→ Runga Kutta Fourth Order Method :-

$k_1 = h f(x_n, y_n)$.

$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$

$k_3 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$

$x_1 = x_0 + h$.

$k_4 = h f(x_n + h \;\; y_n + k_3)$.

$y_1 = y_0 + k$.

$$\boxed{k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

→ QUESTION :-

Given $\frac{dy}{dx} = x + y^2$ with initial condition $y(0) = 1$ by Runga Kutta method $x = 0$ to $x = 0.2$; $h = 0.1$

→ $x_0 = 0$ $y_0 = 1$.

→ $x_1 = 0.1$ $y_1 = ??$

→ $x_2 = 0.2$ $y_2 = ??$

$k_1 = h f(x_n, y_n)$.

$k_1 = 0.1 f(0, 1)$

$= 0.1 (0 + 1^2) = 0.2$

$\boxed{k_1 = 0.2}$

$k_2 = h f\left(x_n + h/2, y_n + \frac{k_1}{2}\right)$.

$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right)$.

$= 0.1 f(0.05, 1.05)$

$= 0.1 (0.05 + 1.05^2)$.

$\boxed{k_2 = 0.1152}$

$\rightarrow k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right).$

$= 0.1 f\left(0.005, 1 + \frac{0.1152}{2}\right)$

$0.1 f$

$\boxed{k_3 = 0.1168}$

$\rightarrow k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.1 f(0+0.1, 1+0.1168).$

$= 0.1\left(0.1 + (1.1168)^2\right)$

$\boxed{k_4 = 0.1347}$

$\rightarrow k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}\left(0.1 + 2(0.1152) + 2(0.1168) + 0.1347\right)$

$\boxed{k = 0.1169}.$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y_1 = y_0 + k = 1 + 0.1168 = 1.1169$

$\boxed{x_1 = 0.1}.$ $\boxed{y_1 = 1.1169}.$

Now $k_1 = hf(x_1, y_1)$.

$= 0.1 f(0.1, 1.1169)$.

$= 0.1 (0.1 + 1.1169^2)$.

$\boxed{k_1 = 0.1347}$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$

$= 0.1 f(0.1 + \frac{0.1}{2}, 1.1169 + \frac{0.1347}{2})$.

$= 0.1 f(0.15, 1.18425)$.

$= 0.1 (0.15 + (1.18425)^2)$

$\boxed{k_2 = 0.1552}$.

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$.

$= 0.1 f(0.1 + \frac{0.1}{2}, 1.1169 + \frac{0.1552}{2})$.

$= 0.1 (0.15 + 1.1929^2)$.

$\boxed{k_3 = 0.157}$.

$k_4 = hf(x_1 + h, y_1 + k_3)$.

$= 0.1 f(0.1 + 0.1, 1.1169)$.

$\boxed{k_4 = 0.1823}$.

$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$.

$= \frac{1}{6}(0.1347 + 2(0.1552) + 2(0.157) + 0.1823)$.

$\boxed{k = 0.1572}$.

$x_2 = x_1 + h = 0.1 + 0.1 \boxed{= 0.2}$

$y_2 = y_1 + 0.1572 = 1.1169 + 0.1572 = \boxed{1.2732}$

→ Euler's Method :- (Runga kutt method of first order)

$y_{n+1} = y_n + h f(x_n, y_n)$. Here $y_n = W_n$

$\downarrow W_{n+1} = \downarrow W_n$

a) $y' = te^{3t} - 2y$ $\quad 0 \le t \le 1$, $y(0) = 0$; $h = 0.5$

for $n = 0$ $\quad t_0 = 0$

$y_1 = y_0 + h f(t_0, y_0)$. $\quad y_0 = 0$

$y_1 = 0 + 0.5 f(0, 0)$

$y_1 = 0 + 0.5 (0 e^{3(0)} - 2(0))$

$\boxed{y_1 = 0}$ $\quad t_1 = t_0 + h$

$\boxed{t_1 = 0.5}$

for $n = 1$.

$y_2 = y_1 + h f(t_1, y_1)$.

$= 0 + 0.5 f(0.5, 0)$

$= 0 + 0.5 (0.5 e^{3(0.5)} - 2(0))$.

