Assignment 5

Handout 20.05.2024 – Return 27.05.2024 – Discussion 30./31.05.2024

Exercise 1 [7 points]: Lindstedt method for the Van der Pol oscillator

Consider the Van der Pol oscillator

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0\,. \tag{1}$$

1. Give arguments why this equation should display self-sustained oscillations. *(1 point)*

2. Perform a Lindstedt expansion via $\tau = \Omega t$ (note that $\Omega_0 = 1$ here). Apart from the trivial oscillator equation in order $\mathcal{O}(\epsilon^0)$, in the next order you should get

   $$\mathcal{O}(\epsilon^1)\ :\ \ddot{x}_1 + x_1 = -2\Omega_1\ddot{x}_0 + (1 - x_0^2)\dot{x}_0\,.$$

   Solve for the initial condition $\dot{x}(0) = 0$. You will get two secular terms (why?). Draw your conclusions from the periodicity condition $x(\tau) = x(\tau + 2\pi)$. *(3 points)*
   HINT: $\sin(3\tau) = 3\sin\tau - 4\sin^3\tau$

3. What is the zero order trajectory of the limit cycle? Calculate also $x(\tau)$ to order ϵ (note: one parameter stays undetermined, it can be fixed by considering the order ϵ^2 only). *(2 points)*

4. Calculate the time average of the dissipative term in Eq. (1) to zero order. What does one learn from it? *(1 point)*

Exercise 2 [4 points]: Van der Pol oscillator: relaxation oscillations

In the limit of large ϵ, the Van der Pol oscillator displays strongly anharmonic oscillations. The Lindstedt perturbation approach (cf. previous problem) is then not useful anymore, but the following approach can be used to rationalize the behavior:

1. Show that the Van der Pol oscillator,

   $$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0$$

   is equivalent to the 2D system

   $$\begin{aligned} \dot{x} &= \epsilon\left(y - f(x)\right)\,, \\ \dot{y} &= -x/\epsilon\,. \end{aligned} \tag{2}$$

   and calculate $f(x)$. *(2 points)*

2. Plot $y = f(x)$, which is the nullcline for x, and use it to rationalize the trajectory of the limit cycle for large ϵ. *(2 points)*
   HINT: Discuss Eqs. (2) (and their ratio) in the limit of large ϵ.

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Exercise 3 [4 points]: SIR model

A famous model for the spread of an infectious desease, employed extensively during the Corona crisis, is the so-called SIR model given by

$$\begin{aligned} \dot{S} &= -\beta IS\,, \\ \dot{I} &= +\beta IS - \gamma I\,, \\ \dot{R} &= +\gamma I\,. \end{aligned} \tag{3}$$

Here S are those people that are uninfected yet, but "susceptible" to getting infected; I are the "infected" people; and R are "removed" (depending on the interpretation/the desease they are "recovered" and immune, or dead).

1. Interpret β and γ (note: they have different units). Then convince yourself that the total number of the population $N = S + I + R$ is constant and that therefore one can restrict the discussion to the two equations for S and I. *(1 point)*

2. Consider the equation for I and find the value of S when I is maximum, $I_{max}(S)$. Assuming the scenario of an outbreak, $S(t = 0) \simeq N$ and very few infected people, sketch a trajectory in the diagram I vs. S. Use the maximum condition to get the dimensionless threshold for the epidemic outbreak, defining the "basic reproduction rate" or $\mathcal{R}$-value.
   Obtain the $\mathcal{R}$-value by studying the growth rate of I.
   Sketch the curves $S(t)$, $I(t)$, $R(t)$. *(2 points)*

3. Find I_{max} as a function of the $\mathcal{R}$-value and N. *(1 point)*
   HINT: Consider the ratio $\dot{I}/\dot{S}$.