Dated:

CHAP #5 INITIAL VALUE PROBLEMS FOR ORDINARY D.E

EULER'S METHOD Ex: 5.2

ODE (ORDINARY DE):
Equation containing only ordinary derivatives of one or more dependent variable with respect to a single independent variable.

$$\frac{dy}{dx} + 5y = e^x, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y$$

IVP

$$\frac{dy}{dx} = f(x,y) \qquad \frac{d^2y}{dx^2} = f(x,y,y')$$

$$y(x_0) = y_0 \qquad y(x_0) = y_0, \; y'(x_0) = y_1.$$

Q. $(1+x)\,dy - y\,dx = 0$

$$y\,dx = (1+x)\,dy$$

$$\int \frac{1}{(1+x)}\,dx = \int \frac{1}{y}\,dy$$

$$\ln(1+x) + c_1 = \ln(y)$$

$$y = e^{\ln(1+x)+c_1} = e^{\ln|1+x|} \cdot e^{c_1}$$

$$y = c(1+x)$$

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Dated:

Q. $\frac{dy}{dx} = -\frac{x}{y}$, $y(4) = -3$

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$x^2 + y^2 = c^2$ → General solution

$x = 4$, $y = -3$,

$16 + 9 = c^2$

$16+19 = 25$

$x^2 + y^2 = 25$ → Particular solution.

EULER METHOD
(To app. IVP)

$y' = f(t,y)$, $a \le t \le b$, $y(a) = \alpha$.

$y_{n+1} = y_n + h\, f(x_n, y_n)$ $x_n = x_0 + nh$, $n = 0, 1, 2, \ldots$

$h = \frac{b-a}{N}$ → step size.

or

$w_0 = y_0$

$w_{i+1} = w_i + hf(t_i, w_i)$ Bander Jaris

Dated:

Ex 1 :

$y' = 0.1\sqrt{y} + 0.4x^2$, $y(2) = 4$, Use Euler's to obtain app. of $y(2.5)$, first $h = 0.1$, then $h = 0.05$.

$f(x,y) = 0.1\sqrt{y} + 0.4x^2$

$y_{n+1} = y_n + h\, f(x_n, y_n)$ $\qquad x_n = x_0 + nh.$

$h = 0.1$, $x_0 = 2$, $y_0 = 4$, $n = 0$

$y_1 = y_0 + h(0.1\sqrt{y_0} + 0.4(x_0)^2)$

$= 4 + (0.1)(0.1\sqrt{4} + (0.4)(2)^2)$

$= 4 + (0.1)(0.2 + 1.6)$

$y_1 = 4.18$

$h = 0.1$, $y_1 = 4.18$, $x_1 = 2.1$, $n = 1$

$y_{1+1} = y_1 + h(0.1\sqrt{y_1} + 0.4(x_1)^2)$

$= 4.18 + (0.1)(0.1\sqrt{4.18} + 0.4(2.1)^2)$

$= 4.18 + (0.1)(0.20445 + 1.764)$

$= 4.18 + 0.196845$

$y_2 = 4.3768$

Dated:

$h = 0.1$

	x_n	y_n
0	2.00	4.0000
1	2.10	4.1800
2	2.2	4.3768
3	2.3	4.5914
4	2.4	4.8244
5	2.5	5.0768

with $h = 0.05$

x_n	y_n
2.00	4.0000
2.05	4.0900
2.10	4.1842
2.15	4.2826
2.20	4.3854
2.25	4.4927
2.30	4.6045
2.35	4.7210
2.40	4.8423
2.45	4.9686
2.50	5.0997.

