CSC591- HW2 (Group: ALL_12)

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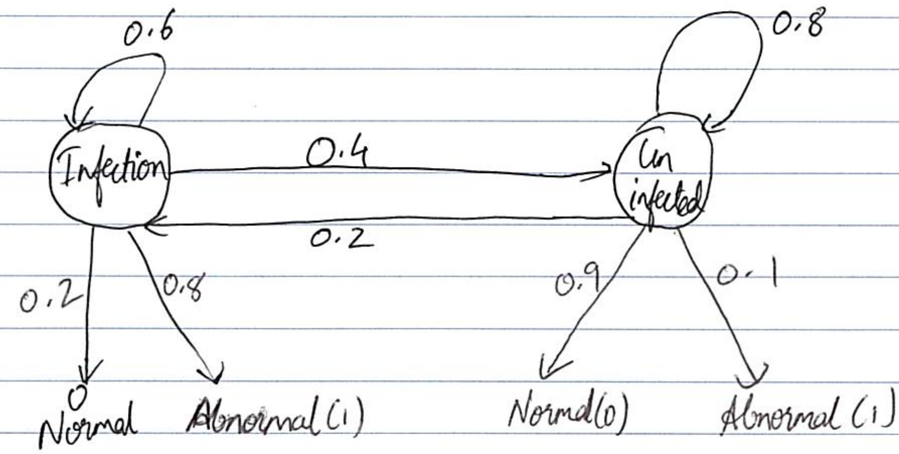

a) N = # of hidden states. Here $N = 2$.
M = # of possible observations of all states. Here $M = 2$.

There are 3 probabilities:

π: initial probability. There are N states so N parameters are required.

a_{ij}: Transitional probability. We can transition from any N state to any of N states. So, $N \times N$ i.e N^2 parameters are required.

$b_i(K)$: Emission or observation probability. Each of N state can emit M observations. So, $N \times M$ ie NM parameters are required.

So, total number of parameters required are
$N + N^2 + NM \Rightarrow 2 + 2^2 + 2 \times 2 \Rightarrow \underline{\underline{10}}$

b)

Initial probability:

State (i)	$\pi(i)$
Infected (1)	0.75
~~Non~~ Un infected (2)	0.25

Transition probability: $a_{ij} \Rightarrow$

from | To	infected (1)	uninfected (2)
infected (1)	0.6	0.4
uninfected (2)	0.2	0.8

Observation or emission probability: $b_i(K) = P(O_t = K \mid q_t = s_i)$

state | observation	Normal (0)	Abnormal (1)
infected (1)	0.2	0.8
uninfected (2)	0.9	0.1

c) Observation : $\{ \overset{O_1}{\underset{\text{normal}}{0}} , \overset{O_2}{\underset{\text{abnormal}}{1}} , \overset{O_3}{\underset{\text{abnormal}}{1}} \}$

Forward algo : initial : $\alpha_1(i) = b_i(O_1) \cdot \pi_i$

at a time : $\alpha_{t+1}(i) = b_i(O_{t+1}) \cdot \sum_j a_{ji} \cdot \alpha_t(j)$

So,

$\alpha_1(1) = b_1(O_1) \cdot \pi_1 \Rightarrow b_1(\text{normal}) \cdot \pi_1 \Rightarrow 0.2 \times 0.75 \Rightarrow 0.15$

$\alpha_1(2) = b_2(O_1) \cdot \pi_2 \Rightarrow 0.9 \times 0.25 \Rightarrow 0.225$

$\underset{(t=1, i=1, j=1,2)}{\alpha_2(1)} = b_1(O_2) \cdot \sum [a_{11} \cdot \alpha_1(1) + a_{21} \cdot \alpha_1(2)]$

$\Rightarrow 0.8 \times [0.6 \times 0.15 + 0.2 \times 0.225] \Rightarrow 0.108$

$\underset{(t=1, i=2, j=1,2)}{\alpha_2(2)} = b_2(O_2) \cdot \sum [a_{12} \cdot \alpha_1(1) + a_{22} \cdot \alpha_1(2)]$

