"Table of values"

x	x_0	x_1	x_2	$\cdots$	x_k	$\cdots$	x_{n-2}	x_{n-1}	x_n
$y=f(x)$	$y_0=f(x_0)$	$y_1=f(x_1)$	$y_2=f(x_2)$	$\cdots$	$y_k=f(x_k)$	$\cdots$	$y_{n-2}=f(x_{n-2})$	$y_{n-1}=f(x_{n-1})$	$y_n=f(x_n)$

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"Difference Operators"

1): Forward difference Operator Δ :-

The first order forward difference operator is denoted by Δ and defined by:

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\Delta y_2 = y_3 - y_2$$
$$\Delta y_{n-1} = y_n - y_{n-1} \quad (\text{What about } \Delta y_n = y_{n+1} - y_n)$$

The second order forward difference operator is defined by:

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$
$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1.$$
$$\cdots\cdots\cdots\cdots$$
$$\Delta^2 y_{n-2} = \Delta y_{n-1} - \Delta y_{n-2}$$

In the same way, the 3rd order forward difference operator Δ^3 is defined as follows:

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$
$$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$$

$$\Delta^3 y_{n-3} = \Delta^2 y_{n-2} - \Delta^2 y_{n-3}$$

In general,

$$\boxed{\Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k}$$

"Forward difference table"

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0	Δy_0			
x_1	y_1	Δy_1	$\Delta^2 y_0$		
x_2	y_2	Δy_2	$\Delta^2 y_1$	$\Delta^3 y_0$	$\Delta^4 y_0$
x_3	y_3	Δy_3	$\Delta^2 y_2$	$\Delta^3 y_1$	
x_4	y_4				

Important Note :-

In forward difference table the upper diagonal is conserved w.r.t. initial value y_0.

EX # 1 :-

Construct forward difference table for the following data:

x	0	1	2	3	4	5
$y=f(x)$	12	15	20	27	39	52

Solution :-

"Forward differemce table"

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	12	3				
1	15	5	2	0		
2	20	7	2	3	3	-10
3	27	12	5	-4	-7	
4	39 ?	13	1			
5	52					

We have, $y_0 = 12$, $\Delta y_0 = 3$, $\Delta^2 y_0 = 2$, $\Delta^3 y_0 = 0$, $\Delta^3 y_2 = -4$, $\Delta^4 y_0 = 3$ and $\Delta^5 y_0 = -10$.

EX # 2 :-

Express $\Delta^2 y_0$ and $\Delta^3 y_0$, in terms of the value of the function y.

Solution :-

We have

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$$

$$\Rightarrow \Delta^2 y_0 = y_2 - 2y_1 + y_0 \longrightarrow (1)$$

and

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (\Delta y_2 - \Delta y_1) - (\Delta y_1 - \Delta y_0)$$

$$\Rightarrow \Delta^3 y_0 = (y_3 - y_2) - (y_2 - y_1) - (y_2 - y_1) + (y_1 - y_0)$$

$$\Rightarrow \Delta^3 y_0 = y_3 - y_2 - y_2 + y_1 - y_2 + y_1 + y_1 - y_0$$

$$\Rightarrow \Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0 \longrightarrow (2)$$

Now,

from (1) and (2), we arrive at the following results:

$$\Delta^n y_0 = y_n - \binom{n}{1} y_{n-1} + \binom{n}{2} y_{n-2} - \binom{n}{3} y_{n-3} + \cdots + (-1)^n y_0$$

wher $\binom{n}{K} = {}^nC_K = \dfrac{n!}{K!(n-K)!}$

2): Backward difference Operator ∇ :-

The first order backward difference operator is denoted by ∇ and defined by:

$$\nabla y_1 = y_1 - y_0$$

$$\nabla y_2 = y_2 - y_1$$

$$\cdots\cdots\cdots$$

$$\nabla y_n = y_n - y_{n-1} \quad (\text{What about } \nabla y_0 = y_0 - y_{-1})$$

The second order backward difference operator is defined by:

$$\nabla^2 y_2 = \nabla . y_2 - \nabla y_1$$

$$\nabla^2 y_3 = \nabla y_3 - \nabla . y_2$$

$$\cdots\cdots\cdots\cdots$$

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

In the same way 3rd order backward differenc

Operator is defined by:

$$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$$
$$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$$
$$\cdots\cdots\cdots\cdots$$
$$\nabla^3 y_n = \nabla^2 y_n - \nabla^2 y_{n-1}$$

In general,

$$\boxed{\nabla^n y_K = \nabla^{n-1} y_K - \nabla^{n-1} y_{K-1}}$$

"Backward difference table"

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Important note :-

In backward difference table, the lower diagor is conserved w.r.t. final value y_n.