$h = 0.05, \quad x_0 = 2, \quad y_0 = 4, \quad n = 0$

$$y_1 = y_0 + 0.05\left(0.1\sqrt{4} + 0.4(2)^2\right)$$

$$= 4 + 0.05(0.2 + 1.6)$$

$$= 4 + 0.05(1.8) \qquad = 4 + 0.09$$

$$y_1 = 4.09$$

$h = 0.05, \quad x_1 = 2.05, \quad y_1 = 4.09, \quad n = 1$

$$= 4.09 + 0.05\left(0.1\sqrt{4.09} + 0.4(2.05)^2\right)$$

$$y_2 = 4.1842$$

Dated:

APPROXIMATE AND ACTUAL VALUE

Actual or true values are calculated from the known solution.

Abs. = |Actual - Approx|

Rel. = $\frac{\text{Abs.}}{\text{actual Value}}$

Example # 2.

$y' = 0.2xy$ $\quad y(1) = 1.$ Use Euler's to approx.

$y(1.5)$, $h = 0.1$ and $h = 0.05$.

Actual: $y = e^{0.1(x^2-1)}$

$f(x,y) = 0.2xy$

$y_{n+1} = y_n + h f(x_n, y_n)$, $x_n = x_0 + nh$.

$h = 0.1$, $n = 0$, $y_0 = 1$, $x_0 = 1$

$y_{0+1} = y_0 + 0.1\ 0.2(x_n y_n)$

$= 1 + 0.1(0.2 \times 1 \times 1)$

$= 1 + 0.02$

$y_1 = 1.02$

$y_2 = y_1 + 0.1(0.2(1.10 \times 1.02))$

$= 1.02 + 0.02244$

$y_2 = 1.0424.$

$h = 0.1$

x_n	y_n	Actual	Abs. Error	Rel.
1	1.0000	1.0000	0	
1.10	1.0200	1.0212	0.0012	0.12
1.20	1.0424	1.0450	0.0026	0.24
1.30	1.0675	1.0714	0.0039	0.37
1.40	1.0952	1.1008	0.0056	0.50
1.50	1.1259	1.1331	0.0073	0.64

Dated:

Q. $y' = y - t^2 + 1$, $0 \le t \le 2$, $y(0) = 0.5$.

$h = 0.2$. $y(t) = (t+1)^2 - 0.5e^t$

$y(2) = ?$

$$w_{i+1} = w_i + h f(t_i, w_i)$$

$$f(t, y) = y - t^2 + 1.$$

$$w_0 = y_0$$

$$w_0 = 0.5$$

$$w_1 = w_0 + 0.2\left(w_0 - (t_0)^2 + 1\right)$$

$$= 0.5 + 0.2\left(0.5 - (0.0)^2 + 1\right)$$

$$w_1 = 0.5 + 0.3 = 0.80000$$

$$w_2 = w_1 + 0.2\left(w_1 - (t_1)^2 + 1\right)$$

$$= 0.8000 + 0.2\left(0.8000 - (0.2)^2 + 1\right)$$

$$w_2 = 1.152.$$

t_i	w_i	$y_i = y(t_i)$	Abs.
0.0	0.5000		
0.2	0.8000		
0.4	1.1520		
0.6	1.5504		
0.8	1.98848		
1.0	2.458176		
1.2	2.9498112		
1.4	3.4517734		
1.6	3.9501281		
1.8	4.4281538		
2.0	4.8657845		

Dated:

EX 5.4

MIDPOINT METHOD

Max 5 iterations

$w_0 = \alpha$

$w_{i+1} = w_i + h f\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right)$, for $i = 0, 1, \ldots N-1$

or

$w_{i+1} = y_i + hf\left(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i)\right)$.

MODIFIED EULER METHOD

$w_0 = \alpha$

$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + h f(t_i, w_i))]$

or

$w_{i+1} = y_i + \frac{h}{2}\left(f(t_i, y_i) + f(t_i + h, y_i + h \cdot f(t_i, y_i))\right)$

also, consider $k_1 = h f(t_i, y_i)$ $\quad k_2 = h f(t_{i+1}, y_i + k_1)$

$w_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)$.

