Dated:

NUMERICAL COMPUTING

- Numerical computing is study of algorithms that are used to obtain approximate sol. of a mathematical problem.

Need of NC

i) No analytical solution exists.
ii) An analytical solution is difficult to obtain.

IEEE 754 Floating Point Standard.

Single Precision (32 bit)

- 23 bits used for significant digits
- 8 bit used for store exponent
- 1 bit used to store sign (+, -).

Double Precision (64 bit)

- 52 for significant digits
- 11 bit exponent.
- 1 for sign.

Accuracy and Precision

- Accuracy is related to the closeness to the true value.
- Precision is related to the closeness to the estimated value.

Rounding and Chopping

- Rounding is replacing no by nearest machine number.

E.g:
$\pi = 3.1415926$
$\pi = 3.1416 \longrightarrow$ Rounding

Dated:

- Chopping is throwing all or dropping the extra digits.
  E.g. $\pi = 3.\,1415926$
  $\pi = 3.1415$ → Chopping

No increasing in chop.

ERROR

- Difference b/w an approximation of no used in computation and its exact value.

Error = True Value − Approx Value.
$= 3.1415926 - 3.1416 = -0.000074 = 7.4\times10^{-5}$

- It can be both −ve or +ve, but if absolute to +ve.
- Better to write in significant digits.

ERROR ANALYSIS

TRUNCATION ERROR: are when an iterative method is terminated or mathematical procedure is approximated and approximate sol differs from exact solution.

DISCRETIZATION ERROR: are committed when a solution of discrete problem doesnot coincide with solution of continuous problem.

ERROR IN COMPUTING METHOD

1) TRUE ERROR:
can be computed if true value is known.

Absolute Error = ~~Ae~~ = |true value − approximation|

Dated:

Absolute relative Error = ARE = $\left|\frac{\text{True value} - \text{Approximation}}{\text{True Value}}\right|$

2) Estimated Error:

- When true value is not known
- Diff b/w two consecetive values.

Estimated Absolute Error = AE = |current est − prev esti|

$$ARE = \left|\frac{\text{current est} - \text{prev est}}{\text{curret est}}\right|$$

LOSS OF SIGNIFICANCE

- Occurs in numerical calculations when too many significant digits cancel.
- Often involves subtracting two almost equal numbers.

Loss of PRECISION THEOREM.

let x, y be normal floating point no with $x > y > 0$. If $2^{-p} \le 1 - \frac{y}{x} \le 2^{-q}$ for some +ve int p and q, then at most p and atleast q significant binary digits are lost in subtraction $x-y$.

Q. $x - y = 37.593621 - 37.584216$, How many sig bits are lost?

$$1 - \frac{y}{x} = 0.0002501754$$

2^{-12} and 2^{-11}, hence '11' at list and 12 at most are lost.

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AVOIDING LOSS OF SIGNIFICANCE

i. Rationalizing:

$$f(x) = \sqrt{(x^2+1)} - 1$$

- Near zero there is potential loss of ~~zeros~~ significance.
- so we ~~re~~write function by ~~form~~ of rationalizing.

$$f(x) = \left(\sqrt{x^2+1} - 1\right)\left(\frac{\sqrt{(x^2+1)}+1}{\sqrt{x^2+1}+1}\right)$$

$$f(x) = \frac{x^2}{\sqrt{x^2+1}+1}$$

ii. Using Series Expansion:

$$f(x) = x - \sin x$$

we uses taylor series for sin to avoid loss of significance.

For x near zero, the series converges quite rapidly

$$\sin x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \dots\right)$$

iii. Using trignometric Identities:

$$f(x) = \cos^2(x) - \sin^2(x)$$

Lose of significance at $x = \pi/4$.

$\cos^2(x) - \sin^2(x) = \cos(2x) \rightarrow$ can be solved by trigno identities.

Dated:

e.g:

$$e^{-5} = 1 + \frac{(-5)}{1!} + \frac{(-5)^2}{2!} + \frac{(-5)^3}{3!} + \dots$$

In this case, there will be loss of significance in calculation of sum.

so,

$$e^{-5} = \frac{1}{e^5} = \frac{1}{\text{series for } e^5}$$ → not involving +ve −ve terms.

INTERMEDIATE VALUE THEOREM

If $f \in C[a,b]$ and k is any no b/w $f(a)$ and $f(b)$, then there exists a number 'c' in (a,b) for which $f(c) = k$.

Ex: $x^5 - 2x^3 + 3x^2 - 1 = 0$ has sol in interval $[0, 1]$.

$f(0) = -1 \qquad f(1) = 1$

$0 < x < 1$

It is used to determine when sol to certain problems exists.

TAYLOR'S THEOREM. $(n \to \infty)$

$$f(x) = P_n(x) + R_n(x)$$

where,

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)(x - x_0)^2}{2!} + \frac{f'''(x_0)(x - x_0)^3}{3!} + \dots + \frac{f^{(n)}(x_0)(x - x_0)^n}{n!}$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x - x_0)^k}{k!}$$

Taylor Polynomial.

Remainder Term $R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}$.

Depends on value of 'x' at which $P_n(x)$ is being evaluated.

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MACLAURIN POLYNOMIAL ($x_0 = 0$)

- When $x_0 = 0$, Taylor series called Maclaurin Polynomial.

