Assignment 10

Handout 24.06.2024 – Return 01.07.2024 – Discussion 04./05.07.2024

Exercise 1 [7 points]: Onset of convection in a Rayleigh-Bénard cell (part II)

On the last sheet we showed that the equations describing Rayleigh-Bénard convection can be written in non-dimensionalized form as

$$\begin{aligned} \partial_t \Delta\psi + [(\partial_z\psi)\partial_x - (\partial_x\psi)\partial_z)]\Delta\psi &= \mathrm{P}\left(-\partial_x\theta + \Delta^2\psi\right) && (1)\\ \partial_t\theta + [(\partial_z\psi)\partial_x - (\partial_x\psi)\partial_z]\theta &= \Delta\theta - \mathrm{Ra}\,\partial_x\psi\,, && (2) \end{aligned}$$

with $\mathrm{P} = \frac{\nu}{\chi}$ the so-called Prandtl number and $\mathrm{Ra} = \frac{\alpha g d^3}{\chi\nu}(T_1 - T_2)$ the Rayleigh number, which is the control parameter. Here $\psi(x,z)$ with $v_x = \partial_z\psi$ and $v_z = -\partial_x\psi$ is the stream function and $\theta(x,z)$ is the deviation from the linear temperature profile in the base state.

1. Linearize equations (1) and (2) and argue that the ansatz

   $$\theta(x,z,t) = \bar{\theta}\sin(\pi z)\sin(qx)e^{\sigma t}\,, \quad \psi(x,z,t) = \bar{\psi}\sin(\pi z)\cos(qx)e^{\sigma t}$$

   solves these linearized equations together with the boundary condition for the temperature (temperature T_1 at $z = 0$ and T_2 at $z = 1$) and $\partial_x\psi = 0$ and $\partial_z^2\psi = 0$ at $z = 0, 1$; note that the distance between the two plates, d, has been scaled away. Which boundary condition does the latter imply for the velocity field? *(2 points)*

2. Use the ansatz to calculate the linear growth rates $\sigma(q)$ via the eigenvalue problem. Determine the critical wave number, q_c, and the critical value Ra_c for the onset of convection. *(4 points)*
   HINT: Put $\sigma = 0$ and solve for $\mathrm{Ra}(q)$. Determine the minimum with respect to q (yielding q_c) and calculate $\mathrm{Ra}_c = \mathrm{Ra}(q_c)$.

3. Sketch the neutral curve $\mathrm{Ra}(q)$. Discuss and compare to the Swift Hohenberg model. *(1 point)*

Exercise 1 [8 points]: Pattern competition in 2D : stripes vs. squares

Consider the coupled homogeneous amplitude equations for a stripe in x-direction, $Ae^{iq_cx} + c.c.$, and a stripe in y-direction, $Be^{iq_cy} + c.c.$ (where $c.c.$ means complex conjugate):

$$\begin{aligned} \partial_t A &= \epsilon A - |A|^2 A - c|B|^2 A\,,\\ \partial_t B &= \epsilon B - |B|^2 B - c|A|^2 B\,. && (3) \end{aligned}$$

Here A and B are complex, ϵ and c are real.

1. Show that square solutions (orthogonal stripes with equal amplitude $|A| = |B|$) exist for $c > -1$ and determine their amplitude. *(1 point)*

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2. Show that a stripe solution with $A \neq 0$ and $B = 0$ becomes unstable against growth of B for $c < 1$. *(2 points)*

3. Show that, in turn, the square solution calculated in part 1 is stable for $c < 1$.
   HINT: You will get a 4×4 system of equations in $\delta A, \delta A^*, \delta B, \delta B^*$. To solve it, decouple it into two 2×2 systems by considering $C_+ = \delta A + \delta B$ and $C_- = \delta A - \delta B$. *(3 points)*

4. Now study the spatially inhomogeneous case, i.e. Eqs. (3) become

$$\partial_t A = \epsilon A - |A|^2 A - c|B|^2 A + \left(\partial_x - \frac{i}{2q_c}\partial_y^2\right)^2 A\,,$$
$$\partial_t B = \epsilon B - |B|^2 B - c|A|^2 B + \left(\partial_y - \frac{i}{2q_c}\partial_x^2\right)^2 B\,. \tag{4}$$

   For the case $c > 1$, determine the new instability border due to the instability to a stripe in the direction orthogonal to the existing stripe. (This is the *cross-roll* instability boundary mentioned in the lecture). *(2 points)*
   HINT: Consider $A = \sqrt{\epsilon - Q^2}$ and study the growth of $\delta B \propto e^{iq_1 x + iq_2 y}$ to get an instability border of the type $\epsilon = \alpha Q^2$.