ANUSHA FATIMA ALAM (Student # 1009056539)

MAT392 – Complex Analysis (Homework 1)

**Question 1:** Describe the locus of points $z$ satisfying the equation $|z-4| = 4z$

Let $z = x+iy$ ; $\text{Re}(z) = x$ and $\text{Im}(z) = y$

$$\left[\,|z|^2 - 8\,\text{Re}(z) + 16\right] = 16|z|^2$$

$$15|z|^2 + 8\,\text{Re}(z) - 16 = 0 \longrightarrow \text{Substitute } \text{Re}(z) = x$$

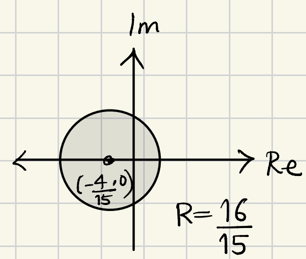
$$15|z|^2 + 8x = 16 \longrightarrow \text{Substitute } |z|^2 = x^2 + y^2$$

$$15(x^2+y^2) + 8x = 16$$

$$15x^2 + 8x + 15y^2 = 16$$

$$15\left(x^2 + \frac{8}{15}x + \frac{16}{225}\right) + 15y^2 = 16 + \frac{16}{15}$$

$$15\left(x + \frac{4}{15}\right)^2 + 15y^2 = \frac{256}{15}$$

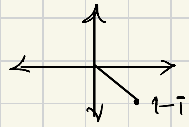



Locus: Circle given by $\left(x + \frac{4}{15}\right)^2 + y^2 = \frac{256}{225}$

This is a circle of $r = \frac{16}{15}$ centered at $-\frac{4}{15} + i(0)$

Question 2 – Find all solutions of the equation $(z+1)^4 = 1-i$

$$(z+1)^2(z+1)^2 = 1-i \qquad |r| = \sqrt{1^2+(-1)^2} = \sqrt{2}$$




$$\psi = \arctan\left(\frac{-1}{1}\right) = \arctan(-1) = -\frac{\pi}{4}$$

$$\theta_K = \frac{\psi}{n} + \left(\frac{2\pi}{n}\right)K \text{ , where } K = 0,1,2,\dots n-1.$$ n in this case equals 4

$$\theta_K = \frac{\left(-\frac{\pi}{4}\right)}{4} + \frac{2\pi}{4}K = \left(-\frac{\pi}{16} + \frac{\pi}{2}K\right) \quad \text{for } K = 0,1,2,3$$

$$(z+1)^4 = \sqrt{2}\,(\cos(\theta_K) + i\sin(\theta_K))$$

$$z+1 = (\sqrt{2})^{1/4}\,(\cos(\psi) + i\sin(\psi))^{1/4}$$

General Solution.

$$\boxed{Z_K = \left[2^{1/8}\left(\cos\left(-\frac{\pi}{16} + \frac{\pi}{2}K\right) + i\sin\left(-\frac{\pi}{16} + \frac{\pi}{2}K\right)\right)\right] - 1 \qquad \text{for } K = 0,1,2,3 \dots}$$

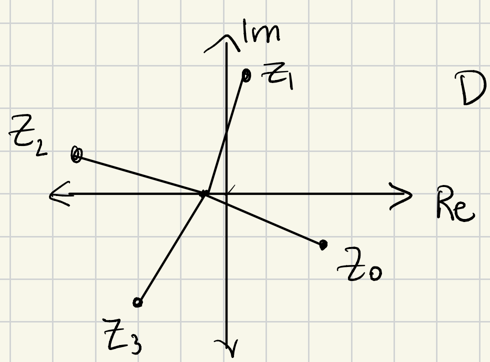
4 Exact Solutions in the domain $[0, 2\pi]$:

$$Z_0 = 2^{1/8}\left(\cos\left(-\frac{\pi}{16}\right) + i\sin\left(-\frac{\pi}{16}\right)\right) - 1$$
$$= 1.0695539 - i\,0.212748 - 1 = 0.0695539 - i\,0.212748$$