Ex 3: $f(x) = \cos x$ and $x_0 = 0$

a) Second Taylor Polynomial
b) Third " ".

a) $n = 2$ $f'(x) = -\sin x$ $f''(x) = -\cos x$ $f'''(x) = \sin x$

$$= f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(\xi(x))}{3!}(x)^3$$

$$= 1 + \left(-\frac{1}{2}\right)x^2 + \frac{1}{6}x^3 \sin \xi(x).$$

Ex#1.2

Que #1.

Compute Absolute Error and Relative errors of p by p^*

a. $p = \pi$ $p^* = \frac{22}{7}$

$AE = \left|\pi - \frac{22}{7}\right| = 1.2644 \times 10^{-3}$

$ARE = \left|\frac{\pi - 22/7}{\pi}\right| = 4.0249 \times 10^{-4}$.

b. $p = \pi$, $p^* = 3.1416$

$AE = |\pi - 3.1416| = 7.3464 \times 10^{-6}$

$ARE = |\pi - 3.1416|/\pi = 2.338 \times 10^{-6}$

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c. $p = e$, $p^* = 2.718$

$AE = |e - 2.718| = 2.818 \times 10^{-4}$

$ARE = |e - 2.7181| / e = 1.0367 \times 10^{-4}$

g. $p = 8!$, $p^* = 39900$

$AE = |8! - 39900| = 420$

$ARE = 0.0104 = 10.416 \times 10^{-3}$

Que # 4

i) exactly ii) 3-digit chopping iii) 3-digit rounding iv) error in (ii) (AE)

a) i) $\frac{4}{5} + \frac{1}{3} = \frac{17}{15} = 1.133 = 1.133 \times 10^{0}$

ii) $1.133\boxed{333} = 1.133$ iii) 1.133 iv) 0.0003, $= 300 \times 10^{-6}$

b) $\frac{4}{5} \cdot \frac{1}{3}$

i) $4/15 = 0.2666$ ii) $0.26666667 = 0.266$ iii) 0.267

c) $\left(\frac{1}{3} - \frac{3}{11}\right) + \frac{3}{20}$

i) $\frac{139}{660} = 0.210606061$ ii) 0.210 iii) $0.2106 = 0.211$

iv) 2.9×10^{-3} and 1.9×10^{-3}

Que # 5

a. $133 + 0.921$

$= 133.9210$

$AE = |133.9210 - 134| = 79 \times 10^{-3}$

$ARE = \left|\frac{133.9210 - 134}{133.9210}\right| = 0.0005899 = -589.9 \times 10^{-6}$

Dated:

CHAP #2 SOLUTION (ROOT) OF EQUATIONS IN ONE VARIABLE

$x^2 = e^x$

$x^2 - 4x + 5 = 0$ → Polynomial $P_2(x)$

$f(x) = x - e^x + \cos x + \cos x$ → Transcendtial function

$P_n(x) = a_0 + a_1 x^2 + \dots a_n x^n$

1st, 2nd

$P_0(x) = 1$ $P_1(x) = 1 + x$ $P_2(x) = 1 + x + \frac{x^2}{2}$ $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ Third

$P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^2}{24}$ fourth degree polynomial

Q. $x^2 - 5x + 6 = 0$ (To give roots)

$(x-3)(x-2) = 0$

$x = 3, 2$ ⇒ x-intercept / roots.

- Function $f(x) = 0$, b/c it is on x-intercept.

SOLUTIONS

Bracketing Method

- Bisection
- Regular Falsie

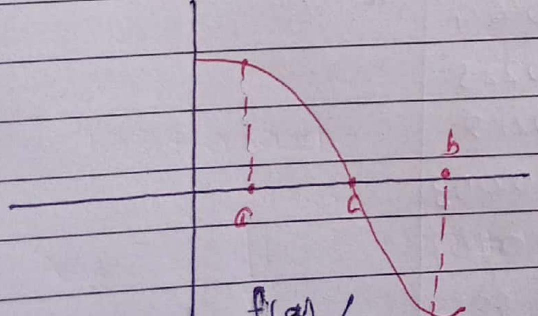
Open Method $x_0 = 0$

- Newton
- Secant
- Fixed point

Bisection: Solution should exists in the given interval so product of both should be less than zero. $[a, b]$

Newton:
- Single value is given as initial value

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ Newton



$f(a) \cdot f(b) < 0$

Dated:

BISECTION METHOD

- Tolerance value (near to zero)

Solve $f(x) = x^2 - 2$; $[\overset{a}{1}, \overset{b}{2}]$

$f(1) = 1 - 2 = -1$ (-ve) $f(2) = 4 - 2 = 2$ (+ve)

$f(1) f(2) = (-1)(+2) = -2 < 0$

Bisection Method: / Binary search Method

$C = \frac{a+b}{2} = \frac{1+2}{2} = \frac{3}{2} = 1.5 \in [1, 2]$

$f(c) = f(1.5) = 2.25 - 2 = 0.25$ (+ve)

- If next value i.e $f(c)$ is +ve so previous +ve will be replaced by it, if $f(c)$ is -ve replaced by -ve.

