Assignment 6

Handout 27.05.2024 – Return 03.06.2024 – Discussion 06./07.06.2024

Exercise 1 [6 points]: More on limit cycles

1. Construct a 2D system that displays two limit cycles, one surrounding the other, and discuss their stability. *(1.5 points)*

2. Consider the following dynamical system with $x, y \in \mathbb{R}$:

$$\begin{aligned}\dot{x} &= y^2 - x\,,\\ \dot{y} &= y + x^2 + yx^3\,.\end{aligned}$$

Show that this system does *not* have a limit cycle. *(1.5 points)*

3. Consider the following dynamical system with $x, y \in \mathbb{R}$:

$$\begin{aligned}\dot{x} &= y - 8x^3\,,\\ \dot{y} &= 2y - 4x - 2y^3\,. \end{aligned} \tag{1}$$

Argue that $(0, 0)$ is the only fixed point and check that it is oscillatory. Then use Poincaré-Bendixson theorem to show that there is a limit cycle in the system. *(3 points)*
HINT: Do not make it complicated, as the confining region you may consider a sufficiently large rectangle.

Exercise 2 [4 points]: SIR model with demography

Consider the following variant of the SIR model:

$$\begin{aligned}\dot{S} &= \mu N - \beta I S - \mu S\,,\\ \dot{I} &= \beta I S - \gamma I - \mu I\,,\\ \dot{R} &= \gamma I - \mu R\,.\end{aligned}$$

1. Interpret the new terms. Argue why one can still consider only S and I, i.e. restrict the analysis to two equations. *(0.5 points)*

2. Show that the equations can be rescaled to *(1 point)*

$$\begin{aligned}\dot{S} &= -\beta I S + \mu(1 - S),\\ \dot{I} &= (\beta S - 1) I\,. \end{aligned} \tag{2}$$

3. Determine the fixpoints of Eqs. (2) and their stability and plot the phase space I vs. S as well as a typical trajectory for the infected population $I(t)$. *(2.5 points)*

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Exercise 3 [5 points]: Lorenz model

Motivated by the Rayleigh-Bénard hydrodynamic convection system, Lorenz proposed in 1963 his famous equations (where x, y, z are modes of the velocity and temperature field)

$$\begin{aligned} \dot{x} &= \sigma(y-x)\,, \\ \dot{y} &= Rx - y - xz\,, \\ \dot{z} &= xy - bz\,, \end{aligned} \tag{3}$$

and found for large R chaotic solutions. Here the parameters σ, R and b are all positive. $R \propto \Delta T$ is proportional to the temperature difference across the fluid layer and is used as the *control parameter*.

1. Determine the non-trivial fixed points. *(1 point)*
2. Study the stability of the trivial fixed point $(0, 0, 0)$, losing stability at a R_1^c. *(2 points)*
   HINT: Use the structure of the matrix.
3. Study the stability of (one of) the non-trivial fixed points.
   HINT: You will get a third order polynomial that is tricky to solve. However, to determine the instability, it is enough to look for the solution with vanishing real part, i.e. one can assume $\lambda = i\omega$ (why?). Using the independence of the real and imaginary parts then allows to determine the critical R_2^c. *(2 points)*