$$Z_1 = 2^{1/8}\left(\cos\left(-\frac{\pi}{16} + \frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{16} + \frac{\pi}{2}\right)\right) - 1$$
$$= 0.212748 + i\,1.069554 - 1 = -0.787252 + i\,1.069554$$

$$Z_2 = 2^{1/8}\left(\cos\left(-\frac{\pi}{16} + \pi\right) + i\sin\left(-\frac{\pi}{16} + \pi\right)\right) - 1$$
$$= -1.0695539 + i\,0.212748 - 1 = -2.0695539 + i\,0.212748$$

$$Z_3 = 2^{1/8}\left(\cos\left(-\frac{\pi}{16} + \frac{3\pi}{2}\right) + i\sin\left(-\frac{\pi}{16} + \frac{3\pi}{2}\right)\right) - 1$$
$$= -0.212748 - i\,1.0695539 - 1 = -1.212748 - i\,1.0695539$$

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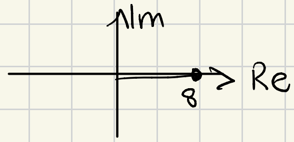
Question 3 - Find all solutions of the equation $z^3 = 8$

$z^3 = 8$




$\psi = 0°$ $\qquad |r| = \sqrt{8^2} = 8$

$\theta_k = \frac{\psi}{n} + \frac{2\pi k}{n}$ for $k = 0, 1, 2, \dots n-1$.

In this case $n = 3$ ; $\theta_k = 0, \frac{2\pi}{3} + \frac{4\pi}{3}$

$z^3 = 8^{1/3}\left[\cos(\psi) + i\sin(\psi)\right]^{1/3}$

$$\boxed{z_k = 2\left[\cos\left(\frac{2\pi k}{3}\right) + i\sin\left(\frac{2\pi k}{3}\right)\right] \text{ for } k = 0, 1, 2}$$

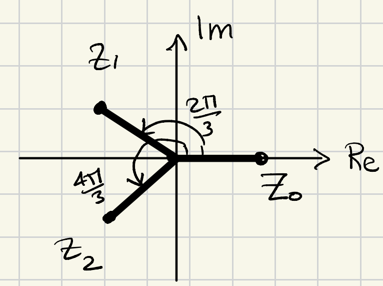
General Solution

3 Exact Solutions (in the domain $[0; 2\pi]$):

$z_0 = 2\left[\cos(0) + i\sin(0)\right] = 2 + i(0)$

$z_1 = 2\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] = -1 + i\,1.732$

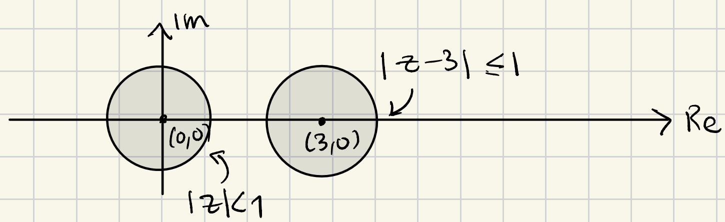
$z_2 = 2\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right] = -1 - i\,1.732$




Drawn not to scale.

Question 4 – For the following set, describe (i) the interior and the boundary, (ii) state whether the set is open, or closed, or neither open nor closed, (iii) state whether the interior of the set is connected (if it has an interior.

$$\{z \in \mathbb{C} : |z| < 1 \text{ or } |z-3| \leq 1$$




(i) Interior : $\forall z_0 : |W_0| < 1$ or $|W_0 - 3| \leq 1$, $W_0 \in \mathbb{C}$

(i.e. any point in the shaded region.

Boundary : $|z| = 1$ or $|z-3| = 1$, $z \in \mathbb{C}$

(ii) The Set is neither opened nor closed

- $|z-3| \leq 1$ Contains a boundary point ($\therefore$ not opened)
- The set's complement is not open

  $|z| \geq 1$ Contains boundary point ($\therefore$ not closed)

(iii) The interior is not Connected.