EX # 1 :-

Construct backward difference table for the following d

x	0	1	2	3	4
$y=f(x)$	2	3	12	35	78

Solution :-

"Backward difference table"

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0	2				
		1			
1	3		8		
		9		6	
2	12		14		0
		23		6	
3	35		20		
		43			
4	78				

We have (since $y_n = y_4$ here), $y_n = 78$, $\nabla y_n = 43$, $\nabla^2 y_n = 20$, $\nabla^3 y_n = 6$, $\nabla^4 y_n = 0$.

EX # 2 (HW) :-

Construct :

a): Forward difference table for :

x	0	2	4	6	8	10
$y=f(x)$	40	51.68	67.04	86.56	110.72	130.25

b): Backward difference table for :

x	10	20	30	40	50
$y=f(x)$	1	1.3010	1.4771	1.6021	1.6990

3): Central difference Operator δ :-

The first order central difference operator is denoted by δ and defined by:

$$\delta y_{1/2} = y_1 - y_0$$
$$\delta y_{3/2} = y_2 - y_1$$
$$\delta y_{5/2} = y_3 - y_2$$
$$\cdots\cdots\cdots$$
$$\delta y_{n-1/2} = y_n - y_{n-1}$$

Similarly higher order central differences are defined as follows :

$$\delta^2 y_{k+1/2} = \delta y_{k+1} - \delta y_k$$
$$\delta^3 y_{k+1/2} = \delta^2 y_{k+1} - \delta^2 y_k$$
$$\cdots\cdots\cdots\cdots$$
$$\delta^n y_{k+1/2} = \delta^{n-1} y_{k+1} - \delta^{n-1} y_k$$

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"Central difference table"

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^2 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

4): Shift Operator E (or Translation or Displacement):-

The shift operator E is defined by

$$\boxed{Ef(x) = f(x+h) \quad \text{or} \quad Ey(x) = y(x+h)}$$

Similarly higher order shift operator is defined by

$\Rightarrow E^2 f(x) = E[Ef(x)] = Ef(x+h) = f(x+h+h)$

$\Rightarrow E^2 f(x) = f(x+2h)$

$\Rightarrow E^3 f(x) = f(x+3h)$

In general

$$\boxed{E^n f(x) = f(x+nh)}$$

We have

$Ey_0 = Ey(x_0) = y(x_0+h) = y(x_1) = y_1 \quad \because x_1 = x_0 + h$

$\Rightarrow E y_1 = y_2 , \; E y_2 = y_3 , \ldots , \; E y_{n-1} = y_n$

$\Rightarrow E^2 y_0 = y_2$

$\Rightarrow E^4 y_0 = y_4$

$\Rightarrow E^n y_0 = y_n$

and

$E^2 y_2 = y_4 , \quad E^3 y_2 = y_5$ and so on.

The inverse operator E^{-1} is defined as:

$$\boxed{\begin{aligned} E^{-1} f(x) &= f(x-h) , \text{ and} \\ E^{-n} f(x) &= f(x-nh) \end{aligned}}$$

5): Average Operator μ :-

The average operator μ is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

Similarly,

$$\mu y(x) = \frac{1}{2}\left[y\left(x+\frac{h}{2}\right) + y\left(x-\frac{h}{2}\right)\right]$$

6): Differential Operator D :-

The differential operator D is defined a

$$D f(x) = \frac{d f(x)}{d x} = f'(x)$$

$$D^2 f(x) = \frac{d^2 f(x)}{d x^2} = f''(x)$$

$$D^3 f(x) = \frac{d^3 f(x)}{d x^3} = f'''(x)$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$D^n f(x) = \frac{d^n f(x)}{d x^n} = f^{(n)}(x)$$

7): Unit Operator 1 :-

The unit operator 1, is such that

$$\boxed{1 \cdot f(x) = f(x)}$$

"Properties of Operators"