$a = 1$, ~~b = 0.25~~ $b = 1.5$ $\in$ $[1, 1.5]$

$c = \frac{a+b}{2} = \frac{1+1.5}{2} = 1.25$ (+ve) $\in [1, 1.5]$

$f(1.25) = 1.5625 - 2 = -0.4375$ (-ve)

$[1.25, 1.5]$

$c = \frac{1.25 + 1.5}{2} = 1.375 \in [1.25, 1.5]$

a	b	c = a+b/2	f(a)	f(b)	f(c)
1	2	1.5	+ve	+ve	0.25
1	1.5	1.25	-ve	+ve	-0.4375
1.25	1.5	1.375	"	"	-0.1094
1.375	1.5	1.4375	"	"	0.0664
1.375	1.4375	1.4063	"	"	-0.0225
1.4063	1.4375	1.4219	"	"	0.0218
1.4063	1.4219	1.4141	"	"	-0.0003
1.4141	1.4219	1.4180	"	"	0.0107
"	1.4180	1.4161			0.0052

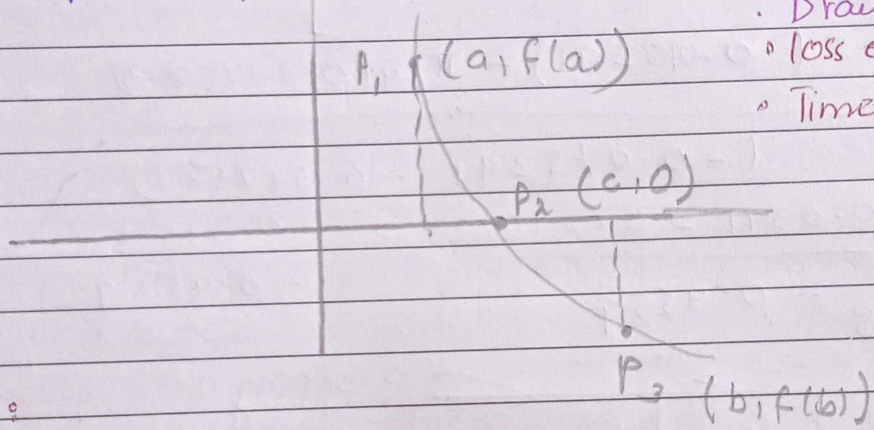
Dated:

. Stoping Criteria :

1) Number of iteration is given
2) $f(c) = 0$
3) $|f(c)| < e$ e = Tolerance Value
4) $|c_n - c_{n-1}| < E \rightarrow AE$
5) $\left|\frac{c_n - c_{n-1}}{c_n}\right| < e \rightarrow ARE$

- consective terms ka diff bhi lyty jao to error.

REGULAR FALSE METHOD



. Drawbacks :
- loss of significance
- Time consuming

Slope :

P_1 and P_3 $\quad m_1 = \dfrac{f(b) - f(a)}{b - a}$

P_2 and P_3 $\quad m_2 = \dfrac{f(b) - \cancel{0}}{b - c}$

For c :

$$m_1 = m_2$$

$$\frac{f(b) - f(a)}{b - a} = \frac{f(b)}{b - c}$$

$$\boxed{c = \frac{f(b)(b-a)}{f(b) - f(a)}}$$

$$\boxed{c = b - \frac{f(b-a)f(b)}{f(b) - f(a)}}$$

Dated:

Example 3:

Use false position method to find a solution to $\cos x = x$ and compare approximation.

$a = 0.5$, $b = \frac{\pi}{4}$

$f(x) = \cos x - x$, $[0.5, \pi/4]$

$f(0.5) = 0.3776 > 0$ $\qquad \epsilon = 10^{-6}$

$f(\pi/4) = -0.078291 < 0$

$f(0.5) f(\frac{\pi}{4}) < 0 \quad \exists \quad [0.5, \pi/4]$

$$f = \frac{0.5(-0.078291) - \pi/4(0.377583)}{(-0.078291 - 0.377583)}$$

$$c = \frac{-0.0391 - 0.2966}{-0.4559} = \frac{-0.3357}{-0.4559} = 0.7364$$

$f(c) = 0.5$

Answer

stopping and updating (a, b)

n	a	b	c	f(c)
1	0.5	~~0.5~~ $\pi/4$	0.7364	0.004577
2	0.7364	$\pi/4$	0.7391	0.0000004517
3	0.7391	$\pi/4$	0.7391	0.000000451 < ϵ

$\boxed{n = 3 \quad 0.7390848638}$

* Algo:

1) → Start
2) → Define $f(x)$
3) → Input a, b, ϵ, n go to step 3
4) → check if $f(a) f(b) > 0$ print "Initial value is not correct"

Dated:

OPEN METHODS

NEWTON-RAPHSON

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)} \quad , \quad n = 0, 1, 2, 3, \ldots$$

If $n = 0$

$$P_1 = P_0 - \frac{f(P_0)}{f'(P_0)}$$

If $n = 1$

$$P_2 = P_1 - \frac{f(P_1)}{f'(P_1)}$$

⋮

$$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})}$$

• Drawback :

If derivative possible to possible, if derivative zero then infinite ky loop mei chalajayega

Continuous

Differentiation   Integrable.

Q6. a) $e^x + 2^{-x} + 2\cos x - 6 = 0$, $1 \leq x \leq 2$ $\epsilon = 10^{-5}$

$$y' = a^x \ln a$$

$$y = 2^{-x}$$

$$y' = -2^{-x} \ln 2$$

$$y' = \frac{-\ln 2}{2^x}$$

$$f(x) = e^x + 2^{-x} + 2\cos x - 6$$

$$f'(x) = e^x - \frac{\ln 2}{2^x} - 2\sin x$$

Initial Value :

$$P_0 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5 \text{ (Mid value)}$$

$$P_0 = 1.5$$

$f(1) = -ve$

$f(2) = +ve$

$f(1)f(2) < 0 \quad x \in [1, 2]$

Dated:

x	$f(x)$	Interval
-2	-ve	[-2, -1]
-1	+ve	[0, 1]
0	+ve	[-2, 0]
1	-ve	[-1, 1]
2	-ve	