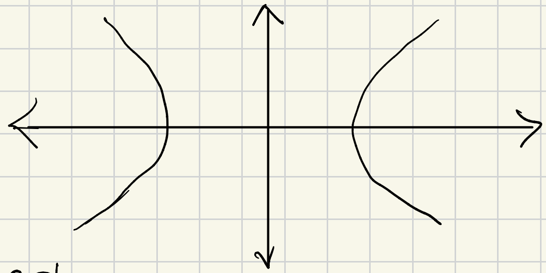
$z_1 = \frac{1}{4} + i\,0 \quad |z_1| < 1$

$z_2 = 2 + i\,0 \quad |z_2 - 3| \leq 1$

$z_1$ and $z_2$ cannot be connected polygonal curve (in the set).
As such it is not connected.

Question 5 – For the following set, describe (i) the interior and the boundary, (ii) state whether the set is open, or closed, or neither open nor closed, (iii) state whether the interior of the set is connected (if it has an interior.

$$\{ z \in \mathbb{C} : \mathrm{Re}(z^2) = 4 \}$$



(i) Interior: is not defined

Boundary: $\mathrm{Re}(z^2) = 4, z \in \mathbb{C}$ $\quad \{x+iy : x^2 - y^2 = 4\}$

Let $z = x+iy$

(ii) Not Opened But Closed

- Contains Boundary
- The set complement is open as it contains no boundary $\rightarrow$ $\therefore$ Closed

(iii) Interior Does Not Exist

Question 6 – Find $\lim_{z\to 2} (z-2)\log|z-2|$ or explain why it doesn't exist.

This limit can be rewritten as $\lim_{z\to 2} \frac{\log|z-2|}{(z-2)^{-1}}$

We can then apply L'Hopital's Rule:

Let $f(z) = \log|z-2|$ and $g(z) = (z-2)^{-1}$

$$\lim_{z\to 2} \frac{f'(z)}{g'(z)} = \lim_{z\to 2} \frac{\left((z-2)\ln(10)\right)^{-1}}{-(z-2)^{-2}} = \lim_{z\to 2} \frac{\cancel{(z-2)^{-1}}(\ln(10))^{-1}}{-\cancel{(z-2)^{-1}}(z-2)^{-1}}$$

$$= \lim_{z\to 2} \frac{(\ln(10))^{-1}}{-(z-2)^{-1}} = \lim_{z\to 2} -\frac{(z-2)}{\ln(10)} = 0$$

Hence the limit of the function above approaches zero

Question 7 – Find all the points where the following function is continuous.

$$f(z) = \begin{cases} \dfrac{z^4-1}{z-i}, & z \neq i \\ 4i, & z = i \end{cases}$$

We first need to determine where the function is discontinuous (or has jump discontinuities). We can do so by evaluating the limit of the first part at $z=i$.

$$\lim_{z\to i} \frac{z^4-1}{z-i} = \lim_{z\to i} \frac{(z^2+1)(z^2-1)}{z-i} = \lim_{z\to i} \frac{(z^2+1)(z+1)(z-1)}{z-i}$$

Writing $z^2+1$ in terms of its complex roots:

$$\lim_{z\to i} \frac{(z+i)\cancel{(z-i)}(z+1)(z-1)}{\cancel{z-i}}$$

$$= \lim_{z\to i} (z+i)(z+1)(z-1) = (2i)(i+1)(i-1)$$

$$= 2i(i^2-1) = 2i(-2) = -4i$$

This means the function is <u>dis</u>continuous at the point $z=i$ and is continuous everywhere else.

Thus continuous at $z \neq i$.

