PSET 0

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Problem 0

I certify that I fully read the mathematical background chapter.

Problem 1

1.1
$$\varphi(x) = (\neg x_0 \wedge \neg x_1) \vee (\neg x_0 \wedge \neg x_2) \vee (\neg x_1 \wedge \neg x_2).$$

This satisfies the given requirements, as when the majority of the variables are false, $\varphi(x)$ will be true.

Problem 2

2.1 An expression $\psi(n,k)$ such that for every natural numbers $n, k, \psi(n,k)$ is true if and only if k divides n.
$$\forall_{n,k \in \mathbb{N}}\ \psi(n,k) \wedge (\exists_{m \in \mathbb{N}}(k \times m = n)).$$

2.2 An expression $\psi(n)$ such that for every natural number n, $\psi(n)$ is true if and only if n is a power of three.

We first define the following function:

power_of_three(n) := 'There is a k such that if k divides n, then either $k = 1$ or $k = 3$.

$$\forall_{n \in \mathbb{N}}\ \psi(n) \wedge (\text{power\_of\_three(n)}).$$

Problem 3

3.1 $S = \{x \in \{0,1\}^{100} : \forall_{i \in \{0,\ldots,98\}} x_i = x_{i+1}\}$:

This is the set of sequences of length 100 consisting of only 0's or 1's. Therefore, this set contains only two (2) elements, namely $x = 000 \cdots 0$ and $x = 111 \cdots 1$.

3.2 $T = \{x \in \{0,1\}^* : |x| > 1 \text{ and } \forall_{i \in \{2,\ldots,|x|-1\}} \forall_{j \in \{2,\ldots,|x|-1\}} i \cdot j \neq |x|\}$:

This is the set of finite length strings such that the string lengths are prime numbers.

Problem 4

4.1 Since there are 64 terms in set S, and only 25 terms in set T, there cannot exists a one-to-one relation between these sets.

4.2 S has 100^3 distinct elements, while T has 2^{100} distinct elements. A function from T to S is an onto function if and only if for every element $y \in S$, there is at least one $x \in T$ such that $f(x) = y$. Since $2^{100} > 100^3$, $|T| > |S|$, which means we can map every element in S to an element in T. Therefore, we conclude that there is a one to one function from T to S.

4.3 The number of elements in set S is 2^{10^6}, while there are infinite number of functions that can map $\{0,1\}^{100}$ to $\{0,1\}$. Since a function has a unique output, we need at least as many elements in S as the number of elements in T. Therefore, there cannot exist an onto relation from S to T.

Problem 5

5.1 From the principle of inclusion and exclusion (PIE), we find,

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|) \qquad (1) \end{aligned}$$

Since $(|A \cap B| + |A \cap C| + |B \cap C| \geq |A \cap B \cap C|)$, we find that

$$(|A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|) \geq 0.$$

Therefore, Equation 1 reduces to,

$$|A \cup B \cup C| \leq |A| + |B| + |C|$$

5.2 As in the case of Problem 5.1, using the principle of inclusion and exclusion (PIE), we find,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \qquad (2)$$

Since the last term on the RHS of Equation 2, $|A \cap B \cap C| \geq 0$, we conclude that

$$|A \cup B \cup C| \geq |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \quad \square$$

Problem 6

6.1 If G is a directed acyclic graph (DAG), then it has at least one topological ordering. Since G has n vertices, and G is a DAG, it has $n-1$ edges. It is given that the path length from u to v is $n-1$, which implies that the vertex u can be considered as the first vertex in the topological ordering. Then the vertex v is the last vertex in the topological ordering, with the remaining $n-2$ vertices found along the path from u to v. Therefore, the above unique topological ordering of G contains all the n vertices, the first vertex u has no in-neighbors. $\square$

6.2 Consider an initially empty set S. Let us remove a vertex v and its neighbors from the given graph G. Since each vertex in G has degree at most 4, v has at most 4 neighbors, which implies that we removed at most 5 vertices from G. Add v to S. Repeat this step until there is no more vertices left in G.

Since we removed at most 5 vertices from G during each step in the above process, and since G contains 1000 vertices, we would have repeated the above step at least 1000/5 = 200 times. During each of these at least 200 steps, we added one vertex to S, which implies that at the end of the above steps, S will contain at least 200 vertices.

Having established that S contains at least 200 vertices, we now claim that no two vertices in S are neighbors of one another. We prove this claim by contradiction as follows. Consider any two arbitrary vertices v_1 and v_2 in S and assume that they are neighbors of one another in G. Let us say, without loss of generality, that we removed v_1, along with its neighbors, from G first. Since v_2 is a neighbor of v_1, we would have also removed it from G. Now, according to the steps of building S described in the first paragraph, we only add v_1 to S and not its neighbors. This means if v_1 was added to S, then v_2 was removed from G, but not added to S. Therefore it must be that v_1 and v_2 are not neighbors in G.

Problem 7

7.1 $f(n) = n(\log n)^3$ and $g(n) = n^2$.

- $f = O(g)$.
- $g \neq O(f)$.

7.2 $f(n) = n^{\log n}$ and $g(n) = n^2$.

- $f \neq O(g)$.
- $g = O(f)$.

7.3 $f(n) = \binom{n}{\lceil 0.2n \rceil}$ and $g(n) = 2^{0.1n}$.

- $f \neq O(g)$.
- $g = O(f)$.