1st Iteration:

$$P_1 = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{-1.02328}{2.24163}$$

$$= 1.95649065$$

2nd Iteration:

$$P_2 = 1.95649065 - \frac{f(1.956)}{f'(1.956)}$$

$$= \text{"} \quad \frac{-0.0579694391}{4.87480871}$$

n	P	Error
0	1.5	-
1	1.956489	0.456489
2	1.341533	
3	1.3295060	
4	1.8295060	
5	1.829383615	
6	1.829383	

n	P	Error
0	1	-
1	3.469798011	0.74367
2	2.726126469	
3	2.197294484	
4	1.914273084	
5	1.834995797	

Dated:

Q. Find $\sqrt[3]{29}$

let $x = \sqrt[3]{29}$

$x^3 = 29$

$f(x) = x^3 - 29 = 0$

$f'(x) = 3x^2$

P_0 = Initial value.

$\sqrt[3]{29} = 3.0723$ (Agar '4' lia to value slowly diverge bhi karskti or converge bhi)

$P_0 = 3$

$P_1 = 3 - \frac{f(3)}{f'(3)}$

$P_1 = 3.0740$

$P_2 = 3.07231783$

$P_3 =$

Dated:

2) # SECANT- METHOD

Newton Formula:

$$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})} \quad \text{for } n \geq 1$$

- Diff b/w regular falsi and secant is that in falsi we have to check a, b to find new 'c', but no in secant.

Secant Formula: for $n \geq 2$

$$P_n = P_{n-1} - \frac{f(P_{n-1})}{\frac{f(P_{n-1}) - f(P_{n-2})}{P_{n-1} - P_{n-2}}}$$

- Diff b/w Newton and secant is that derivative is replaced by $\frac{y_2 - y_1}{x_2 - x_1}$

- Newton we need single value in secant do values.

OR

$$P_n = P_{n-1} - \frac{f(P_{n-1})(P_{n-1} - P_{n-2})}{f(P_{n-1}) - f(P_{n-2})}$$

$n = 2$

$$\Rightarrow P_2 = P_1 - \frac{f(P_1)(P_1 - P_0)}{f(P_1) - f(P_0)} = \frac{P_0 f(P_1) - P_1 f(P_0)}{f(P_1) - f(P_0)}$$

For new iteration $P_0 = P_1$, $P_1 = 2$ (updation)

$$P_3 = \frac{P_2 - f(P_2)(P_2 - P_1)}{f(P_2) - f(P_1)} = \frac{P_1 f(P_2) - P_2 f(P_1)}{f(P_2) - f(P_1)}$$

- Newton draw tangent on graph and secant draw secant on graph.

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Dated:

Q.5 (c) Repeat Ex '5' with secant.

$x - \cos x = 0$ $\quad [0, \pi/2]$ $\quad E = 10^{-5}$

P_0, P_1 (You have to show errors)

$f(x) = x - \cos x$

~~f(0)~~

Let $P_0 = 0$ and $P_1 = \frac{\pi}{2}$

$f(0) = 0 - \cos 0 = -1$

$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2}$

For $n = 2$:

$$P_2 = \frac{P_0 f(P_1) - P_1 f(P_0)}{f(P_1) - f(P_0)}$$

$$= \frac{(0)\left(\frac{\pi}{2}\right) - \frac{\pi}{2}(-1)}{\frac{\pi}{2} - (-1)} = \frac{+\pi/2}{\pi/2 + 1} = \frac{1.5707}{2.570796}$$

$P_2 = 0.61100\ldots8947$

For next iteration: $(n = 3)$

$P_0 = \frac{\pi}{2}$ $\quad P_1 = 0.6110154704$

$f(P_0) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2}$

$f(P_1) = 0.6110154704 - \cos(0.6110154)$

$= 0.6110154704 - 0.819065$

$f(P_1) = -0.208050395$

$$= \frac{\pi/2\,(-0.208050395) - (0.6110154704)(\pi/2)}{-0.208050395 - \pi/2} = 0.7232883445$$

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Dated:

For Calculator:

$$c = \frac{a(b-\cos b) - b(a-\cos a)}{(b-\cos b) - (a-\cos a)}$$

$$c = \frac{\cancel{ab} - a\cos b - \cancel{ab} + b\cos a}{(b - a - \cos b + \cos a)}$$

a. A. Error $< 10^{-5}$

b. R. Error $< 10^{-5}$

$$c = \frac{b\cos a - a\cos b}{(b - a - \cos b + \cos a)}$$

n	a	b	$c = \frac{af(b) - bf(a)}{f(b) - f(a)}$	Error
1	0	$\pi/2$	0.6110154704	
2	$\pi/2$	0.6110154	0.72326954	
3	0.6110154	0.72326954	0.73956707	
4	0.72326954	0.73956707	0.7390834365	
5			0.739085133	

Q5(d). $f(x) = x - 0.8 - 0.2\sin x$ $[0, \pi/2]$

Q6 (e) $f(x) = \sin x - e^{-x} = 0$, $[0,1]$ P_0, P_1, $[3,4]$ P_0, P_1 $[6,7]$ P_0, P_1

Q5(d)

n	a	b	$c = \frac{af(b) - bf(a)}{f(b) - f(a)}$	Error
1.	0	$\pi/2$	0.916720476 2	
2.	$\pi/2$		0.9615513264	
3.	0.91672	0.96155	0.9643460851	
4.	0.96155	0.96434	0.9643338845	

Q6(e) $[0,1]$

n	a	b	c	Error
1	0	1	0.678614	
2	1	0.6786	0.5690622	
3.	0.6786	0.56906	0.5892596	
4.	0.56906	0.5892	0.5885383	
5.	0.5892	0.58853	0.5885327	

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Dated:

* Formula for no of iterations required for given tolerance:

$$|P-P_n| \leq \frac{b-a}{2^n} \leq \epsilon,$$

then taking the logarithms of both sides yields

$$n \geq \frac{\log\left(\frac{b-a}{\epsilon}\right)}{\log(2)}$$

If $n = 3.24$ round to 3, if $n = 3.7$ round to 4.