Question 8 Does the following series converge or diverge? $\sum_{n=1}^{\infty} \frac{1}{2+i^n}$

$$\sum_{n=1}^{\infty} \frac{1}{2+i^n} = \frac{1}{2+i} + \frac{1}{2+i^2} + \frac{1}{2+i^3} + \dots + \frac{1}{2+i^\infty}$$

$$\sum_{n=1}^{\infty} \frac{1}{2+i^1} = \frac{1}{2+i} + \frac{1}{2-1} + \frac{1}{2-i} + \frac{1}{2+1} + \dots \dots \frac{1}{2+i^\infty}$$

$$= \frac{1}{2+i} + 1 + \frac{1}{2-i} + \frac{1}{3} + \frac{1}{2+i} + \dots + \frac{1}{1+i^\infty}$$

This infinite series will be an infinite summation of terms of sequence $1, \frac{1}{3}, \frac{1}{2+i}$ and $\frac{1}{2-i}$. As such this series will diverge.

Alternatively, we know that a summation will diverge if $\lim_{n\to\infty} a_n \neq 0$. We know that limit of the infinite series above is non-zero and hence divergent.

Question 9 – Show that each of the following series converges for all of $z$.

(a) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ (b) $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$ (c) $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$

(a) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ → Apply the ratio test for $z = x+iy$, $z \in \mathbb{C}$

$$\lim_{n\to\infty} \left| \frac{a^{n+1}}{a^n} \right| = \lim_{n\to\infty} \left[ \left( \frac{z^{n+1}}{(n+1)!} \right) \cdot \left( \frac{(n)!}{z^n} \right) \right] = \lim_{n\to\infty} \left| \left( \frac{\cancel{z^n} \cdot z}{(n+1)\cancel{n!}} \right) \cdot \left( \frac{\cancel{n!}}{\cancel{z^n}} \right) \right|$$

$$= \lim_{n\to\infty} \left| \frac{z}{n+1} \right| = \lim_{n\to\infty} \frac{|z|}{|n+1|} = 0$$ (<1 Therefore absolutely convergent for all of $z$)

Let $z = x+iy \Rightarrow \sqrt{x^2+y^2} = |z|$ where $x, y \in \mathbb{R}$

$$= \lim_{n\to\infty} \frac{|x+iy|}{|n+1|} = \lim_{n\to\infty} \frac{\sqrt{x^2+y^2}}{n+1} = 0 \text{ as } n \to \infty$$

This infinite summation converges to $e^{(z)}$ by definition its maclaurin series

(b) $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{2n!}$ → Apply the root test. for $z = x+iy$, $z \in \mathbb{C}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} z^{2(n+1)}}{(2n+2))!} \cdot \frac{2n!}{(-1)^n z^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \left[ \frac{\cancel{(-1)^n}(-1) \cdot \cancel{z^{2n}} \cdot z}{(2n+2)(2n+1)\cancel{(2n)!}} \right] \cdot \left[ \frac{\cancel{2n!}}{\cancel{(-1)^n} \cancel{z^{2n}}} \right] \right|$$

$$= \lim_{n\to\infty} \left| \frac{-z}{(2n+2)(2n+1)} \right| = \lim_{n\to\infty} \frac{|z|}{(2n+2)(2n+1)} = 0$$ thus it converges for all of $z$

(c) $\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ → Apply the ratio test $z = x+iy$, $z \in \mathbb{C}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \left[ \frac{z^{2(n+1)+1}}{(2(n+1)+1)!} \right] \cdot \left[ \frac{(2n+1)!}{z^{2n+1}} \right] \right| = \lim_{n\to\infty} \left| \frac{z^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{z^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \left[ \frac{\cancel{z^{2n+1}} \cdot z^2}{(2n+3)(2n+2)\cancel{(2n+1)!}} \right] \cdot \left[ \frac{\cancel{(2n+1)!}}{\cancel{z^{2n+1}}} \right] \right| = \lim_{n\to\infty} \frac{|z|^2}{(2n+3)(2n+2)} = 0$$

Therefore this is convergent for all of $z$.