4)

$Ax = b \qquad A \in \mathbb{R}^{m \times n} \qquad b \in \mathbb{R}^m$

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\therefore Ax = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$

that is lineare combination of columns

So if $b \in$ Column space of A, then a solution exists.

If, $a_1, \ldots, a_n$ form a basis of $\mathbb{R}^m$, then b can be written as a unique linear combination of a_i. So solution will then be unique.

If $b \notin$ Column Space, No solution exists.

Let the rank of A be r,

So there are r linearly independent columns,

Let. $a_1, \ldots, a_r$ be independent

$\therefore a_{r+1}, \ldots, a_n$ can be expressed in terms of $a_1, \ldots, a_r$

If b can be expressed as linear combination of $\{a_1, \ldots, a_n\} \Rightarrow b$ can be expressed as a linear combination of $\{a_1, \ldots a_r\}$

$\therefore$ Rank $\{a_1, a_2, \ldots, a_n, b\}$ must be r,

$\therefore$ Rank $\{a_1, a_2, \ldots, a_n\} =$ Rank $\{a_1, a_2, \ldots, a_n, b\}$

for solution to exist

Here Rank $\{\ldots\}$ denotes max number of L.I. vectors

$\therefore$ Rank$(A \mid b)$ = Rank(A) for solution to exist.
(A augmented with b)

Now suppose solution exists and the above rank $= r$

$\therefore$ $r \le n$ and $r \le m$

Case I: For $r < n$

$\therefore$ $a_1, a_2, \ldots, a_r$ are L.I.

So b can be represented uniquely in terms of $\{a_1, \ldots, a_r\}$

However considering $\{a_1, \ldots, a_n\}$

Let $b = x_1 a_1 + \cdots + x_r a_r + 0 \cdot a_{r+1} + \cdots + 0 \cdot a_n$

This is possible since, $a_1, \ldots, a_r$ are L.I.,

so $x_1 \ldots x_r$ are unique

$x_{r+1} = \cdots = x_n = 0$

Consider $Ax = 0$, $\because$ $a_1, \ldots, a_n$ are not LI, This will have infinite sets of solutions with dimension of solution set $= n - r$

[By Rank Nullity Theorem]

If

$Ax = b$ $\quad A(x + \alpha y) = b$

where $\underline{Ay = 0}$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}$$

So solution

= particular sol^n + general sol^n

$= x + \alpha$

Where $y \in Y$ (Solution set of $Ay = 0$, with $\dim = n - r$)

Case 2 If $r = n$, $\{a_1, \cdots, a_n\}$ are L.I
and b can be uniquely represented
in terms of $\{a_1, \cdots, a_n\}$

So unique Solution

Summarising, For $Ax = b$

If $\mathrm{Rank}(A|b) = \mathrm{Rank}(A) = r$,
Solution exists

If $r < n$, Infinite Sol^{n}s
with dim of Sol^n
space $= n - r$

Else Unique Sol^n

If $\mathrm{Rank}(A|b) \neq \mathrm{Rank}(A) \Rightarrow$ No Sol^n
exists.

Additional Observations,

1) $\because$ $r \leq n$ and $r \leq m$

$$\begin{bmatrix} & & \\ & & \end{bmatrix}_{m \times n} \begin{bmatrix} \\ \end{bmatrix}_{n \times 1} = \begin{bmatrix} \\ \end{bmatrix}_{m \times 1}$$

If solution exists,

for uniqueness, $r = n$
$\Rightarrow n \leq m$

So A should be square or tall

m; Number of equations
n: No. of variables

So No of varibles $\leq$ Number of equations
for unique solution to exist

2) Also $\Rightarrow$ If $n > m$ (Wide matrix)

$\Rightarrow$ Solution Cannot be unique,

In fact if $n > m$,

In this case if rows are LI , i.e $Rank(A) = m$,

then $Span(a_1, \dots, a_n) = \mathbb{R}^m$, & $b \in$ Column Space (A)

So solution exists, But not unique

Solution Space has dimension $n - m$