Ex: 2 $\quad N = 10, \; h = 0.2, \; t_i = 0.2i, \quad w_0 = 0.5.$

$y' = y - t^2 + 1, \; 0 \le t \le 2, \quad y(0) = 0.5$.

using Midpoint:

$w_i = w_0 + hf(t_0 + h/2, w_0 + h/2 \, f(t_0, w_0))$

$= 0.5 + 0.2 \, f\left(0 + \frac{0.2}{2}, 0.5 + \frac{0.2}{2} f(0, 0.5)\right)$

$= 0.5 + 0.2 \, f(0.1, 0.5 + 0.1(1.5))$

$= 0.5 + 0.2 \, f(0.1, 0.65)$

Dated:

$= 0.5 + 0.2[0.65 - (0.1)^2 + 1]$

$= 0.5 + 0.328$

$w_1 = 0.828$

$w_2 = w_1 + h f(t_1 + \frac{h}{2}, w_1 + \frac{h}{2} f(t_1, w_1))$

$= 0.828 + 0.2 f(0.2 + \frac{0.2}{2}, 0.828 + \frac{0.2}{2} f(0.2, 0.828))$

$= 0.828 + 0.2 f(0.3, 0.828 + 0.1[0.828 - (0.2)^2 + 1])$

$(0.3, 0.828 + 0.1788)$

$= 0.828 + 0.2 f(0.3, 1.0068)$

$= 0.828 + 0.2[(1.0068) - (0.3)^2 + 1]$

$= 0.828 + 0.38336$

$w_2 = 1.21136$

Table.

Dated:

Using Modified Euler Method

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + h f(t_i, w_i))\right]$$

$$w_1 = w_0 + \frac{h}{2}\left[f(t_0, w_0) + f(t_1, w_0 + h f(t_0, w_0))\right]$$

$$= 0.5 + \frac{0.2}{2}\left[f(0, 0.5) + f(0.2, 0.5 + 0.2 f(0, 0.5))\right]$$

$$= 0.5 + 0.1\left[f(0, 0.5) + f(0.2, 0.5 + 0.2\left[0.5 - (0)^2 + 1\right])\right]$$

$$= 0.5 + 0.1\left[1.5 + f(0.2, 0.5 + 0.2(1.5))\right]$$

$$= 0.5 + 0.1\left[1.5 + f(0.2, 0.5 + 0.3)\right]$$

$$= 0.5 + 0.1\left[1.5 + f(0.2, 0.8)\right]$$

$$f(0.2, 0.8) = y - t^2 + 1$$

$$= 0.8 - (.2)^2 + 1$$

$$= 1.76$$

$$= 0.5 + 0.1\left[1.5 + 1.76\right]$$

$$= 0.5 + 0.326$$

$$w_1 = 0.826$$

Dated:

HEUN'S METHOD

$w_0 = \alpha$ Que from assignment #2.

$$w_{i+1} = w_i + \frac{h}{4}\left(f(t_i, w_i) + 3f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\right)\right)\right).$$

$$k_1 = f(t_i, w_i)$$

$$k_2 = f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} k_1\right)$$

$$k_3 = f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} k_2\right)$$

$$w_{i+1} = w_i + \frac{h}{4}\left(k_1 + 3k_3\right)$$

RUNGE KUTTA METHOD

$w_0 = \alpha$ Que from Assigment #2.

$$k_1 = h f(t_i, w_i)$$

$$k_2 = h f\left(t_i + \frac{h}{2}, w_i + \frac{1}{2} k_1\right)$$

$$k_3 = h f\left(t_i + \frac{h}{2}, w_i + \frac{1}{2} k_2\right)$$

$$k_4 = h f\left(t_{i+1}, w_i + k_3\right)$$

$$w_{i+1} = w_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Dated:

CHAP # 6

Ex : 6.5

Revision:

1- Gauss Elimination 2. Gauss Jordan 3. Cramer Rule's

4. Inversion Method ($X = A^{-1}b$)

$$\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ \vdots & & \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}$$

