Time Series Models: From Statistics to AI

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Exercise Sheet 2

Regulations

Please submit your solutions via Moodle in teams of **2 students**, before the exercise group on **Wednesday, May 7th, 2025**. Each submission must include **exactly** two files:

- A `.pdf` file containing both your Jupyter notebook and solutions to analytical exercises. The Jupyter notebook can be exported to pdf by selecting **File → Download as → pdf** in JupyterLab. If this method does not work, you may print the notebook as a pdf instead. Your analytical solutions can be either scanned handwritten solutions or created using LaTeX.
- A `.ipynb` file containing your code as Jupyter notebook.

Both files must follow the naming convention:
`Lastname1-Lastname2-sheet02.pdf`
`Lastname1-Lastname2-sheet02.ipynb`

Task 1. Univariate AR models
In file `ex2file1.mat`, you will find four time-series obtained from human fMRI recordings from the dorsolateral prefrontal cortex (DLPFC) and the parietal cortex (Parietal), obtained during a working memory task. For task 1, consider the first time series of DLPFC (termed `'DLPFC1'`).

1. Write down the formula for an AR(4) model and explain how it can be rewritten as a VAR(1) model. Using this, how would you determine an estimate for the parameters $\{a_i\}_{0:4}$? Write down as a formula and compute numerically.

2. Compute the log-likelihood of an AR(4) model. Please write the derivation down explicitly.

3. Plot the residuals of the model in a histogram. What do they look like? What do you expect?

4. Compute the log-likelihood of an AR(n) model, with n ranging from 1...5. How does the likelihood change when you increase the order of the model?

5. Increasing the capacity of the model is likely to increase its explanatory power, but it is important to explore the tradeoff between this and the increase in model parameters. Determine the optimal order p of the AR model by computing the log-likelihood-ratio test statistic. Start with Wilk's D, which simplifies here to

$$D = -2[\log \Sigma_p - \log \Sigma_{p-1}]$$

   Plug D into a χ^2 distribution with appropriate degrees of freedom (`scipy.stats.chi2.cdf()` in Python). Start with the comparison between an AR(2) vs. AR(1) model, and then keep on computing log-likelihood-ratio test statistics for models of consecutive orders up to order $p = 5$, estimating what the best order model is.

Task 2. Multivariate (vector) AR (=VAR) processes
For this task, use all four time series contained in the data file (`'DLPFC1'`, `'DLPFC2'`, `'Parietal1'`, `'Parietal2'`).

1. Estimate a VAR(1) model by performing multivariate regression on the 4-variate time series. What do the coefficients in matrix A tell you about the coupling between the DLPFC and parietal cortex? Is the resulting VAR(1) model stationary or not?

Task 3. AR Poisson processes

1. Recall the equations for a Poisson process. Write them down and explain the reason behind the use of the link function, the logarithm.

2. Create your own second order Poisson time series with $T = 1000$ time steps and the following given parameters:

$$A_1 = \begin{pmatrix} 0.2 & -0.1 \\ 0.1 & 0.1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.1 & -0.1 \\ 0.1 & 0.1 \end{pmatrix}, \quad \mu_0 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \quad a_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

3. Now we want to think about the log-likelihood landscape (log likelihood dependent on more than one parameter). What form do you expect theoretically? Why can it be important to know the form of this landscape?

4. Given the data generated in (a), vary the parameters $A_1(1,1)$ and $A_2(2,1)$ between 0 and 0.4 with 0.01 increments. For each parameter value pair, compute the log-likelihood of the data (keeping all other parameters fixed!). Plot the log-likelihood landscape surface as a function of these two parameters. Does the real parameter pair value correspond (or is close) to an extreme point in the approximate log-likelihood landscape? What kind of an extreme point is it?