# Assignment 9

Handout 17.06.2024 – Return 24.06.2024 – Discussion 27./28.06.2024

### Exercise 1 [5 points]: Eckhaus instability for higher nonlinearities

In the lecture we discussed the stability region of stripe patterns, which above the neutral curve is bounded by several instability curves, in 1D just by the Eckhaus instability. This result is general for pitchfork bifurcations close to threshold, where the nonlinearity is cubic. In other systems or further away from the threshold, other nonlinearities may come into play. Here we exemplifiy how the stability region changes via nonlinearities.
Consider the following amplitude equation

$$\partial_t A = \epsilon A + \partial_x^2 A - |A|^{2n} A \tag{1}$$

for $n = 2$. Calculate the amplitude $F$ of stationary stripe patterns, $A = Fe^{iQx}$. Then investigate their stability via the ansatz

$$A = e^{iQx}\left(F + w_1 + w_2^*\right) = e^{iQx}\left(F + v_1 e^{\sigma t + ikx} + v_2^* e^{\sigma^* t - ikx}\right) .$$

How does the region of stable stripes change for $n = 2$ compared to $n = 1$ (pitchfork) ?

BONUS QUESTION: By considering Eq. (1) with $|A|^2 A$, then $|A|^4 A$, etc. you can figure out the general expression for the nonlinearity $|A|^{2n} A$, $n \geq 1$. If you study this general case, you get **4 extra points**.

### Exercise 2 [4 points]: Inhomogeneous solutions of the Ginzburg-Landau equation

Consider the Ginzburg-Landau (GL) equation, for simplicity with real-valued amplitude $A$,

$$\partial_t A = \epsilon A + \xi_0^2 \partial_x^2 A - gA^3 \, . \tag{2}$$

1. Show explicitly that the GL equation can be derived from a "potential", i.e. there is a functional $F = \int f(A, \partial_x A) dx$ with

$$\partial_t A = -\frac{\delta F}{\delta A},$$

   where the r.h.s. is the functional derivative that you know from analytical mechanics (cf. Euler-Lagrange equation). Argue from the obtained potential why one expects – except for the known homogeneous solutions $A_1 = 0$ and $A_{2,3} = \pm\sqrt{\frac{\epsilon}{g}}$ – also inhomogeneous solutions that dscribe "domains" (so-called kink-solutions). *(1.5 points)*

2. Calculate a stationary inhomogeneous solution that fulfills $A(x) = A_3$ at $x = -\infty$ and $A(x) = A_2$ at $x = +\infty$. Argue what is the meaning of $\xi_0$ in Eq. (2). *(2.5 points)*
   HINT: Look for solutions of tanh-type and note that $\frac{\partial}{\partial_x} \tanh(x) = 1 - \tanh^2(x)$.

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**Exercise 3 [6 points]: Onset of convection in a Rayleigh-Bénard cell (part I)**

An incompressible fluid (i.e. for which $\nabla \cdot \mathbf{v} = 0$ holds) heated from below is governed by the following equations, if the so-called Boussinesq approximation has been applied:

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} &= -\frac{1}{\rho_0}\nabla p + \nu\Delta\mathbf{v} - \frac{\rho(T)}{\rho_0} g\hat{z}\,, & (3) \\ \frac{\partial}{\partial t}T + (\mathbf{v}\cdot\nabla)T &= \chi\Delta T\,. & (4) \end{aligned}$$

Here the first equation is the Navier-Stokes equation, where $\rho(T) = \rho_0[1 - \alpha(T - T_0)]$ with $\alpha$ the thermal expansion coefficient describes that hotter fluid expands and hence has smaller density $\rho$. $\hat{z}$ is the unit vector in $z$-direction, where gravity $g$ acts. The second equation describes heat diffusion. The fluid is sandwiched between two parallel plates (of distance $d$). The lower one (at $z = 0$) is kept at a temperature $T_1$ and the upper one (at $z = d$) at $T_2 < T_1$.

1. What is the steady base state of the system for $T_1 \neq T_2$? *(1 point)*

2. Non-dimensionalize the equations by introducing the distance $d$ as the typical length scale, $\chi/d$ as the typical velocity scale and $d^2/\chi$ as the typical time scale. *(2 points)*

3. Assume that the system is translationally invariant in $y$-direction (i.e. you can ignore $v_y$). Introduce the stream function $\psi(x, z)$ with $v_x = \partial_z\psi$ and $v_z = -\partial_x\psi$ and use the $x$ and $z$-component of equation (3) to get a single equation for $\psi$.

   For the temperature equation, Eq. (4), split off the base state via $T = T_{\text{base}} + \theta(x, z, t)$.

   Show that then you arrive at the following (non-dimensionalized) system of equations:

$$\begin{aligned} \partial_t\Delta\psi + [(\partial_z\psi)\partial_x - (\partial_x\psi)\partial_z)]\Delta\psi &= \mathrm{P}\left(-\partial_x\theta + \Delta^2\psi\right)\,, & (5) \\ \partial_t\theta + [(\partial_z\psi)\partial_x - (\partial_x\psi)\partial_z]\theta &= \Delta\theta - \mathrm{Ra}\,\partial_x\psi\,, & (6) \end{aligned}$$

   with $\mathrm{P} = \frac{\nu}{\chi}$ the so-called Prandtl number and $\mathrm{Ra} = \frac{\alpha g d^3}{\chi\nu}(T_1 - T_2)$ the Rayleigh number, which is the control parameter. *(3 points)*