Applied Combinatorial Optimization

Exercise sheet 1

Dr. Bogdan Savchynskyy <bogdan.savchynskyy@iwr.uni-heidelberg.de>
Sebastian Stricker <sebastian.stricker@iwr.uni-heidelberg.de>

Linear program relaxation

Each exercise costs 0.5 point.

0.1 Labeling problem

Exercise 1.1 [Local polytope repaxation is not tight in general] Consider the local polytope relaxation of the labeling problem. Show that it is not tight in general. To this end

1. Consider the labeling problem with three nodes as in Figure 1(a) and its local polytope relaxation. Assume that all unary costs are zero, pairwise costs corresponding to solid lines are equal to 0, and costs corresponding to dashed lines are $\alpha > 0$.
2. Show that the depicted relaxed solution is feasible and optimal.
3. Show that the depicted relaxed solution is unique.
4. Prove that the respective local polytope has non-integer vertices and, therefore, does not coincide with the marginal polytope (the integer hull of the non-relaxed problem).

Answer. See Example 4.3 in [1]

[1] B. Savchynskyy "Discrete graphical models – An optimization perspective" □

Exercise 1.2 [Deterministic rounding] Let μ' be a relaxed solution of the labeling problem and let its coordinates corresponding to the labels in some of the nodes be assigned fractional values. For such a "fractional-valued" node u it holds that $\mu'_u \notin \{0,1\}^{\mathcal{Y}_u}$, or, in other words, $\mu_u(s) \in (0,1)$ for at least two labels $s \in \mathcal{Y}_u$. A labeling $\mathbf{y}'$, which is an approximate integer solution to the labeling problem can be obtained by rounding of μ' in the "fractional-valued" nodes:

$$y'_u := \arg\max_{s\in\mathcal{Y}_u} \mu'_u(s),\ u \in \mathcal{V}. \tag{1}$$

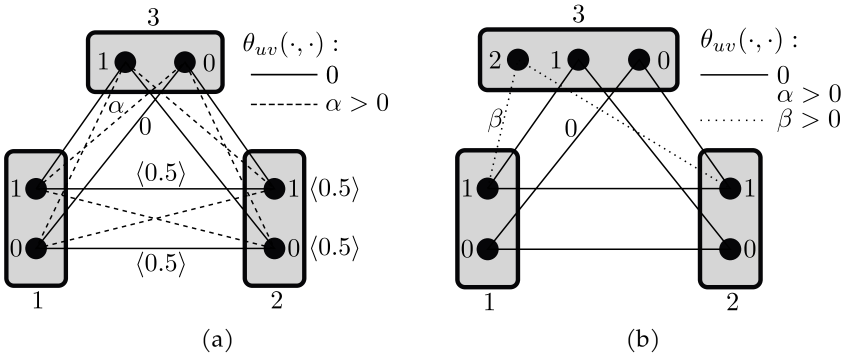

Figure 1: (**a**) Illustration to Exercise 1. The fully-connected labeling problem contains three nodes 1, 2, 3. Unary costs as well as the pairwise costs corresponding to solid lines are equal to 0 . Pairwise costs corresponding to dashed lines are equal to $\alpha > 0$. Angular brackets $\langle\rangle$ are used for coordinates of μ. For the unique relaxed solution μ it holds that $\mu_u(0) = \mu_u(1) = 0.5$ for all u. The same value 0.5 is assigned to, the coordinates corresponding to the solid lines, *i.e.*, $\mu_{12}(0,0) = \mu_{12}(1,1) = \mu_{13}(1,1) = \mu_{13}(0,0) = \mu_{32}(1,0) = \mu_{32}(0,1) = 0.5$. The remaining coordinates are zero. (**b**) Illustration for Exercise 2. Similar to (a), but the node 3 has 3 labels. All unary costs are zero. All label pairs not connected by a line, have cost $\alpha > 0$, dotted lines correspond to the pairwise cost $\beta > 0$, solid lines to the cost 0. It holds $\alpha > 2\beta$.

In particular, from equation (1) it follows that the label s, is assigned to the node u if $\mu_u(s) = 1$, *i.e.*, if the node u is already integer-valued.

Show that the deterministic rounding can be arbitrary bad, *i.e.*, the value $d := E(y';\theta) - \min_{y\in\mathcal{Y}_\mathcal{V}} E(y;\theta) \geq 0$ can be arbitrary large.

To this end

- Consider the labeling problem in Figure 1(b);
- Compute the relaxed solution of this problem instance and its deterministic rounding;
- Consider the case when $\alpha - 2\beta > 0$ and compute an optimal integer solution;
- Compare the costs of the optimal and the rounded solutions, compute d as a function of α and β.

Answer. See Example 4.5 in [1]

[1] B. Savchynskyy "Discrete graphical models – An optimization perspective" □

Exercise 1.3 [Partial integrality does not imply partial optimality] Most rounding schemes implicitly assume that integer-valued coordinates of the relaxed solution belong to an optimal solution of the non-relaxed problem. In other words, they assume that these coordinates are *partially optimal*.

This observation indeed often holds in practice, and, moreover, there are exact algorithms for the labeling problem, which efficiently utilize it.

