Assignment 3

Handout 06.05.2024 – Return 13.05.2024 – Discussion 16./17.05.2024

Exercise 1 [8 points]: A mechanical example for the saddle-node bifurcation

Consider the mathematical pendulum (with mass m, length l) including friction (coefficient 2λ). In addition, at the suspension point of the pendulum, a small motor is acting, creating a torque fl (with f a force).

1. Motivate that the system is described by the equation

   $$\ddot{\theta} + 2\lambda\dot{\theta} + \frac{g}{l}\sin\theta - \frac{f}{ml} = 0.$$

   Make the equation dimensionless by rescaling t, λ and f to get *(2 points)*

   $$\ddot{\theta} + 2\mu\dot{\theta} + \sin\theta - \nu = 0.$$

2. Choose the dimensionless external force ν as the control parameter and determine the stationary solutions (how many are there?) as a function of ν. Perform a linear stability analysis of the stationary solutions. Consider the case of high friction and give the leading order eigenvalues. Plot the bifurcation diagram. *(4 points)*
   HINT: High friction simplifies the square root term for the eigenvalue pair.

3. Derive the evolution equation close to the bifurcation point by writing $\theta = \frac{\pi}{2} + \psi$. Find the normal form for the deviation ψ. *(2 points)*
   HINT: No need to make a formal expansion, just argue which terms should be small and can be neglected/expanded.

Exercise 2 [7 points]: Subcritical bifurcation and ubiquity of saddle-nodes

In the lecture we already discussed the subcritical pitchfork bifurcation given by

$$\dot{x} = \epsilon x + x^3 - x^5\,. \tag{1}$$

1. Introduce a potential such that $\dot{x} = -\frac{dV}{dx}$. Use it to determine the stationary solutions analytically. Depending on the value of ϵ, there are three different cases. Sketch the potential for these cases and finally the bifurcation diagram. *(4 points)*

2. You will find that the branches bifurcating backwards from $x = 0$ at $\epsilon = 0$ turn forwards again at certain points (x^T, ϵ^T). Show that the normal form in the neighborhood of such a "turning point" T is the one of a saddle-node bifurcation. *(3 points)*
   HINT: Expand Eq. (1) around this point to the first non-vanishing order in deviations from x and ϵ.