$\boxed{y_2 = 1.12}$ $\quad t_2 = t_1 + h = 0.5 + 0.5$

$\boxed{t_2 = 1}$.

for $n = 2$.

$y_3 = y_2 + h f(t_2, y_2)$.

$= 1.12 + 0.5 f(1, 1.12)$.

$= 1.12 + 0.5 (1e^{3} - 2(1.12))$

$y_3 = 17.8$

b) $y' = 1 + (t-y)^2$; $2 \le t \le 3$; $y(2) = 1$; with $h = 0.5$.

$\therefore y_0 = 1$ $\qquad$ $\therefore y_{n+1} = y_n + h f(t_n, y_n)$.

$t_0 = 2$

→ For $n = 0$:

$y_1 = y_0 + h f(t_0, y_0)$

$y_1 = 1 + 0.5 f(2, 1)$.

$y_1 = 1 + 0.5 (1 + (2-1)^2)$.

$= 1 + 0.5 (1+1)$

$y_1 = 1 + 0.5 (2)$.

$\boxed{y_1 = 2}$

$t_1 = t_0 + h$.

$t_1 = 2 + 0.5$

$\boxed{t_1 = 2.5}$

→ For $n = 1$

$y_2 = y_1 + h f(t_1, y_1)$.

$= 2 + 0.5 f(2.5, 2)$.

$= 2 + 0.5 (1 + (2.5-2)^2)$

$\boxed{y_2 = 3}$

$t_2 = t_1 + h$.

$t_2 = 2.5 + 0.5$

$\boxed{t_2 = 3}$

c) $y' = 1 + \frac{y}{t}$; $1 \le t \le 2$; $y(1) = 2$; with $h = 0.25$

$y_0 = 2$.

$t_0 = 1$.

For $n = 0$.

$y_1 = y_0 + h f(t_0, y_0)$

$= 2 + 0.25 f(1, 2)$.

$= 2 + 0.25 \left(1 + \frac{2}{1}\right)$.

$\boxed{y_1 = 2.75}$

$t_1 = t_0 + h$.

$t_1 = 1 + 0.25$.

$\boxed{t_1 = 1.25}$.

For $n = 1$.

$y_2 = y_1 + h f(t_1, y_1)$.

$= 2.75 + 0.25 f(1.25, 2.75)$.

$= 2.75 + 0.25 \left(1 + \frac{2.75}{1.25}\right)$

$\boxed{y_2 = 3.55}$

$t_2 = t_1 + h$.

$t_2 = 1.25 + 0.25$.

$\boxed{t_2 = 1.5}$.

For $n = 2$.

$y_3 = y_2 + h f(t_2, y_2)$.

$y_3 = 3.55 + 0.25 \left(1 + \frac{3.55}{1.5}\right)$.

$\boxed{y_3 = 4.26}$

$t_3 = t_2 + h$.

$t_3 = 1.5 + 0.25$.

$\boxed{t_3 = 1.75}$

For $n = 3$:

$y_4 = y_3 + h f(t_3, y_3)$.

$= 4.26 + 0.25 \left(1 + \frac{4.26}{1.75}\right)$

$\boxed{y_4 = 5.11}$

$t_4 = t_3 + h$

$t_4 = 1.75 + 0.25$.

$\boxed{t_4 = 2}$

d = $y' = \cos 2t + \sin 3t$ $\quad 0 \le t \le 1 \quad y(0) = 1$ with $h = 0.25$

for $n = 0$

$\to y_0 = 1$

$\to t_0 = 0$.

$y_1 = y_0 + h f(t_0, y_0)$.

$y_1 = 1 + 0.25 (\cos 2t_0 + \sin 3t_0)$

$\boxed{y_1 = 1.25}$

$t_1 = t_0 + h$.

$t_1 = 0 + 0.25$.

$\boxed{t_1 = 0.25}$

for $n = 1$.

$y_2 = y_1 + h f(t_1, y_1)$.

$= 1.25 + 0.25 \{f(0.25, 1.25)$.

$= 1.25 + 0.25 (\cos 2(0.25) + \sin 3(0.25))$

$\boxed{y_2 = 1.58}$

$t_2 = t_1 + h. = 0.25 + 0.25$.

$\boxed{t_2 = 0.5}$

for $n = 2$.