$\Rightarrow 0.1 \times [0.4 \times 0.15 + 0.8 \times 0.225] \Rightarrow 0.024$

$\underset{(t=2, i=1, j=1,2)}{\alpha_3(1)} = b_1(O_3) \cdot \sum [a_{11} \cdot \alpha_2(1) + a_{21} \cdot \alpha_2(2)]$

$= 0.8 \times [0.6 \times 0.108 + 0.2 \times 0.024] \Rightarrow 0.05568$

$\underset{(t=2, i=2, j=1,2)}{\alpha_3(2)} = b_2(O_3) \cdot \sum [a_{12} \cdot \alpha_2(1) + a_{22} \cdot \alpha_2(2)]$

$= 0.1 \times [0.4 \times 0.108 + 0.8 \times 0.024] \Rightarrow 0.00624$

P(observed sequence) $= 0.05568 + 0.00624$

$\left(P(O|\lambda) = \sum_{i=1}^{N} \alpha_T(i) \right)$ $= \underline{\underline{0.06192}}$

d) Observation : {0, 1, 1}

Backward algo : $\sum_j a_{ij}\, b_j(O_{t+1}) \cdot \beta_{t+1}(j)$

So,

$\beta_3(1) = 1$

$\beta_3(2) = 1$

$\beta_2(1) = [a_{11} \cdot b_1(O_3) \cdot \beta_3(1) + a_{12} \cdot b_2(O_3) \cdot \beta_3(2)]$
$= [0.6 \times 0.8 \times 1 + 0.4 \times 0.1 \times 1] \Rightarrow 0.52$

$\beta_2(2) = [a_{21} \cdot b_1(O_3) \cdot \beta_3(1) + a_{22} \cdot b_2(O_3) \cdot \beta_3(2)]$
$= [0.2 \times 0.8 \times 1 + 0.8 \times 0.1 \times 1] \Rightarrow 0.24$

$\beta_1(1) = [a_{11} \cdot b_1(O_2) \cdot \beta_2(1) + a_{12} \cdot b_2(O_2) \cdot \beta_2(2)]$
$= [0.6 \times 0.8 \times 0.52 + 0.4 \times 0.1 \times 0.24] \Rightarrow 0.2592$

$\beta_1(2) = [a_{21} \cdot b_1(O_2) \cdot \beta_2(1) + a_{22} \cdot b_2(O_2) \cdot \beta_2(2)]$
$= [0.2 \times 0.8 \times 0.52 + 0.8 \times 0.1 \times 0.24] \Rightarrow 0.1024$

$P(\text{Observation}) = P(O|\lambda) = \sum_{i=1}^{N} \pi_i\, \beta_{0_{\text{initial}}}(i)$

$= \sum [\pi_1 \cdot \beta_1(1) + \pi_2 \cdot \beta_1(2)]$

$= 0.15 \times 0.2592 + 0.225 * 0.1024$
$= \underline{\underline{0.06192}}$

The forward & backward algorithm gives the same probability.

e) Forward Backward $(\alpha_t(i) \cdot \beta_t(i))$

$\alpha_1(1) \cdot \beta_1(1) = 0.15 \times 0.2592 \Rightarrow 0.03888$

$\alpha_1(2) \cdot \beta_1(2) = 0.225 \times 0.1024 \Rightarrow 0.02304$

$\alpha_2(1) \cdot \beta_2(1) = 0.108 \times 0.52 \Rightarrow 0.05616$

$\alpha_2(2) \cdot \beta_2(2) = 0.024 \times 0.24 \Rightarrow 0.00576$

$\alpha_3(1) \cdot \beta_3(1) = 0.05568 \times 1 \Rightarrow 0.05568$

$\alpha_3(2) \cdot \beta_3(2) = 0.00624 \times 1 \Rightarrow 0.00624$

Since for every place $\alpha_t(1) \cdot \beta_t(1) > \alpha_t(2) \cdot \beta_t(2)$ we can say the most likely sequence is infected, infected, infected.

