Institute for Theoretical Physics | Heidelberg University
PD Dr. Falko Ziebert
Nonlinear dynamics | Summer term 2024

Assignment 4

Handout 13.05.2024 – Return 22.05.2024 – Discussion 23./24.05.2018

Exercise 1 [6 points]: Lotka-Volterra Model

Consider the following predator-prey model (proposed by Lotka and Volterra): given is a habitat with two species. The prey animals $x(t)$ are vegetarians, food is abundantly present. The second species, the predators $y(t)$, lives upon the first species.

1. Interpret the model equations

$$\dot{x} = ax - bxy\,, \;\; \dot{y} = -dy + cxy\,,$$

where $a, b, c, d > 0$. *(0.5 points)*

2. Determine the fixed points and their stability. *(1.5 points)*

3. For the harmonic oscillator, $\dot{x} = y\,, \;\; \dot{y} = -x\,$, we know that the quantity $\phi(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ is a constant of motion. Show that, quite similarly, $\phi(x, y) = -a \ln y + by - d \ln x + cx$ is a constant of motion for the Lotka-Volterra model. Discuss the behavior around the nontrivial fixpoint. Plot the trajectories and the flow field with *python*. Does the model display a stable limit cycle? *(2 points)*

4. Derive $\phi(x, y)$ from the equations of motion. *(2 points)*

Exercise 2 [5 points]: A stable limit cycle

Consider the following dynamical system:

$$\begin{aligned}
\dot{x} &= -y + \frac{x}{\sqrt{x^2 + y^2}}(1 - x^2 - y^2)\,, \\
\dot{y} &= x + \frac{y}{\sqrt{x^2 + y^2}}(1 - x^2 - y^2)\,.
\end{aligned}$$

1. Transform to polar coordinates ($x = r\cos\theta, y = r\sin\theta$). *(2 points)*
   HINT: Consider $x\dot{x} + y\dot{y}$ and $x\dot{y} - y\dot{x}$.

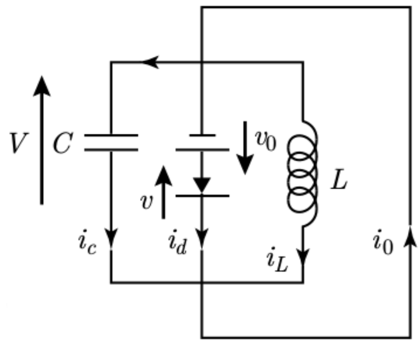
2. Solve analytically for $r(t)$. *(2 points)*
   HINT: Separation of variables and partial fraction decomposition.

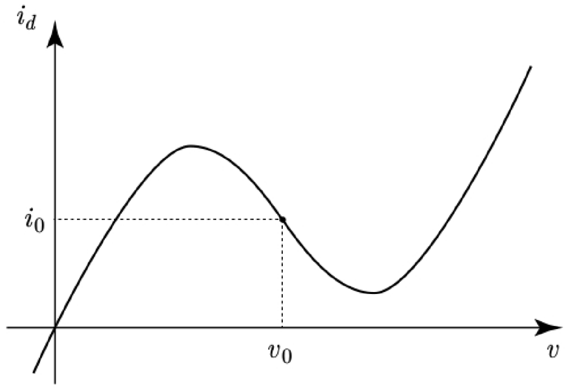
3. Sketch the three possible types of trajectories. Alternatively use *python* to plot both the trajectories and the flow field. *(1 point)*

– turn page –

Exercise 3 [4 points]: Electrical circuit for the Van der Pol oscillator

Consider the circuit shown below, i.e. a parallel circuit of a capacitor, an inductance and of an element with a tunnel diode and a generator (voltage v_0) in series. Unlike a classical, ideal diode where the current is blocked in one direction, the tunnel diode allows the current to flow in both directions, but with a nonlinear characteristics $i_d(v)$ as also shown below.





1. From current conservation (Kirchhoff's rule), derive the equation for the voltage $V = v - v_0$.
   HINT: Expand the given shape $i_d(v)$ up to third order around (i_0, v_0). *(2 points)*

2. Rescale and show that you get the so-called Van der Pol oscillator, i.e.

$$\ddot{x} - (\epsilon - x^2)\dot{x} + x = 0\,. \tag{1}$$

   What does ϵ correspond to (i.e. how is it related to the electric parameters)? *(2 points)*
   NOTE: You can find an argument why Eq. (1) is identical to the version of the Van der Pol equation analyzed (for convenience) in the next problems, i.e. with ϵ *outside* the bracket.