$y_3 = y_2 + h f(t_2, y_2)$

$= 1.58 + 0.25 f(0.5, 1.58)$

$\boxed{y_3 = 1.83}$

$t_3 = t_2 + h$

$\boxed{t_3 = 0.75}$

→ for $n = 3$

$y_4 = y_3 + h f(t_3, y_3)$.

$y_4 = 1.83 + 0.25 f(0.75, 1.83)$

$\boxed{y_4 = 2.09}$

$t_4 = t_3 + h$

$t_4 = 0.75 + 0.25$

$\boxed{t_4 = 1}$

→ Modified Euler's Method :- (Ranga kulla second order)

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

$\left.\begin{matrix} y = w \\ x = t \end{matrix}\right\}$ for book solution

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right]$$

where $t_i = a + ih$

Consider $k_1 = hf(t_i, y_i)$.

$k_2 = hf(t_{i+1}, y_i + k_1)$.

$$w_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)$$

a) $y' = te^{3t} - 2y$; $(0 \leq t \leq 1)$; $y(0) = 0$

exact solution $h = 0.5$

$$y(t) = \frac{1}{5} te^{3t} - \frac{1}{25} e^{3t} + \frac{1}{25} e^{(-2t)}$$

$\therefore t_0 = 0$; $w_0 = 0$.

$$\therefore w_{n+1} = w_n + \frac{h}{2}\left[f(t_n, w_n) + f(t_{n+1}, w_n + hf(t_n, w_n))\right].$$

→ for $n = 0$. $t_1 = t_0 + h = 0.5$

$$w_1 = w_0 + \frac{h}{2}\left[f(t_0, w_0) + f(t_1, w_0 + hf(t_0, w_0))\right].$$

$$= 0 + \frac{0.5}{2}\left[f(0,0) + f(0.5, 0 + 0.5 f(0,0))\right].$$

$$= 0 + \frac{0.5}{2}\left[0 + (0.5e^{3(0.5)} - 2(0))\right].$$

$\boxed{w_1 = 0.56}$ $t_2 = t_1 + h = 0.5 + 0.5$

$\boxed{t_2 = 1}$.

for $n = 1$.

$$w_2 = w_1 + \frac{h}{2}\left[f(t_1, w_1) + f(t_2, w_1 + hf(t_1, w_1))\right.$$

$$= 0.56 + \frac{0.52}{2}\left[f(0.5, 0.56) + f(1, 0.56 + 0.5(0.5, 0.56)\right]$$

$$= 0.56 + \frac{0.52}{2}\left[(0.5e^{1.5} + 2(0.56)) + f(1, 0.56 + 0.5(0.5e^{1.5} + 2(0.56))\right]$$

$$= 0.56 + \frac{0.52}{2}\left[1.120 + f(1, 1.120)\right].$$

$w_2 = 0.56 + \frac{0.5}{2}\left[1.120 + \left(1 \times e^{3\times 1} - 2(1.120)\right)\right].$

$\boxed{w_2 = 5.29}$ $\boxed{t_2 = 1}$

c)

$y' = 1 + \frac{y}{t}$ $1 \le t \le 2$; $y(1) = 2$; h 0.25

$\Rightarrow$ $w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i\right.$

Here: $t_0 = 1$ $h = 0.25$.

$y_0 = w_0 = 2$.

$\therefore$ For $i = 0$. $t_1 = t_0 + h = \boxed{1.25}$

$w_1 = w_0 + \frac{h}{2}\left[f(t_0, w_0) + f(t_1, w_0 + hf(t_0, w_0))\right]$

$w_1 = 2 + \frac{0.25}{2}\left[f(1, 2) + f(1.25, 2 + 0.25 f(1, 2))\right]$

$w_1 = 2 + \frac{0.25}{2}\left[\left(1 + \frac{2}{1}\right) + f(1.25, 2 + 0.25(1+2))\right]$