Q. $f(x) = x^3 + 4x^2 - 10 = 0$ $\quad a_1 = 1, b_1 = 2.$

$n = 9.96 = 10$

FIXED-POINT ITERATION

The number p is a fixed point for a given function g if $g(p) = p$. Single value required.

Example 1:

$$g(x) = x^2 - 2$$
$$p = g(p)$$
$$p = p^2 - 2$$
$$p^2 - p - 2 = 0$$

$p = -1$, $p = 2 \Rightarrow$ fixed point

Example 2: $g(x) = (x^2 - 1)/3$ has a unique fixed point on interval $[-1, 1]$

$$p = g(p) = \frac{(p^2 - 1)}{3}$$

$$p = \frac{p^2 - 1}{3}$$

Dated:

$p^2 - 3p - 1 = 0$

$P_1 = 3.30277 = 3 + \sqrt{13}/2$

$P_2 = \frac{3 - \sqrt{13}}{2} = -0.30277756377 \in [-1, 1]$

$\frac{3-\sqrt{13}}{2}$ is called unique fixed point.

$|g'(x)| = \left|\frac{2x}{3}\right| \leq \frac{2}{3}$ for all $x \in (-1, 1)$

* Fixed Point Iteration

Ex :

$f(x) = x^2 - 2x - 3 = 0$

$x = g(x) \rightarrow$ Is format mai laO.

$x_1 = g(x_0)$

⋮

$x_{n+1} = g(x_n)$

$f(x) = x^2 - 2x - 3$

$x^2 = 2x + 3$

$2x = x^2 - 3$

$x = \frac{1}{2}(x^2 - 3)$

Suppose $x_0 = 4$

$x_1 = g(x_0) = g(4) = 1/2(16 - 3) = 6.5$

$x_2 = g(x_1) = g(6.5) = 1/2[(6.5)^2 - 3] = 19.625$

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Dated:

n	x_n	Error
0	4	–
1	6.5	2.5
2	19.625	13.125
3	191.070	
4	18252.43	
5	166575638.4	
6	1.3873×10^{16}	
7	9.624×10^{31}	
8	4.631076×10^{63} (overflow)	

$$f(x) = x^2 - 2x - 3 = 0$$

$$x(x-2) = 3$$

$$x = \frac{3}{x-2} = g(x)$$

n	x_n	Error	(single value jahan aski)
0			
1			
2			
⋮			
n.		$x = 3$	or -1

ex 2.2

Q5. $x^4 - 3x^2 - 3 = 0 \Rightarrow P_0 = 1 \Rightarrow \epsilon = 10^{-2}$

Q6. $x^3 - x - 1 = 0 \Rightarrow P_0 = 1 \Rightarrow \epsilon = 10^{-2}$

$$f(x) = x^4 - 3x^2 - 3 = 0$$

$$\cancel{x^4} = \cancel{3x^2 + 3}$$

$$x^4 - 3x^2 = 3$$

$$x^2(x^2 - 3) = 3$$

$$\sqrt{x^2} = \sqrt{\frac{3}{(x^2-3)}}$$

Dated:

$$x = \pm\sqrt{\frac{3}{(x^2-3)}}$$

n	X_n	Error
0	..	Math Error

$x^4 = 3+3x^2$

$x = \sqrt[4]{3+3x^2}$

$- \cancel{3x^2} = 3 - x^4$

$- x^2 = 3 - x^4$

$x = \sqrt[4]{3+3x^2}$

n	x_n	Error
0	1.56	
1	1.793	
2	1.885	
3	1.922	
4	1.937	
5	1.9433	
6	1.945	

Q9. $\sqrt{3}$ with 10^{-4}

let

$x^2 = (\sqrt{3})^2$

$x^2 - 3 = 0$

$x \cdot x = 3$

$$\boxed{x = \frac{3}{x} = g(x)}$$

(repeat).

Adding and -ve

$x, x = 3$

$x \cdot x + x - x = 3$

$x^2 + x = 3 + x$

$x(x+1) = 3 + x$

$$\boxed{x = \frac{3+x}{x+1}}$$

$P_1 = g(P_0)$

$P_1 = \frac{3}{P_0} = \frac{3}{2} = 1.5$ → $P_6 = 2$

$P_2 = 2$

$P_3 = 1.5$

$P_4 = 2$

$P_5 = 1.5$

Dated:

$$x = \frac{3+x}{x+1}$$

$P_1 = g(P_0)$

$$P_1 = \frac{3+P_0}{P_0+1}$$

$P_0 = 2$

	P_0	Error
0	1.666	0.09
1	1.75	0.09
2	1.72	0.03
3	1.733	0.013
4	1.73	3×10^{-3}
5	1.732	2×10^{-3}
6	1.732	2×10^{-4}

1.732

Q10

$\sqrt[3]{25}$

$x = \sqrt[3]{25}$

$x^3 = 25$

$x^2 \cdot x = 25$

$$x = \frac{25}{x^2}$$ (Not right arrangement)