Ex 1 : Gauss Elimination

Partial Pivoting

$$6x_1 + 2x_2 + 8x_3 = 26$$
$$3x_1 + 5x_2 + 2x_3 = 8$$
$$8x_2 + 2x_3 = -7$$

$\frac{6}{2} = -3$

$\frac{8}{2} = 4$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{array}\right] \xRightarrow{-\frac{1}{2}R_1 + R_2} \left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 4 & -2 & -5 \\ 0 & 8 & 2 & -7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 4 & -2 & -5 \end{array}\right]$$

$$-\frac{1}{2}R_2 + R_3 \left[\begin{array}{ccc|c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & -3 & -3/2 \end{array}\right]$$

$\frac{+7}{2} - 5$

$\frac{7-10}{2}$

$$x_1 = \frac{1}{6}(26 - 2x_2 - 8x_3)$$

$$\boxed{x_1 = 4}$$

$$-3x_3 = -\frac{3}{2}$$

$$\boxed{x_3 = 1/2}$$

$$x_2 = \frac{1}{8}(-7 - 2x_3) = \frac{1}{8}(-7 - 2(1/2))$$

$$\boxed{x_2 = 1/8(-8) = -1}$$

Dated:

Guass Jordan Elimination.

$x + 2y + z = 8$

$2x + 3y + 4z = 20$

$4x + 3y + 2z = 16$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \\ 16 \end{bmatrix}$$

$-2R_1 + R_2$
$-4R_1 + R_3$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & -5 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -16 \end{bmatrix}$$

$2R_2 + R_1$
$-5R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & -12 \end{bmatrix}\begin{bmatrix} \ \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ -36 \end{bmatrix} \Rightarrow \frac{-1}{12}R_3 \begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$x = 1, \; y = 2, \; z = 3$

Cramer's Rule :

$0.3x_1 + 0.52x_2 + x_3 = -0.01$

$0.5x_1 + x_2 + 1.9x_3 = 0.67$

$0.1x_1 + 0.3x_2 + 0.5x_3 = -0.44$

$x_1 = \frac{D_1}{D}$ $\quad x_1 = \frac{D_1}{|D|}, \quad x_2 = \frac{D_2}{|D|}, \quad x_3 = \frac{D_3}{|D|}$

$D = -0.0022$

$$D_1 = \begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.64 & 1 & 1.9 \\ -0.44 & 0.3 & 0.5 \end{vmatrix} = -14.9$$

Similarly, x_2, x_3

D

Dated:

LU FACTORIZATION

$A = LU$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

lower Triangular

$$L = \begin{bmatrix} L_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

Upper Triangular Matrix

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$L \cdot U = A \Rightarrow \begin{bmatrix} L_{11}U_{11} & L_{11}U_{12} & L_{11}U_{13} \\ L_{21}U_{11} & L_{21}U_{12}+L_{22}U_{22} & L_{21}U_{13}+L_{22}U_{23} \\ L_{31}U_{11} & L_{31}U_{12}+l_{32}U_{22} & l_{31}U_{13}+l_{32}U_{23}+l_{33}U_{33} \end{bmatrix} = A$$

$L_{11}U_{11} = A_{11}$ $\quad L_{11}U_{12} = A_{12}$ $\quad L_{11}U_{13} = A_{13}$

$L_{21}U_{11} = A_{21}$ $\quad L_{21}U_{12} + l_{22}U_{22} = A_{22}$ $\quad l_{21}U_{13} + L_{22}U_{23} = A_{23}$

$L_{31}U_{11} = A_{31}$ $\quad L_{31}U_{12} + l_{32}U_{22} = A_{32}$ $\quad l_{31}U_{13} + L_{32}U_{23} + l_{33}U_{33} = A_{33}$

Unknown = 12 $\quad$ Equations = 9

We have two Methods to solve these.