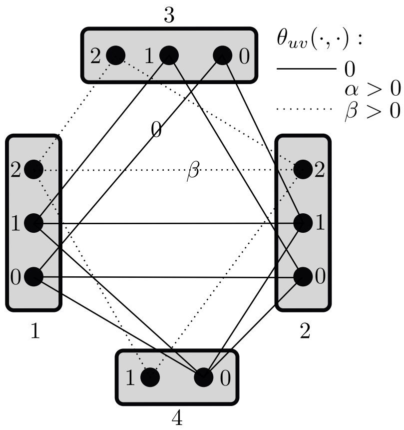

Figure 2: Illustration of Exercise 3. Notation is similar to Figure 1(b): unary costs are 0, as well as pairwise costs associated with solid lines. Dotted lines are associated with pairwise costs $\beta > 0$, not shown lines with the cost $\alpha > 0$. It holds $\alpha > 5\beta$.

Show that this assumption is, however, wrong in general.

To this end

- Consider the labeling problem in Figure 2.
- Compute the optimal solution of the local polytope relaxation. What coordinates of the solution are integral?
- Compute an optimal solution of the non-relaxed problem.
- Compare the relaxed and the optimal solutions. Do their coordinates with the value 1 coincide in at least one node?

Answer. See Example 4.6 in [1]

[1] B. Savchynskyy "Discrete graphical models – An optimization perspective" □

0.2 Max-weight independent set problem (MWIS)

Exercise 1.4 [Edge relaxation of MWIS is not tight in general] Consider the max-weight independent set problem from the Section "Integer linear programs" of the lecture draft (Example 2.51):

$$\max_{\mathbf{x}\in\{0,1\}^{\mathcal{V}}} \langle \mathbf{c}, \mathbf{x} \rangle \qquad (2)$$
$$\text{s.t. } x_i + x_j \le 1\,,\ \{i,j\} \in \mathcal{E}\,.$$

The following proposition shows that the set of vertices of its natural LP relaxation includes fractional ones:

Proposition 0.1. Let the graph $\mathcal{G}$ be fully-connected, all costs positive, $c_j > 0$, $j \in \mathcal{V}$, and $c_i < \sum_{j\in\mathcal{V}\setminus\{i\}} c_j$ for $i = \operatorname{argmax}_{j\in\mathcal{V}} c_j$. Then the LP relaxation

$$\begin{aligned} &\max_{\mathbf{x}\in[0,1]^{\mathcal{V}}} \langle \mathbf{c}, \mathbf{x} \rangle && (3)\\ &\text{s.t. } x_i + x_j \leq 1\,,\ \{i,j\} \in \mathcal{E}\,. \end{aligned}$$

of the max-weight independent problem (2) has a unique solution $\underbrace{(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})}_{n}$.

Uniqueness of the solution implies that it corresponds to a vertex of the respective feasible set.

Prove the proposition by using the following

Lemma 0.2. All vertices if the feasible set of (2) are half-integral, *i.e.*, they are vectors with coordinates taking values in the set $\{0, 1, 1/2\}$.

Answer. Due to half-integrality, it is sufficient to consider only half-integer feasible solutions. Due to the cost non-negativity the feasible solution $\mathbf{y} = \underbrace{(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})}_{n}$ is the best among those with coordinates $1/2$ and 0 only, as it contains the maximal number of non-zero coordinates. The cost of this solution is $\langle \mathbf{c}, \mathbf{y} \rangle = \frac{1}{2}\sum_{j\in\mathcal{V}} c_j$.

Consider now feasible solutions that are at least one coordinate equal to 1. Since the graph is fully-connected, there are only $|\mathcal{V}|$ of them, *e.g.*, those of the form $x^i = (\underbrace{0, \dots, 0}_{i-1}, i, 0 \dots, 0)$ with the total costs equal to c_i, respectively. The condition of the proposition implies

$$c_i < \sum_{j\in\mathcal{V}\setminus\{i\}} c_j = \sum_{j\in\mathcal{V}} c_j - c_i \tag{4}$$

and, therefore, $c_i < \frac{1}{2}\sum_{j\in\mathcal{V}} c_j = \langle \mathbf{c}, \mathbf{y} \rangle$. □

0.3 Binary knapsack problem

Exercise 1.5 [LP relaxation of binary knapsack is not tight in general] Consider the binary knapsack problem described in the Section "Integer linear programs" of the lecture draft (Example 2.50):

$$\max_{x\in\{0,1\}^n} \langle c, x \rangle \tag{5}$$
$$\text{s.t. } \langle a, x \rangle \leq b\,. \tag{6}$$

Show that its natural LP relaxation

$$\max_{x\in[0,1]^n} \langle c, x \rangle \tag{7}$$
$$\text{s.t. } \langle a, x \rangle \leq b\,. \tag{8}$$

is not tight in general. Consider the case $a_i \leq b$ for $i \in [n]$.

Answer. Consider $n = 2$, $c_1 = 10$, $a_1 = 2$, $c_2 = 1$, $a_2 = 2$, $b = 3$. The LP solution of $(1, 0.5)$ with the total cost 10.5, the integer solution is $(1, 0)$ with the total cost 10. □

0.4 Integer solutions as vertices of the feasible set of the LP relaxation

Exercise 1.6 Show that any integer solution is a vertex if the feasible set of the LP relaxation for the labeling problem, MWIS and binary knapsack.

Answer. Let $\mathbf{x} \in \mathbb{R}^n$ be the integer solution. Consider $c_i = 1$, if $x_i = 1$ and $c_i = -1$ if $x_i = 0$. □