Let $P_0 = 2$

P_0	Error
6.25	~~5~~
0.64	5.61
61.03	60.39
6.71×10^{-3}	61.02

55511.68123 55511.5456

10^{-4}

~~$x^3 = 25$~~

~~$x^2 \cdot x = 25$~~

~~$x^2 \cdot x + x - x = 25$~~

~~$x(x+1) = 25$~~

~~$x+1-1 = \frac{25}{x}$~~

~~$x^2 \cdot x + x = 25 + x$~~

~~$x(x^2+1) = 25 + x$~~

~~$x = \frac{25+x}{x^2+1}$~~

9	
3.4	5.6
6.4545	3.0545
4.2195	2.235
5.5981	1.3786
4.6373	0.9608
5.2985	0.6612
4.8355	0.423
5.1127	0.2772

5.4	4 -
1.007	4.393
12.900	11.893
0.226	12.674
23.996	23.77
0.084	23.912
24.93	24.846
0.080	
24.91	
0.080	

Dated:

$$x^3 + x^2 - x^2 - 25 = 0$$

$$x^3 + x^2 = 25 + x^2$$

$$x^2(x + 1) = 25 + x^2$$

$$x^2 = \frac{25 + x^2}{x + 1}$$

$$x = \sqrt{\frac{25 + x^2}{x + 1}}$$

Dated:

Q4 a)

$$x = \frac{2 - e^{x} + x^{2}}{3} \qquad 10^{-5}$$

$[0, 2]$

Q14. $x = \tan x$

$x + x - x = \tan x$

$x + x = \tan x + x$

$2x = \frac{\tan x + x}{\cancel{2}}$; $x = \frac{\tan x + x}{2}$

$\boxed{x = \tan x}$

$P_0 = 4$

P_0

4.289

4.807

2.62

$P_0 = 5$

4.323

4.888

3.7425

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INTERPOLATION AND POLYNOMIAL APPROXIMATION

Interpolation: $x \to$ given $f(x)$ unknown $P_2(x) = x^2 - 7x + 4$

- When we predict values that fall within the range of data points taken it is called interpolation.

Extraporation:

- When we predict values for points outside the range of data taken it is called extrapolation.

Inverse-Interpolation: $x \to$ unknown $f(x) \to$ given

- Process of finding value of the argument corresponding to a given value of function lying b/w two tabulated functional values.

$f(x_k) = ?$ → function is unknown.
↳ given

x	$f(x)$
x_0	$f(x_0) = f_0$
x_1	$f(x_1) = f_1$
⋮	⋮
x_n	$f(x_n) = f_n$

$x_0 < x_k < x_n \to$ Interpolation

If $\left.\begin{array}{l} x_k > x_n \\ x_k < x_0 \end{array}\right\}$ Extrapolation

INTERPOLATION

Lagrange ↙ ↘ Newton interpolation formula

- Taylor's Theorem: 1.14

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x -$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

With the help of taylor and maclaurin we can decompose any type of function into polynomial

Dated:

Π | Pi | Product
Σ | Sigma | Sum

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{n} \frac{x^k}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots - \sum_{k=1}^{n} (-1)^{(k+1)} \frac{x^{(2k-1)}}{(2k-1)!}$$

$$\prod_{i=1}^{3} (x - x_i) = (x-x_1)(x-x_2)(x-x_3)$$

Initial value

Weierstrass Approximation Theorem

Suppose that 'f' is defined and continuous on $[a,b]$. For each $\varepsilon > 0$, there exists a polynomial $P(x)$ defined on $[a,b]$, with property that

$$|f(x) - P(x)| < \varepsilon, \quad \text{for all } x \in [a,b]$$

Q.

$$e^{1.1} = ?$$

$$P_2(1.1) = 1 + 1.1 + \frac{1.1^2}{2!} \qquad = 1 + x + \frac{x^2}{2!}$$

Re $\qquad = 2.705$

$$P_3(1.1) = 2.705 + \frac{1.1^3}{3!} = 2.9268$$

$$P_4(1.1) = 2.9268 + \frac{1.1^4}{4!} = 2.987$$

$$e^{1.1} = 3.0041$$

Dated:

Lagrange Theorem:

$f(x_k) = P(x_k)$ for $k = 0, \text{—}, n$

$$P(x) = f(x_0) L_{n,0}(x) + \cdots + f(x_n) L_{n,n}(x) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x)$$

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

$$= \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}$$

$$L_{n,k}(x) = L_k(x) = \sum_{k=0}^{n} f(x_k) L_k(x)$$

$$= \sum_{k=0}^{n} \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)} f(x_k)$$

Case $n=1$:

$$P_1(x) = \sum_{k=0}^{1} \prod_{\substack{i=0 \\ i\neq k}}^{1} \frac{(x-x_i)}{(x_k-x_i)} f(x_k)$$

$$P_1(x) = \frac{x-x_1}{x_0-x_1} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1)$$ → linear Lagrange Theorem.

$$P_2(x) = \sum_{k=0}^{2} \prod_{\substack{i=0 \\ i\neq k}}^{1} \frac{(x-x_i)}{(x_k-x_i)} f(x_k)$$

$$= \frac{x-x_2}{x_0-x_2} f(x_0) + \frac{x-x_0}{x_1-x_0} f(x_1) + \frac{x-x_1}{x_2-x_1} f(x_2)$$

Dated:

Dated:

- Quadractic lagrange Inter-Polation:

$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

- Cubic langrange:

$$P_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_2) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

