Anurat Bhattacharya Assign 1

Q.1) To prove that $P_n(x)$ is a real vector space

(i) With respect to addition '+'

a) Closure Let $P_1, P_2 \in P_n(x)$

$\therefore P_1 + P_2 = \sum_{i=0}^{n} (a_{1i} x^i) + \sum_{i=0}^{n} (a_{2i} x^i)$

$= \sum_{i=0}^{n} (a_{1i} + a_{2i}) x^i \in P_n(x)$

Hence Closure holds

b) Commutativity, Let $P_1, P_2 \in \mathbb{P}_n(x)$

$\therefore P_1 + P_2 = \sum_{i=0}^{n} (a_{1i} + a_{2i}) x^i = \sum_{i=0}^{n} (a_{2i} + a_{1i}) x^i = P_2 + P_1$

$\forall P_1, P_2 \in \mathbb{P}_n(x)$

So Commutative

c) Associativity, Let $P_1, P_2, P_3 \in \mathbb{P}_n(x)$

$\therefore (P_1 + P_2) + P_3 = \left(\sum_{i=0}^{n} (a_{1i} + a_{2i}) x^i \right) + \sum_{i=0}^{n} a_{3i} x^i$

$= \sum_{i=0}^{n} (a_{1i} + a_{2i} + a_{3i}) x^i = \sum_{i=0}^{n} a_{1i} x^i + \sum_{i=0}^{n} (a_{2i} + a_{3i}) x^i$

$= P_1 + (P_2 + P_3) \quad \forall$ such $P_1, P_2, P_3 \in \mathbb{P}_n(x)$

Hence Associativity holds

d) Additive Identity $= 0$

$\therefore$ For any $p_1 \in \mathbb{P}_n(x)$,

$p_1 + 0 = p_2 + 0 = p_2$, so Additive Identity exists

e) Additive Inverse : Let $p_1 \in \mathbb{P}_n(x)$,

$p_1 = \sum_{i=0}^{n} (a_{1i}) x^i$ Let $p_2 = \sum_{i=0}^{n} (-a_{1i}) x^i$

$\therefore$ $p_1 + p_2 = p_2 + p_1 = 0$ and $p_2 = -p_1$

Hence Additive Inverse exists.

With respect to scalar multiplication

a) Closure : $\alpha p_1 = \alpha \sum_{i=0}^{n} (a_{1i} x^i) = \sum_{i=0}^{n} (\alpha a_{1i} x^i)$

$\in \mathbb{P}_n(x)$ for any $p_1 \in \mathbb{P}_n(x)$

Hence closure holds

b) Left Distribution :

$(\alpha + \beta) p_1 = (\alpha + \beta) \sum_{i=0}^{n} (a_{1i} x^i)$

$= \alpha \sum_{i=0}^{n} a_{1i} x^i + \beta \sum_{i=0}^{n} \alpha_{1i} x^i = \alpha p_1 + \beta p_1$

Hence left distribution holds

c) Associativity:, For $p_1 \in \mathbb{P}_n(x)$,

$$(\alpha\beta)\, p_1 = \alpha\beta \sum_{i=0}^{n} (a_{1i}\, x^i) = \alpha \sum_{i=0}^{n} (\beta a_{1i}\, x^i)$$

$$= \alpha(\beta p_1)$$

Hence associativity holds.

d) Right Distribution, Let $p_1, p_2 \in \mathbb{P}_n(x)$

$$\therefore \alpha(p_1 + p_2) = \alpha \sum_{i=0}^{n} \left[a_{1i}\, x^i + a_{2i}\, x^i \right]$$

$$= \sum_{i=0}^{n} \alpha\, a_{1i}\, x^i + \sum_{i=0}^{n} \alpha\, a_{2i}\, x^i$$

$$= \alpha p_1 + \alpha p_2$$

Hence Right distributions holds.

e) Multiplicative Identity exists and $= 1$

$$1 \cdot p = p \cdot 1 = p \quad \forall\, p \in \mathbb{P}_n(x)$$

Hence $\mathbb{P}_n(x)$ is a vector space.

ii) $$F(p_1(x)) = \frac{d}{dx} p_1(x) \Big|_{x=0} = \frac{d}{dx} \left[\sum_{i=0}^{n} a_{1i}\, x^i \right] \Big|_{x=0}$$

for some $p_1 \in \mathbb{P}(x)$

$$= a_{11}$$

$\therefore F(p_1(x)) = a_{11}$

Let p_1 & $p_2 \in \mathbb{P}_n(x)$

$$\therefore F(p_1 + p_2) = \frac{d}{dx}\bigg|_{x=0} \left[\sum_{i=0}^{n} a_{1i} x^i + \sum_{i=0}^{n} a_{2i} x^i \right]$$

$$= a_{11} + a_{21} = F(p_1) + F(p_2)$$

$\therefore F(p_1 + p_2) = F(p_1) + F(p_2)$ holds

Again $F(\alpha p_1) = \frac{d}{dx}\bigg|_{x=0} \left[\alpha \sum_{i=1}^{n} a_{1i} x^i \right] = \alpha a_{11}$

$$= \alpha (a_{11})$$

$$= \alpha F(p_1) \quad \forall\, p_1 \in \mathbb{P}_n(x)$$

then $F(\alpha p_1) = \alpha F(p_1)$

Hence F is linear functional.

c) Let us represent some $p \in \mathbb{P}_n(x)$

as a vector $\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}$ a_i denoting coefficient of x^i

$\therefore F(p) = a_1$, Hence for $u = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}$,

$u^T p = F(p)$

[We select only the coefficient of x^1]