f) Viterbi algorithm : $\delta_1(i) = \pi_i \cdot b_i(O_1)$

At time t+1 : $\delta_{t+1}(j) = \max_{1 \le i \le N}[\delta_t(i) \cdot a_{ij}] \cdot b_j(O_{t+1})$

Observation : (0, 1, 1)

$\delta_1(1) = \pi_1 \cdot b_1(O_1) \Rightarrow 0.75 \times 0.2 \Rightarrow 0.15$

$\delta_1(2) = \pi_2 \cdot b_2(O_1) \Rightarrow 0.25 \times 0.9 \Rightarrow 0.225$

$\delta_2(1) = \max[\delta_1(1) \cdot a_{11}, \delta_1(2) \cdot a_{21}] \cdot b_1(O_2)$
$(t=1, j=1, i=1,2)$

$= \max[0.15 \times 0.6, 0.225 \times 0.2] \times 0.8 \Rightarrow 0.072$

$\delta_2(2) = \max[\delta_1(1) \cdot a_{12}, \delta_1(2) \cdot a_{22}] \cdot b_2(O_2)$
$(t=1, j=2, i=1,2)$ $= \max[0.15 \times 0.4, 0.225 \times 0.8] \times 0.1 \Rightarrow 0.018$

$\delta_3(1) = \max[\delta_2(1) \cdot a_{11}, \delta_2(2) \cdot a_{21}] \cdot b_1(O_3)$
$(t=2, j=1, i=1,2)$ $= \max[0.072 \times 0.6, 0.018 \times 0.2] \times 0.8 \Rightarrow 0.03456$

$\delta_3(2) = \max[\delta_2(1) \cdot a_{12}, \delta_2(2) \cdot a_{22}] \cdot b_2(O_3)$
$(t=2, j=2, i=1,2)$ $= \max[0.072 \times 0.4, 0.018 \times 0.8] \times 0.1 \Rightarrow 0.00288$

Since $\delta_1(2) > \delta_1(1)$ uninfected

This is because the value came from 2.

$\delta_2(1) > \delta_2(2)$ infected

This is because the highest value came from 1.

$\delta_3(1) > \delta_3(2)$ infected

This is because the highest value came from 1.

The most probable sequence will be uninfected, infected, infected.

2 HMM

a) The output sequence of length 2 that cannot be generated by the given HMM is 5, 4.

This is because if S_1 gets 5 ~~then~~ first then there is no way of getting 4 since the S_1 has probability of $P(x=4) \Rightarrow 0$. Also, it cannot go to S_2 since there is no transition probability.

Now, for ~~if~~ S_2 getting 5 first probability is $P(x=5) \Rightarrow 0$. ~~So~~ it cannot have 5 in first place.

So, this sequence cannot be obtained.

(b) 1st toss —

$$P(S1 \mid x=1) = \frac{P(S_1 \cap x=1)}{P(x=1)} = \frac{0.01 \times 0.5}{0.33 \times 0.5} = \frac{0.005}{0.33 \times 0.5} = 0.0303$$

$$P(S2 \mid x=1) = \frac{P(S2 \cap x=1)}{P(x=1)} = \frac{0.32 \times 0.5}{0.33 \times 0.5} = \frac{0.16}{0.33 \times 0.5} = 0.9696$$

$$P(x=1) = P(x=1 \mid S1)\,P(S1) + P(x=1 \mid S2)\,P(S2)$$

$$= 0.01 \times 0.5 + 0.32 \times 0.5$$

$$= 0.33 \times 0.5$$

∴ ~~After~~ 1st toss most likely by Player 2.