Q. $f(3) = ?$

$f(1.0) = 14.2$ $f(2.7) = 17.8$ $f(3.2) = 22,$ $f(4.8) = 38.3$

	x	$f(x)$	
$x_0 \leftarrow$	1	14.2	$\rightarrow f_0$
$x_1 \leftarrow$	2.7	17.8	$\rightarrow f_1$
$x_2 \leftarrow$	3.2	22	$\rightarrow f_2$
$x_3 \leftarrow$	4.8	38.3	$\rightarrow f_3$

$$\Rightarrow \frac{(x-2.7)(x-3.2)(x-4.8)}{(1-2.7)(1-3.2)(1-4.8)}(14.2) + \frac{(x-1)(x-3.2)(x-4.8)}{(2.7-1)(2.7-3.2)(2.7-4.8)}(17.8)$$

$$+ \frac{(x-1)(x-2.7)(x-4.8)}{(3.2-1)(3.2-2.7)(3.2-4.8)}(22) + \frac{(x-1)(x-2.7)(x-3.2)}{(4.8-1)(4.8-2.7)(4.8-3.2)}(38.3)$$

Dated:

$$= \frac{(0.3)(-0.5)(-1.8)}{(-1.7)(-2.2)(-3.8)}(14.2) + \frac{(2)(-0.2)(-1.8)}{(1.7)(-0.5)(-2.1)}(17.8) + \frac{(2)(0.3)(-1.8)}{(2.2)(0.5)(-1.6)}(22)$$

$$+ \frac{(2)(0.3)(-0.2)}{(3.8)(2.1)(1.6)}(38.3)$$

$\Rightarrow$ $0.26977 + 7.17983 + 13.5 + (-0.359962)$

$\Rightarrow$ 20.589638 (can come b/w $f(2.7)$ and $f(3.2)$)

Example : 2

x	$f(x) = 1/x$
2	$1/2 = 0.5$
2.75	$1/2.75 = 0.3636$
4	0.25

node = x ki value.

a) lagrange ~~Theorem~~ Polynomial :

Dated: Compare = Calculate Error = Discuss = Comments

Ex 2: $f(1.5)$

(x)	$f(x)$
1.0	0.7651977
1.3	0.6200860
1.6	0.4554022
1.9	0.281816
2.2	0.1103623

1.5 b/w

$$P_1(x) = l_0(x) f_0 + L_1(x) f_1 = \frac{x-x_1}{x_0-x_1} \cdot f_0 + \frac{x-x_0}{x_1-x_0} \cdot f_1$$

$$= \frac{(1.5-1.6)}{(1.3-1.6)}(0.6200860) + \frac{(1.5-1.3)}{(1.6-1.3)}(0.4554022)$$

$$= 0.2066953 + 0.3036014$$

$$= 0.5102967$$ Linear

2nd degree

$$P_2(1.5) = \frac{(1.5-1.6)(1.5-1.9)}{(1.3-1.6)(1.3-1.9)}(0.6200860) + \frac{(1.5-1.3)(1.5-1.9)}{(1.6-1.3)(1.6-1.9)}$$

Degree of polynomial increases error decreases.

$$= 0.5112857$$

Dated:

ERROR BOUNDS

- Obtained from lagrange Error Formula.

$$E = |f_x - P_x| = f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

- Only be calculated when function is given

Q3. Use Theorem to find an error bound from each polynomial.
$x_0 = 0, x_1 = 0.6, x_2 = 0.9$ where $f(x) = \cos x$, $f(0.45) = ?$

Let

$n = 1$

$x_0 = 0, \quad x_1 = 0.6$

~~$f(x) = \cos(\theta)$~~ -

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$$E \leq \frac{f''(\xi)}{(1+1)!}(x-x_0)(x-x_1)$$

Maximum value of cos x

$$\leq \frac{-\cos x}{2!}(0.45-0)(0.45-0.6)$$

$$\leq \frac{-\cancel{\cos(0.45)}\cancel{(0.45)}(-0.15)}{2}$$

$$\leq \left| \frac{\cancel{(0.900447)}}{2}\cancel{(0.0675)} \right|$$

x	$\cos x$
0	1
0.6	0.823
0.9	0.6216

Maximum

$\leq$ we will put '1'

$$\leq \left| \frac{1}{2}(0.45)(0.15) \right| = \left| \frac{1}{2}(0.0675) \right|$$

$$E \leq 0.03375$$

Dated:

Let $n = 2$

$x_0 = 0$, $x_1 = 0.6$ $x_2 = 0.9$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$$E \leq \frac{f'''(\xi)}{(2+1)!} (x-x_0)(x-x_1)(x-x_2)$$

$$E \leq \frac{\sin \xi}{6} (0.45-0)(0.45-0.6)(0.45-0.9)$$

$$\leq \frac{\sin \xi}{6} (0.45)(-0.15)(-0.45)$$

$$E \leq \frac{\sin \xi}{6} (0.03)$$

• ξ : It varies, hamsha Max value uthaty hain

$\sin x$	x	$\cos x$
0	0.	1
0.5646	0.6	0.825
0.7833	0.9	0.6216

$$E \leq \sin \xi \frac{(0.03)}{6}$$

$$E \leq \sin(0.4) \frac{(0.03)}{6}$$

$$E \leq (0.7833)(5 \times 10^{-3})$$

$$E \leq 3.91 \times 10^{-3}$$

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Dated:

DIVIDED DIFFERENCES (NEWTON'S)

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \dots x_k](x - x_0) \dots (x - x_{k-1})$$

