Linearity II Final Project

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1 Objective

For our final Linearity II project, we are exploring both the wave equation and its application to the study of transmission lines in theoretical and experimental cases. By completing a rigorous theoretical study and examining use cases with real signals and electronics, we hope to gain a deeper understanding of the wave equation and its applications in signal processing.

2 The Wave Equation

2.1 Background

The wave equation models the movement and behavior of waves, or disturbances in continuous media that propagate with constant velocity and shape. The wave equation can be used to quantify the behavior of elements in systems including physical systems and electromagnetic systems. It is a second-order partial differential equation of a scalar function u, time variable t, one or more space variables (x, y, z), and application-specific value(s) c. The general wave equation is stated below.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

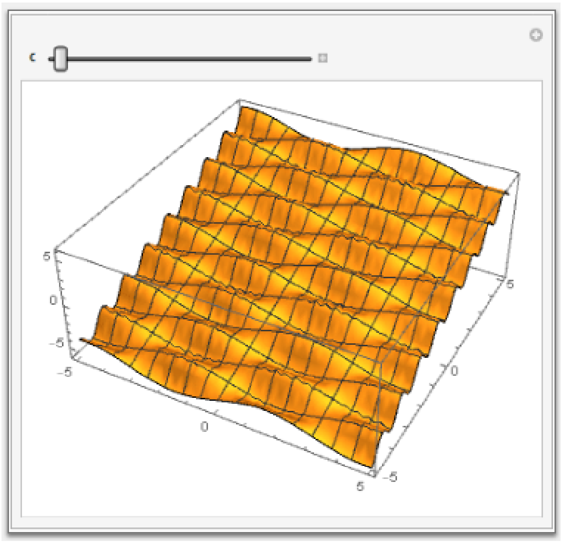
2.2 General Solution in One Dimension

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

The one-dimensional wave equation describes the motion of the system under study in terms of displacement of x and time t. The result, u, depending on the application, is generally expressed in the y-direction. In the one-dimensional case, c often refers to the speed of propagation.
Solutions that satisfy this relationship are generally linear combinations of functions of the form stated below.