2nd toss —

$$P(x=1) = P(x=1 \mid S1)\,P(S1) + P(x=1 \mid S2)\,P(S2)$$

$$= 0.01 \times [\underset{S1\ 1st}{0.5} \times \underset{S1\ 2nd}{1} + \underset{S2\ 1st\ toss}{0.5} \times \underset{S1\ 2nd\ toss}{0.01}] + 0.32\,[\underset{S1}{0.5} \times \underset{S2}{0} + \underset{S2}{0.5} \times \underset{S2}{0.99}]$$

$$= 0.15505$$

$$P(S1 \mid x=1) = \frac{P(S1 \cap x=1)}{P(x=1)} = \frac{0.00505}{0.15505} = 0.032$$

$$P(S2 \mid x=1) = \frac{P(S2 \cap x=1)}{P(x=1)} = \frac{0.15}{0.15505} = 0.9674$$

2nd toss most likely by player 2

But if we look at the probability, player 2's decreases while player1's increases. And if the observation increases till $20{,}601^{2020}$, Player 2's probability of getting $x=1$ decreases while Player 1's increases. Hence, without knowing if the values converge or not, it is most likely that <u>S1</u> is the hidden state.

c) Viterbi : $\delta_1(i) = \pi_i \cdot b_i(O_1)$

$\delta_{t+1}(j) = \max_{1 \le i \le N} [\delta_t(i) \cdot a_{ij}] \cdot b_j(O_{t+1})$

Observation : (1,1)

$\delta_1(1) = \pi_1 \cdot b_1(O_1) \Rightarrow 0.5 \times 0.01 \Rightarrow 0.005$

$\delta_1(2) = \pi_2 \cdot b_2(O_1) \Rightarrow 0.5 \times 0.32 \Rightarrow 0.16$

$\delta_2(1) = \max[\delta_1(1) \cdot a_{11}, \delta_1(2) \cdot a_{21}] \cdot b_1(O_2)$
$(t=1, j=1, i=1,2)$
$= \max[0.005 \times 1, 0.16 \times 0.01] \times 0.01$
$\Rightarrow 0.00005$

$\delta_2(2) = \max[\delta_1(2) \cdot a_{22}, 0] \cdot b_2(O_2)$
$(t=1, j=2, i=1,2)$
$= 0.16 \times 0.99 \times 0.32 \Rightarrow 0.0506$

Since $\delta_1(2) > \delta_1(1)$ we select S_2. Since the high value came from S2.
& $\delta_2(2) > \delta_2(1)$ we select S_2 since again the value came from S_2.
The most likely sequence will be S_2, S_2 if this is the observation given.

d) Output sequence {1, 1, 3}

→ As we found in "c" for {1, 1} observation the states were S_2, S_2 hereby, in this as well it will be S_2, S_2 for first two states.

→ However, for 3 we have shown our calculations & ~~wt~~ the state which we get is below.

Using Viterbi

$\delta_3(1) = \max[\delta_2(1) \cdot a_{11}, \delta_2(2) \cdot a_{21}] \cdot b_1(O_3)$
$\left(\substack{t=2, j=1, \\ i=1,2}\right) = \max[0.00005 \times 1, 0.0506 \times 0.01] \times 0.3$
$= 0.00015$

$\delta_3(2) = \max[\delta_2(2) \cdot a_{22}, 0] \cdot b_2(O_3)$
$\left(\substack{t=2, j=2, \\ i=1,2}\right) = 0.0506 \times 0.99 \times 0.14$
$= 0.00701$

Since $\delta_3(2) > \delta_3(1)$ & the highest value came from S_2 we see S_2 as output state.

So, for the given output sequence {1, 1, 3} we see S_2, S_2, S_2.

Q3 (a) Optimal policy –

S1 : Right

S2 : Right

S3 : Right

S4 : Right

S5 : Left.

(b) Using the Bellman equation –

$$V_{S5} = 2 + 0.8\left(1 * V_{S4}\right) \text{ —— (1)}$$

V_{S4}: Right – $$V_{S4} = 20 + 0.8\left(1 * V_{S5}\right) \text{ —— (2)}.$$

Solving (1) & (2) –

$$0.8\,\cancel{V_{S4}} = V_{S5} - 2$$

$$-\left(\cancel{V_{S4}} = 0.8V_{S5} + 20\right)0.8$$

$$0 = 0.36\,V_{S5} - 18 \Rightarrow V^{*}_{S5} = \frac{18}{0.36} = 50.$$

$$\boxed{V^{*}_{S5} = 50}$$

(c) $\alpha = 0.5$

Start from S3

	L	R
S1		0.9
S2	1	
S3	1	
S4		
S5		

$$Q(S3, L) = (1-\alpha)\overset{0}{\cancel{Q(S3,L)}} + \underset{0.5}{\alpha}\Big[\underset{2}{r(S3,L)} + \underset{0}{\cancel{\gamma Q(S2)}}\Big]$$