1. Crout's Method:

set

$U_{11} = U_{22} = U_{33} = 1$

$$LU = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

2. Dolittle's Method:

set

$L_{11} = L_{22} = L_{33} = 1$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

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Dated:

3. LDL^t form:

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

Q. $A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ Factorize given Matrix $A = LU$, using Do little Method

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$A = LU$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{21}U_{13}+U_{23} \\ L_{31}U_{11} & L_{31}U_{12}+l_{32}U_{22} & l_{31}U_{13}+l_{32}U_{23}+U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$\boxed{U_{11} = 1}$ $\boxed{U_{12} = 3}$ $\boxed{U_{13} = 8}$

$L_{21}U_{11} = A_{21} \Rightarrow L_{21}U_{11} = 1$ $\quad l_{21} = \frac{1}{U_{11}}$ $\boxed{l_{21} = 1}$

$L_{21}U_{12} + U_{22} = 4$

$(1)(1) + U_{22} = 4$

$\boxed{U_{22} = 3}$

$L_{21}U_{13} + U_{23} = 3$

$(1)(8) + U_{23} = 3$

$\boxed{U_{23} = -5}$

$L_{31}U_{11} = 1$

$\boxed{l_{31} = 1}$

$l_{31}U_{12} + l_{32}U_{22} = 3$

$(1)(3) + l_{32}(3) = 3$

$3 + 3l_{32} = 3$

$\boxed{L_{32} = 0}$

$L_{31}U_{13} + l_{32}U_{23} + U_{33} = 4$

$(1)(8) + (0)(0) + U_{33} = 4$

$8 + U_{33} = 4$

$\boxed{U_{33} = -4}$

Dated:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 3 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

Q. Do same Que using Crout's Method

$$LU = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} L_{11} & L_{11}U_{12} & L_{11}U_{13} \\ L_{21} & L_{21}U_{12} + l_{22} & L_{21}U_{13} + l_{22}U_{23} \\ L_{31} & L_{31}U_{12} + L_{32} & L_{31}U_{13} + l_{32}U_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$\boxed{L_{11} = 1}$

$L_{11}U_{12} = 3$

$\boxed{U_{12} = 3}$

$L_{11}U_{13} = 8$

$\boxed{U_{13} = 8}$

$\boxed{L_{21} = 1}$

$L_{21}U_{12} + l_{22} = 4$

$(1)(3) + l_{22} = 4$

$\boxed{l_{22} = 1}$

$L_{21}U_{13} + l_{22}U_{23} = 3$

$(1)(8) + (1)U_{23} = 3$

$8 + U_{23} = 3$

$\boxed{U_{23} = -5}$

$\boxed{L_{31} = 1}$

$L_{31}U_{12} + l_{32} = 3$

$(1)(3) + l_{32} = 3$

$\boxed{l_{32} = 0}$

$L_{31}U_{13} + l_{32}U_{23} + l_{33} = 4$

$(1)(8) + 0 + l_{33} = 4$

$\boxed{l_{33} = -4}$

$$L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 3 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

Dated:

EXERCISE 6.6
USING LU FACTORIZATION TO SOLVE $Ax = b$

$Ax = LUx = b \rightarrow (1)$

$y = Ux \rightarrow (2)$

$Ly = b \rightarrow (3)$

Q. Solve $Ax = b$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -2 \\ 3 & -2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 13 \\ 7 \\ -5 \end{bmatrix}$$

We have to factorize L and U matrix by do little or Crout's method if not given.

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0.5 & 0 \\ 3 & -3.5 & -25 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 0.5 & 1.5 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix}$$

Now,

$LUX = B \Rightarrow LY = B$

$LY = B$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & .5 & 0 \\ 3 & -3.5 & -25 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \\ -5 \end{bmatrix}$$

Solve by

$Y_1 = \dfrac{B_1}{L_{11}} \qquad Y_2 = \dfrac{B_2 - l_{21}Y_1}{L_{22}} \qquad Y_3 = \dfrac{B_3 - L_{31}Y_1 - L_{32}}{L_{33}}$

Dated:

$Y_1 = 6.5 \quad Y_2 = 1 \quad Y_3 = 0.84.$

$\cancel{Y} = UX$

$$\begin{bmatrix} 1 & .5 & 1.5 \\ 0 & 1 & .7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 6.5 \\ 1 \\ 0.84 \end{bmatrix}$$

$$X_3 = \frac{Y_3}{U_{33}} \qquad X_2 = \frac{Y_2 - U_{23} X_3}{U_{22}} \qquad X_1 = \frac{Y_1 - U_{12} \cdot X_2 - U_{13} \cdot X_3}{U_{11}}.$$

$$\cancel{X} = \begin{bmatrix} 1.8 \\ 6.88 \\ 0.84 \end{bmatrix}$$

Example: using an LU decomposition and Crout Method

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ 2x_1 - 4x_2 + 6x_3 = 18 \\ 3x_1 - 9x_2 - 3x_3 = 6 \end{cases}$$

1) $Ax = b$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}$$

2) Find LU.

$A = LU$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Dated:

3) $AX = B$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix} \Rightarrow$$

$LUX = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = B$$

4) Now, let $Y = UX$, then solve

$$\underbrace{LUX}_{} = B$$

$$\Rightarrow LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}$$

$y_1 = 5$, $y_2 = -1$, $y_3 = 2$

5) Now.

$$Y = UX$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$$

$x_3 = 2$

$x_2 = -1$

$x_1 + 2x_2 + 3x_3 = 5 \Rightarrow x_1 = 1$.

Gene. steps:

1. Write eq in Matrix form
2. Factorize LU $\Rightarrow A = LU$
3. Solve $LUX = B \Rightarrow$ let $UX = Y$ and $LY = B$ and get y
4. Solve $UX = Y$ and get x

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Dated: $A^t = -A \rightarrow$ skew sym.

$A^t = A \Rightarrow$ Symmetric Matrix

In LA, a tridiagonal Matrix is a "band Matrix" that has non zero elements on Main diagonal.

EXERCISE # 6.6

Diagonally Dominant Matrices: $n \times n$ matrix when $|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$

-ve value ko +ve lengy condition check karty hoye.

$$B = \begin{bmatrix} 6 & \boxed{4 \quad -3} \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

~~Sum of~~ diagonal $\geq$ sum of Other element of first row

~~$6+2+1 \leq 4+3$~~ $\quad 6 \geq 4+3$

~~B&~~ ~~....~~ Not diagonally Dominant

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix}$$

$7 \geq 2+0 \rightarrow$ Yes, Diagonally Dominant

Positive Definite Matrix: Matrix is positive definite if it is symmetric and if $x^t A x > 0$ for every n-dimensional vector $x \neq 0$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = A^t = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Now,

$$\overbrace{x^t}^{} A x = [x_1, x_2, x_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1, x_2, x_3] \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{bmatrix}$$

$$= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2$$

Dated:

Rearrange,

$$x^t A x = x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + x_3^2$$

$$\Rightarrow x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0$$

Another Def of +ve Definite:

if only if each of its leading principle submatrices has the determinant.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\det A_1 = \det[2] = 2 > 0$$

$$\det A_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$$

$$\det A_3 = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2(3) + (-2) = 6 - 2 = 4 > 0.$$

Another : Matrix A is +ve definite if and only if A can be factored in the form LDL^t, where L is lowertriangular with 1's on diagonal and D is diagonal Matrix with the dia entries :

Cholesky : Matrix A is +ve definite if and if A can be factored in form LL^t, where L lowertriangular with non zero diagonal.

If Matrix (A) is symmetric but not the definite then Cholesky not Valid, we'll use LDL^T.