$$f(z, t) = f(z - vt, 0) = g(z - vt)$$

Below is an animation of the 1D wave equation and the Mathematica code that was used to create it. In the animation, the value of c is changing. This animation shows the effect of changing c and its effects on movement.



```
ClearAll[u, t, x];

pde = D[u[x, t], {t, 2}] == c^2D[u[x, t], {x, 2}];

ic = {u[x, 0] == Sin[x], Derivative[0, 1][u][x, 0] == 1};

sol = DSolve[{pde, ic}, u[x, t], {x, t}]
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$$\left\{\left\{u[x,t] \to t + \tfrac{1}{2}\left(-\mathrm{Sin}\left[\sqrt{c^2}t - x\right] + \mathrm{Sin}\left[\sqrt{c^2}t + x\right]\right)\right\}\right\}$$

$$\textbf{Manipulate}\left[\textbf{Plot3D}\left[t + \tfrac{1}{2}\left(-\textbf{Sin}\left[\sqrt{c^2}t - x\right] + \textbf{Sin}\left[\sqrt{c^2}t + x\right]\right), \{x, -5, 5\}, \{t, -5, 5\}\right], \{c, -5, 5\}\right]$$

2.3 General Solution in Two Dimensions

The general solution to the two-dimensional wave equation scales to include an additional dimension as stated below. Use cases for the two-dimensional case include quantifying the behavior of stretched, elastic membranes.

$$c^2 \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

2.4 General Solution in Three Dimensions

Again, in three dimensions, the solution scales to incorporate another dimension as stated below. Use cases for the three-dimensional case include quantifying the behavior of plane waves.

$$c^2 \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

2.5 Boundary Conditions

A solution to a problem with expressed boundary conditions must fall within the boundary conditions and satisfy the initial equation. An example of boundary conditions is the endpoints of a given finite line carrying some signal (i.e. a transmission line or a oscillating string). Solving the wave equation while accounting for these boundary conditions requires the use of either numerical methods or exploiting the eigenvalues of the function.

3 Transmission Lines

3.1 Background

Transmission lines refer to cables used to conduct a signals at a nontrivial frequency such that the wave nature of the signal must be accounted for. Because the wavelength of the signal decreases as the frequency increases, the cable becomes a significant portion of the wavelength for high-frequency signals. Generally, any cable where the length of the cable is greater than ten percent of the wavelength of the signal is a transmission line.

To account for the wave nature of the transmission line, consider the time delay associated with the propogation of the wave along the line. An ideal transmission line carries a signal $V(t)$ from a starting point on the line to an ending point on the line with some time delay τ such that the signal at that point will be $V(t-\tau)$. However, a real transmission line contains distributed inductances, resistances, and capacitances. We can calculate this value across the entire line and then account for it with some value δ. This implies that the signal at some ending point will be $V(t-\delta x)$. The behavior of signals across a real transmission line can be quantified by the wave equation.

3.2 Derivation of the Wave Equation for a Real Transmission Line in the Time Domain

Consider some real transmission line in the context of Ohm's Law, $V = IR$.

$$\begin{aligned} V(x) - V(x+\delta x) &= \frac{\partial I(x)}{\partial t} L\delta x + I(x)R\delta x \\ -\frac{\partial V}{\partial x}\delta x &= \frac{\partial I}{\partial t} L\delta x + IR \\ \frac{\partial V}{\partial x} &= -L\frac{\partial I}{\partial t} - IR \end{aligned}$$

To calculate the value of $\frac{\partial I}{\partial t}$, consider the rate of change of the voltage.

$$\begin{aligned} I(x) - I(x+\delta x) &= \frac{\partial V(x+\delta x)}{\partial t} c\delta x \\ -\frac{\partial I(x)}{\partial x}\delta x &= \frac{\partial V(x)}{\partial t} c\delta x + \frac{\partial^2 V(x)}{\partial x \partial t}\delta x c \delta x \\ \lim_{\delta x \to 0} \frac{\partial I}{\partial x} &= -c\frac{\partial V}{\partial t} \end{aligned}$$

