Optimal Transport and Applications
(SoSe 2025)
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Exercise Sheet 1
Submission deadline: 08.05.2025

General Remarks

- Every two weeks, a new exercise sheet is released on Moodle. The submission deadline is 1400hrs on each respective date mentioned on the sheet (upload via Moodle).
- 50% of the available points and presentation of at least one solution in the tutorial are required for admission to the exam.
- Working together in groups of up to three people is allowed and encouraged. One upload per group is sufficient, but group members must be clearly specified on each submission: – by using last name(s) as the file name.
  - **File and naming conventions:** Please upload a single PDF for the theoretical exercises and a single ZIP file for the programming exercise, both with `last1last2...` as the basename. Further requirements will be communicated in the tutorial as needed.
- **Note on self-graded assignments:** tba with @nikolai
- Please note that we may selectively check additional solutions of your group if we notice strong discrepancies between your answers and the graded points.
- Contact Persons: `purusharth.saxena`[1], `nikolai.koehler`[2]

Exercise 1.1 [2pt]

A bakery supply chain consists of two bakeries and two customers. The supply available at the bakeries is given by:

¡change to real reasonable bakery stuff¿

$$\alpha = (0.4, 0.6)$$

where the first bakery has 0.4 units of supply and the second bakery has 0.6 units. The demand at the customers is:

$$\beta = (0.5, 0.5)$$

where each customer requires 0.5 units of bread.

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Determine a feasible transportation plan r_{ij} (a transport matrix) that satisfies the following supply and demand constraints:

- Each bakery can only supply up to its available amount.
- Each customer must receive the exact amount they demand.

Find a feasible transportation plan r_{ij} that satisfies these conditions. Also show that it satisfies all the given constraints.

Exercise 1.2 `[4pt]`

Find two probability measures $\mu, \nu \in \mathcal{P}(\mathbb{R})$ such that there exists a map (or a plan) from μ to ν that is optimal for the linear cost

$$c(x, y) \;=\; |x - y|$$

but not for the quadratic cost

$$c(x, y) \;=\; \tfrac{1}{2}\,|x - y|^2.$$

Exercise 1.3 `[3pt]`

Find an example of two probability measures on $\mathbb{R}^d$ for some $d > 0$ such that there is more than one optimal transport map with respect to the quadratic cost, pushing one into the other.

Exercise 1.4 `[5pt]`

Derive the optimal formula for P in the entropy-regularized optimal transport problem defined in Equation (1). **Hints:** Rewrite the marginal constraints using Lagrange multipliers.

Try to arrive at a solution that does *not* depend on the marginals a and b themselves, i.e. only depend on C, ϵ, and the Lagrange multipliers.

$$\begin{aligned} \arg\min_{P} \quad & \langle P, C\rangle + \epsilon\langle P, \log P\rangle \\ s.t. \quad & P\mathbb{1} = a \\ & P^T\mathbb{1} = b \end{aligned} \tag{1}$$

Exercise 1.5 `[6pt]`

For one-dimensional optimal transport problems, the computational solutions turns out to be surprisingly simple.

Starting with sets u and v and their corresponding probability weights w_u and w_v (corresponding to the marginals), we can simply sort them and assign always assign the larger amount of weight to be transported between two the current indices in the transport matrix P. The pseudocode for this procedure is given below. Note, that it requires the sum of the marginals w_u and w_v to be equal.

Algorithm 1 1D OT Solver

$i \leftarrow 0$; $j \leftarrow 0$
$w_i \leftarrow w_i^u$, $w_j \leftarrow w_j^v$
$P \leftarrow [0]_{n \times m}$ ▷ $n \times m$ coupling matrix
if $w_i < w_j$ **or** $j == (|v| - 1)$ **then**
  $P_{ij} \leftarrow w_i$
  $i++$
  if $i == |u|$ **then**
    return P ▷ Reached the end of u
  end if
  $w_j \leftarrow w_j - w_i$ ▷ Update remaining weight
  $w_i \leftarrow w_i^u$ ▷ Set "new" weight for i
else ▷ Analogous to if-case
  $P_{ij} \leftarrow w_j$
  $j++$
  if $j == |v|$ **then**
    return P
  end if
  $w_i \leftarrow w_i - w_j$
  $w_j \leftarrow w_j^v$
end if

Implement this procedure in the code framework provided in `ot_solver.py`. Make sure that you read the docstring and add your implementation inside the function within the provided API.

To check whether your implementation is correct, we provide a few tests in the main section that you can run by calling `python ot_solver.py`. **Important Note:** You will *not* be graded solely based on whether these tests pass!

Exercise 1.6 (Bonus) `[5pt]`

- Let $T : X \to Y$ satisfy $T_{\#\mu} = \nu$. Consider the map $I_d \times T : X \to Y, i.e, x \longmapsto (x, T(x))$ and define

$$\gamma_T := (I_d \times T)_{\#\mu} \in \mathcal{P}(X \times Y)$$

  Show that $\gamma_T \in \Gamma(\mu, \nu)$
  Hint: Start with $\gamma_T := (I_d, T)_{\#\mu}$

- Show that γ_T belongs to the set $\Pi_{(\mu,\nu)}$ iff T satisfies

$$\min\{\int c(x,T(x))d\mu(x) : T_{\#\mu} = \nu\}$$