Dated: LDL^t

Q. Determine LDL^t factorization of the definite Matrix.

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}$$

$$A = LDL^t$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & d_1 l_{21} & d_1 l_{31} \\ d_1 l_{21} & d_2 + d_1 l_{21}^2 & d_2 l_{32} + d_1 l_{21} l_{31} \\ d_1 l_{31} & d_1 l_{21} l_{31} + d_2 l_{32} & d_1 l_{31}^2 + d_2 l_{32}^2 + d_3 \end{bmatrix}$$

Equate and find value

$$A = LDL^t = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & 0.75 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -.25 & .25 \\ 0 & 1 & .75 \\ 0 & 0 & 1 \end{bmatrix}$$

Procedure to solve $Ax = b$ using LDL^t

· Step 1 :

$$[A] = [L][D][L]^T$$

Step 2 : Forward sol and diagonal scaling phase.

$$[L][D][L]^T[x] = [b]$$

$$[L]^T[x] = [Y]$$

$$\begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

Dated:

$[D][Y] = [Z]$

$$\begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$[L][z] = [b]$

$$\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Summary.

1. $A = LDL^t$
2. $Lz = b \rightarrow z$
3. $DY = z \rightarrow Y$
4. $L^t X = Y \rightarrow X$

Dated:

CHOLESKY'S METHOD

For a symmetric, +ve definite Matrix A $(A = A^t,\ x^t A x > 0$ $x \neq 0)$ (compulsory)

solving $Ax = b$ based on this factorization $A = LL^T$ is called Cholesky's Method.

Example;

$$A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} \rightarrow LL^T = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix}$$

Symmetric $A = A^t$

Procedure to find U_{ii} of LL^t.

$$A = \begin{bmatrix} U_{11} & 0 & 0 \\ U_{12} & U_{22} & 0 \\ U_{13} & U_{23} & U_{33} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$U_{11} = \sqrt{a_{11}}$ $\quad U_{12} = \dfrac{a_{12}}{U_{11}}$ $\quad U_{13} = \dfrac{a_{13}}{U_{11}}$

$U_{22} = (a_{22} - U_{12}^2)^{1/2}$ $\quad U_{23} = \dfrac{a_{23} - U_{12}U_{13}}{U_{22}}$

$U_{33} = \sqrt{(a_{33} - U_{13}^2 - U_{23}^2)}$

Q Compute cholesky factorization for symmetric.

$$A = \begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix}$$ symmetric

$U_{11} = \sqrt{a_{11}} = \sqrt{6} = 2.44949$

$U_{12} = \dfrac{a_{12}}{U_{11}} = \dfrac{15}{\sqrt{6}} = 6.123724$

Dated:

$$U_{13} = \frac{a_{13}}{U_{11}} = \frac{55}{2.44949} = 22.45366$$

$$U_{22} = \sqrt{a_{22} - U_{12}^2} = \sqrt{55-(6.123724)^2} = 4.1833$$

$$U_{23} = \frac{a_{23} - U_{12}U_{13}}{U_{22}} = 20.9165$$

$$U_{33} = \sqrt{a_{33} - U_{12}^2 - U_{23}^2} = 6.110101$$

$$U = \begin{bmatrix} 2.44949 & 6.1237 & 22.45366 \\ & 4.1833 & 20.9165 \\ & & 6.110101 \end{bmatrix}.$$

Examples from pdf.

Procedure to solve $Ax = b$.

1) Check $A = A^t$ and the definite
2) Factorize, $A = LL^t$
3) Solve $LY = B$
4) Solve $L^t X = Y$

Dated:

CHAP # 7 → Numerical Methods
(ITERATIVE METHOD)

· JACOBYS METHOD :

- System should be diagonaly dominant

Q. Solve Sys by Jacobi's Iterative Method (4 Iteration)

$8x - 3y + 2z = 20 \rightarrow 1$

$4x + 11y - z = 33 \rightarrow 2$

$6x + 3y + 12z = 35 \rightarrow 3$

$8x = 20 + 3y - 2z \rightarrow 1)$

$x = \frac{1}{8}(20 + 3y - 2z)$

$y = \frac{1}{11}(33 - 4x + z)$

$z = \frac{1}{12}(35 - 6x - 3y)$

Initial App $x_0 = y_0 = z_0 = 0$

First Iteration.