Algo 3.2.

$$P_n(x) = f[x_0] + (x - x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, \dots x_2]$$
$$+ (x-x_0)(x-x_1)(x-x_2) f[x_0 \dots x_3] + \dots + (x-x_0)$$
$$(x-x_1) \dots (x - x_{n-1}) f[x_0 \dots x_n]$$

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0) \dots (x - x_{k-1})$$

$f(x_0)$ $f(x_1)$ Table 3.9

Example 1: Complete the divided diff table for data.

n-1 colums

	x	$f(x)$ a_0	1DD a_1	2DD	3DD	4DD
0	1.0	0.765	-0.483	a_2		
				$-\frac{0.550+0.483}{1.6-1} = -0.112$	a_3	
1	1.3	0.620	-0.550		$\frac{-0.050+0.112}{1.9-1}$	
					$= 0.069$	
2	1.6	0.455	-0.580	$\frac{-0.580+0.550}{1.9-1.3}$		a_4
				$= -0.050$		$\frac{0.069-0.069}{2.2-1} = 0$
3	1.9	0.282			$\frac{0.012+0.050}{2.2-1.3}$	
				$\frac{-0.573+0.580}{2.2-1.6}$	$= 0.069$	
4	2.2	0.110	-0.573	$= 0.012$		

Interpolating Polynomial:

first value of every column is 'a'.

$$P_4(x) = 0.765 + (-0.483)(x-1.0) + (-0.112)(x-1.0)(x-1.3) + (0.069)$$
$$(x-1.0)(x-1.3)(x-1.6) + (0)(x-1.0)(x-1.3)(x-1.6)$$
$$(x-1.9)$$

4/6

Dated:

Q) Construct DDT b) compute f(9.2) from the values.

x_j	$f_j = f(x_j)$	$f[x_j, x_{j+1}]$ 1DD	$f[x_j, x_{j+1}, x_{j+2}]$ 2DD	$f[x_j, \dots, x_{j+3}]$ 3DD
8.0	2.079442 (x_0)	0.117783 (a_1)	a_2 -0.006433	
9.0	2.197225	0.108134		0.000411
9.5	2.251292		-0.005200	
11.0	2.397895	0.097735		

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f(x_0, x_1)$$

$$a_2 = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$P_n = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots + a_n(x - x_0)$$

$$f(9.2) = 2.079442 + 0.117783(9.2-8) + (-0.006433)(9.2-8)(9.2-9) + (0.000411)(9.2-8)(9.2-9)(9.2-9.5)$$

$$= 2.079442 + 0.141340 + (-0.001544) - 0.000030$$

$$f(9.2) = 2.219208$$

Dated:

a) find $f(9.2)$ using first degree polynomial.

$P_1(9.2) = a_0 + a_1(x - x_0)$

$= 2.079442 + 0.117783(9.2 - 8) = 2.220816$

b) find 2nd degree.

$P_2(9.2) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$

$= 2.219238$ (6D).

Error (True) $\quad f(9.2) = \ln 9.2 = 2.219203$

$\Rightarrow |f(9.2) - P_1(9.2)| = 0.001579$

$\Rightarrow |f(9.2) - P_2(9.2)| = 0.000035$

$\Rightarrow |f(9.2) - P_3(9.2)| = 0.000005$

Absolute Error

$\Rightarrow |P_2(9.2) - P_1(9.2)|$

Dated:

Equal spacing (h = common)

NEWTON FORWARD DIFFERENCE FORMULA

$$P_n(x) = f_0 + sh\,a_1 + s(s-1)h^2 a_2 + s(s-1)(s-2)h^3 a_3 + \dots$$

where;

$$s = \frac{x - x_0}{h}, \quad h = x_{i+1} - x_i$$

if $i = 0$: $h = x_1 - x_0$ if $i = 1$: $h = x_2 - x_1$

Examp1 DDT ka table hai.

a) $f(1.1)$ use Newton forward difference, using DDT.

$x = 1.1 \quad h = 0.3$

$P_n(x) =$ $$s = \frac{1.1 - 1.0}{1.3 - 1.0} = 0.333333$$

~~= 0.7651977 + (0.333333)(0.3)(0.7651977) + (0.33)(-0.666667) (+) (0.3)^2(-0.483)~~

$= (0.7651977) + (0.333333)(0.3)(-0.483057) + (0.333333)(-0.666667)(0.3)^2 (-0.1087339) + (0.333333)(-0.666667)(-1.666667)(0.065878$

$= (0.7651977) + (-0.048371) + ~~0.020000~~\ 0.002175 + 0.000106$

$= 0.719107..$

Dated:

NEWTON BACKWARD DIFFERENCE FORMULA

$$P_n(x) = f_n + sh\,a_1 + s(s+1)h^2 a_2 + s(s+1)(s+2)h^3 a_3 + \dots$$

$\rightarrow$ last entries of each column

where

$$s = \frac{x - x_n}{h}, \quad h = x_{i+1} - x_i$$

b) $x = 2$

$h = 0.3$

$$s = \frac{x - x_n}{h} = \frac{2 - 2.2}{0.3} = \frac{-0.2}{0.3} = -\frac{2}{3} = -0.666667$$

$$= 0.1103623 + (-0.666667)(0.3)(-0.5715210) + (-0.666667)(0.3333)(0.3)^2(0.0118183) + (-0.666667)(0.3333)(1.333333)(0.3)^3(0.068068)$$

$$= 0.1103623 + 0.114304 + (-0.000236) - 0.000544$$

$$= 0.223886$$