Assignment 7

Handout 03.06.2024 – Return 10.06.2024 – Discussion 13./14.06.2024

Exercise 1 [6 points]: On chaos: Lorenz and Rössler attractors

1. Consider again the Lorenz model

$$\begin{aligned} \dot{x} &= \sigma(y-x)\,, \\ \dot{y} &= Rx - y - xz\,, \\ \dot{z} &= xy - bz\,. \end{aligned} \tag{1}$$

   Last time we found that for $R > R_2^c$ there are no fixed points anymore. What does the system do there? To get an idea, consider a sphere in phase space of radius ρ around $(0, 0, R + \sigma)$, i.e.

$$x^2 + y^2 + (z - (R+\sigma))^2 = \rho^2\,.$$

   Take the time derivative to get an equation for the change in radius of the considered phase space sphere. Insert the equations of motion, Eqs. (1), to get a closed equation. Study the case of large and small (x, y, z) and discuss. *(2 points)*

2. Make a plot (using python, mathematica etc.) of the Lorenz attractor. Typical parameters are $\sigma = 10$ and $\beta = 8/3$. *(1 point)*

3. For the Lorenz model, the shape of the attractor is quite complicated. Rössler proposed a simpler model (which however does not correspond to any physical system), that also displays chaos:

$$\begin{aligned} \dot{x} &= -y - z\,, \\ \dot{y} &= x + ay\,, \\ \dot{z} &= b + (x-c)z\,. \end{aligned} \tag{2}$$

   Make a plot (using python, mathematica etc.) of the Rössler attractor. Typical parameters are $a = 0.1$ and $b = 0.1$, $c = 18$. Varying c, can you also find periodic solutions? *(2 points)*

4. The behavior can be understood as follows: consider a trajectory that starts in the x-y-plane, i.e. $z \simeq 0$. What is the resulting dynamics and how does it relate to the attractor? *(1 point)*

Exercise 2 [5 points]: Bifurcations in the discrete logistic map

Consider the discrete logistic map

$$x_{n+1} = rx_n(1 - x_n) = f(x_n). \tag{3}$$

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1. Determine the fixed points, i.e. the $x^\star$ with $x^\star_{n+1} = x^\star_n = x^\star$, of this equation and their stability. The stability is given by $|f'(x^\star)| < 1$ (why?). Explain graphically (plotting x_{n+1} over x_n, $f(x_n)$ and a few steps in n) how the nontrivial fixed point arises. *(2 points)*

2. You will find that the nontrivial fixed point loses stability at $r = 3$, but no other fixed points are around. To understand what happens, consider "period-2 solutions" which fulfill $x^\star_{n+2} = x^\star_n$. Calculate these new solutions. *(2 points)*
   HINT: Use the fact that the fixed points from above obviously are also period-2 solutions.

3. Can you guess/calculate the stability of the period-2 solutions? Plot all (so far) obtained stable solutions as a function of r. *(1 point)*

Exercise 3 [4 points]: Swift-Hohenberg model – warm up

Consider the Swift-Hohenberg (SH) model discussed in the lecture, where $u(x,t)$ obeys periodic BCs:

$$\partial_t u = \left[\epsilon - (q_0^2 + \partial_x^2)^2\right] u - u^3 \,. \tag{4}$$

1. From the linear growth rates $\sigma(q)$ of small perturbations $u \propto e^{iqx+\sigma t}$ determine the so-called neutral curve $\epsilon(q)$, i.e. the curve defined by the condition of neutral stability, $\sigma(q,\epsilon) = 0$. Show that close to q_0, the neutral curve is a parabola. *(1 point)*

2. Nonlinear stationary solutions of the SH model can be obtained approximately by the following procedure, called Galerkin method: insert the ansatz $u(x) = A\cos(qx)$ into Eq. (4). The problem obviously is that the cubic nonlinearity leads to higher modes (via trigonometric identities). One then restricts the analysis to the subspace of the studied mode by projecting on it: multiply the obtained equation by $\cos(qx)$ and integrate over one period $\int_0^{2\pi/q} dx$. What is the obtained closed equation for A? *(2 points)*

3. Discuss the result: Which wave numbers q are allowed? How does the amplitude of these solutions behave in the region of existence (cf. the neutral curve obtained above)? What kind of bifurcation does the SH model show? *(1 point)*