$x_1 = \frac{1}{8}(20 + 3(0) - 2(0)) = 2.5$

$y_1 = \frac{1}{11}(33 - 4(0) + 0) = 3$

$z_1 = \frac{1}{12}(35 - 6(0) - 3(0)) = 2.916667$

Second Iteration

$x_2 = \frac{1}{8}(20 + 3(3) - 2(2.916667)) = 2.895833$

$y_2 = \frac{1}{11}(33 - 4(2.5) + 2.916667) = 2.3560606$

$z_2 = \frac{1}{12}[35 - 6(2.5) - 3(3)] = 0.916666$

Dated:

Third Iteration

$x_3 = \frac{1}{8}[20 + 3(2.3560606) - 2(0.9166666)] = 3.1543561$

$y_3 = \frac{1}{11}[33 - 4(2.8958333) + 0.9166666] = 2.030303$

$z_3 = \frac{1}{12}[35 - 6(2.8958333) - 3(2.3560606)] = 0.8797348$

Fourth Iteration

$x_4 =$

$y_4 =$

$z_4 =$

If sys not diagonally dominant, then we'll arrange equations.

$3x_1 + 7x_2 + 13x_3 = 76$
$x_1 + 5x_2 + 3x_3 = 28$
$12x_1 + 3x_2 - 5x_3 = 1$

Arrange

$12x_1 + 3x_2 - 5x_3 = 1$
$x_1 + 5x_2 + 3x_3 = 28$
$3x_1 + 7x_2 + 13x_3 = 76$

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Dated:

GAUSS-SEIDEL METHOD: (Diagonally dominant)

- Most recently updated values of unknowns are used at each step, even if updating occurs in current step.

Q. $3U+V=5, \quad U+2V=5$

$$U = \frac{5-V}{3} \qquad V = \frac{5-U}{2}$$

$U_0 = V_0 = 0 \rightarrow$ Initial

$$U_{k+1} = \frac{1}{3}[5-V_k]$$

$$V_{k+1} = \frac{1}{2}[5-U_{k+1}]$$

$$\begin{bmatrix} U_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} 5-V_0/3 \\ 5-U_1/2 \end{bmatrix} = \begin{bmatrix} 5-0/3 \\ 5-5/3/2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5/3 \end{bmatrix}$$

$$\begin{bmatrix} U_2 \\ V_2 \end{bmatrix} = \begin{bmatrix} 5-V_1/3 \\ 5-U_2/2 \end{bmatrix} = \begin{bmatrix} 5-5/3/3 \\ 5-10/9/2 \end{bmatrix} = \begin{bmatrix} 10/9 \\ 35/18 \end{bmatrix}$$

$$\begin{bmatrix} U_3 \\ V_3 \end{bmatrix} = \begin{bmatrix} 5-V_2/3 \\ 5-U_3/2 \end{bmatrix} = \begin{bmatrix} 5-35/18/3 \\ \frac{5-55/54}{2} \end{bmatrix} = \begin{bmatrix} 55/54 \\ 215/108 \end{bmatrix}$$

Q. $$\begin{bmatrix} 3 & 1 & -1 \\ 2 & 4 & 1 \\ -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$U_{K+1} = \frac{4-V_K+W_K}{3} \qquad V_{K+1} = \frac{1-2U_{K+1}-W_K}{4}$$

$$W_{K+1} = \frac{1+U_{K+1}-2V_{K+1}}{5}$$

Scanned with CamScanner

Dated:

Relative Error:

$$\varepsilon_x = \left| \frac{New - Prev}{New} \right| \times 100$$

$$\varepsilon_x = \left| \frac{New - Prev}{New} \right| \times 100$$

continue untill all of error values are less than tolerance value.

Table

Iteration	x_1	$\lvert\varepsilon_{x_1}\rvert \times 100$	x_2	$\lvert\varepsilon_{x_2}\rvert\%$	x_3	$\lvert\varepsilon\rvert_{x_3}\%$