1): Linear Property :-

The operators $\Delta, \nabla, \delta, E, \mu, D$ and 1 are all linear i.e. they satisfy:

$$\Delta[\alpha f(x) + \beta g(x)] = \alpha \Delta f(x) + \beta \Delta g(x), \quad \alpha, \beta \in \mathbb{R}$$

Examples :-

Differential and integral operators are linear. Since

1): $\dfrac{d}{dx}[\alpha f(x) + \beta g(x)] = \alpha \dfrac{df(x)}{dx} + \beta \dfrac{dg(x)}{dx}$

2): $\int [\alpha f(x) + \beta g(x)]\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$

2): Distributive over addition Property :-

These operators satisfy distributive over addition Property, that is

$$\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x) = \Delta^n \Delta^m f(x)$$

"Relation among the Operators"

1): Relation between E and Δ :-

By the definition of forward difference operator, we have

$$\Delta f(x) = f(x+h) - f(x) \quad \because \Delta y_0 = y_1 - y_0$$

$$\Rightarrow \Delta f(x) = E f(x) - f(x)$$

$$\Rightarrow \Delta f(x) = (E - 1) f(x)$$

Thus

$$\boxed{\begin{array}{c} \Delta = E - 1 \\ \hline E = \Delta + 1 \end{array}}$$

Here 1 is unit operator.

(a)

2): Relation between E and ∇ :-

By the definition backward difference operator we have

$$\nabla f(x) = f(x) - f(x-h) \qquad \because \nabla y_0 = y_0 - y_{-1}$$

$$\Rightarrow \nabla f(x) = f(x) - E^{-1} f(x)$$

$$\Rightarrow \nabla f(x) = (1 - E^{-1}) f(x)$$

Thus

$$\boxed{\begin{array}{c} \nabla = 1 - E^{-1} \\ \hline E^{-1} = 1 - \nabla \\ \hline E = (1 - \nabla)^{-1} \end{array}} \qquad \begin{array}{l} \because (D^{-1})^{-1} = \int^{-1} = D \\ \because (E^{-1})^{-1} = E \end{array}$$

where 1 is the unit operator.

3): Relation between E and δ :-

By the definition of central difference operator, we have

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

$$\Rightarrow \delta f(x) = E^{1/2} f(x) - E^{-1/2} f(x)$$

$$\Rightarrow \delta f(x) = (E^{1/2} - E^{-1/2}) f(x)$$

Thus

$$\boxed{\delta = E^{1/2} - E^{-1/2}}$$

Now

$$\boxed{\begin{array}{c} \delta = E^{-1/2}(E - 1) = E^{-1/2} \Delta \\ \delta = E^{1/2}(1 - E^{-1}) = E^{1/2} \nabla \end{array}}$$

and

Hence

$$\boxed{\delta = E^{1/2} - E^{-1/2} = E^{-1/2}\Delta = E^{1/2}\nabla}$$

4): Relation between E and μ :-

By the definition of average operator, we have

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

$$\Rightarrow \mu f(x) = \frac{1}{2}\left[E^{1/2} f(x) + E^{-1/2} f(x)\right]$$

$$\Rightarrow \mu f(x) = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x)$$

Thus

$$\boxed{\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]}$$

5): Relation between D and Δ :-

By the definition of differential operator we have

$$D f(x) = \frac{d f(x)}{dx} = f'(x)$$

$$D^2 f(x) = \frac{d^2 f(x)}{dx^2} = f''(x)$$

and so on.

Using Taylor's series expansion, we have

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) \ldots$$

$$\Rightarrow E f(x) = f(x) + \frac{h}{1!} D f(x) + \frac{h^2}{2!} D^2 f(x) + \frac{h^3}{3!} D^3 f(x) \ldots$$

$$\Rightarrow E f(x) = \left[1 + \frac{hD}{1!} + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \cdots\right] f(x)$$

$$\Rightarrow \quad Ef(x) = e^{hD} f(x) \quad \because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

Thus

$$\boxed{\begin{aligned} E &= e^{hD} \\ \Rightarrow \quad E &= 1 + \Delta = e^{hD} \end{aligned}}$$

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Again

$$e^{hD} = E = 1 + \Delta$$

$$\Rightarrow \quad hD = \ln E = \ln(1+\Delta)$$

Now, consider

$$hD = \ln(1+\Delta)$$

Since $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$$\Rightarrow \quad hD = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots$$

$$\Rightarrow \quad \boxed{D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \cdots\right]}$$