Then, differentiate the voltage across the inductive element with respect to x.

$$\frac{\partial^2 V}{\partial x^2} = -L\frac{\partial^2 I}{\partial t \partial x} - R\frac{\partial I}{\partial x}$$

Differentiate the value of $\frac{\partial I}{\partial t}$ with respect to t.

$$\frac{\partial^2 I}{\partial x \partial t} = -c\frac{\partial^2 V}{\partial t^2}$$

Substitute the value of $\frac{\partial I}{\partial t}$ to yield the complete transmission line equation.

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + RC\frac{\partial V}{\partial t}$$

Note that the value of R is often 0, so the equation often simplifies to only consider the voltage.

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}$$

3.3 Telegrapher's Equations: Frequency-Domain Analysis

Telegrapher's Equations, at their most basic components, describe the effect of reflection of electromagnetic waves along a wire. From the distributed element model, we assume the components of a transmission line (i.e. resistance, capacitance, inductance) are spread evenly and throughout the line. When evaluating these components, we look at them as infinitesimally small segments (δ). We assume with this model that it has impedance due to the fact that there is non-uniform voltage and current. This model is best used at high frequencies, and it accounts for impedance because of the nonuniform voltage and current.

When evaluating a transmission line, first model the system composed of different distributed components along the line. The distributed components are resistance R, inductance L, capacitance C, conductance G. Below is a schematic showing the components on a circuit diagram.

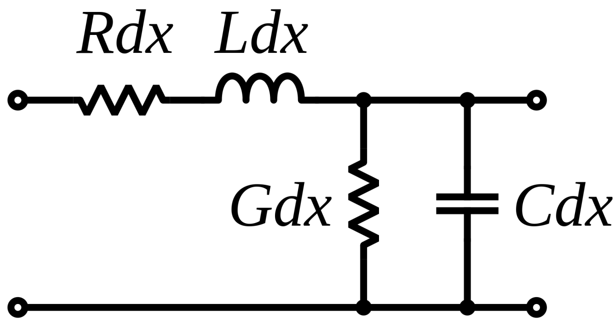

Figure 1: Schematic of infinitesimally small region of transmission line. Source: Wikipedia

The telegrapher's equations are stated below.

$$\frac{\partial}{\partial x}V(x,t) = -L\frac{\partial}{\partial t}I(x,t) - RI(x,t)$$
$$\frac{\partial}{\partial x}I(x,t) = -C\frac{\partial}{\partial t}V(x,t) - GV(x,t)$$

These two equations can be manipulated to get a second order partial differential equation. The derivation is stated below.

$$
\begin{aligned}
\frac{\partial}{\partial x}V(x,t) &= -L\frac{\partial}{\partial t}I(x,t) - RI(x,t) \\
\frac{\partial}{\partial x}I(x,t) &= -C\frac{\partial}{\partial t}V(x,t) - GV(x,t) \\
RI(x,t) &= -L\frac{\partial}{\partial t}I(x,t) - \frac{\partial}{\partial x}V(x,t) \\
I(x,t) &= \frac{-L}{R}\frac{\partial}{\partial t}I(x,t) - \frac{1}{R}\frac{\partial}{\partial x}V(x,t) \\
\frac{\partial}{\partial x}(\frac{-L}{R}\frac{\partial}{\partial t}I(x,t) - \frac{1}{R}\frac{\partial}{\partial x}V(x,t)) &= -C\frac{\partial}{\partial t}V(x,t) - GV(x,t) \\
\frac{-L}{R}\frac{\partial}{\partial t}\frac{\partial}{\partial x}I(x,t) - \frac{1}{R}\frac{\partial^2}{\partial x^2}V(x,t) &= -C\frac{\partial}{\partial t}V(x,t) - GV(x,t) \\
\frac{-L}{R}\frac{\partial}{\partial t}(-C\frac{\partial}{\partial t}V(x,t) - GV(x,t)) - \frac{1}{R}\frac{\partial^2}{\partial x^2}V(x,t) &= -C\frac{\partial}{\partial t}V(x,t) - GV(x,t) \\
\frac{LC}{R}\frac{\partial^2}{\partial t^2}V(x,t) + \frac{GL}{R}\frac{\partial}{\partial t}V(x,t) - \frac{1}{R}\frac{\partial^2}{\partial x^2}V(x,t) &= -C\frac{\partial}{\partial t}V(x,t) - GV(x,t) \\
\frac{LC}{R}\frac{\partial^2}{\partial t^2}V(x,t) - \frac{1}{R}\frac{\partial^2}{\partial x^2}V(x,t) &= -C\frac{\partial}{\partial t}V(x,t) - GV(x,t) - \frac{GL}{R}\frac{\partial}{\partial t}V(x,t) \\
LC\frac{\partial^2}{\partial t^2}V(x,t) - \frac{\partial^2}{\partial x^2}V(x,t) &= -CR\frac{\partial}{\partial t}V(x,t) - GRV(x,t) - GL\frac{\partial}{\partial t}V(x,t) \\
-LC\frac{\partial^2}{\partial t^2}V(x,t) + \frac{\partial^2}{\partial x^2}V(x,t) &= CR\frac{\partial}{\partial t}V(x,t) + GRV(x,t) + GL\frac{\partial}{\partial t}V(x,t)
\end{aligned}
$$

When solving for voltage, this yields the final second-order partial differential equation below.

$$\frac{\partial^2}{\partial x^2}V(x,t) - LC\frac{\partial^2}{\partial t^2}V(x,t) = (RC + GL)\frac{\partial}{\partial t}V(x,t) + GRV(x,t)$$