$$= 1$$

$$Q(S2, L) = (1-\alpha)\overset{0}{\cancel{Q(S2,L)}} + 0.5\big[2 + \overset{0}{\cancel{\gamma Q(S1)}}\big]$$

$$= 1$$

$$Q(S1, R) = (1-\alpha)\overset{0}{\cancel{Q(S1,R)}} + 0.5\Big[1 + 0.8 \max\big[\underset{0}{Q(S2,R)}, \underset{1}{Q(S2,L)}\big]\Big]$$

$$= 0.5\big[1 + 0.8 \times 1\big] = 0.9$$

$$Q(S2, L) = (1-\alpha)Q(S2,L) + 0.5\big[2 + 0.8 \times 0.9\big]$$

$$= 0.5 \times 1 + 0.5(2 + 0.72)$$

gets updated.

10 pairs

(S3, L), (S2, L), (S1, R), (S2, L), (S1, R), (S2, L), (S1, R), (S2, L), (S1, R), (S2, L)

(d)

	V_{S1}	V_{S2}	V_{S3}	V_{S4}	V_{S5}
t	20	30	20	30	10
$t+1$	25	21.2	23.6	36	26

$Q(S1, R) = 1 + 0.8(1 \times 30) = 25$ R

$Q(S2, R) = 1 + 0.8(0.7 \times 20 + 0.3 \times 30) = 19.4$

$Q(S2, L) = 2 + 0.8(0.4 \times 30 + 0.6 \times 20) = 21.2$

$\Big\}$ 21.2 L

$Q(S3, R) = 2 + 0.8(0.8 \times 20 + 0.2 \times 30) = 19.6$

$Q(S3, L) = 2 + 0.8(0.7 \times 30 + 0.3 \times 20) = 23.6$

$\Big\}$ 23.6 L

$Q(S4, R) = 20 + 0.8(0.5 \times 30 + 0.5 \times 10) = 36$

$Q(S4, L) = 3 + 0.8(0.6 \times 20 + 0.4 \times 30) = 22.2$

$\Big\}$ 36 R

$Q(S5, L) = 2 + 0.8(1 \times 30) = 2 + 24 = 26$ - L

(e) (1) Based on the value function at time t —

The values of Q for each state is from part (d)

$Q(S_1, R) = 25$ — S1: Right

$Q(S2, R) = 19.4$
$Q(S2, L) = 21.2$ } 21.2 — S2: Left.

$Q(S3, R) = 19.6$
$Q(S3, L) = 23.6$ } 23.6 — S3: Left.

$Q(S4, R) = 36$
$Q(S4, L) = 22.2$ } 36 — S4: Right.

$Q(S5, L) = 26$ — S5: Left.

⇒ The optimal policy is —

(S1: Right), (S2: Left), (S3: Left), (S4: Right), (S5: Left).

(2) Based on value function at time t+1 —

Q(S1,R)= 1 + 0.8(1*21.2)=17.96

$Q(S_1, R) = 17.96$ — Right

$Q(S2, R) = 19.3$
$Q(S2, L) = 20.78$ } 20.78 — Left.

Q(S2,R) = 1+0.8(0.3*21.2+0.7*23.6)=19.3

$Q(S3, R) = 22.86$
$Q(S3, L) = 19.5$ } 22.86 — Right.

Similarly done for rest states.

$Q(S4, R) = 44.8$
$Q(S4, L) = 25.8$ } 44.8 — Right

$Q(S5, L) = 30.8$ — Left.

All Q values are calculated at t+1

⇒ The optimal policy is —

(S1: Right), (S2: Left), (S3: Right), (S4: Right), (S5: Left)

(3) Yes, the two policies are different.