EX # 1 :-

Prove that :

$$hD = -\ln(1-\nabla) = \sinh^{-1}(\mu\delta)$$

$-\ln(1-\nabla)$
$-\ln(E)$
$\ln(E^{-1})^{-1}$
$\ln E$
$\ln e^{hD}$
hD ✓

Proof :-

We Know that :

$$hD = \ln E = \ln(E^{-1})^{-1} = -\ln E^{-1}$$

Hence

$$\boxed{hD = -\ln(1-\nabla)} \qquad \because E^{-1} = 1 - \nabla$$

Now

$$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1})$$

$$\Rightarrow \quad \mu\delta = \frac{1}{2}(e^{hD} - e^{-hD}) \qquad \because E = e^{hD}$$

$$\Rightarrow \quad \mu\delta = \sinh(hD) \qquad \because \sinh z = \frac{e^z - e^{-z}}{2}$$

$$\Rightarrow \boxed{hD = \sinh^{-1}(\mu\delta)}$$

Hence

$$\boxed{hD = -\ln(1-\nabla) = \sinh^{-1}(\mu\delta)}$$

EX # 2 :-

Show that the operators μ and E commutative.

Proof :-

By the definition of operators μ and E, we have

$$\mu E y_0 = \mu y_1 = \frac{1}{2}\left[y_{3/2} + y_{1/2}\right] \longrightarrow (1)$$

$$\because \mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

While

$$E\mu y_0 = E\,\frac{1}{2}\left[y_{1/2} + y_{-1/2}\right] = \frac{1}{2}\left[y_{3/2} + y_{1/2}\right] \longrightarrow (2)$$

Equaling (1) and (2), we have

$$\mu E y_0 = E\mu y_0$$

Thus

$$\boxed{\mu E = E\mu}$$

Therefore, the operators μ and E commute.

EX # 3 :-

Prove that :

1): $1 + \mu^2\delta^2 = \left(1 + \frac{\delta^2}{2}\right)^2$ 2): $E^{1/2} = \mu + \frac{\delta}{2}$

3): $\Delta = \frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}}$ 4): $\mu\delta = \frac{\Delta}{2} + \frac{\Delta E^{-1}}{2}$

5): $\mu\delta = \frac{\Delta + \nabla}{2}$

Proof (1) :-

By the definition of operators, we have

$$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1})$$

Therefore

$$1 + (\mu\delta)^2 = 1 + \frac{1}{4}(E - E^{-1})^2 = 1 + \frac{1}{4}(E^2 - 2 + E^{-2})$$

$$\Rightarrow 1 + \mu^2\delta^2 = \frac{4 + E^2 - 2 + E^2}{4} = \frac{E^2 + 2 + E^{-2}}{4}$$

$$\Rightarrow 1 + \mu^2\delta^2 = \frac{1}{4}(E + E^{-2})^2 \longrightarrow (1)$$

Now, Consider

$$1 + \frac{\delta^2}{2} = 1 + \frac{1}{2}(E^{1/2} - E^{-1/2})^2 = \frac{2 + E - 2 + E^{-1}}{2}$$

$$\Rightarrow 1 + \frac{\delta^2}{2} = \frac{1}{2}(E + E^{-1})$$

$$\Rightarrow \left(1 + \frac{\delta^2}{2}\right)^2 = \frac{1}{4}(E + E^{-1})^2 \longrightarrow (2)$$

Equating (1) and (2), we get

$$1 + \mu^2\delta^2 = \left(1 + \frac{\delta^2}{2}\right)^2$$

Proof (2) :-

By the definition of operators, we have

$$\mu + \frac{\delta}{2} = \frac{1}{2}(E^{1/2} + E^{-1/2}) + \frac{1}{2}(E^{1/2} - E^{-1/2})$$

$$\Rightarrow \boxed{\mu + \frac{\delta}{2} = E^{1/2}}$$

Proof (3) :-

By the definition of operator, we have

$$\frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}}$$

$$= \frac{(E^{1/2} - E^{-1/2})^2}{2} + (E^{1/2} - E^{1/2})\sqrt{1 + \frac{(E^{1/2} - E^{-1/2})^2}{4}}$$

$$= \frac{E - 2 + E^{-1}}{2} + (E^{1/2} - E^{-1/2})\sqrt{\frac{4 + E - 2 + E^{-1}}{4}}$$