$= 2 + \frac{0.25}{2}\left[3 + f(1.25, 2.75)\right].$

$= 2 + \frac{0.25}{2}\left[3 + \left(1 + \frac{2.75}{1.25}\right)\right].$

$\boxed{w_1 = 2.775}$

$\therefore$ For $i=1$. $\quad t_2 = t_1 + h = 1.25 + 0.25 = \boxed{1.50}$

$$w_2 = w_1 + \frac{h}{2}\left[f(t_1, w_1) + f(t_2, w_1 + h f(t_1, w_1))\right].$$

$$w_2 = 2.775 + \frac{0.25}{2}\left[f(1.25, 2.775) + f\left(1.50, 2.775 + 0.25 f(1.25, 2.775)\right)\right]$$

$$= 2.775 + \frac{0.25}{2}\left[\left(1 + \frac{2.775}{1.25}\right) + f\left(1.50, 2.775 + 0.25\left(1 + \frac{2.775}{1.25}\right)\right)\right].$$

$$= 2.775 + \frac{0.25}{2}\left[3.22 + f(1.50, 3.58)\right].$$

$$= 2.775 + \frac{0.25}{2}\left[3.22 + \left(1 + \frac{3.58}{1.50}\right)\right]$$

$$\boxed{w_2 = 3.6008}$$

for $i=2 \rightarrow$, $t_3 = t_2 + h = 1.50 + 0.25 = 1.75$.

$$w_3 = w_2 + h\left[f(t_2, w_2) + f(t_3, w_2 + h f(t_2, w_2))\right].$$

$$= 3.6008 + \frac{0.25}{2}\left[f(1.50, 3.60) + f(1.75, 3.6008 + 0.25 f(1.50, 3.60))\right]$$

$$= 3.6008 + \frac{0.25}{2}\left[\left(1 + \frac{3.6008}{1.50}\right) + f\left(1.75, 3.6008 + 0.25\left(1 + \frac{3.6008}{1.50}\right)\right)\right].$$

$$= 3.6008 + \frac{0.25}{2}\left[3.4005 + f(1.75, 4.45)\right].$$

$$= 3.6008 + \frac{0.25}{2}\left[3.4005 + \left(1 + \frac{4.45}{1.75}\right)\right].$$

$$\boxed{w_3 = 4.46}$$

→ for $i=3$: $t_4 = t_3 + h = 1.75 + 0.25 = 2.$

$$w_4 = w_3 + \frac{h}{2}\Big[\big(f(t_3, w_3) + f(t_4, w_3 + h f(t_3, w_3))\big)\Big].$$

$$w_4 = 4.46 + \frac{0.25}{2}\Big[f(1.75, 4.46) + f(2, 4.46 + 0.25 f(1.75, 4.46$$

$$= 4.46 + \frac{0.25}{2}\Big[\Big(1 + \frac{4.46}{1.75}\Big) + f\Big(2, 4.46 + 0.25\Big(1 + \frac{4.46}{1.75}\Big)\Big)\Big].$$

$$= 4.46 + \frac{0.25}{2}\big[3.54 + f(2, 5.347)\big].$$

$$= 4.46 + \frac{0.25}{2}\Big[3.54 + \Big(1 + \frac{5.347}{2}\Big)\Big]..$$

$$\boxed{w_4 = 5.36}$$

→ Heun's Method:-

$$w_{i+1} = w_i + \frac{h}{4}\left(f(t_i, w_i) + 3 f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\right)\right)\right).$$

$k_1 = f(t_i, w_i)$

$k_2 = f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} k_1\right).$

$k_3 = f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} k_2\right).$

$w_{i+1} = w_i + \frac{h}{4}(k_1 + 3k_3).$

→ Mid Point Method:-

$w_{i+1} = w_i + h f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right).$

a) $y' = te^{3t} - 2y$ $\quad 0 \le t \le 1 \quad y(0) = 0;\ h = 0.5.$

By using mid point method:-

$$w_{i+1} = w_i + h\left[f\left(t_i + \frac{h}{2},\ w_0 + \frac{h}{2} f(t_i, w_i)\right)\right].$$

→ $t_0 = 0$

→ $w_0 = y_0 = 0$

→ $h = 0.5$

→ for $i = 0$.

