mapping image)

$$\partial D = \{ g \in \mathbb{R}^m \colon g_j = \frac{\partial D}{\partial \lambda_j} = 1 - \sum_{i \in \bar{K}_j} x_i^* , \; x^* \in \operatorname{conv}\left(\operatorname{sign}(c^{\lambda}) \right) \} \, . \tag{12}$$

According to Lemma 2.3.4 (Convex hull of a Cartesian product)

$$\operatorname{conv}\left(\operatorname{sign}(c^\lambda)\right) = \left\{ x \in \mathbb{R}^n \colon x_i = \begin{cases} 0, & \text{if } c < 0 \\ 1, & \text{if } c > 0 \\ \in [0,1], & \text{if } c = 0 \end{cases} \right\} = \operatorname*{argmax}_{x \in [0,1]^{n+m}} \langle c^\lambda, x \rangle \,. \tag{13}$$

The subgradient-based optimality condition is the existence of $\mathbf{0}$ in ∂D, which is equivalent to the existence of $x^* \in \operatorname{argmax}_{x \in [0,1]^{n+m}} \langle c^\lambda, x \rangle$ such that $1 - \sum_{i \in \bar{K}_j} x_i^* = 0$ for all $j \in [m]$. The latter is precisely the dual optimality condition given in the lecture. □