$$= \frac{E - 2 + E^{-1}}{2} + (E^{1/2} - E^{-1/2})\sqrt{\frac{(E^{1/2} + E^{-1/2})^2}{4}}$$

$$= \frac{E - 2 + E^{-1}}{2} + \frac{1}{2}(E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2})$$

$$= \frac{E - 2 + E^{-1} + E - E^{-1}}{2}$$

$$= E - 1 = \Delta$$

Hence

$$\boxed{\frac{\delta^2}{2} + \delta\sqrt{1 + \frac{\delta^2}{4}} = \Delta}$$

Proof (4) :-

We have

$$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1})$$

$$\mu\delta = \frac{1}{2}(1 + \Delta - E^{-1}) = \frac{\Delta}{2} + \frac{1}{2}\left[1 - \frac{1}{E}\right]$$

$$\Rightarrow \mu\delta = \frac{\Delta}{2} + \frac{1}{2}\,\frac{E-1}{E} = \frac{\Delta}{2} + \frac{\Delta E^{-1}}{2}$$

Proof (5) :-

We have

$$\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1})$$

$$\Rightarrow \mu\delta = \frac{1}{2}(1 + \Delta - 1 + \nabla) = \frac{1}{2}(\Delta + \nabla)$$

EX # 4 :-

Find :

a): $\Delta \log x$ b): $\Delta \tan^{-1} x$

Proof (a) :-

By the definition of forward operator, we have

$\Delta \log x = \log(x+h) - \log x \quad \because \Delta y_0 = y_1 - y_0$

$$\Rightarrow \Delta \log x = \log \frac{x+h}{x} = \log\left[1 + \frac{h}{x}\right]$$

Proof (b) :-

By the definition of forward operator, we have

$\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$

$$\text{Since } \tan^{-1} A \pm \tan^{-1} B = \tan^{-1}\left[\frac{A \pm B}{1 \mp AB}\right]$$

$$\Rightarrow \Delta \tan^{-1} x = \tan^{-1}\left[\frac{x+h-x}{1+(x+h)x}\right]$$

$$\Rightarrow \Delta \tan^{-1} x = \tan^{-1}\left[\frac{h}{1+hx+x^2}\right]$$

EX # 5 :-

Evaluate (Taking 1 as the interval of differencing)

a): $\Delta\left[\frac{5x+12}{x^2+5x+6}\right]$ b): $\Delta^n\left[\frac{1}{x}\right]$

Sol (a) :-

By the definition of forward operator, we have

$$\Delta\left[\frac{5x+12}{x^2+5x+6}\right] = \Delta\left[\frac{2(x+3)+3(x+2)}{(x+2)(x+3)}\right]$$

$$\Rightarrow = \Delta\left[\frac{2}{x+2} + \frac{3}{x+3}\right]$$

$$\Rightarrow = \Delta\left[\frac{2}{x+2}\right] + \Delta\left[\frac{3}{x+3}\right]$$

$$\Rightarrow = \left[\frac{2}{x+1+2} - \frac{2}{x+2}\right] + \left[\frac{3}{x+1+3} - \frac{3}{x+3}\right]$$

Since interval of differencing is 1 mean's $h = 1$.

$$\Rightarrow \Delta\left[\frac{5x+12}{x^2+5x+6}\right] = \frac{-2}{(x+2)(x+3)} - \frac{3}{(x+3)(x+4)}$$

Sol (b) :-

By the definition of forward operator, we have

$$\Delta\left[\frac{1}{x}\right] = \frac{1}{x+1} - \frac{1}{x} = \frac{(-1)\,1!}{x(x+1)}$$

Then

$$\Rightarrow \Delta^2\left[\frac{1}{x}\right] = \frac{-1}{(x+1)(x+1+1)} + \frac{1}{x(x+1)}$$

$$\Rightarrow \qquad = \frac{-1}{(x+1)(x+2)} + \frac{1}{x(x+1)}$$

$$\Rightarrow \qquad = \frac{(-1)^2\, 2!}{x(x+1)(x+2)} \qquad \frac{-1(x)+x+2}{x(x+1)(x+2)}$$

Similarly, we can get

$$\Rightarrow \boxed{\Delta^n\left[\frac{1}{x}\right] = \frac{(-1)^n \cdot n!}{x(x+1)(x+2)\cdots(x+n)}}$$

EX # 6 :-

Evaluate : $\left(\frac{\Delta^2}{E}\right)x^3$.