$$w_1 = w_0 + h\left[f\left(t_0 + \frac{h}{2},\ w_0 + \frac{h}{2} f(t_0, w_0)\right)\right].$$

$$w_1 = 0 + 0.25\left[f\left(0 + \frac{0.25}{2},\ 0 + \frac{0.25}{2} f(0,0)\right)\right]$$

$$w_1 = 0 + 0.25\left[f(0.125,\ 0.125(0)\right].$$

$$w_1 = 0.25\left[f(0.125, 0)\right].$$

$$w_1 = 0.25\left(0.125 e^{3 \times 0.125} - 2(0)\right).$$

$$\boxed{w_1 = 0.264} \qquad t_1 = t_0 + h = 0 + 0.5 = \boxed{0.5}$$

→ For $i = 1$

$$w_2 = w_1 + h\left[f\left(t_1 + \frac{h}{2},\ w_1 + \frac{h}{2} f(t_1, w_1)\right)\right].$$

$$= 0.264 + 0.5\left[f\left(0.5 + \frac{0.5}{2},\ 0.264 + \frac{0.5}{2} f(0.5, 0.264\right)\right.$$

$$= 0.264 + 0.5\left[f\left(0.75,\ 0.264 + \frac{0.5}{2}\left(0.5e^{3 \times 0.5} - 2(0.264)\right)\right)\right]$$

$$= 0.264 + 0.5\left[f(0.75, 0.692)\right].$$

$$= 0.264 + 0.5\left[0.75e^{3 \times 0.75} - 2(0.692)\right].$$

$$\boxed{w_2 = 3.129} \qquad t_2 = t_1 + 0.5 = 0.5 + 0.5$$

$$\boxed{t_2 = 1}$$

→ By Heun's Method :- (First method)

$$w_{i+1} = w_i + \frac{h}{4}\left[f(t_i, w_i) + 3f\left(t_i+\frac{2h}{3},\ w_i+\frac{2h}{3} f\left(t_i+\frac{h}{3},\ w_i+\frac{h}{3} f(t_i,w_i)\right)\right)\right]$$

For $n=0$.

$$w_1 = w_0 + \frac{h}{4}\left[f(t_0, w_0) + 3f\left(t_0+\frac{2h}{3},\ w_0+\frac{2h}{3} f\left(t_0+\frac{h}{3},\ w_0+\frac{h}{3} f(t_0,w_0)\right)\right)\right]$$

$$= 0 + \frac{0.5}{4}\left[f(0,0) + 3f\left(0+\frac{2\times 0.5}{3},\ 0+\frac{2\times 0.5}{3} f\left(\frac{0+0.5}{3},\ 0+\frac{0.5}{3} f(0,0)\right)\right)\right].$$

$$= \frac{0.5}{4}\left[0 + 3f\left(0.33,\ 0.33 f(0.166, 0)\right)\right]$$

$$= \frac{0.5}{4}\left[3f\left(0.33,\ 0.33\left(0.166\, e^{3\times 0.166} - 2(0)\right)\right)\right].$$

$$= \frac{0.5}{4}\left[3f(0.33,\ 0.090)\right].$$

$$= \frac{0.5}{4}\left[3\left(0.33\, e^{3\times 0.33} - 2(0.090)\right)\right]$$

$\boxed{w_1 = 0.264}$ $\qquad t_1 = t_0 + h = 0 + 0.5 = \boxed{0.5}$

→ Find w_2 by second method:-

$K_1 = f(t_i, w_i)$.

$K_2 = f\left(t_i + \frac{h}{3}, w_i + \frac{hK_1}{3}\right)$.

$K_3 = f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3}K_2\right)$

$w_{i+1} = w_i + \frac{h}{4}(K_1 + 3K_3)$.

For $i = 1$

$\therefore K_1 = f(t_1, w_1)$.

$K_1 = f(0.5, 0.264)$

$K_1 = 0.5e^{3 \times 0.5} - 2(0.264)$.

$\boxed{K_1 = 1.712}$ $\qquad t_2 = t_1 + h = \boxed{1}$.

$K_2 = f\left(0.5 + \frac{0.5}{3}, 0.264 + \frac{0.5}{3}(1.712)\right)$

$K_2 = f(0.666, 0.549)$.

$K_2 = 0.666e^{3 \times 0.666} - 2(0.549)$.

$\boxed{K_2 = 3.813}$.

$K_3 = f\left(0.5 + \frac{2 \times 0.5}{3}, 0.264 + \frac{2 \times 0.5}{3}(3.813)\right)$.

$K_3 = f(0.83, 1.535)$.

$K_3 = 0.83e^{3 \times 0.83} - 2(1.532)$

$\boxed{K_3 = 7.047}$