Additionally, solving for current yields the partner equation below.

$$\tfrac{\partial^2}{\partial x^2}I(x,t) - LC\tfrac{\partial^2}{\partial t^2}I(x,t) = (RC + GL)\tfrac{\partial}{\partial t}I(x,t) + GRI(x,t)$$

3.3.1 The Skin Effect and the Proximity Effect

The real values of R and L vary from ideal approximations due to to the Skin Effect and the Proximity Effect. The Skin Effect describes the physical tendency for high frequency currents to direct towards the surface of the conductor. In contrast, the Proximity Effect is the change in distribution of current along the line. It is caused by the alternating magnetic field produced from the alternating current traveling through the line. In a practical sense, this means that when applying the Telegrapher's Equations, it is necessary to validate the exact R and L values for the given frequency.

3.4 Characteristic Impedance

3.4.1 Background

Characteristic impedance is the ratio of the voltage and the current of a given transmission line at a given frequency, assuming sinusoidal steady state. On a finite line, the signal is subject to input impedance, or resistance to current flow. Additionally, because the line has a termination point, the signal will travel along the line and then be reflected back, and this will add to the original impedance. Infinite lines do not have any input impedance because the signal never reaches the end of the line.

The characteristic impedance of a line is the value at which the impedance along the line and the input are equal. Therefore, a line terminated at its appropriate characteristic impedance, there is no meaningful reflection on the line, so there is no voltage or current loss.

3.4.2 Characteristic Impedance of a Transmission Line For a Sinusoidal Signal on a Lossless Line

The characteristic impedance of a transmission line without any reflection (i.e. one that has been terminated at the value of its characteristic impedance) is a straightforward function of the inductance and capacitance present in the given circuit.
Note from the time-domain analysis that

$$Given\ R = 0$$
$$\frac{\partial I}{\partial t} = -\frac{1}{L}\frac{\partial V}{\partial x}$$

We can then determine the relationship between the voltage and the current for some sinusoidal signal $V(x,t)$ with frequency ω and amplitude α for a circuit with inductance L and capacitance C.

$$V(x,t) = a\sin\omega t - \omega\sqrt{LC}x$$
$$\frac{\partial I}{\partial t} = -\frac{1}{L}\frac{\partial a\sin(\omega t - \omega\sqrt{LC}x}{)}\partial x$$
$$\frac{\partial I}{\partial t} = -\frac{1}{L}(-\omega\sqrt{LC})a\cos(\omega t - \omega\sqrt{LC}x)$$
$$\frac{\partial I}{\partial t} = a\omega\sqrt{\frac{C}{L}}\cos(\omega t - \omega\sqrt{LC}x)$$
$$I(x,t) = a\sqrt{\frac{C}{L}}\sin(\omega t - \omega\sqrt{LC}x)$$

Dividing both equations yields z.

$$\sqrt{\frac{L}{C}} \equiv z$$

z is the characteristic impedance of the transmission line. Note that because this model assumes that no voltage or current is lost along the line, this approximation has no imaginary components and no frequency-dependent impedance. This is generally a reasonable assumption because, at high frequencies, the values of resistance and conductance are essentially negligible.

3.4.3 Characteristic Impedance on a Lossy Line

Incorporating additional frequency components requires considering the original relationship between the voltage and the current and adding in the necessary frequency-dependent components.

In the case of the voltage, any series resistance in the circuit will also affect the impedance, and then the inductance will have a frequency-dependent component. Therefore, the value that was earlier simplified to L is actually $R + j\omega L$. In the case of the current, any parallel resistance in the line will also affect the impedance, and then the capacitance will have a frequency-dependent component. Therefore the value that was earlier simplified to C is actually $G + j\omega C$.

From there, values can be incorporated into the original equation.