Solution :-

Let h be the interval of differencing, then

$$\left(\frac{\Delta^2}{E}\right)x^3 = (\Delta^2 E^{-1})x^3 = (E-1)^2 E^{-1} x^3$$

$$\Rightarrow \qquad = (E^2 - 2E + 1)E^{-1}x^3 = (E - 2 + E^{-1})x^3$$

$$\Rightarrow \qquad = Ex^3 - 2x^3 + E^{-1}x^3$$

$$\Rightarrow \qquad = (x+h)^3 - 2x^3 + (x-h)^3$$

$$\Rightarrow \qquad = x^3 + h^3 + 3x^2h + 3xh^2 - 2x^3 + x^3 - h^3 - 3x^2h + 3xh^2$$

$$\Rightarrow \boxed{\left(\frac{\Delta^2}{E}\right) x^3 = 6xh^2}$$

$\Delta f(3)$ 17

$\Delta f(2) = f(2$

$f(3) = \Delta f(2$

Note :-

If $h = 1$, then $\left(\frac{\Delta^2}{E}\right) x^3 = 6x$

EX # 7 :-

Prove that : $f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$

Proof :-

By the definition of forward operator, we have

$f(4) - f(3) = \Delta f(3)$ $\quad \because \Delta y_0 = y_1 - y_0$

$\Rightarrow \quad = \Delta[f(2) + \Delta f(2)]$ $\quad \because \Delta f(2) = f(3) - f(2)$

$\Rightarrow \quad = \Delta f(2) + \Delta^2 f(2)$ $\quad \because \Delta f(1) = f(2) - f(1)$

$\Rightarrow \quad = \Delta f(2) + \Delta^2[f(1) + \Delta f(1)]$

$\Rightarrow f(4) - f(3) = \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$

Hence

$$\boxed{f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)}$$

EX # 8 :-

Given $u_0 = 1$, $u_1 = 11$, $u_2 = 21$, $u_3 = 28$ and $u_4 = 29$. Find $\Delta^4 u_0$.

Solution :-

We have

$\Delta^4 u_0 = (E - 1)^4 u_0$ $\quad \because \Delta = E - 1$

$$\Rightarrow \Delta^4 u_0 = \left[E^4 - \binom{4}{1} E^3 + \binom{4}{2} E^2 - \binom{4}{3} E + 1\right] u_0$$

$$\Rightarrow \Delta^4 u_0 = \left[E^4 - 4E^3 + 6E^2 - 4E + 1\right] u_0$$

$\Rightarrow \Delta^4 U_0 = E^4 U_0 - 4E^3 U_0 + 6E^2 U_0 - 4E U_0 + U_0$

$\Rightarrow \Delta^4 U_0 = U_4 - 4U_3 + 6U_2 - 4U_1 + U_0$

$\Rightarrow \Delta^4 U_0 = 29 - 112 + 126 - 44 + 1$

$\Rightarrow \boxed{\Delta^4 U_0 = 0}$

EX # 9 :-

Given $U_0 = 3$, $U_1 = 12$, $U_2 = 81$, $U_3 = 200$, $U_4 = 100$ and $U_5 = 8$. Find $\Delta^5 U_0$.

Solution :-

We have

$\Delta^5 U_0 = (E - 1)^5 U_0$

$\Rightarrow \Delta^5 U_0 = \left[E^5 - \binom{5}{1}E^4 + \binom{5}{2}E^3 - \binom{5}{3}E^2 + \binom{5}{4}E - 1\right]U_0$

$\Rightarrow \Delta^5 U_0 = \left[E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1\right]U_0$

$\Rightarrow \Delta^5 U_0 = E^5 U_0 - 5E^4 U_0 + 10E^3 U_0 - 10E^2 U_0 + 5E U_0 - U_0$

$\Rightarrow \Delta^5 U_0 = U_5 - 5U_4 + 10U_3 - 10U_2 + 5U_1 - U_0$

Substituting values, we get

$\Rightarrow \Delta^5 U_0 = 8 - 500 + 2000 - 810 + 60 - 3 = 755$

Hence

$\boxed{\Delta^5 U_0 = 755}$