Q3)

$$\|x\|_w = \sqrt{\sum_{i=1}^{n} w_i^2 x_i^2}$$

To prove this defines a norm (weighted norm)

$$\text{Let } W = \begin{bmatrix} w_1 & 0 & \cdots \\ 0 & w_2 & \\ \vdots & & \ddots & w_n \end{bmatrix}$$

$\therefore \|x\|_w = \|Wx\|$ (Converting to equivalent Euclidean Norm)

Properties

(i) Non negative homogenity:

[By Prop of Euclidian norm

$$\|\alpha x\|_w = \|\alpha W x\| = |\alpha| \|Wx\| = |\alpha| \|x\|_w$$

Hence nonnegative homogenity holds

(ii) Δ Inequality:

$$\|x+y\|_w = \|W(x+y)\| = \|Wx + Wy\|$$

$$\Rightarrow \|Wx + Wy\| \leq \|Wx\| + \|Wy\| = \|x\|_w + \|y\|_w$$

$\therefore \|x+y\|_w \leq \|x\|_w + \|y\|_w$ holds

iii) Non negativity:

$\|x\|_w = \|Wx\| \geq 0 \Rightarrow \|x\|_w \geq 0$ holds

iv) Definiteness

$$\|x\|_W = 0 \Rightarrow \|Wx\| = 0 \Rightarrow Wx = 0$$

$$\Rightarrow \begin{bmatrix} w_1 x_1 \\ w_2 x_2 \\ \vdots \\ w_n x_n \end{bmatrix} = 0 \Rightarrow w_i x_i = 0 \quad \forall\, 1 \le i \le n$$

$$\Rightarrow x_i = 0 \quad \forall\, 1 \le i \le n \text{ since } w_i > 0$$

$$\Rightarrow x = 0_n$$

$\therefore$ $\|x\|_W = 0 \Rightarrow x = 0_n$ holds

So $\|\cdot\|_W$ defines a norm