$$\frac{V(0,t)}{I(0,t)} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \equiv z$$

3.5 Standing Waves

Standing waves are waves formed by the superposition of the incident and reflected wave in a given transmission line. Ideally, if the reflected and transmitted wave are equal, there is no impact on signal transmission. However, due to losses along the line (i.e. those created by impedance mismatch) partial standing waves occur, and they can result in large losses along the line. The metric standing wave ratio quantifies the impact of reflections through the ratio of the amplitude at the maximum and the minimum value of the voltage. When the impedances are matched, the standing wave ratio is one.

4 Application: Impedance Matching Validation

4.1 Description

To explore how reflection due to impedance mismatch and standing waves can impact power transfer over a given line, we created a small demonstration where we modeled a system with an initial source of signal with some load impedance, ran it over some transmission line, and then measured the voltage following the transmission line with some matched load impedance. By varying the resistances across the source, load, and line (and alternating whether or not they were all matched), we empirically showed how the reflectance present on unmatched lines can lead to suboptimal signal transfer.

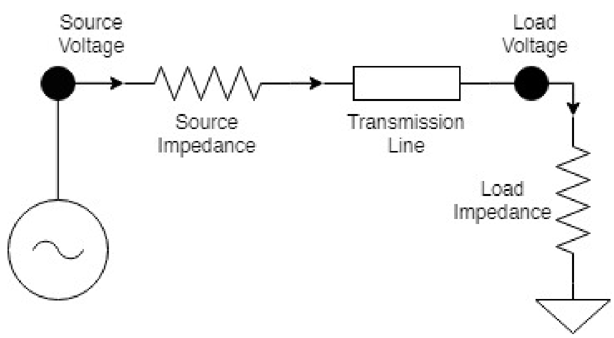

Figure 2: Block Diagram of Impedance Matching System

This conclusion is fairly intuitive; varying the resistances of the medium during transmission will impact the efficiency of power transfer. To extend this work, we then attempted to determine the closeness of an impedance match between a wireless transmitter and antenna - however, because of issues including not having access to a proper transmitter hardware (lab function generators do not produce signals of necessary frequency, and small radio modules do not allow for straightforward voltage measurements at the source point), we were not able to prove this in a meaningful way. At the same time, the conclusion remains representative of the broader thesis of impedance matching.

4.2 Circuit Diagram

We created a circuit to facilitate quick iteration and accurate measurement of voltage differences due to changes in source/load impedance and transmission lines. This circuit can also facilitate the use of a transmitter and antenna, had we had the proper transmitter equipment available. We used a simple measurement circuit with opeational amplifiers to clean up the signal and an Arduino Uno V3 to measure and compare the voltages.

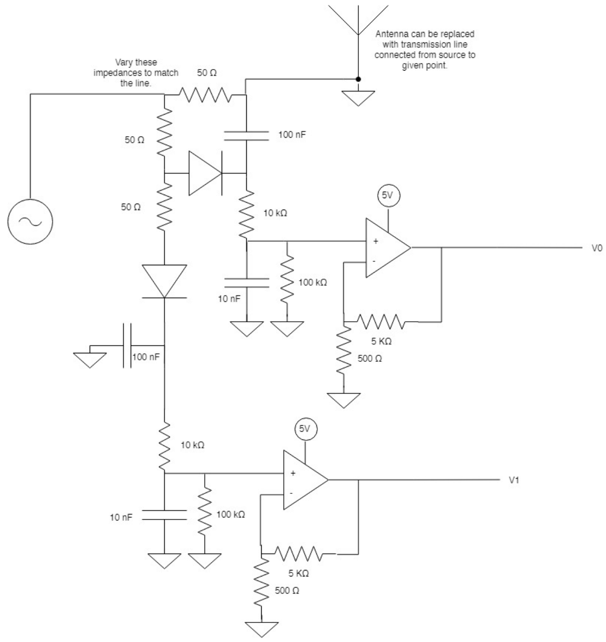

Figure 3: Circuit Diagram of Impedance Matching System for Antenna Analysis

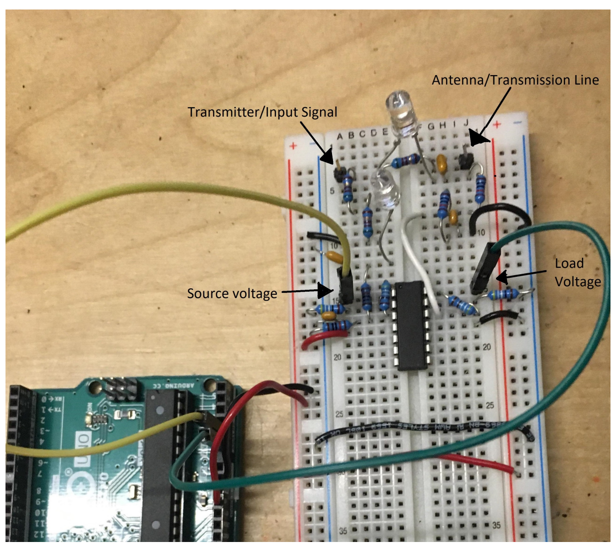

Figure 4: Labeled photo of Impedance Matching System

4.3 Results and Reflection

While this quick experiment was a useful way to view the change in voltages associated with unmatched impedance, it mainly served as a validation of otherwise intuitive or mathematically clear conclusions: we could have just as easily attached two strings of different materials or widths to each other and examined the difficulties in wave generation, especially at the connection point. While increasing the functionality to support radio transmitters and antennas would have been an interesting addition, most of the problems were electrical and not mathematical, so we chose not to pursue them in favor of understanding the mathematics more deeply.

5 Sources

1. MIT handout on the wave equation
   Basic discussion of the wave equation and its solutions.

2. UCSS Transmission Lines Lecture
   Introduction to theory behind transmission lines.

3. Brandeis Transmission Line Analysis Lab
   Paper connecting the wave equation to transmission lines and it's relation to impedance.

4. University of Denver The Telegrapher's Equation Handout
   Description of transmission Lines and the constant linear parameters that govern them.

5. Griffith's Introduction to Electrodynamics
   Electricity and Magnetism textbook.